CAS Exam MAS-1

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Stochastic Models

The slides are in the same order as the sections of my study guide.

Section #	Section Name
1	Introduction
2	Exponential Distribution
3	Homogeneous Poisson Processes
4	Interevent Times, Poisson Processes
5	Thinning & Adding Poisson Processes
6	Mixing Poisson Processes
7	Negative Binomial Distribution
8	Gamma Distribution
9	Gamma-Poisson
10	Compound Poisson Processes
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14	Nonhomogeneous Poisson Processes
15	Interevent Times, Nonhomogeneous Poisson Processes
16	Thinning & Adding Nonhomogeneous Poisson Processes
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18	Homogeneous Markov Chains
19	Cashflows While in States, Markov Chains
20	Cashflows, Transitioning States, Markov Chains
21	Stationary Distributions of Markov Chains
22	Classification of States of Markov Chains
23	Expected Time Spent in Transient States
24	Limiting Distribution, Markov Chains
25	Random Walks
26	Gambler's Ruin
27	Branching Processes
28	Time Reversible Markov Chains

At the end, there are some additional questions for study.
After the slides is my section of important ideas and formulas.

Introduction

A **stochastic process {X(t), t ∈ T} is a collection of random variables.**

There are four common types of stochastic processes; the first two are on this exam:

1. Poisson Processes, including the nonhomogeneous and compound cases.
2. Markov Chains.
3. Markov Processes.
4. Brownian Motion.

Poisson Distribution

Support: x = 0, 1, 2, 3...

Parameters: $\lambda > 0$

$$\mathbf{f(x) = \frac{\lambda^x e^{-\lambda}}{x!}}$$

Mean = λ Variance = λ

The sum of two independent Poisson Distributions is another Poisson Distribution,
with the sum of the means.

$f(x) \Leftrightarrow p_k$.

Exponential Distribution

Support: $x > 0$ Parameters: $\theta > 0$

$F(x) = 1 - e^{-x/\theta}$. **$f(x) = e^{-x/\theta} / \theta$.**

Mean = θ. **Variance = θ^2.**

Moments: $E[X^n] = n!\ \theta^n$ second moment = $2\theta^2$.

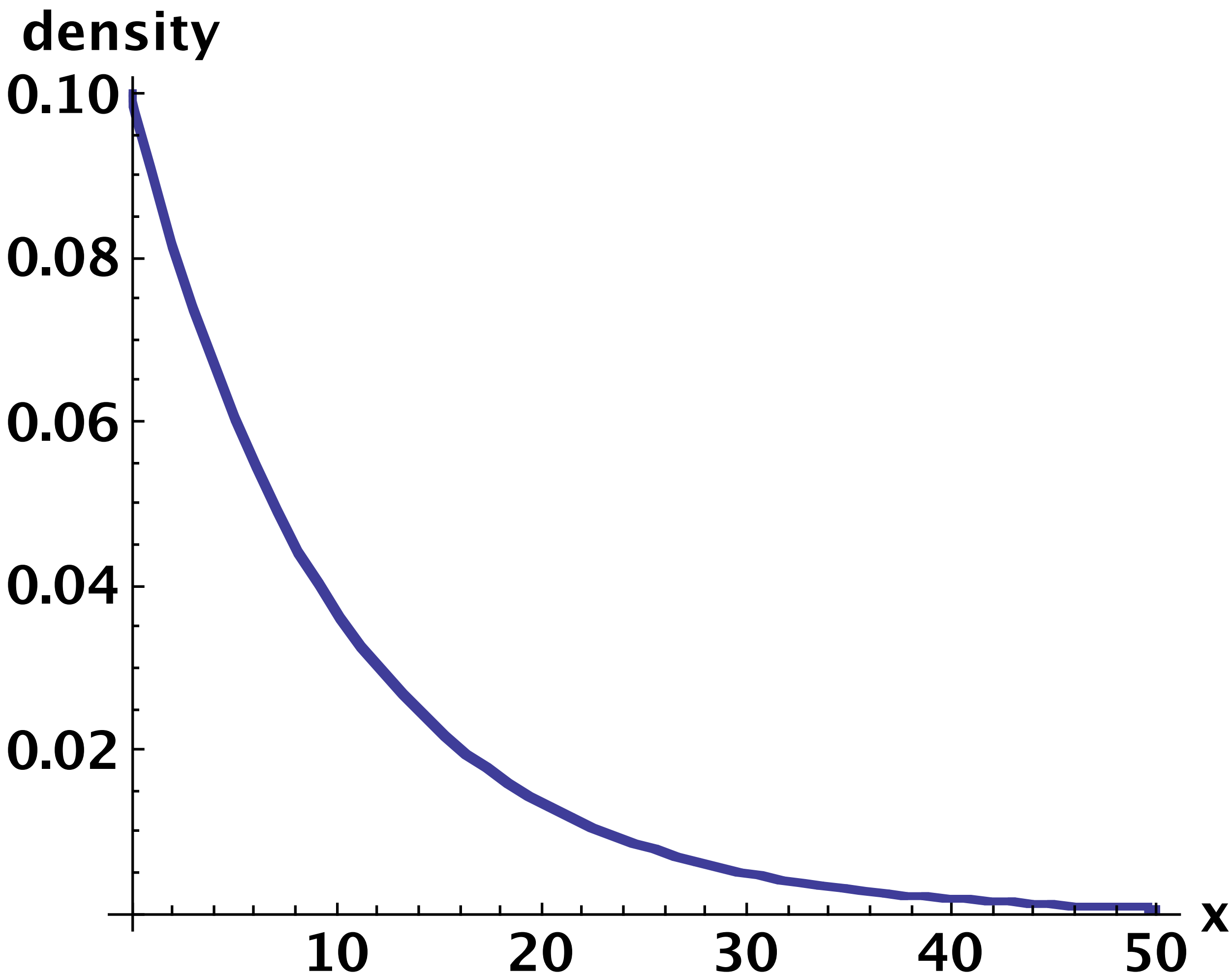
$F(x) = 1 - e^{-\lambda x}$.

$\lambda = 1/\theta$.

λ = hazard rate = force of mortality.

$e(x) = \theta$ constant mean residual life

An Exponential Distribution with mean 10:



Mode = 0

Let X be an exponentially distributed random variable, the probability density function of which is: $f(x) = 8\ e^{-8x}$, $x \geq 0$

2.1 (1 point) What is the mean of X?

2.4 (1 point)
What is the chance that X is greater than 0.3?

2.5 (1 point) What is the variance of X?

2.1. An exponential with $\theta = 1/8$;
mean = θ = **0.125**.

2.4. An exponential with $1/\theta = 8$;
$F(x) = 1 - e^{-x/\theta} = 1 - e^{-8x}$.
$1 - F(0.3) = e^{-(8)(0.3)} = e^{-2.4}$ = **0.0907**.

2.5. An exponential with $1/\theta = 8$;
variance = θ^2 = **0.015625**.

Page 22 Failure Rate:

The failure rate, also called the hazard rate or the force of mortality, is: r(x) = f(x) / S(x).

For the Exponential Distribution with mean θ, the failure rate is: $(e^{-x/\theta}/\theta) / e^{-x/\theta} = 1/\theta$.

The Exponential Distribution has a constant failure rate: λ = 1/mean.

P.23 **Memoryless Property**:

A random variable is memoryless if
Prob[X > s + t | X > t] = Prob[X > s].

The Exponential Distribution is memoryless.

Exercise: Your time waiting to hail a cab is Exponentially distributed with mean 3 minutes. What is the probability you need to wait more than 5 minutes?

Solution: $S(5) = e^{-5/3} = 18.9\%$.

Exercise: If you have already waited 4 minutes, what is the probability you need to wait more than an additional 5 minutes?

Due to the memoryless property,
if you have already waited 4 minutes,
that does not affect your future waiting time,
so there is an 18.9% chance of waiting more than an additional 5 minutes.

The mean residual life is defined as:
e(d) = E[X - d | X > d].
Thus for a life aged 60, the mean residual life is the expected future lifetime.

The Exponential Distribution has a constant Mean Residual Life.

2.21 (1 point) The exponential distribution with a mean of 28,700 hours was used to describe the hours to failure of a fan on diesel engines.
A diesel engine fan has gone 10,000 hours without failing.
Determine the probability of this fan lasting at least an additional 5000 hours.

2.21. Due to its memoryless property, the future lifetime follows the original Exponential.

$S(5000) = e^{-5000/28,700}$ = **84.0%**.

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If the ground-up losses follow
an Exponential,
then with a deductible
the non-zero payments follow
the same Exponential Distribution.

Mixtures of Exponentials:

Let us assume that the size of each claims has a 60% chance of being a random draw from an Exponential Distribution with mean 100 and a 40% chance of being a random draw from an Exponential Distribution with mean 200, independent of the size of any other claim.
An example of a mixture of two Exponentials.

Exercise: For this mixture, what is the probability that a claim is of size less than 150?

For the first Exponential:

$F(150) = 1 - e^{-150/100} = 77.69\%$.

For the second Exponential:

$F(150) = 1 - e^{-150/200} = 52.76\%$.

For the mixture, F(150) =

(60%)(77.69%) + (40%)(52.76%) = 67.72%.

The Distribution Function of the mixture is the mixture of the Distribution Functions.

Exercise: For this mixture, what is the mean?
[Solution: (60%)(100) + (40%)(200) = 160.]

The mean of the mixture is
the mixture of the means.

**The n^{th} moment of the mixture is
the mixture of the n^{th} moments**.

The sum of n independent, identically Exponential Distributions each with mean θ, is a Gamma Distribution with parameters α = n and θ.

Exercise: What is the distribution of the sum of four independent, identically distributed Exponentials each with hazard rate 0.1?

Solution: A Gamma Distribution with $\alpha = 4$ and $\theta = 1/0.1 = 10$.

Comparing Independent Exponentials:

If X_1 and X_2 are independent Exponentials with failure rates λ_1 and λ_2,
then the probability that $X_1 < X_2$ is: $\frac{\lambda_1}{\lambda_1 + \lambda_2}$.

Exercise: Claims from theft are Exponentially Distributed with mean 1000.
Claims from vandalism are Exponentially Distributed with mean 500.
Determine the probability that a random vandalism claim is bigger than a random theft claim.

$$\text{Prob}[X_1 < X_2] = \frac{\lambda_1}{\lambda_1 + \lambda_2} = \frac{1/1000}{1/1000 + 1/500}$$

$$= \frac{500}{500 + 1000} = 1/3.$$

If one has n independent Exponentials with failure rates λ, then the minimum of a sample consisting of a random draw from each Exponential is another Exponential with the sum of the failure rates.

2.37 (2 points) Bob has two friends Howard and Jerry, who are stopping by to visit today.
The time until Howard arrives is Exponential with mean 30 minutes.
The length of Howard's visit is Exponential with mean 70 minutes.
The time until Jerry arrives is Exponential with mean 50 minutes.
The length of Jerry's visit is Exponential with mean 80 minutes.
Determine the probability that Howard arrives first and leaves last.
A. 9% B. 11% C. 13% D. 15% E. 17%

2.37. E. The probability that Howard arrives first is: 50 / (30 + 50) = 5/8.

Given that Howard shows up first, the probability that Jerry shows up before Howard leaves is: 70 / (50 + 70) = 7/12.

Given that both friends are there, the probability that Jerry leaves first is: 70 / (70 + 80) = 7/15.

Thus the probability that Howard arrives first and leaves last is: (5/8) (7/12) (7/15) = **17.0%**.

Section 3, Homogeneous Poisson Processes

counting process ⇔ claims frequency process.

$N(t) \geq 0$

N(t) is an integer

If $s < t$, then $N(s) \leq N(t)$.

For $s < t$, $N(t) - N(s)$ is the number of events in the interval (s, t].

The increment over time interval s to t is the change in the process over that time interval: $N(t) - N(s)$.

Independent increments ⇔
increments over disjoint intervals are independent.
(touching at endpoints is OK.)

The Poisson Process is a counting process with independent increments.

A homogeneous **Poisson Process has a constant claims intensity** λ.

The number of claims observed in time interval (t, t + h) is given by the Poisson Distribution with mean h λ**.**

3.39. CAS3, 5/05, Q.39 Longterm Insurance Company insures 100,000 drivers who have each been driving for at least five years.

Each driver gets "violations" at a Poisson rate of 0.5/year.

Currently, drivers with 1 or more violations in the past three years pay a premium of 1000.
Drivers with 0 violations in the past three years pay 850.

Your marketing department wants to change the pricing so that drivers with 2 or more violations in the past five years pay 1,000 and drivers with zero or one violations in the past five years pay X.

Find X so that the total premium revenue for your firm remains constant when this change is made.

CAS3, 5/05, Q.39. A. The number of violations over three years is Poisson with mean 1.5.

Prob[0 in 3 years] = $e^{-1.5}$ = 0.2231.

⇒ Current Average Premium is:
(0.2231) (850) + (1 - 0.2231) (1000) = 966.54.

The number of violations over five years is Poisson with mean 2.5.

Prob[0 or 1 in 5 years] = $e^{-2.5} + 2.5 e^{-2.5}$ = 0.2873.

0.2873 X + (1 - 0.2873) (1000) = 966.54.
⇒ X = **883.5**.

<u>Comment</u>: The total number of drivers is not used.

For a tyrannosaur with 10,000 calories stored:
(i) The tyrannosaur uses calories uniformly at a rate of 10,000 per day. If his stored calories reach 0, he dies.
(ii) The tyrannosaur eats scientists (10,000 calories each) at a Poisson rate of 1 per day.
(iii) The tyrannosaur eats only scientists.
(iv) The tyrannosaur can store calories without limit until needed.

3.31. 3, 11/01, Q.10. Calculate the probability that the tyrannosaur dies within the next 2.5 days.

3.32. 3, 11/01, Q.11. Calculate the expected calories eaten in the next 2.5 days.

3, 11/01, Q.10. C. If the tyrannosaur does not eat a scientist within the first day it exhausts its initial store of calories and dies.

Prob[death @ t = 1] =

Prob[0 scientists by time = 1] = e^{-1}.

If the tyrannosaur survives at time = 1, it dies at t = 2 if it has not eaten at least a total of 2 scientists by time 2. This requires that the tyrannosaur eats 1 scientist from time 0 to 1, and no scientists from time 1 to 2.

Prob[death @ t = 2] =

Prob[1 sc. for $0 < t \le 1$] Prob[0 sc. for $1 < t \le 2$] =

$(1\ e^{-1})\ (e^{-1}) = e^{-2}$.

If the tyrannosaur survives to t = 2 it will survive to t = 2.5.

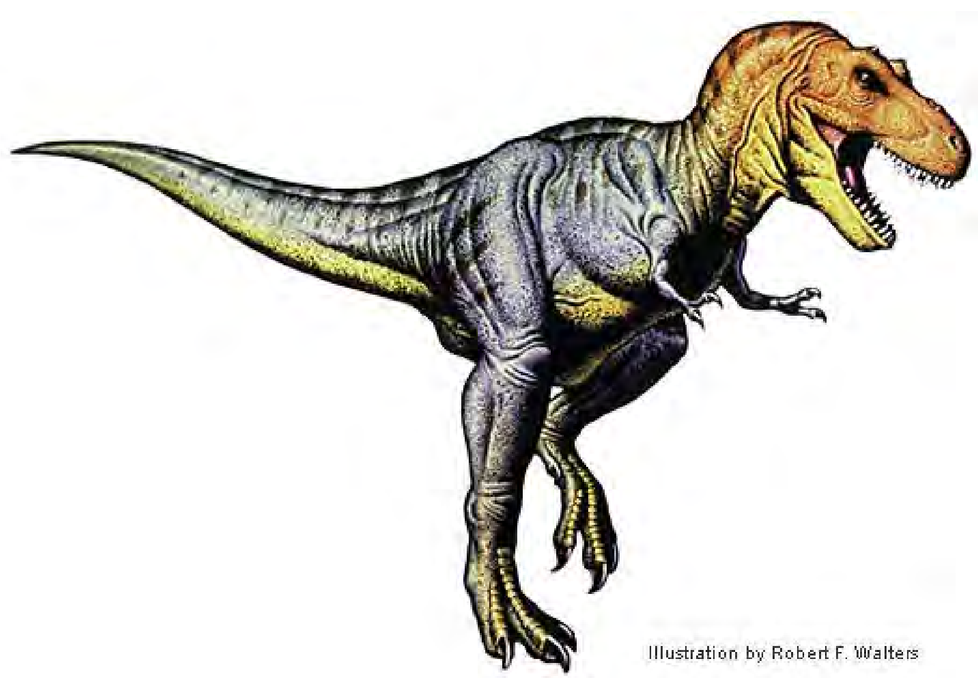
The probability that the tyrannosaur dies within the next 2.5 days is: $e^{-1} + e^{-2}$ = **0.503**.

3, 11/01, Q.11. B. A living tyrannosaur is expected to eat 1 scientist or 10,000 calories per day.
Day 1: 10,000.

Day 2: $(10{,}000)(1 - e^{-1})$.

First half of Day 3: $(10{,}000/2)(1 - e^{-1} - e^{-2})$.
Expected calories:

$10{,}000\{1 + 1 - e^{-1} + (1 - e^{-1} - e^{-2})/2\}$ = **18,805**.
Comment: Over the first 2.5 days, a tyrannosaur survives an average of 1.8805 days.



Section 4, Interevent Times and Waiting Times

What is the probability that the waiting time until the first claim is less than or equal to 10 for a Poisson Process with $\lambda = 0.03$?

The number of claims by time 10 is Poisson with mean: (10) (0.03) = 0.3.
Waiting time is ≤ 10 ⇔ At least 1 claim by t = 10.
Prob[0 claims by time 10] = $e^{-0.3}$.
Prob[≥ 1 claim by time 10] = 1 - $e^{-0.3}$ = 25.9%.

The distribution function of the waiting time until the first claim is: $F(t) = 1 - e^{-0.03t}$,
an Exponential Distribution with mean
$1/0.03 = 1/\lambda$.

Waiting time to first claim has an Exponential Distribution with mean $1/\lambda$, and hazard rate λ:

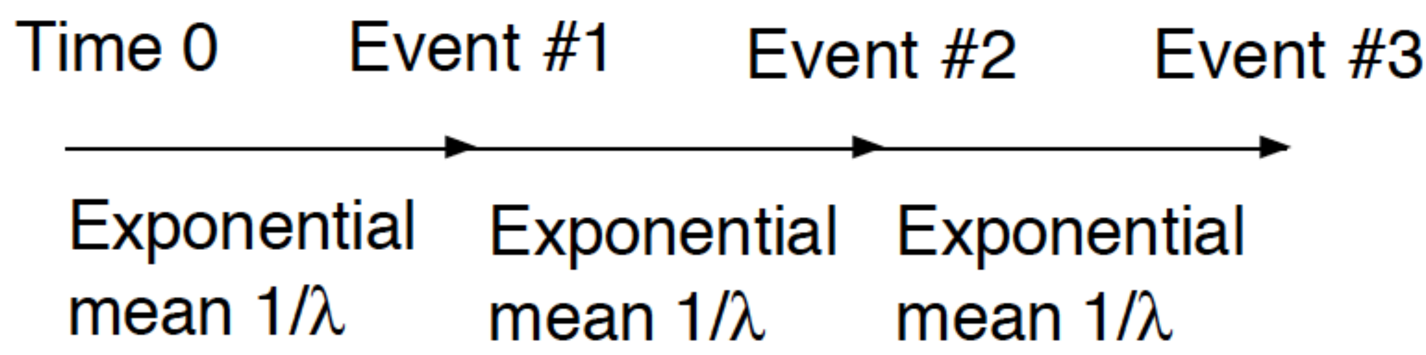
$F(t) = 1 - e^{-\lambda t}$.

V_j = time from the j-1th event to the jth event.

Due to constant independent increments,
the interevent time between
the 1st and 2nd claims
has the same distribution
as the waiting time until the first claim.

(We can start a new Poisson Process
when the 1st claim arrives;
there is no memory.)

Interevent times are independent Exponential Distributions each with mean $1/\lambda$.



T_3 = time of the 3rd event = $V_1 + V_2 + V_3$.

T_n = time of the nth event = $V_1 + V_2 + ... + V_n$.

The Poisson Process is a random process.

Fewer claims showing up. ⇔ Event time is larger.

More claims showing up. ⇔ Event time is smaller.

Assume we have a Poisson Process with $\lambda = 0.03$. What is the probability that we have observed at least 4 claims by time 100?

N(100) is Poisson with mean (0.03) (100) = 3.
Prob[N(100) ≥ 4] = 1 - Prob[N(100) < 4] =
$1 - \{e^{-3} + 3e^{-3} + 3^2 e^{-3}/2 + 3^3 e^{-3}/6\} = 0.353$.

$T_k \leq t. \Leftrightarrow N(t) \geq k$.

Prob[$T_k \leq t$] = Prob[$N(t) \geq k$].

Prob[$T_4 \leq 100$] = Prob[N(100) ≥ 4] = 0.353.

Fourth event time less than or equal to 100.
⇔ At least 4 claims by time 100.

4.5.

A Poisson Process has a claims intensity of 0.05. What is the probability that the time from the eighth claim to the tenth claim is greater than 50?

4.5. B. Over a time period of length 50,
the number of claims is Poisson with mean:
(0.05) (50) = 2.5.

Prob[time from the 8th claim to the 10th > 50] =
Prob[0 or 1 claim in time period of length 50] =
Prob[0 claims from a Poisson with mean of 2.5] +
Prob[1 claim from a Poisson with mean of 2.5]
= $e^{-2.5} + 2.5\, e^{-2.5}$ = **0.287**.

0 or 1 claim →

time of 8th claim — 50 + time of 8th claim

For example, assume the 8th occurs at time 200.
Then does the 10th occur by: 200 + 50 = 250?

Number of claims between times 200 and 250	Does the 10th claim occur by time 250?
0	No
1	No
2 or more	Yes

Comment: Due to the memoryless property, one can just start a new Poisson Process whenever the 8^{th} claim shows up.
Then the given question is equivalent to:

A Poisson Process has a claims intensity of 0.05. What is the probability that the time until the second claim is greater than 50?

Sec. 5, Thinning and Adding Poisson Processes

If we select at random a fraction of the claims from a Poisson Process, we get a new Poisson Process, with smaller claims intensity.

This is called **thinning a Poisson Process**.

λ = 10. Frequency and severity are independent.
30% of claims are of size less than $10,000,
50% of claims are of size
between $10,000 and $25,000,
and 20% of the claims are of size
greater than $25,000.

The frequencies for the claims of different sizes are three independent Poisson Processes.

Claims of a size less than $10,000,
have a claims intensity of 3.
Claims of size between $10,000 and $25,000,
have a claims intensity of 5.
Claims of size greater than $25,000,
have a claims intensity of 2.

5.98. 3, 5/00, Q.2.

Lucky Tom finds coins on his way to work at a Poisson rate of 0.5 coins / minute.

The denominations are randomly distributed:

(i) 60% of the coins are worth 1;

(ii) 20% of the coins are worth 5; and

(iii) 20% of the coins are worth 10.

Calculate the conditional expected value of the coins Tom found during his one-hour walk today, given that among the coins he found exactly ten were worth 5 each.

3, 5/00, Q.2. C.

The finding of the three different types of coins are independent Poisson processes.

The number of pennies or dimes
expected to be found
does not depend on
the number of nickels Tom found.

Over the course of 60 minutes,
Tom expects to find:
(0.6) (0.5) (60) = 18 coins worth 1 each and
(0.2) (0.5) (60) = 6 coins worth 10 each.

If Tom finds 10 coins worth 5 each,
then the expected worth of the coins he finds is:
(18) (1) + (10) (5) + (6) (10) = **128**.

5.102. 3, 11/02, Q.9.

Lucky Tom finds coins on his way to work at a Poisson rate of 0.5 coins/minute.
The denominations are randomly distributed:
(i) 60% of the coins are worth 1 each
(ii) 20% of the coins are worth 5 each
(iii) 20% of the coins are worth 10 each.
Calculate the probability that in the first ten minutes of his walk he finds at least 2 coins worth 10 each, and in the first twenty minutes finds at least 3 coins worth 10 each.

(A) 0.08 (B) 0.12 (C) 0.16 (D) 0.20 (E) 0.24

3, 11/02, Q.9. D. Coins of worth 10 are found via a Poisson Process with mean (20%)(0.5) = 0.1 per minute or 1 per 10 minutes.

Prob[≥ 2 in 1st 10 min. & ≥ 3 in 1st 20 min.] =

Prob[2 in 1st 10 min.] Prob[≥ 1 in next 10 min.] + Prob[at least 3 in the first 10 minutes] =

$(1^2\ e^{-1}/2)\ (1 - e^{-1}) + (1 - e^{-1} - 1\ e^{-1} - 1^2\ e^{-1}/2) =$ **0.197**.

First 10 minutes	Next 10 minutes
0	No Good
1	No Good
2	Need at least 1
3	OK
4	OK
etc.	OK

(0 ,10) overlaps with (0, 20).
(0, 10) is disjoint from (10, 20).

Alternately, Tom <u>fails</u> if either:
0 or 1 dimes in the first 10 minutes, or
2 dimes in first 10 minutes and none in the next 10.

Thus the probability that Tom fails is:
$(e^{-1} + 1\ e^{-1}) + (1^2\ e^{-1}/2)(e^{-1}) = 0.803$.

Thus the desired probability that Tom succeeds is:
1 - 0.803 = **0.197**.

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If one adds two independent Poisson Processes, one gets a new Poisson Process, with claims intensity the sum of the two individual claims intensities.

See for example, ST, 5/14, Q.2.

Claims from illness are a Poisson Process with claims intensity 13.
Claims from accident are a Poisson Process with claims intensity 7.
The two processes are independent.
What is the probability of at least 3 claims by time 0.1?

Claims are a Poisson Process with:
$\lambda = 13 + 7 = 20$.
Thus the number of claims by time 0.1 is Poisson with mean 2.

The probability of at least 3 claims is:
$1 - \{e^{-2} + 2e^{-2} + 2^2e^{-2}/2\} = 0.323$.

Section 6, Mixing Poisson Processes

Assume there are two types of risks,
each with claims given by a Poisson Process:

Type	Annual Rate	Probability
A	0.3	60%
B	0.5	40%

For an insured of unknown type, what is the probability of one claim next year?

For type A, $f(1) = 0.3\ e^{-0.3} = 0.222$.

For type B, $f(1) = 0.5\ e^{-0.5} = 0.303$.

$(60\%)(0.222) + (40\%)(0.303) = 0.254$.

The density function of
the mixed distribution,
is the mixture of the density function for specific values of the parameter that is mixed.

The mean of a mixed distribution is the mixture of the means.

Mean = (60%) (0.3) + (40%) (0.5) = 0.38.

The second moment of a mixed distribution is the mixture of the second moments.

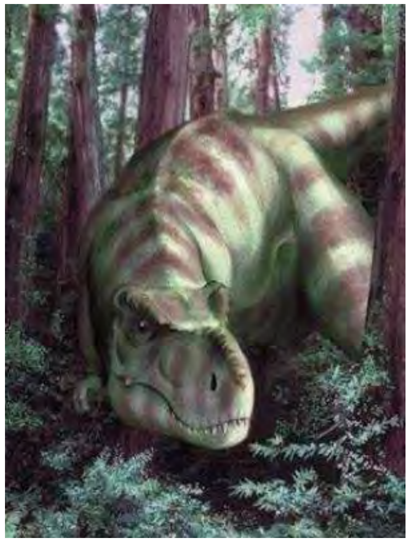
For type A, the second moment is:
variance + mean^2 = 0.3 + 0.3^2 = 0.39.

For type B, the second moment is:
variance + mean^2 = 0.5 + 0.5^2 = 0.75.

Therefore the second moment of the mixture is:
(60%) (0.39) + (40%) (0.75) = 0.534.

Variance of this mixed distribution is:
0.534 - 0.38^2 = 0.3896.

The nth moment of a mixture is the mixture of the nth moments.



6.14. A tyrannosaur eats only scientists. It eats scientists at a Poisson rate that varies by day:
On a warm day the tyrannosaur eats an average of 1.5 scientists.
On a cold day the tyrannosaur eats an average of 0.5 scientists.
On a future day that is equally likely to be warm or cold, what is the probability that the tyrannosaur eats exactly two scientists?

6.14. C.

Prob[2 scientists | warm] = $1.5^2 e^{-1.5} / 2$ = 25.10%.

Prob[2 scientists | cold] = $0.5^2 e^{-0.5} / 2$ = 7.58%.

(50%) (25.10%) + (50%) (7.58%) = **16.3%**.

Comment: The mean of the mixed distribution is: (1.5 + 0.5)/2 = 1.
However, $1^2 e^{-1} / 2$ = 18.4% ≠ 16.3%.
One should average the Poisson densities at 2 for the different types of days.

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Mixing involves different types of insureds or risks.

For example: Good and Bad, each of which follows a Poisson Process, but with $\lambda = 0.03$ for Good insureds and $\lambda = 0.07$ for Bad insureds.

In order to simulate this mixture:
First simulate the type of insured.
Then simulate how many claims that insured had and when they occurred.

Thinning involves different types of claims or events.

For example, Lucky Tom finds coins at a Poisson rate. The dimes, nickels, and pennies are three independent Poisson processes.

In order to simulate this example of thinning:
First simulate when the coin is found.
Then simulate the type of coin.

Section 7, Negative Binomial Distribution

$$\mathbf{f(x) = \frac{r\,(r+1)\ldots(r+x-1)}{x!}\;\frac{\beta^x}{(1+\beta)^{r+x}}}$$

$$\mathbf{= \binom{r+x-1}{x}\frac{\beta^x}{(1+\beta)^{r+x}}.}$$

$$f(0) = \frac{1}{(1+\beta)^r}.$$

$$f(1) = \frac{r\,\beta}{(1+\beta)^{r+1}}.$$

$$f(2) = \frac{r\,(r+1)}{2}\;\frac{\beta^2}{(1+\beta)^{r+2}}.$$

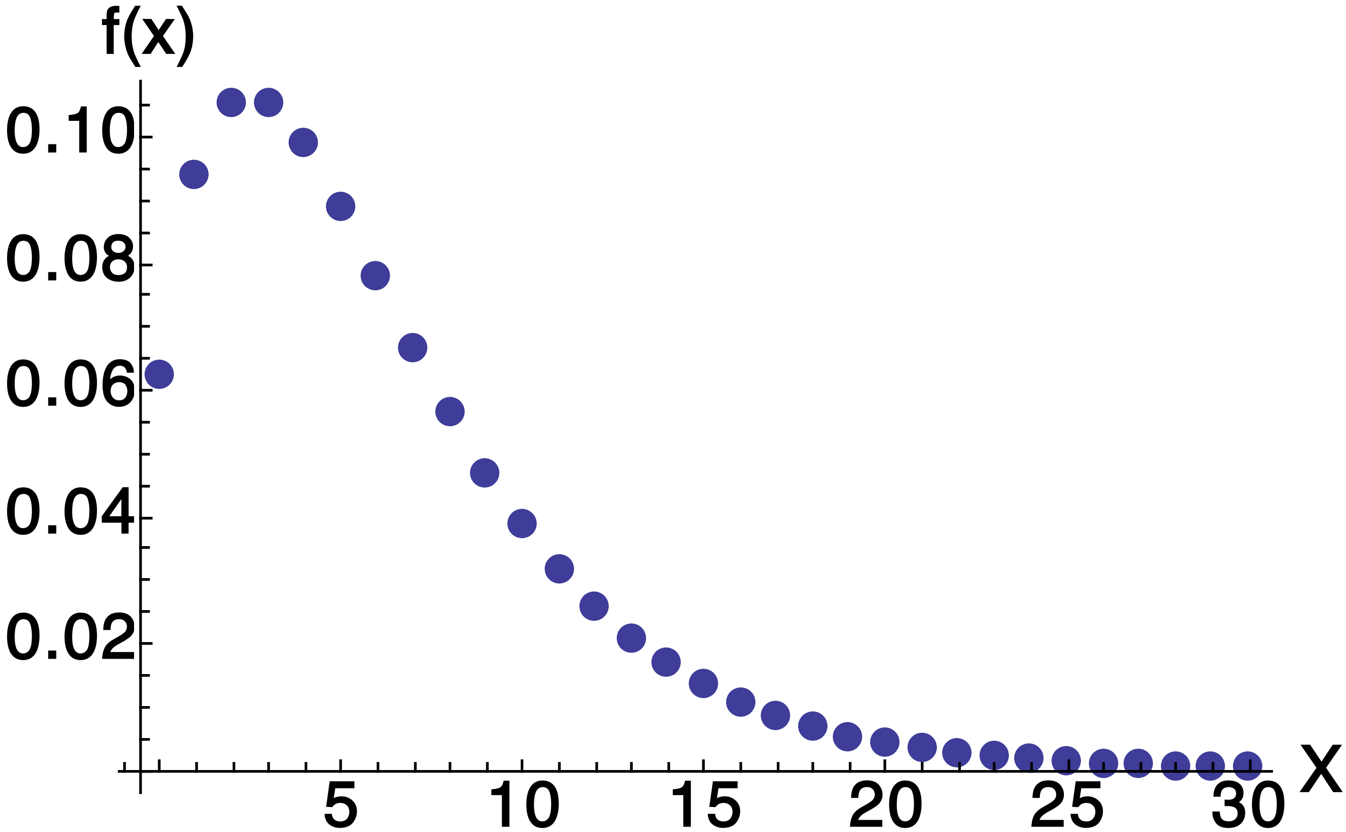
$$f(3) = \frac{r\,(r+1)\,(r+2)}{6}\;\frac{\beta^3}{(1+\beta)^{r+3}}$$

Mean = $r\beta$

Variance = $r\,\beta(1+\beta)$ > Mean.

Negative Binomial Distrib. with r = 2 and β = 4:

$$f(x) = \frac{(2)(3)\ldots(x+1)}{x!} \frac{4^x}{5^{x+2}} = (x+1)\,(0.04)\,(0.8^x):$$



A Negative Binomial with r = 1 is a Geometric.

7.5. For a Negative Binomial distribution with $\beta = 0.8$ and $r = 1.4$, what is the density at 4?

$$f(x) = \frac{r\,(r+1)\ \ldots\ (r+x-1)}{x!}\ \frac{\beta^x}{(1+\beta)^{r+x}}.$$

7.5. B. $f(x) = \frac{r (r+1) \ldots (r+x-1)}{x!} \frac{\beta^x}{(1+\beta)^{r+x}}$.

$$f(4) = \frac{r (r+1) (r+2) (r+3)}{4!} \frac{\beta^4}{(1 + \beta)^{r+4}}$$

$$= \frac{(1.4)(2.4)(3.4)(4.4)}{24} \frac{0.8^4}{1.8^{5.4}} = \mathbf{3.6\%}.$$

Section 8, Gamma Distribution

A continuous distribution with two parameters alpha and theta.

Mean = $\alpha\,\theta$ **Variance = $\alpha\,\theta^2$**

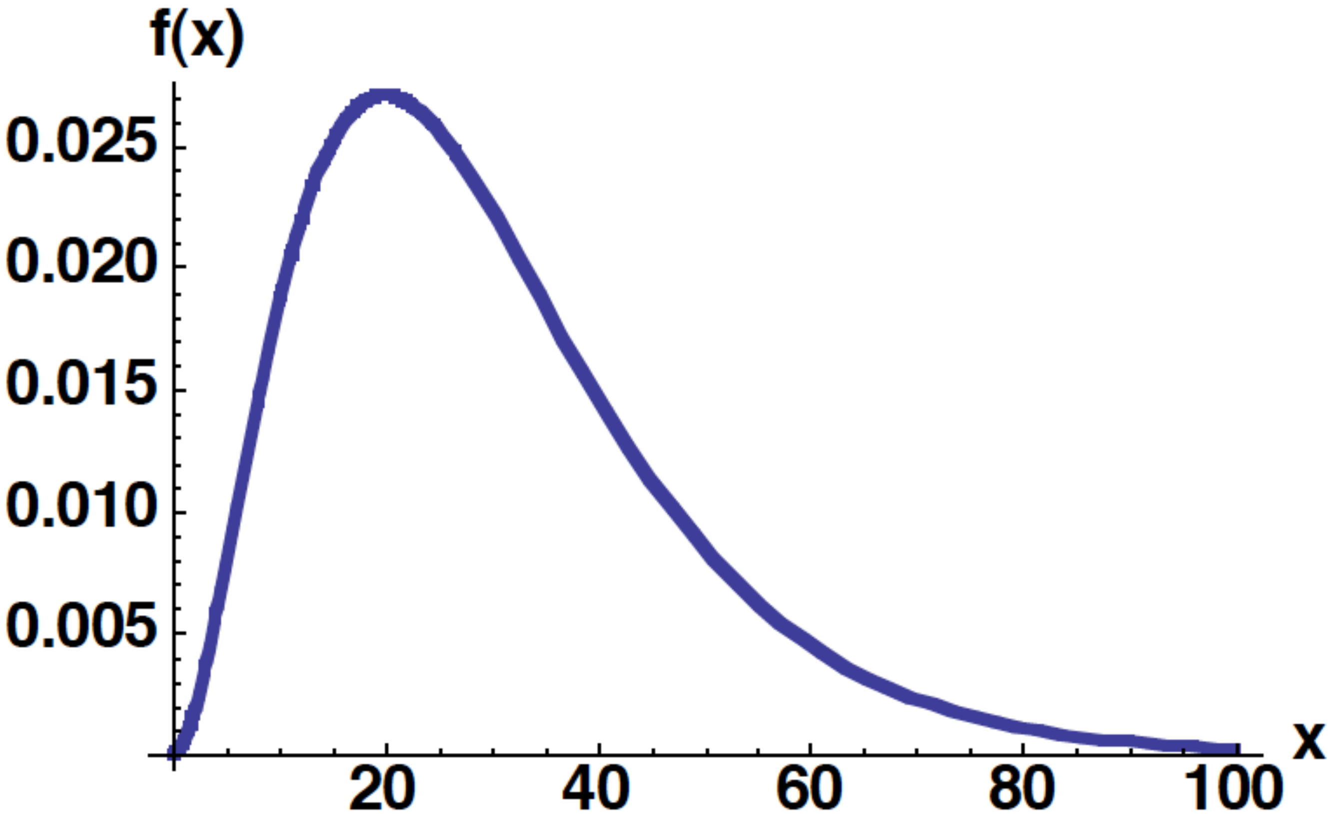
$$f(x) = \frac{x^{\alpha-1}\, e^{-x/\theta}}{\Gamma(\alpha)\,\theta^{\alpha}},\ x > 0.$$

where for alpha integer, $\Gamma(\alpha) = (\alpha-1)!$.

$\Gamma(5) = 4! = 24$.

For $\alpha = 1$ the Gamma Distribution is the Exponential Distribution.

The density function of a Gamma Distribution with $\alpha = 3$ and $\theta = 10$:

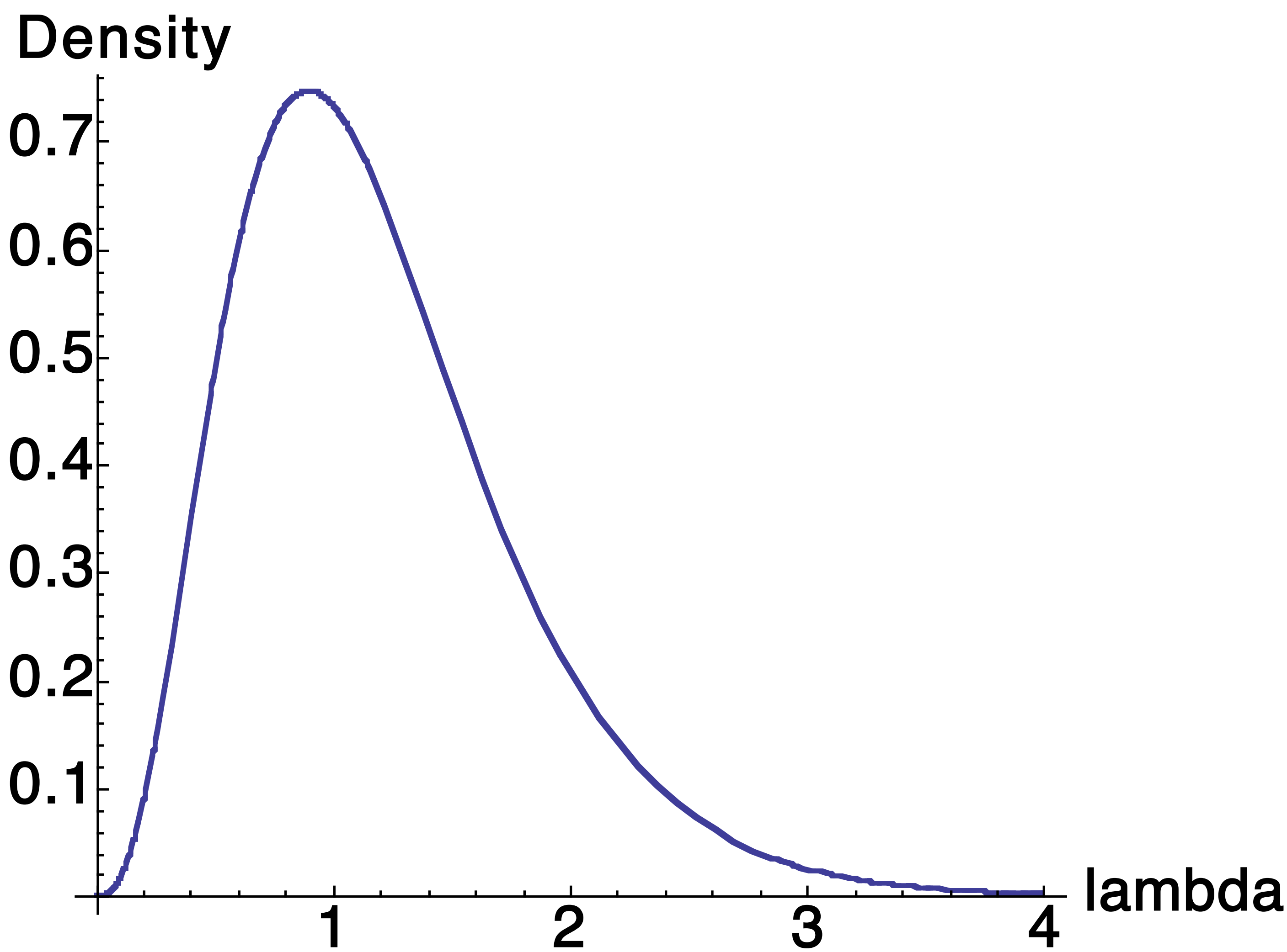


Section 9, Gamma-Poisson

The number of claims
for a particular policyholder
is a homogenous Poisson Process
with rate λ.

The λ **values over the portfolio of policyholders are Gamma distributed**.

Given the value of λ, N is a homogeneous Poisson process with rate λ, but λ is itself random, with a Gamma Distribution with $\alpha = 4$ and $\theta = 0.3$:

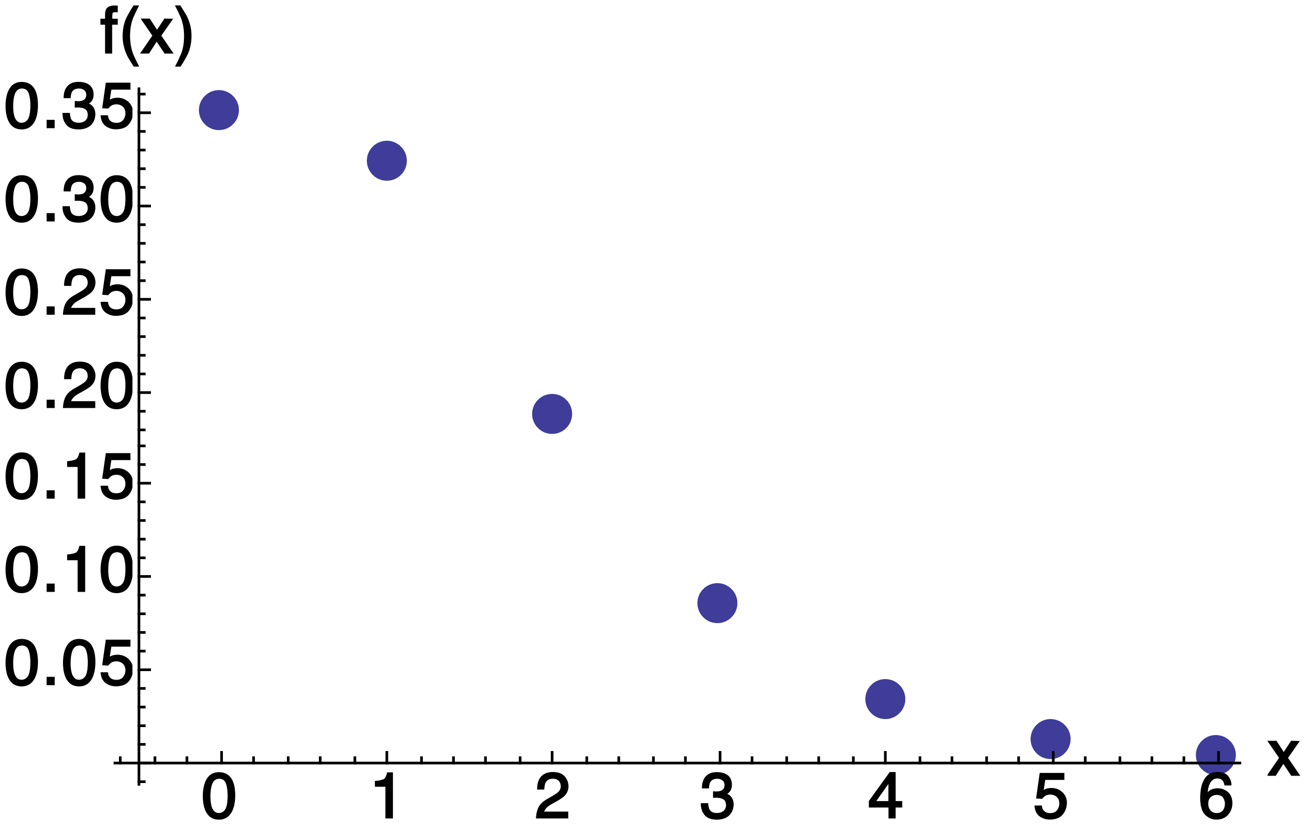


Mean of this Gamma is: (4) (0.3) = 1.2.

Variance of this Gamma: (4) (0.3^2) = 0.36.

Over a period of length one,
for a homogenous Poisson process
mixed by a Gamma Distribution,
the mixed distribution is Negative Binomial
with $r = \alpha$ and $\beta = \theta$.

The distribution of the number of claims next year from an insured picked at random is Negative Binomial, with r = α = 4 and β = θ = 0.3:



The distribution of the number of claims next year from an insured picked at random is Negative Binomial, with r = α = 4 and β = θ = 0.3.

An insured is picked at random. The probability of 2 claims from this insured next year is:

$$\frac{r\,(r+1)}{2}\ \frac{\beta^2}{(1+\beta)^{r+2}} = \frac{(4)\,(5)}{2}\ \frac{0.3^2}{1.3^6} = 18.6\%.$$

The distribution of the number of claims next year from an insured picked at random is Negative Binomial, with $r = \alpha = 4$ and $\beta = \theta = 0.3$.

Mean of this Neg. Binomial is: $(4)(0.3) = 1.2$.

Variance of this Neg. Bin. is: $(4)(0.3)(1.3) = 1.56$.

The Exponential-Poisson is a special case of the Gamma-Poisson.

Over a period of length one, for a homogenous Poisson process mixed by an Exponential Distribution, the mixed distribution is Negative Binomial with $r = \alpha = 1$ and $\beta = \theta$, in other words a Geometric Distribution.

9.4. Given the value of λ, N is a homogeneous Poisson process with rate λ, but λ is itself random, with a Gamma Distribution with
$\alpha = 2.5$ and $\theta = 1/2$.
What is Pr[N(1) > 2]?

Negative Binomial: $f(x) = \frac{r\,(r+1)\,...\,(r+x-1)}{x!}\,\frac{\beta^x}{(1+\beta)^{r+x}}$.

9.4. B. Over one year, the mixed distribution is Negative Binomial with r = 2.5 and β = 0.5.

$$f(0) = \frac{1}{(1+\beta)^r} = \frac{1}{1.5^{2.5}} = 36.29\%.$$

$$f(1) = \frac{r\,\beta}{(1+\beta)^{r+1}} = \frac{(2.5)\,(0.5)}{1.5^{3.5}} = 30.24\%.$$

$$f(2) = \frac{r\,(r+1)}{2}\,\frac{\beta^2}{(1+\beta)^{r+2}} = \frac{(2.5)\,(3.5)}{2}\,\frac{0.5^2}{1.5^{4.5}}$$

$$= 17.64\%.$$

Pr[N(1) > 2] = 1 - f(0) - f(1) - f(2) = **15.83%**.

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If one has a homogenous Poisson Process mixed by a Gamma Distribution with parameters α and θ,
then over a period of length Y,
the mixed distribution is Negative Binomial with $r = \alpha$ and $\beta = Y\theta$.

Over a period of length Y,
for a homogenous Poisson process
mixed by an Exponential Distribution,
the mixed distribution is Negative Binomial
with $r = \alpha = 1$ and $\beta = Y\theta$,
in other words a Geometric Distribution.

9.25. N | Λ is a homogeneous Poisson Process with rate Λ. Λ is a Gamma random variable with parameters α = 0.5 and θ = 3.
Determine Pr[N(3) = 1].

Negative Binomial: $f(x) = \frac{r\,(r+1)\,...\,(r+x-1)}{x!}\,\frac{\beta^x}{(1+\beta)^{r+x}}$.

9.25. The mixed distribution over three years is Negative Binomial with r = α = 0.5 and β = 3 θ = 9.

Probability that there is exactly 1 event by time 3 =

$$\Pr[N(3) = 1] = \frac{r\,\beta}{(1 + \beta)^{r+1}} = \frac{(0.5)\,(9)}{10^{1.5}} = \mathbf{14.23\%}.$$

Section 10, **Compound Poisson Process**

- A Poisson Process, with intensity λ, determines when events occur.
- Given an event, there is a random draw from some distribution in order to determine an amount attached to that event.

S(t) is the total of the amounts from all the events by time t.

If the events are claims and the amounts are sizes of claims, then S(t) is the aggregate loss by time t.

Mean of S(t) =

λt (Mean of the Amount Distribution)

Variance of S(t) =

λt (2nd Moment of the Amount Distrib.)

Second Moment of S(t) = $E[S(t)^2]$

$= Var[S(t)] + E[S(t)]^2$.

Claims follow a Poisson Process with $\lambda = 0.4$.

Only three sizes of claims: 100 with chance 25%, 200 with chance 60%, and 300 with chance 15%.

The mean size of claim is:
$(100)(0.25) + (200)(0.60) + (300)(0.15) = 190$.

E[S(5)] = the expected dollars by time 5 is:
$(0.4)(5)(190) = 380$.

The second moment of severity is:
$(100^2)(0.25) + (200^2)(0.60) + (300^2)(0.15) = 40{,}000$.

$\text{Var}[S(5)] = (0.4)(5)(40{,}000) = 80{,}000$.

As t approaches infinity,
S(t) is approximately Normally Distributed.

$$\text{Prob}[S(5) > 400] \cong 1 - \Phi\left[\frac{400 - 380}{\sqrt{80{,}000}}\right]$$

$$= 1 - \Phi[0.07] = 1 - 0.5279 = 47.21\%.$$

10.47. 3, 5/01, Q.4.

Lucky Tom finds coins on his way to work at a Poisson rate of 0.5 coins per minute.

The denominations are randomly distributed:

(i) 60% of the coins are worth 1;

(ii) 20% of the coins are worth 5;

(iii) 20% of the coins are worth 10.

Calculate the variance of the value of the coins Tom finds during his one-hour walk to work.

3, 5/01, Q.4. E. Over one hour, the mean frequency is (0.5) (60) = 30.

The second moment of the severity is:

$(0.6)(1^2) + (0.2)(5^2) + (0.2)(10^2) = 25.6.$

For a compound Poisson, variance =
(mean frequency) (2nd moment of severity) =
(30) (25.6) = **768.**

Dimes are Poisson with mean:
(20%) (60) (0.5) = 6.
Thus the <u>number</u> of dimes has variance 6.
<u>Value</u> of dimes found is: (10) (# of dimes found).
<u>Value</u> of dimes found has variance: $(10^2)(6) = 600$.

Value of nickels found has variance: $(5^2)(6) = 150$.

Value of pennies has variance: $(1^2)(18) = 18$.

The three processes are independent, so the variance of the total value of coins found is:
18 + 150 + 600 = **768**.

10.52. CAS3, 11/03, Q.30. Speedy Delivery Company makes deliveries 6 days a week. Accidents involving Speedy vehicles occur according to a Poisson process with a rate of 3 per day and are independent.

In each accident, damage to the contents of Speedy's vehicles is distributed as follows:

Amount of damage	Probability
$0	1/4
$2,000	1/2
$8,000	1/4

Using the normal approximation, calculate the probability that Speedy's weekly aggregate damages will not exceed $63,000.

CAS3, 11/03, Q.30. D. For a week, frequency is Poisson with mean: (6) (3) = 18.
The mean severity is:
(1/4) (0) + (1/2) (2000) + (1/4) (8000) = 3000.
The 2nd moment of severity is:
(1/4) (0^2) + (1/2) (2000^2) + (1/4) (8000^2)
= 18,000,000.
Mean aggregate loss = (18) (3000) = 54,000.
Variance of aggregate loss = (18) (18,000,000) = 324,000,000.
Prob[Aggregate ≤ 63,000] ≅
$\Phi[\frac{63,000 - 54,000}{\sqrt{324,000,000}}] = \Phi[0.50] =$ **0.6915**.

Comment: Do not use the continuity correction for aggregate losses, which are assumed to be continuous.

Section 11, Comparing Poisson Processes

For two independent Poisson Processes with claims intensities λ_1 and λ_2,

then the chance that a claim from the first process appears before a claim from the second process is $\frac{\lambda_1}{\lambda_1+\lambda_2}$.

You are sitting by the side of a country road watching cars pass in one direction.
Cars pass with a Poisson Process
Assume that cars are only red or blue.
The red cars pass with claims intensity 0.02, while blue cars pass with claims intensity 0.03.

The chance that the first car to pass is red is:
0.02 / (0.02 + 0.03) = 40%.
The chance that the first car to pass is blue is:
0.03 / (0.02 + 0.03) = 60%.

If one has two independent Poisson Processes with claims intensities λ_1 and λ_2, then the chance that n claims from
the first process occur before k claims from the second process is either

the sum of Binomial terms from n to n + k - 1, with q = $\lambda_1 / (\lambda_1 + \lambda_2)$ and m = n + k - 1,

or F(k-1) for a Negative Binomial Distribution with parameters r = n and $\beta = \lambda_2 / \lambda_1$.

Exercise: Claims from illness are a Poisson Process with claims intensity 13.
Claims from accident are a Poisson Process with claims intensity 7.
The two processes are independent.
What is the probability of 5 claims from illness before 3 claims due to accident?

Each claim has a 13 / (13 + 7) = 0.65 chance of being from illness.
There are 5 claims from illness before 3 claims due to accident if and only if at least 5 of the first 7 claims is from illness. (If fewer than 5 of the first 7 are from illness, then at least 3 are from accident.) The number of the first 7 claims that are from illness is Binomial with q = 0.65 and m = 7.
Desired probability is the sum of the densities at 5, 6, and 7:

$$21(0.65^5)(0.35^2) + 7(0.65^6)(0.35) + 0.65^7$$

$$= 0.2985 + 0.1848 + 0.0490 = 0.5323.$$

Alternately, we want F(3-1) = F(2) for
a Negative Binomial with parameters r = 5 and
β = chance of failure / chance of success
= 0.35 / 0.65 = 7/13.

$f(0) = 1 / (1+ \beta)^r = 1 / (20/13)^5 = 0.1160$,
$f(1) = r\beta / (1+ \beta)^{r+1} = (5)(7/13) / (20/13)^6 = 0.2031$.
$f(2) = r(r+1)\beta^2 / 2(1+ \beta)^{r+2}$
$= (5)(6)(7/13)^2 / 2(20/13)^7 = 0.2132$.

F(2) = 0.1160 + 0.2031 + 0.2132 = 0.5323.

11.24 (CAS3, 5/06, Q.34) (2.5 points)
The number of claims arriving each day in the Montana and Nevada claim offices follow independent Poisson processes with parameters $\lambda_M = 2$ and $\lambda_N = 3$, respectively.
Calculate the probability that the Montana office receives three claims before the Nevada office receives two claims.
A. Less than 0.15
B. At least 0.15 but less than 0.20
C. At least 0.20 but less than 0.25
D. At least 0.25 but less than 0.30
E. At least 0.30

CAS3, 5/06, Q.34. B. Each claim has a probability of being from Montana of: 2 / (2 + 3) = 0.4.
Prob[3 M before 2 N] =
Prob[at least 3 of the first 4 are M] =
Prob[3 of the first 4 are M]
+ Prob[4 of the first 4 are M]
= $(4)(0.4^3)(0.6) + 0.4^4$ = **0.1792**.

Alternately, M ⇔ success, and N ⇔ failure.
The number failures (Ns) before the third success (M) is a Negative Binomial with r = 3 and
β = (chance of failure) / (chance of success)
= 0.6 / 0.4 = 1.5.
Prob[less than 2 failures before the 3rd success] =
$f(0) + f(1) = 1/2.5^3 + (1.5)(3)/(2.5^4)$ = **0.1792**.

Section 12, Known Number of Claims

If we know we have a total of n claims from a Poisson Process on (0, T), then the n claim times are independently uniformly distributed on (0, T), due to the constant independent claims intensity.

Assume we have a Poisson Process on (0, 5) and we observe 4 claims.
Then the 4 times of the claims are independent and each is uniform on (0, 5).

12.6 (2 points) Claims are received by the Symphonic Insurance Company via a Poisson Process with mean 0.017.
During the time interval (0, 100) it receives three claims, one from Beethoven, one from Tchaikovsky, and one from Mahler.
The claim from Beethoven was received at time 27 and the claim from Mahler was received at time 91.
What is probability that the claim from Tchaikovsky was received between the other two claims?
(A) 64% (B) 66% (C) 68% (D) 70% (E) 72%

12.6. A. The time of Tchaikovsky's claim is uniformly distributed over (0, 100), independent of the time of the other two.
The probability it is in the interval (27, 91) is:
(91 - 27) / 100 = **64%**.

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Assume we have a Poisson Process on (0, 10) and we observe 4 claims.
The expected time of the first claim is: (10)(1/5).
The expected time of the second claim is:
(10)(2/5).
The expected time of the 4th claim is: (10)(4/5).
The expected times of the four claims are:
2, 4, 6, and 8;
they divide the time period into 5 equal segments.

If there have been N claims from time 0 to T, the expected time of the i^{th} claim is: $T \frac{i}{N+1}$.

This is an example of a general result:
for a random sample of size N from a uniform distribution, the expected values of the order statistics are spread evenly over the support.

12.21 (2 points) Trucks pass a given spot via a Poisson Process via a rate of 8 per hour.
5 trucks passed between 1 and 2 o’clock.
What is the expected value of the time at which the fourth truck passed?
(A) 1:32 (B) 1:36 (C) 1:40 (D) 1:44 (E) 1:48

12.21. C. Expected time until 4th truck out of 5 is:
(1 hour) 4 / (5+1) = 2/3 hour from start ⇔ **1:40**.

Comment: Similar to CAS S, 11/15, Q.2.

Section 13 *Estimating Software Reliability*

Unlikely to be asked about on your exam.

Section 14

A **Nonhomogeneous Poisson Process** differs from the homogeneous Poisson Process, in that the claims intensity

λ(t) is a function of time,

rather than a constant.

A (homogeneous) Poisson Process is a special case of a nonhomogeneous Poisson Process.

m(x) is the mean number of claims from time 0 to x.

m(x) = $\int_0^x \lambda(t)\, dt$.

λ(t) = m'(t)

= derivative of expected number of claims

λ(t) $\Leftrightarrow$ velocity. m(t) $\Leftrightarrow$ distance.

Number of claims from time a to b is Poisson with mean: m(b) - m(a) = $\int_a^b \lambda(t)\, dt$.

14.30. SOA MLC, 5/07, Q. 25

Subway trains arrive at a certain station according to a nonhomogeneous Poisson process.
$\lambda(t)$, the intensity function (trains per minute), varies with t, the time in minutes after 7:00 AM:

(i) $\lambda(t) = 0.05$, $0 \le t < 10$

(ii) $\lambda(t) = t/200$, $10 \le t < 20$

(iii) $\lambda(t) = 0.10$, $20 \le t$

Calculate the probability that exactly four trains arrive between 7:00 AM and 7:25 AM.
(A) 0.05 (B) 0.07 (C) 0.09 (D) 0.11 (E) 0.13

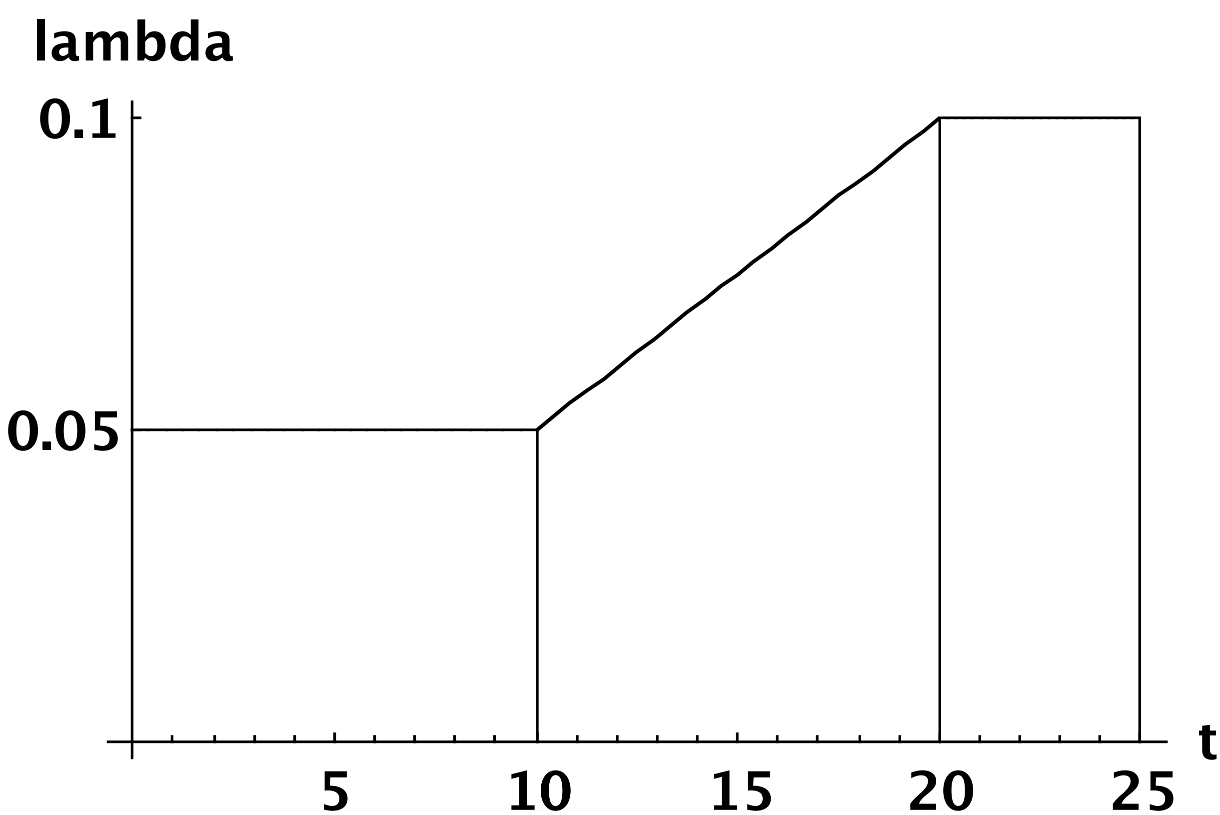
SOA MLC, 5/07, Q. 25. B. The number of trains between t = 0 and 25 is Poisson with mean:

$$\int_0^{25} \lambda(t)\,dt = (10)(0.05) + \int_{10}^{20} t/200\ dt + (5)(0.10)$$

$$= 0.5 + 0.75 + 0.5 = 1.75.$$

$f(4) = 1.75^4\ e^{-1.75}\ /\ 4! =$ **0.068**.

Comment: The mean of the Poisson is the area under the graph of $\lambda(t)$ from 0 to 25:



14.15 (2 points) The number of claims {N(t), t ≥ 0} is a nonhomogeneous Poisson process with intensity function:

$\lambda(t) = 1,\ 0 \le t < 10$

$\lambda(t) = 2,\ 10 \le t < 20$

$\lambda(t) = 3,\ 20 \le t.$

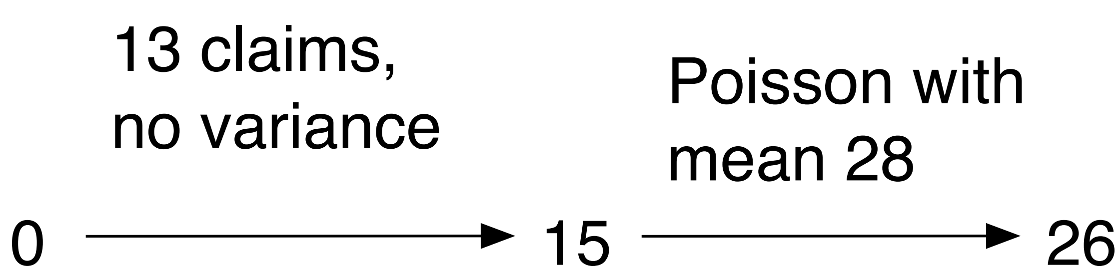
Calculate Var[N(26) I N(15) = 13].

(A) 20 (B) 22 (C) 24 (D) 26 (E) 28

14.15. E. m(10) = 10. m(15) = 10 + (5)(2) = 20.
m(20) = 10 + (10)(2) = 30.
m(26) = 30 + (6)(3) = 48. ⇒ m(26) - m(15) = 28.
⇒ N(26) - N(15) is Poisson with mean 28
and thus variance 28.

Var[N(26) | N(15) = 13] =
Var[N(26) - N(15) + N(15) | N(15) = 13] =
Var[N(26) - N(15) + 13 | N(15) =13] =
Var[N(26) - N(15) | N(15) = 13].

However what happens from time 15 to 26 is independent of what happens from time 0 to 15.
Thus, Var[N(26) - N(15) | N(15) = 13]
= Var[N(26) - N(15)] = 28.
Thus, Var[N(26) | N(15) = 13] = **28**.



Comment: $E[N(26) \mid N(15) = 13] = 13 + 28 = 41$.
$Var[N(26) \mid N(15) = 13] = 0 + 28 = 28$.

Section 15

For a Nonhomogeneous Poisson Process,
the waiting time for the first claim is distributed:

$$\mathbf{F(t) =} 1 - \exp\left[- \int_0^t \lambda(s)\, ds \right] = \mathbf{1 - e^{-m(t)}}.$$

If $\lambda(t) = \lambda$, constant, then $m(t) = \lambda t$, and
$F(t) = 1 - e^{-\lambda t}$, the homogeneous case.

The density of the waiting time for the first claim is:

$$\frac{dF(t)}{dt} = m'(t)\, e^{-m(t)} = \lambda(t)\, e^{-m(t)}.$$

Survival Function = S(t) = 1 - F(t) = $e^{-m(t)}$.

For a distribution with support starting at zero, **mean = the integral of the survival function**, a result that can be derived by integration by parts.

Thus the mean waiting time until the first claim is:

$$\int_0^\infty S(t)\, dt = \int_0^\infty e^{-m(t)}\, dt.$$

15.19. SOA M, 11/06, Q.10

You arrive at a subway station at 6:15.
Until 7:00, trains arrive at a Poisson rate of 1 train per 30 minutes. Starting at 7:00, they arrive at a Poisson rate of 2 trains per 30 minutes. Calculate your expected waiting time until a train arrives.

SOA M, 11/06, Q.10. D.

$\lambda(t) = 1/30,\ t \le 45.\ \ \lambda(t) = 1/15,\ t > 45.$

$m(t) = t/30,\ t \le 45.\ \ m(45) = 45/30 = 1.5.$

$$\text{For } t > 45,\ m(t) = \int_0^{45} \lambda(s)\,ds + \int_{45}^{t} \lambda(s)\,ds$$

$$= 1.5 + (t - 45)/15 = t/15 - 1.5.$$

S(t) = Prob[no train by time t] = exp[-m(t)].

$$\text{mean wait} = \int_0^{\infty} S(t)\,dt = \int_0^{45} S(t)\,dt + \int_{45}^{\infty} S(t)\,dt =$$

$$\int_0^{45} \exp[-t/30]\,dt + \int_{45}^{\infty} \exp[1.5 - t/15]\,dt$$

$$= 30\,(1 - e^{-1.5}) + 15\,e^{1.5}\,e^{-3}$$

$= 30 - 15\,e^{-1.5} =$ **26.65 minutes**.

SOA M, 11/06, Q.10, continued

$\lambda(t) = 1/30,\ t \le 45.$ $\lambda(t) = 1/15,\ t > 45.$

If a train fails to arrive by time 45, then we can start a new Poisson Process with $\lambda = 1/15$, and the average additional wait is 15 minutes.

Thus the mean wait conditional on the train not arriving by time 45 is: 45 + 15 = 60 minutes.

If a train arrives by time 45, then the conditional mean is:

$$\frac{\int_0^{45} t\, f(t)\, dt}{F(45)} = \frac{\int_0^{45} t \exp[-t/30]/30\, dt}{F(45)} =$$

$$\left. -t\, e^{-t/30} - 30\, e^{-t/30} \right]_{t=0}^{t=45} / F(45)$$

$$= (30 - 75\, e^{-1.5}) / F(45) = 13.265 / F(45).$$

$\lambda(t) = 1/30$, $t \le 45$. $\lambda(t) = 1/15$, $t > 45$.

The probability that a train fails to arrive by time 45 is: $e^{-45/30} = e^{-1.5}$.

mean wait =
(mean wait if train arrives <u>by</u> time 45) F(45) +
(mean wait if train arrives <u>after</u> time 45) S(45) =

$\{13.265 / F(45)\}$ F(45) + (45 + 15) $e^{-1.5}$

= **26.65 minutes**.

<u>Comment</u>: I have ignored the vanishingly small probability that the first train does not arrive within 24 hours.
For example, the mean number of trains within 3 hours is: 45 / 30 + 135 / 15 = 10.5.
Probability of no trains within 3 hours is:
$e^{-10.5}$ = 0.003%.

Section 16, Thinning & Adding Nonhomogeneous Poisson Processes:

If one adds independent nonhomogeneous Poisson Processes, on the same time interval, **then one gets a new nonhomogeneous Poisson Process, with the sum of the individual claim intensities.**

If we select at random a fraction of the claims from a nonhomogeneous Poisson Process, we get a new nonhomogeneous Poisson Process, with smaller claims intensity.
This is called **thinning a nonhomogeneous Poisson Process**.

If claims are from a nonhomogeneous Poisson Process, and one divides these claims into subsets in a manner independent of the frequency process, then the claims in each subset are <u>independent</u> nonhomogeneous Poisson Processes.

16.14. 3, 5/01, Q.37.

For a claims process, you are given:

(i) The number of claims $\{N(t), t \geq 0\}$ is a nonhomogeneous Poisson process with intensity function:

$\lambda(t) = 1,\ 0 \leq t < 1$ $\lambda(t) = 2,\ 1 \leq t < 2$ $\lambda(t) = 3,\ 2 \leq t$.

(ii) Claims amounts Y_i are independently and identically distributed random variables that are also independent of N(t) .

(iii) Each Y_i is uniformly distributed on [200,800].

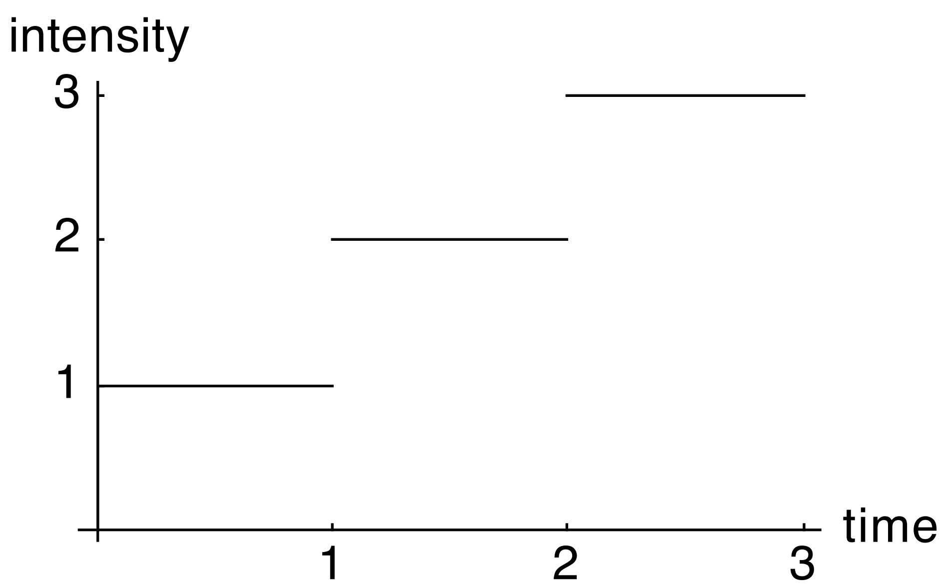
(iv) The random variable P is the number of claims with claim amount less than 500 by time t = 3.

(v) The random variable Q is the number of claims with claim amount greater than 500 by time t = 3.

(vi) R is the conditional expected value of P, given Q = 4.

Calculate R.

3, 5/01, Q.37. C. Thinning, P and Q are <u>independent</u> nonhomogeneous Poisson processes. Thus the conditional expected value of P, given a certain value for Q, is just the unconditional mean of P.



$$m(3) = \int_0^3 \lambda(t)\, dt = 1 + 2 + 3 = 6.$$

Of the original claims: $\frac{500 - 200}{800 - 200} = 1/2$ are P.

Thus, mean of P is: (1/2) (6) = **3**.

Section 17
Comparing Nonhomogeneous Poisson Processes

Unlikely to be asked about on your exam.