To Buyers of CAS Exam MAS-1 Study Guides

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These study guides are split into five volumes.

My Study Aids cover everything on the syllabus of Exam MAS-1.

How much detail is needed and how many problems need to be done varies by person and topic. In order to help you to concentrate your efforts:

1. About 1/6 of the many problems are labeled "highly recommended", while another 1/6 are labeled "recommended."
2. Important Sections are listed in bold in the table of contents. Extremely important Sections are listed in larger type and in bold.
3. Important ideas and formulas are in bold.
4. Each Study Guide has a Section of Important Ideas and Formulas.
5. Each Study Guide has a chart of past exam questions by Section.

My Study Aids are a thick stack of paper.[1] However, many students find they do not need to look at the textbooks. **For those who have trouble getting through the material, concentrate on the introductions and sections in bold**.

Sections and material in italics is less likely to be needed to directly answer exam questions, and should be skipped on the first time through.

Highly Recommended problems (about 1/6 of the total) are double underlined.
Recommended problems (about 1/6 of the total) are underlined.
Do at least the Highly Recommended problems your first time through.
It is important that you **do problems when learning a subject and then some more problems a few weeks later.**

Be sure to do all the recent exam questions at some point.

In October 2023, the CAS made available a practice exam via computer, similar to the computer based testing exam you will take.[2]

I have written some easy and tougher problems.[3] The former exam questions are arranged in chronological order. The more recent exam questions are on average more similar to what you will be asked on your exam, than are less recent exam questions.

In the electronic version use the bookmarks / table of contents in the Navigation Panel in order to help you find what you want..

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[1] The number of pages is not as important as how long it takes you to understand the material. One page in a textbook might take someone as long to understand as ten pages in my Study Guides.
[2] See the MAS-1 page of the CAS webpage. They were charging $99 to take this practice exam.
[3] "Points" in my study guides are based on 100 points = a 4 hour exam.
Questions on your exam are worth the equivalent of about "2.4 points."

My Study Guides are listed below and my estimate of each of their percentages of the exam:[4]

Study Guides for Exam MAS-1

Volume	#	Percent	Study Guide
One	1	17%	Mahler's Guide to Stochastic Models
Two	2	4%	Mahler's Guide to Loss and Frequency Distributions
Three	3	25%	Mahler's Guide to Statistics
Four	4	8%	Mahler's Guide to Regression
Four	5	15%	Mahler's Guide to Generalized Linear Models
Five	6	18%	Mahler's Guide to Statistical Learning
Five	7	5%	Mahler's Guide to Life Contingencies
Five	8	5%	Mahler's Guide to Reliability Theory
Five	9	3%	Mahler's Guide to Simulation

My Practice Exams are sold separately.
My Seminar Style Slides are sold separately.

For the Fall 2023 sitting, the CAS moved Time Series to Exam MAS-2.
Also some material was added to Statistics.[5]

There will be new types of questions for MAS-1 starting with the Fall 2023 sitting:
- Multiple Choice:
  Multiple answer choices are presented after a problem with only one correct answer.
- Multiple Selection (new):
  Multiple answer choices are presented after a problem with more than one correct answer.
- Point and Click (new):
  An image is presented after a problem where the candidate must identify the correct area of the image by clicking on the correct location in the image.
- Fill in the Blank (new):
  A blank section is presented after a problem where the candidate must input the correct value.

"The CAS will be publishing a new sample exam in the Pearson VUE software for candidates to practice the functionality of these new item types."

[4] My estimates should be used with appropriate caution!
In any case, the number of questions by topic varies somewhat from exam to exam.
[5] I have added the following four sections to "Mahler's Guide to Statistics":
Anderson-Darling Test, Multiple Comparisons, Information Matrix and Covariance Matrix, Variance of Functions Multiparameter Case.

Besides many past exam questions from the CAS and SOA, my study guides include some past questions from exams given by the Institute of Actuaries and Faculty of Actuaries in Great Britain. These questions are copyright by the Institute of Actuaries and Faculty of Actuaries, and are reproduced here solely to aid students studying for actuarial exams. These IOA questions are somewhat different in format than those on your exam, but should provide some additional perspective on the syllabus material.

I suggest you buy and try the TI-30XS **Multiview** calculator.
You will save time doing repeated calculations using the same formula.

Download from the CAS website, a copy of the tables to be attached to your exam.[6]

CAS Exam MAS-1 will be administered via Computer Based Testing.[7]
You will be able to use a spreadsheet similar to Excel.[8]

In June 2022, the CAS announced that effective with the Fall 2022 exam sittings,
the guessing penalty for exams MAS-I and MAS-II will be eliminated.
Therefore, you should make sure to choose a letter response for every question.

Some students have reported success with the following guessing strategy.
When you are ready to guess (a few minutes before time is finished for the exam), count up how many you have answered of each letter.
Then fill in the least used letter, at each stage.
For example, if the fewest were A, fill in A's until some other letter is fewest.
Now fill in that letter, etc.
Remember that for every question you should fill in a letter answer.[9]

While studying, you should do as many problems as possible. Going back and forth between reading and doing problems is the only way to pass this exam. The only way to learn to solve problems is to solve lots of problems. You should not feel satisfied with your study of a subject until you can solve a reasonable number of the problems.

Note that In some cases, numerical values shown in one of my spreadsheets are unrounded, while the corresponding value in my text may be rounded.

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[6] For CAS Exam MAS-1 you will be supplied with a Normal Table, Illustrative Life Table, Chi-Square, t-distribution, F-Distribution tables, and abridged Appendices A and B of Loss Models.
Read the statement on the second page regarding AIC and BIC.
http://www.casact.org/admissions/syllabus/MASI_Tables.pdf

[7] See the CAS webpage for details.

[8] Review the CAS testing guide prior to sitting for your exam to note any differences between the Excel and the Pearson testing environment. I reproduce a memo from the CAS at the end of this introduction.

[9] Nothing will be added for an unanswered question and nothing will be subtracted for an incorrect answer.

There are two manners in which you should be doing problems. First you can do problems in order to learn the material. Take as long on each problem as you need to fully understand the concepts and the solution. Reread the relevant syllabus material. Carefully go over the solution to see if you really know what to do. Think about what would happen if one or more aspects of the question were revised.[10] This manner of doing problems should be gradually replaced by the following manner as you get closer to the exam.

The second manner is to do a series of problems under exam conditions, with the items you will have when you take the exam. Take in advance a number of points to try based on the time available. For example, if you have an uninterrupted hour, then one might try either 60/2.5 = 24 points or 60/3 = 20 points of problems. Do problems as you would on an exam in any order, skipping some and coming back to some, until you run out of time. I suggest you leave time to double check your work.

Expose yourself somewhat to everything on the syllabus. Concentrate on sections and items in bold.

Each study guide has a chart of where the past exam questions have been; this may help you to direct your efforts.[11]

Try not to get bogged down on a single topic. On hard subjects, try to learn at least the simplest important idea. The first time through do enough problems in each section, but leave some problems in each section to do closer to the exam. Make a schedule and stick to it. Spend a minimum of one hour every day. I recommend at least two study sessions every day, each of at least 1/2 hour. Most of you will need to spend a total of 300 or more hours of study time on the entire syllabus; this means an average of at least two hour a day for 4 months.

Throughout do Exam Problems and Practice Problems in my study guides**.**
At least 50% of your time should be spent doing problems.
As you get closer to the Exam, the portion of time spent doing problems should increase.
Review the important formulas and ideas sections, at the end of each study guide.
During the last several weeks do my practice exams, sold separately.

Use whatever order to go through the material that works best for you.
Here is a schedule that may work for some people.[12]

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[10] Some may also find it useful to read about a dozen questions on an important subject, thinking about how to set up the solution to each one, but only working out in detail any questions they do not quickly see how to solve.
[11] While this may indicate what ideas questions on your exam are likely to cover, every exam contains a few questions on ideas that have yet to be asked. Each sitting of an exam has its own unique mix of questions.
[12] This is just an example of one possible schedule. Adjust it to suit your needs or make one up yourself.

A 12 week Study Schedule for Exam MAS-1:

1. Start of Stochastic Models (Poisson Processes): sections 1 to 17

2. Loss and Frequency Distributions[13]

3. Start of Statistics: sections 1 to 11

4. Remainder of Stochastic Models (Markov Chains): sections 18 to 29

5. More Statistics: sections 12 to 27

6. Remainder of Statistics: sections 28 to 39.

7. Regression

8. Start of GLMs: sections 1 to 9

9. Remainder of GLMs: sections 10 to 19

10. Reliability, and Simulation

11. Start of Statistical Learning: sections 1 to 9

12. Remainder of Statistical Learning: sections 10 to 15, Life Contingencies

CAS Content Outline versus My Study Guides:[14]

Domain	Study Guides
A	Stochastic Models, Life Contingencies, Reliability Theory, Simulation
B	Loss Distributions, Statistics
C	Statistics, Regression, GLMs, Statistical Learning

[13] There is a lot of material on Loss and Frequency Distributions.
However, based on released exams (which they have stopped doing), there were relatively few questions directly on this material. (Sometimes you need to use this material to answer questions primarily on other material.)
Depending on your background, you may need to spend more than a week on this material.
In any case, for Loss and Frequency Distributions, you could concentrate on the important ideas, the double underlined problems, and being able to use the Tables given to you with your exam.

[14] A "Content Outline" replaced a Syllabus for Fall 2023. The Content Outline is much less detailed, and in my opinion less useful. I relied on which parts of the various readings are listed by the CAS.

Pass Marks and Passing Percentages for Past Exams:[15]

MAS-1	Total Points	Pass Mark	% of Available Points	95th Perc.	75th Perc.	Number of Cand.	Number Passing	Raw Passing Percent	Effective Passing Percent
S 2018	90	53	**58.9%**	72.00	60.00	499	229	45.9%	**50.3%**
F 2018	90	53	**58.9%**	70.00	55.00	846	259	30.6%	**36.4%**
S 2019	90	51.5	**57.2%**	66.00	54.00	948	295	31.1%	**36.4%**
F 2019	90	50.5	**56.1%**	69.50	57.00	992	398	40.1%	**47.3%**
F 2020						1224	541	44.2%	**52.3%**
S 2021						866	371	42.8%	**50.3%**
F 2021						791	437	55.2%	**62.4%**
S 2022						816	415	50.8%	**64.8%**
F 2022						845	440	52.1%	**55.1%**
S-2023						941	507	53.9%	**58.0%**
F 2023						1015	481	47.4%	**49.7%**
S-2024						932	515	55.3%	**60.2%**
F-2024						872	369	42.3%	**48.5%**
S-2025						852	363	42.6%	**50.0%**
Su-2025						446	223	50.0%	**51.4%**

For Spring 2019, for Q.28 both B and E were accepted by the CAS.
For Fall 2019, for Q.35 both B & E were accepted by the CAS.
For Fall 2019, for Q.38 both B & D were accepted by the CAS.

There was no exam given in Spring 2020.

[15] Information taken from the CAS website. Check the website for updated information.

Percent of past (released) Exam MAS-1 questions broken down by my Study Guides:[16] [17] [18]

	Spring 2018	Fall 2018	Spring 2019	Fall 2019	Average	Average Excluding Time Series
Stochastic Models	17.8%	15.6%	17.8%	13.3%	16.1%	**18.1%**
Loss & Freq. Dists.	2.2%	2.2%	6.7%	8.9%	5.0%	**5.6%**
Statistics	17.8%	22.2%	22.2%	17.8%	20.0%	**22.5%**
Regression	6.7%	11.1%	13.3%	4.4%	8.9%	**10.0%**
GLMs	15.6%	8.9%	2.2%	11.1%	9.4%	**10.6%**
Statistical Learning	15.6%	13.3%	15.6%	20.0%	16.1%	**18.1%**
Life Contingencies	4.4%	6.7%	4.4%	4.4%	5.0%	**5.6%**
Reliability	6.7%	6.7%	2.2%	6.7%	5.6%	**6.3%**
Time Series	11.1%	11.1%	11.1%	11.1%	11.1%	
Simulation	2.2%	2.2%	4.4%	2.2%	2.8%	**3.1%**
Total	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%

Past students helpful suggestions and questions have greatly improved these Study Aids.
I thank them! Feel free to send me any questions or suggestions:
Howard Mahler, Email: hmahler@mac.com

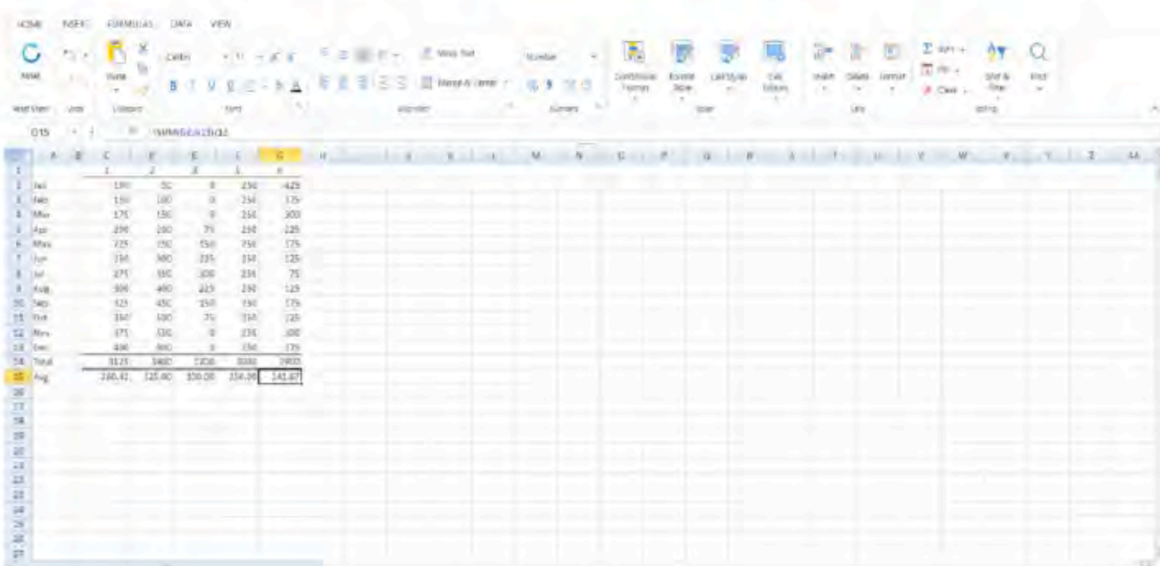
Please send me any suspected errors by Email. hmahler@mac.com
(Please specify as carefully as possible the page, Study Guide, and Exam.)
The errata sheet will be posted on my webpage: www.howardmahler.com/Teaching

[16] Sometimes a question relies on material in more than one of my study guides.
In that case, I have put the question in the study guide in which I think it fits best.
[17] There was no Spring 2020 exam, and the CAS did not release subsequent exams.
[18] Starting with the Fall 2023 sitting, the CAS moved Time Series to Exam MAS-2.
[19] These study aids represent thousands of hours of work.

New Pearson VUE Spreadsheet Functionality for CAS Exams

Pearson VUE has recently enhanced the spreadsheet functionality to include a more fully functioning toolbar—one that more closely resembles typical spreadsheet applications used in everyday settings. Starting with the April/May 2024 exam sitting, candidates can expect to see this updated spreadsheet on **all CAS exams** at Pearson VUE test centers.

Candidates are encouraged to review this document, which outlines the new features and functions, as well as the New Item Type Samples to prepare for the exam environment. If you have any questions, please reach out to casexams@casact.org.



The spreadsheet grid and its toolbar are rendered onscreen within the test item interface. It is not a standalone application in this instance, but rather is embedded in the test driver's user interface.

Menus and Associated Features

Home

- Reset sheet
- Undo (undo, redo)
- Clipboard (cut, copy, paste, paste options menu)
- Fonts (font face, size, styling, border, color, cell fill, and provides access to the format cells dialog)
- Alignment (horizontal and vertical alignment, indent, text angle submenu, merge/center options, text wrap, quick access to format cells dialog)

- Number formatting options (percentage, comma style, decimal places)
- Styles (sub-menus for conditional formatting, formatting tables, cell styles, cell editors)
- Cells (submenus for insertion, deletion and formatting)
- Editing (basic functions with sub-menu options for Sum, Fill, Clear, Sort & Filter, and Find)

Insert

- Tables
- Charts (Column, Line, Bar, Pie, Area, Scatter, Radar, Funnel, etc. and sparklines
- Illustrations (shapes and lines)

Formulas

- Functions Library (categorized lists: AutoSum, Financial, Logical, DateTime, Lookup/Ref, Math & Trig, Statistical, Engineering, Information, Database
- Formula Auditing

Data

- Sort & Filter
- Data Tools
- Outline

View

- Show/Hide (Row & Column headers, Horizontal & Vertical gridlines
- Zoom (zoom pre-set options, 100% reset & zoom to selection)
- Viewport (freeze & unfreeze panes)

Functionality limitations

For use in proctored testing, there are some aspects of this item response type to be aware of:

- **There is no traditional File menu functionality.** The File menu for a typical spreadsheet application allows the user access to the computer's file system in order to do things like save to file, find and open documents or data sets, and import data from external sources. When this spreadsheet is offered as a test delivery response type, the test delivery system manages saving the sheet and the state of the response.
- **There is a Reset function** on the Home view of the toolbar (the left-most function shown in the illustration above). This allows a test-taker to quickly reset the current worksheet to the original state as it was presented to them, prior to having manipulated any cells or values on the sheet. It is guarded by a simple "Are you sure?" warning as a reset cannot be undone.
- **This is not a workbook** (i.e., a related set of worksheets in a single document). The spreadsheet response type offers a single worksheet per instance.
- **There is no searchable Help menu.** Formulas provide standard syntactical guidance via tool tips and the Insert Function Dialog, but there is no help library of examples.

A complete list of the Pearson VUE Spreadsheet Functions is available on the Pearson VUE/CAS website.

Volume One

Mahler's Guide to
Stochastic Models

CAS Exam MAS-1

prepared by
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Study Aid 2026-MAS1-**1**

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Mahler's Guide to Stochastic Models

Information in bold or sections whose title is in bold are more important for passing the exam. Larger bold type indicates it is extremely important.
Information presented in italics (and sections whose titles are in italics) is less likely to be needed to directly answer exam questions and should be skipped on first reading.

Highly Recommended problems are double underlined.
Recommended problems are underlined.[1]

Solutions to the problems in each section are at the end of that section.

From 5/00 to 5/03, the Course 3 Exam was jointly administered by the CAS and SOA.
Starting in 11/03, the CAS and SOA gave separate exams.
Starting in 2012, most of this material was no longer on the syllabus of the SOA exam.
Starting in 2014, Poisson Processes was moved from CAS Exam 3L to new CAS Exam ST.
Starting in 2014, Markov Chains was moved from CAS Exam 3L to new CAS Exam LC.
Starting in Fall 2015, Stochastic Models was moved to new CAS Exam S.
Starting in Spring 2018, Stochastic Models was moved to new CAS Exam MAS-1.

[1] Note that problems include both some written by me and some from past exams. The latter are copyright by the Society of Actuaries and the Casualty Actuarial Society and are reproduced here solely to aid students in studying for exams. The solutions and comments are solely the responsibility of the author; the SOA and CAS bear no responsibility for their accuracy. While some of the comments may seem critical of certain questions, this is intended solely to aid you in studying and in no way is intended as a criticism of the many volunteers who work extremely long and hard to produce quality exams.

The material in my Sections 2, 11-13, and 21-28 were added to the syllabus for the Fall 2015 Exam S. However, most of that material had been on the syllabus at some time in the past. Nonhomogeneous Markov Chains that were formerly on the syllabus were removed for Fall 2015.

Past Exam Questions by Section of this Study Aid[2]

		Exam 3	Exam 3	Exam 3	Exam 3	Exam 3	Exam 3	**CAS 3**	**SOA 3**
Section		Sample	5/00	11/00	5/01	11/01	11/02	11/03	11/03
1									
2				21					
3						10, 11		32	26
4									
5		23	2	23, 29			9, 20	31	11
6									
7								18	
8									
9			4		3, 15	27	5		
10			10		4, 36	19, 30	15	30	20
11				6					
12									
13									
14									
15									
16					37				
17									
18							30		
19		26				29			24
20									
21			40	34	7	22	29	27	25
22								28	
23								29	12
24									
25									
26							7	26	
27									
28									
29			38		21, 22	23	21		

From 5/00 to 5/03, the Course 3 Exam was jointly administered by the CAS and SOA.
Starting in 11/03, the CAS and SOA gave separate exams.
The CAS/SOA did not release the 5/02 and 5/03 exams.

[2] Excluding any questions that are no longer on the syllabus.

		CAS 3	CAS 3	SOA 3	CAS 3	SOA M	CAS 3	SOA M	CAS 3
Section		5/04	11/04	11/04	5/05	5/05	11/05	11/05	5/06
1									
2								14	
3			18, 19	16	39				
4							28		
5		31	17		7, 11	5, 24, 25	29, 31	8	
6			21						32
7									
8									
9		26				6	27	7, 40	
10									
11					12		25		34
12									
13									
14		15, 27		26	14		26		33
15									
16					13				
17									
18		18, 23	15		36	11		23	29
19				14			39		24
20						12		6	
21		24	16						
22		14	13, 14						
23		25		13					
24									
25									
26									
27									
28									
29									

The SOA did not release its 5/04 and 5/06 exams.

		CAS 3	SOA M	CAS 3	SOA MLC	CAS 3	CAS 3L	CAS 3L	CAS 3L
Section		11/06	11/06	5/07	5/07	11/07	5/08	11/08	5/09
1									
2									
3									
4		26						2	
5		27	8	1	5, 26	1, 2			
			9						8, 9
6									
7		23							
8									
9									
10				2	6	3	12	3	10
11					8				
12									
13									
14		28			25		10, 11	1	
15			10						
16									
17									
18			14		16	36	20	19	
19		38			17		25	25	7, 16
20						40			
21									
22									
23									
24									
25									
26									
27									
28									
29									

The SOA did not release its 11/07 and subsequent MLC exams,
until the SOA did release its April 2012 and subsequent MLC exams.

		CAS 3L	**CAS 3L**	**CAS 3L**	**CAS 3L**	**CAS 3L**	**CAS 3L**	**CAS 3L**	**CAS 3L**
Section		11/09	5/10	11/10	5/11	11/11	5/12	11/12	5/13
1									
2									
3			13						
4				10	9	11		10	10
5		10, 11			10		10	9	
6									
7									
8									
9									
10			14	12	11	10	11		11
11									
12									
13									
14			12	11			9	11	9
15									
16									
17									
18		9	11	8, 9	8		8	8	8
19			19	17		16			7
20		16			16			16	
21									
22									
23									
24									
25									
26									
27									
28									
29					8				

		CAS 3L	CAS ST	CAS LC	CAS ST	CAS LC	CAS ST	CAS LC
Section		11/13	5/14	5/14	11/14	11/14	5/15	5/15
1								
2						6		4
3								
4		23						
5		10	2		1		2	
6								
7								
8								
9								
10		11						
11								
12								
13								
14		9	3		3		3	
15								
16			1		2		1	
17								
18		8		10, 11		10		10
19				15		9, 15		15
20		16						
21								
22								
23								
24								
25								
26								
27								
28								
29								

For 2014 the material on Poisson Processes formerly on Exam 3L was moved to new Exam ST.
For 2014 the material on Markov Chains formerly on Exam 3L was moved to new Exam LC.

		Sample S	CAS ST	CAS LC	CAS S	CAS ST	CAS LC
Section		2015	11/15	11/15	11/15	5/16	5/16
1							
2					5		
3							
4			2		1	2	
5					3		
6							
7							
8							
9							
10			3		4		
11							
12					2	1	
13							
14			1			3	
15							
16							
17							
18				9, 10			9, 10
19				14, 15			15
20							
21					10, 11		
22					9		
23							
24							
25							
26		2			8		
27		1					
28							
29							

In Fall 2015, new Exam S was given including: Poisson Processes including some new material, Markov Chains including some new material and excluding Nonhomogeneous Markov Chains, and Markov Processes new to the syllabus.[3]

[3] Markov Processes had been on the syllabus, but were removed from the syllabus after the Spring 2001 Exam 3.

		CAS S	CAS S	CAS S	CAS S	MAS-1	MAS-1	MAS-1	MAS-1
Section		5/16	11/16	5/17	11/17	5/18	11/18	5/19	11/19
1									
2		1, 6	3, 5, 7	1, 6	6, 7	1, 5, 23		4, 6	1
3							1	1	
4								2	
5		2, 4	2	2	2				2
6									
7									
8									
9									
10		5	1, 4		4	4	4	3	3
11				3, 7	1		3		
12						2			
13									
14		3			3	3			
15							2		
16									
17									
18		9	14	11, 12	13		9	8	8
19									
20									
21		10, 11	12, 13		11, 14			9	
22						10			9
23			11						
24				13		11			
25									
26							10	10	10
27					12		11		
28									

There was no Spring 2020 exam, and the CAS will not release subsequent exams.

Section 1, Introduction

The concepts in Introduction to Probability Models by Ross and
"Poisson processes (and mixture distributions)" by James W. Daniel
are demonstrated.[4]

Assume we let X(t) = the surplus of the Sari Insurance Company at time t.
Then as time changes the surplus changes randomly.
This is an example of a **Stochastic Process**.
For each time t, X(t) is a random variable.

If we only look at the surplus at each year end, then this would be a **discrete-time** stochastic process. If instead we were able to examine the surplus at any point in time, this would be a **continuous-time** stochastic process. Generally, continuous-time processes can be approximated by discrete-time processes, by taking very small time intervals in a discrete time process. For example, the difference between being able to examine the surplus of the Sari Insurance Company at the end of each day or at any time is unlikely to be of any practical importance.

A **stochastic process {X(t), t ∈ T} is a collection of random variables.**
T is the index set of the stochastic process.

So if one can only look at the surplus at each year end, we would have T = {1, 2, 3,...} in units of years. We would have corresponding random variables X(1), X(2), X(3) , ..., the observed amounts of surplus at years end. If instead we were able to examine the surplus at any point in time, we would have T = {t > 0} and X(t), t > 0.

In the Surplus example, $-\infty < X(t) < \infty$, since the surplus can (in theory) be any real number.[5]
The set of possible values for the random variables is called the state space of the stochastic process. For the surplus example, the state space is the set of real numbers.[6]

Exercise: The price of stock at the close of business each day is P(t). What type of stochastic process is this? What is the state space? What is the index set?
[Solution: This is a discrete-time process. The state space is the positive real numbers.
The index set is the positive integers (in units of days.)]

There are four common types of stochastic processes; the first two are on this exam:
1. Poisson Processes, including the nonhomogeneous and compound cases.
2. Markov Chains.
3. Markov Processes also called Continuous Time Markov Chains.
4. Brownian Motion

Poisson Processes, Markov Processes, and Brownian Motion are continuous time models.
Markov Chains are discrete time models.

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[4] Sections 9.1-9.6 of Ross on Reliability Theory are covered in a separate study guide.
[5] Small or negative amounts of Surplus would indicate an insurer in serious trouble or insolvent.
[6] Ignoring as usual, that the smallest unit of currency is $0.01.

Poisson Distribution:

Support: x = 0, 1, 2, 3... Parameters: $\lambda > 0$

D. f. : $F(x) = 1 - \Gamma(x+1 ; \lambda)$ *Incomplete Gamma Function*

P. d. f. : $\mathbf{f(x) = \lambda^x e^{-\lambda} / x!}$

Mean = $\boldsymbol{\lambda}$ **Variance =** $\boldsymbol{\lambda}$

Mode = largest integer in λ *(if* λ *is an integer then* $f(\lambda) = f(\lambda-1)$ *and both* λ *and* $\lambda-1$ *are modes.)*

The sum of two independent variables each of which is Poisson with parameters λ_1 **and** λ_2 **is also Poisson, with parameter** $\lambda_1 + \lambda_2$**.**

Geometric Distribution:

Support: x = 0, 1, 2, 3, ... Parameters: $\beta > 0$.

D. f. : $F(x) = 1 - \left(\frac{\beta}{1+\beta}\right)^{x+1}$ P. d. f. : $f(x) = \frac{\beta^x}{(1+\beta)^{x+1}}$

$f(0) = 1/(1+\beta)$. $f(1) = \beta/(1+\beta)^2$.
$f(2) = \beta^2/(1+\beta)^3$. $f(3) = \beta^3/(1+\beta)^4$.

Mean = β Variance = $\beta(1+\beta)$ Variance / Mean = $1 + \beta > 1$.
Mode = 0.

Binomial Distribution:

Support: x = 0, 1, 2, 3..., m   Parameters: $1 > q > 0$, $m \geq 1$.

P. d. f. : $\mathbf{f(x)} = \frac{m!\, q^x\, (1-q)^{m-x}}{x!\,(m-x)!} = \binom{m}{x} q^x (1-q)^{m-x}$.

Mean = mq   **Variance = mq(1-q)**

Mode = largest integer in mq + q (if mq + q is an integer, then f(mq + q) = f(mq + q- 1) and both mq + q and mq + q - 1 are modes.)

A Binomial Distribution with m = 1 is a Bernoulli Distribution.

The sum of m independent Bernoulli Distributions with the same q is a Binomial with parameters m and q.

The sum of two independent Binomials with parameters (m_1, q) and (m_2, q) is also Binomial with parameters $m_1 + m_2$ and q.

Normal Distribution:

Support: $\infty > x > -\infty$ Parameters: $\infty > \mu > -\infty$ (location parameter)
$\sigma > 0$ (scale parameter)

D. f. : $\mathbf{F(x) = \Phi[(x-\mu)/\sigma]}$

P. d. f. : $f(x) = \phi[\frac{x - \mu}{\sigma}]/\sigma = \frac{\exp[-\frac{(x-\mu)^2}{2\sigma^2}]}{\sigma\sqrt{2\pi}}$. $\phi(x) = \frac{\exp[-x^2/2]}{\sqrt{2\pi}}$.

Mean = μ   **Variance = σ^2**

Skewness = 0 (distribution is symmetric)

Mode = μ   Median = μ

Attached to the exam is a table of the Standard Normal Distribution, with $\mu = 0$ and $\sigma = 1$.

Normal Distribution Table

Entries represent the area under the standardized normal distribution from -∞ to z, Pr(Z < z).
The value of z to the first decimal place is given in the left column.
The second decimal is given in the top row.

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767

Table continued on the next page

Entries represent the area under the standardized normal distribution from -∞ to z, Pr(Z < z).
The value of z to the first decimal place is given in the left column.
The second decimal is given in the top row.

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

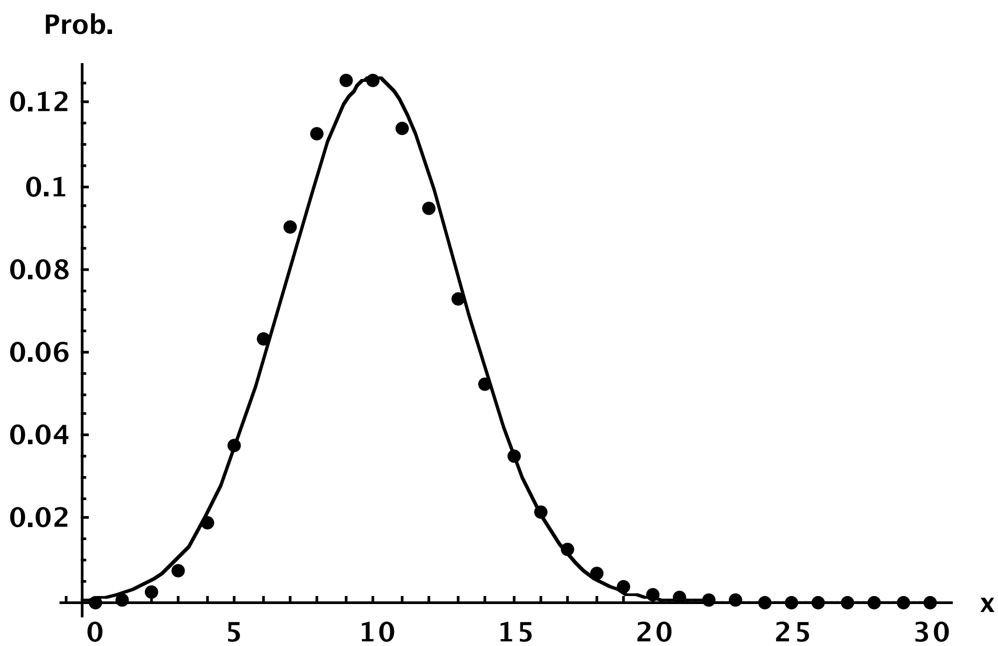
	Values of z for selected values of Pr(Z < z)						
z	0.842	1.036	1.282	1.645	1.960	2.326	2.576
Pr(Z < z)	0.800	0.850	0.900	0.950	0.975	0.990	0.995

The bottom of the Normal Table has selected percentiles of the Standard Normal Distribution.
For example, $\Phi[1.645] = 0.95$; 1.645 is the 95^{th} percentile of the Standard Normal Distribution.
$\Phi[1.960] = 0.975$; 1.960 is the 97.5^{th} percentile of the Standard Normal Distribution.

<u>Normal Approximation</u>:

The Poisson Distribution, when λ is an integer, is the sum of λ independent Poisson variables each with mean of one. Thus by the Central Limit Theorem, a Poisson Distribution can be approximated by a Normal Distribution with the same mean and variance.[7]

For example, here is the graph of a Poisson Distribution with $\lambda = 10$, and the approximating Normal Distribution with $\mu = 10$ and $\sigma = \sqrt{10} = 3.162$:

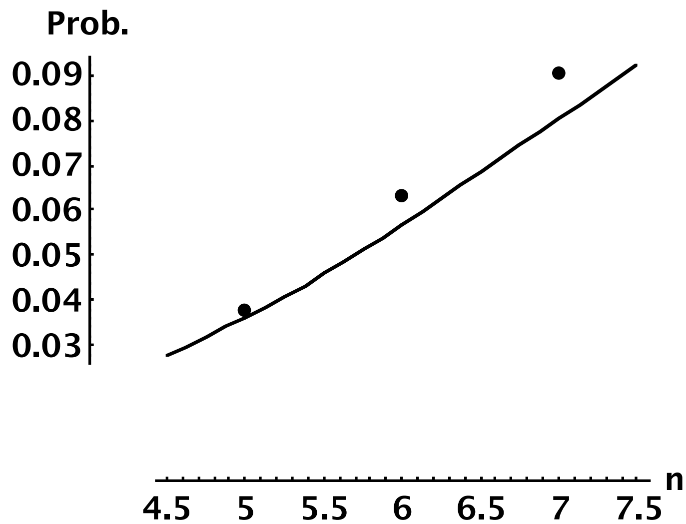


A typical use of the Normal Approximation would be to find the probability of observing a certain range of claims, where adding up the individual densities would be a lot of work. For example, given a certain distribution, what is the probability of at least 10 and no more than 20 claims. However, as an example, let us see how we can approximate the chance of 5, 6, or 7 claims.

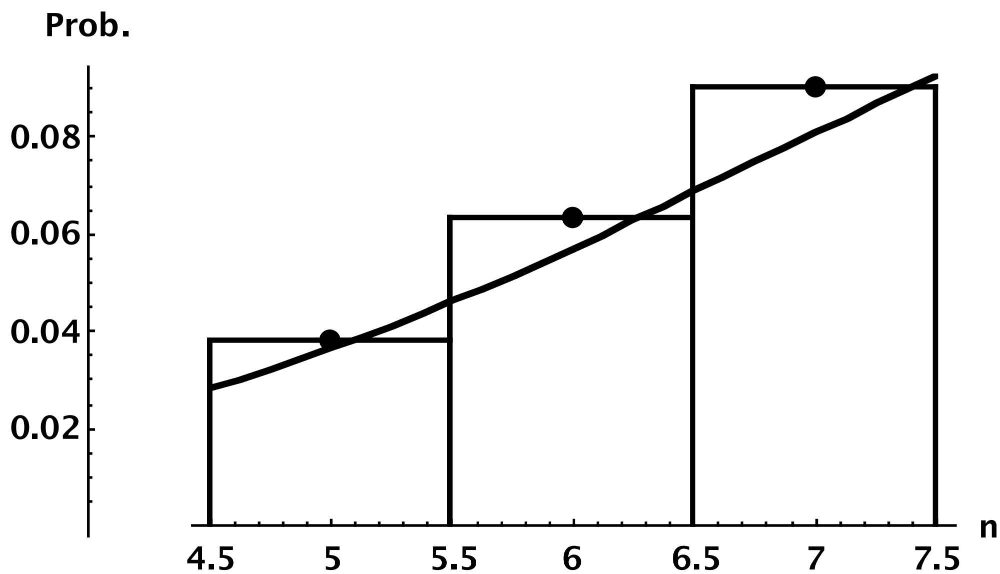
---

[7] One can also approximate via a Normal Distribution the Binomial Distribution, the Negative Binomial Distribution, and Compound Poisson Distributions.

Here is a graph of the densities of the Poisson Distribution with $\lambda = 10$, at 5, 6, and 7, and the approximating Normal Distribution:



I have added rectangles to the above graph:



The first rectangle has height f(5) and width one, and thus area f(5).
The sum of the three rectangles is the exact answer: f(5) + f(6) + f(7).

In order to approximate the area of these three rectangles, I must go from 4.5 to 7.5 on the approximating continuous Normal Distribution: F[7.5] - F[4.5] =
$\Phi[(7.5 - 10)/\sqrt{10}\,] - \Phi[(4.5 - 10)/\sqrt{10}\,] = \Phi[-0.79] - \Phi[-1.74] = 0.2148 - 0.0409 = 0.1739$.

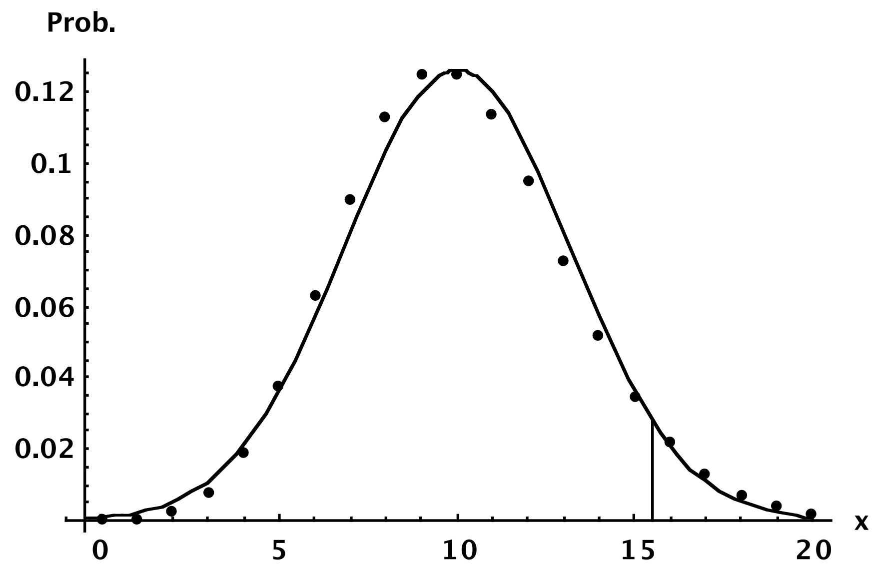
The exact answer is: $10^5 e^{-10}/5! + 10^6 e^{-10}/6! + 10^7 e^{-10}/7! = 0.1910$. In order to use the Normal Approximation, one must translate to the so called "Standard" Normal Distribution.[8] We therefore need to standardize the variables by subtracting the mean and dividing by the standard deviation.

In general, let μ be the mean of the frequency distribution, while σ is the standard deviation of the frequency distribution, then the chance of observing at least i claims and not more than j claims is approximately: $\Phi[\frac{(j + 0.5) - \mu}{\sigma}] - \Phi[\frac{(i - 0.5) - \mu}{\sigma}]$.

One should **use the continuity correction whenever one is using the Normal Distribution in order to approximate the probability associated with a discrete distribution**.
Do not use the continuity correction when one is using the Normal Distribution in order to approximate continuous distributions, such as aggregate distributions or the Gamma Distribution.

Exercise: Use the Normal Approximation in order to estimate the probability of observing at least 16 claims (more than 15 claims) from a Poisson Distribution with $\lambda = 10$.
[Solution: Mean = variance = 10. Prob[# claims ≥ 16] = 1 - Prob[# claims ≤ 15] ≅
$1 - \Phi[(15.5 - 10)/\sqrt{10}] = 1 - \Phi[1.74] = 1 - 0.9591 = 4.09\%$.
Comment: The exact answer is 4.87%.]

The area under the Normal Distribution and to the right of the vertical line at 15.5 is the approximation used in this exercise:



In order to get the probability of at least 16 (more than 15) on the discrete Poisson Distribution, one has to cover an interval starting at 15.5 on the real line for the continuous Normal Distribution.

[8] With mean zero and standard deviation one, in the table attached to the exam.

Diagrams for the Use of the Normal Approximation:

Some of you will find the following simple diagrams useful when applying the Normal Approximation to discrete distributions.

More than 15 claims ⇔ At least 16 claims ⇔ 16 claims or more

15 15.5 16

|⟶

Prob[More than 15 claims] ≅ 1 - Φ[(15.5 - μ)/σ].

Exercise: For a frequency distribution with mean 14 and standard deviation 2, using the Normal Approximation, what is the probability of at least 16 claims?
[Solution: Prob[At least 16 claims] = Prob[More than 15 claims] ≅ 1 - Φ[(15.5 - μ)/σ] =
1 - Φ[(15.5 - 14)/2] = 1 - Φ[0.75] = 1 - 0.7734 = 22.66%.]

Less than 12 claims ⇔ At most 11 claims ⇔ 11 claims or less

11 11.5 12

⟵|

Prob[Less than 12 claims] ≅ Φ[(11.5 - μ)/σ].

Exercise: For a frequency distribution with mean 10 and standard deviation 4, using the Normal Approximation, what is the probability of at most 11 claims?
[Solution: Prob[At most 11 claims] = Prob[Less than 12 claims] ≅ Φ[(11.5 − μ)/σ] =
Φ[(11.5 - 10)/4] = Φ[0.375] = 64.6%.]

At least 10 claims and at most 13 claims ⇔ More than 9 claims and less than 14 claims
9 9.5 10 11 12 13 13.5
14

◀───────────────▶

Prob[At least 10 claims and at most 13 claims] ≅ Φ[(13.5 - μ)/σ] - Φ[(9.5 - μ)/σ].

Exercise: For a frequency distribution with mean 10 and standard deviation 4, using the Normal Approximation, what is the probability of more than 9 claims and less than 14 claims?
[Solution: Prob[more than 9 claims and less than 14 claims] =
Prob[At least 10 claims and at most 13 claims] ≅ Φ[(13.5 - μ)/σ] − Φ[(9.5 - μ)/σ] =
Φ[(13.5 - 10)/4] - Φ[(9.5 - 10)/4] = Φ[0.875] - Φ[-0.125] = 0.809 - 0.450 = 35.9%.]

<u>Survival Function</u>:

We can define the Survival Function: $S(x) = 1 - F(x) = \text{Prob}[X > x]$.

$S'(x) = -f(x) \le 0$.
$0 \le S(x) \le 1$, nonincreasing, right-continuous, starts at 1 and ends at 0.[9]

For example, for the Exponential Distribution, $S(x) = 1 - F(x) = 1 - (1- e^{-x/\theta}) = e^{-x/\theta}$.

The **mean can be written as an integral of the Survival Function from 0 to the infinity**, **for a distribution with support starting at <u>zero</u>**.[10]

$$E[X] = \int_0^\infty S(t)\ dt.$$

In many situations you may find that the survival function is easier for you to use than the distribution function. Whenever a formula has $S(x)$, one can always use $1 - F(x)$ instead, and vice-versa.

[9] See Definition 2.4 in <u>Loss Models</u>.
[10] See formula 3.5.2 in <u>Actuarial Mathematics</u>.
This can be derived via integration by parts.

Section 2, Exponential Distribution[11]

This single parameter distribution is extremely simple to work with and thus appears in many exam questions. Following is a summary of the Exponential Distribution.

Exponential Distribution

Support: $x > 0$ Parameter: $\theta > 0$ (scale parameter)

F(x) = $\mathbf{1 - e^{-x/\theta}}$ **F(x) =** $\mathbf{1 - e^{-\lambda x}}$.

f(x) = $\mathbf{e^{-x/\theta} / \theta}$ **f(x) =** $\mathbf{\lambda e^{-\lambda x}}$.

Moments: $E[X^n] = n!\ \theta^n$ Moment Generating Function: $\frac{1}{1 - \theta t}$, $t < \theta$

Mean = $\boldsymbol{\theta}$ Hazard rate = $f(x) / S(x) = \lambda = 1/\theta$.

Variance = $\boldsymbol{\theta^2}$

Coefficient of Variation = Standard Deviation / Mean = 1.

Skewness = 2 *Kurtosis = 9*

Mode = 0 Median = $\theta \ln(2)$

Limited Expected Value Function: $E[X \wedge x] = \theta(1- e^{-x/\theta})$.[12]

e(x) = Mean Excess Loss = Mean Residual Life = θ.

Method of Moments: $\theta = \bar{X}$.

Method of Maximum Likelihood: $\theta = \bar{X}$, same as method of moments.

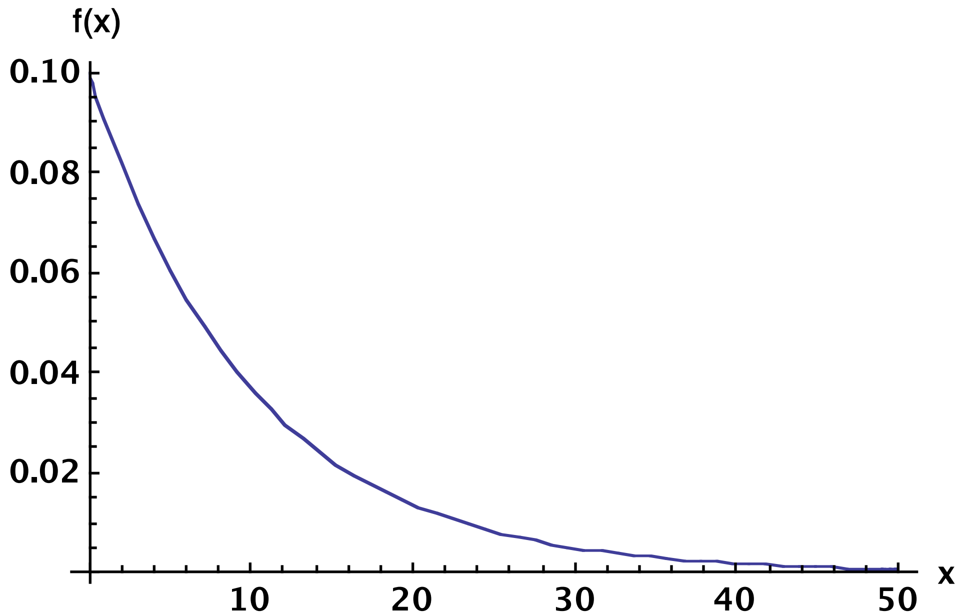
---

[11] See Section 5.2 of Introduction to Probability Models by Ross.

[12] In general to compute the limited expected value at a limit L, first limit each value to L and then take an average.

Exercise: An Exponential Distribution has $\theta = 10$.
Determine the mean, second moment, and variance.
[Solution: Mean = θ = 10. Second Moment = $2\theta^2$ = 200.
Variance = 200 - 10^2 = 100 = θ^2.]

Here's a graph of the density function of an Exponential. All Exponential Distributions look the same except for the scale; in this case the mean is 10. Also note that while I've only shown $x \le 50$, the density is positive for all $x > 0$.



Failure Rate:

The failure rate, also called the hazard rate or the force of mortality, is: r(x) = f(x) / S(x).[13]
For the Exponential Distribution with mean θ, the failure rate is: $(e^{-x/\theta}/\theta) / e^{-x/\theta} = 1/\theta$.

The Exponential Distribution has a constant failure rate: λ = 1/mean;
it is the only continuous distribution with this property.

The failure rate, r(x), determines the survival function and thus the distribution function:[14]
$\frac{d \ln[S(x)]}{dx} = \frac{d S(x)}{dx}$ / S(x) = -f(x) / S(x) = - r(x). Thus $r(x) = -\frac{d \ln[S(x)]}{dx}$.

$\Rightarrow \int_0^x r(t)\, dt = -\ln[S(x)]$. $\Rightarrow$ $\mathbf{S(x) = \exp[-\int_0^x r(t)\, dt\,]}$.

Exercise: The failure rate is: r(x) = 1/10. What is the distribution function?
[Solution: $\int_0^x r(t)\, dt = x/10$. $F(x) = 1 - e^{-x/10}$, an Exponential Distribution with $\lambda = 1/10$.]

Exercise: The failure rate is: r(x) = 3/(10 + x). What is the distribution function?
[Solution: $\int_0^x r(t)\, dt = \int_0^x \frac{3}{10+t}\, dt = 3\,\{\ln(10+x) - \ln(10)\} = 3\ln[(10+x)/10]$.

$F(x) = 1 - \exp[-\int_0^x r(t)\, dt\,] = 1 - \left(\frac{10}{10+x}\right)^3$.

This is a Pareto Distribution with $\alpha = 3$ and $\theta = 10$.]

---

[13] See equation 5.4 in Ross.
Rather than r(x), actuaries commonly use the symbol $\mu(x)$ in the context of a force of mortality, and h(x) in the context of a hazard rate. They are all the same mathematical object.

[14] In Survival Analysis, the integral of the hazard rate $H(x) = \int_0^x h(t)\, dt$ is called the cumulative hazard rate.

Then the survival function is S(x) = exp[-H(x)].

<u>Memoryless Property</u>:[15]

For an Exponential Distribution:

$$\text{Prob}[X > s + t \mid X > t] = \frac{\text{Prob}[X > s + t, \text{ and } X > t]}{\text{Prob}[X > t]} = \frac{\text{Prob}[X > s + t]}{\text{Prob}[X > t]} = \frac{e^{-(s+t)/\theta}}{e^{-t/\theta}}$$

$= e^{-s/\theta} = \text{Prob}[X > s]$.

A random variable is memoryless if Prob[X > s + t | X > t] = Prob[X > s].
The Exponential Distribution is memoryless,
and is the only continuous distribution to be so.

Exercise: Your time waiting to hail a cab is Exponentially distributed with mean 3 minutes. What is the probability you need to wait more than 5 minutes? If you have already waited 4 minutes, what is the probability you need to wait more than an additional 5 minutes?
[Solution: $S(5) = e^{-5/3} = 18.9\%$.
Due to the memoryless property, if you have already waited 4 minutes, that does not affect your future waiting time, so there is an 18.9% chance of waiting more than an additional 5 minutes.]

For t > 0, and s > 0, we can rewrite the memoryless property as:
Prob[X > s + t] = Prob[X > s + t | X > t] Prob[X> t] = Prob[X > s] Prob[X > t].

For an Exponential Distribution: for all s, t ≥ 0, Pr(X > s + t) = Pr(X > s) Pr(X > t).[16]

Exercise: Show that the Exponential Distribution is the only continuous distribution for which for all s, t ≥ 0, Pr(X > s + t) = Pr(X > s) Pr(X > t).
[Solution: For an Exponential with mean θ:
$\Pr(X > s + t) = \exp[-(s+t)/\theta] = \exp[-s/\theta] \exp[-t/\theta] = \Pr(X > s) \Pr(X > t)$.
Conversely, assume that for all s, t ≥ 0, Pr(X > s+ t) = Pr(X > s) Pr(X > t).

$$-f(x) = \frac{dS(x)}{d(x)} = \lim_{\varepsilon \to 0} \frac{S(x+\varepsilon) - S(x)}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{S(x)S(\varepsilon) - S(x)}{\varepsilon} = S(x) \lim_{\varepsilon \to 0} \frac{S(\varepsilon) - 1}{\varepsilon}.$$

Thus the hazard rate is: $r(x) = f(x)/S(x) = -\lim_{\varepsilon \to 0} \frac{S(\varepsilon) - 1}{\varepsilon}$ = a constant independent of x.

Call this constant λ.

Then $S(x) = \exp[-\int_0^x r(t)\, dt] = \exp[-\int_0^x \lambda\, dt] = \exp[-\lambda t]$, and we have an Exponential Distribution.]

[15] See Equation 5.3 in <u>Introduction to Probability Models</u>, by Ross.
[16] See Exam S, 5/17, Q.1.

The mean residual life, also called the mean excess loss, is defined as: $e(d) = E[X - d \mid X > d]$.
Thus for a life aged 60, the mean residual life is the expected future lifetime.

Exercise: Light bulbs have lifetimes that follow an Exponential Distribution with mean 1000 hours. What is the mean residual life at 600?
[Solution: Due to the memoryless property, the distribution of additional hours lived does not depend on how many hours have already been lived. So the distribution of future hours lived is also Exponential Distribution with mean 1000 hours. Thus, e(600) = 1000.]

The Exponential Distribution has a constant Mean Residual Life;
it is the only continuous distribution with this property.

Exercise: Losses prior to any deductible follow an Exponential Distribution with $\theta = 8$.
A policy has a deductible of size 5.
In other words, the insurer pays nothing for a loss of size less than 5,
and pays 5 less than the size of loss for a large loss.
What is the distribution of non-zero payments under that policy?
[Solution: For X > 5, the distribution of non-zero payments is the amount by which X exceeds 5.
By the memoryless property, this is also an Exponential Distribution with $\theta = 8$.
Alternately, the distribution function of the non-zero payments is:

$$1 - \frac{S(x+d)}{S(d)} = 1 - \frac{S(x+5)}{S(5)} = 1 - \frac{e^{-(x+5)/8}}{e^{-5/8}} = 1 - e^{-x/8}.$$

This is also an Exponential Distribution with $\theta = 8$.]

When losses follow an Exponential Distribution,
the non-zero payments excess of a deductible follow the same Exponential Distribution, due to its memoryless property.

Exercise: Use the memoryless property to determine for an Exponential Distribution with mean θ
$E[X^2 \mid X > d > 0]$.
[Solution: For X > d, X - d is Exponential with mean θ and thus second moment $2\theta^2$.
Therefore, $2\theta^2 = E[(X-d)^2 \mid X > d] = E[X^2 \mid X > d] - 2\,d\,E[X \mid X > d] + d^2$
$= E[X^2 \mid X > d] - 2\,d\,E[X - d \mid X > d] - 2d^2 + d^2 = E[X^2 \mid X > d] - 2d\theta - d^2$.
$\Rightarrow E[X^2 \mid X > d] = 2\theta^2 + 2d\theta + d^2$.
Comment: See Exercise 5.3 in Introduction to Probability Models by Ross.]

Mixtures of Exponentials:[17]

Let us assume that the size of each claims has a 60% chance of being a random draw from an Exponential Distribution with mean 100 and a 40% chance of being a random draw from an Exponential Distribution with mean 200, independent of the size of any other claim.
This is an example of a mixture of two Exponentials.[18]

Exercise: For this mixture, what is the probability that a claim is of size less than 150?
[Solution: For the first Exponential: $F(150) = 1 - e^{-150/100} = 77.69\%$.
For the second Exponential: $F(150) = 1 - e^{-150/200} = 52.76\%$.
For the mixture, F(150) = Prob[from first Exponential] Prob[[less than 150 | first Exponential] + Prob[from second Exponential] Prob[[less than 150 | second Exponential] =
(60%)(77.69%) + (40%)(52.76%) = 67.72%.]

The Distribution Function of the mixture is the mixture of the Distribution Functions.[19]
Similarly, the Survival Function of the mixture is the mixture of the Survival Functions.
By differentiation, the density of the mixture is the mixture of the densities.

Exercise: For this mixture, what is the density at 150?
[Solution: For the first Exponential: $f(150) = e^{-150/100} / 100 = 0.002231$.
For the second Exponential: $f(150) = e^{-150/200} / 200 = 0.002362$.
For the mixture, f(150) = (60%)(0.002231) + (40%)(0.002362) = 0.002283.]

Exercise: For this mixture, what is the failure rate at 150?
[Solution: r(150) = f(150) / S(150) = 0.002283 / (1 - 67.72%) = 0.00707.
Comment: The failure rate of the mixture is not the mixture of the failure rates.
The failure rate for the mixture is not constant.]

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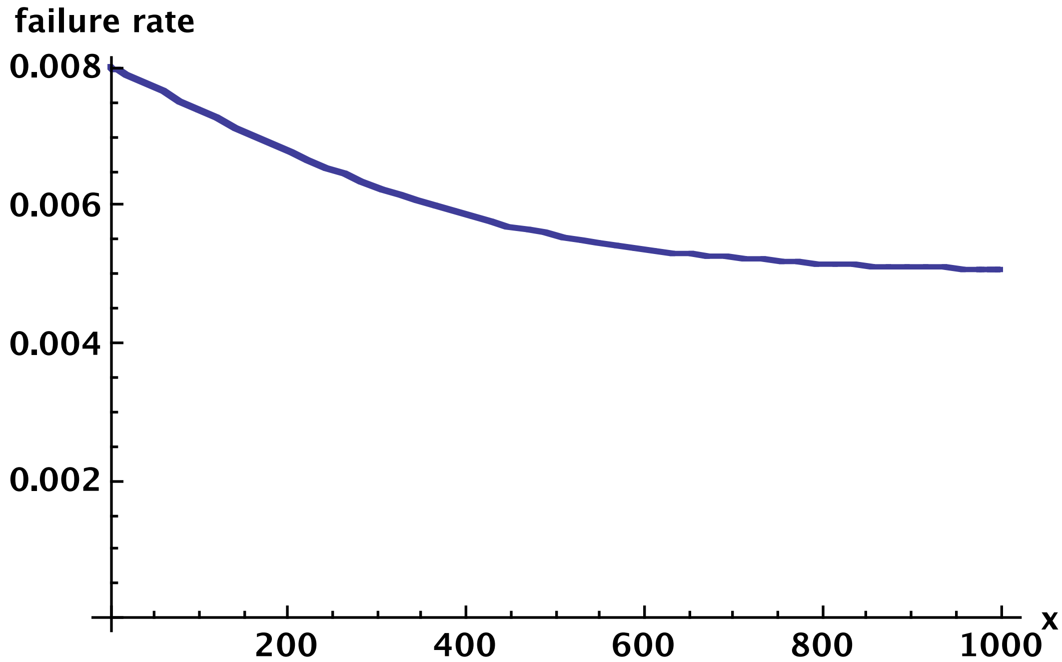
[17] See Example 5.6 in Ross.
Ross refers to the mixtures of Exponential Distributions as a hyperexponential distribution.
[18] In a similar manner one can mix any number of Exponentials together.
Instead one could for example mix a LogNormal and a Pareto Distribution.
[19] This is true in general for mixtures.

Here is a graph of the failure rate as a function of x:



As x approaches infinity, the failure rate of the mixture approaches 1/200 = 0.005, the failure rate of the Exponential with the smallest failure rate. In general for a mixture of n Exponentials with probabilities p_i and hazard rates λ_i, the failure rate for the mixture is: $r(x) = \dfrac{\sum_{i=1}^{n} p_i \lambda_i \exp[-\lambda_i x]}{\sum_{i=1}^{n} p_i \exp[-\lambda_i x]}$.

As x approaches infinity, the failure rate of the mixture approaches the smallest of the λ_i.[20]

Exercise: For this mixture, what is the mean?
[Solution: For the mixture, mean = Prob[from first Exponential] (mean given from first Exponential) + Prob[from second Exponential] (mean given from second Exponential) = (60%)(100) + (40%)(200) = 140.]

The mean of the mixture is the mixture of the means.[21]
Similarly, **the n^{th} moment of the mixture is the mixture of the n^{th} moments**.[22]

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[20] See Example 5.6 in Ross.
As x approaches infinity, $\exp[-\lambda_i x]$ goes to zero more slowly for λ_i smaller.

[21] This is true in general for mixtures.

[22] The n^{th} moment is $E[X^n]$. The second moment is $E[X^2]$.

Variance of Amounts Paid With a Deductible:[23]

Assume that the claim severity distribution is Exponential.
An insurance company will pay the amount of each claim in excess of a deductible.
Then the non-zero payments will follow the same Exponential distribution.
Sometimes we are interested in the amount paid by the insurance company for one claim, including the possibility that the amount paid is zero.

Exercise: You are given the following information:
- The severity of each loss in an insurance policy independently follows an exponential distribution with mean 800.
- The insurance company only pays the amount exceeding the per-loss deductible of 500.

Calculate the variance of the amount paid by the insurance company for one claim, including the possibility that the amount paid is zero.
[Solution: $F(500) = 1 - e^{-500/800} = 46.5\%$.
Due to the memoryless property of the Exponential, the amount the insurance company pays per non-zero claim follows the original Exponential distribution.
The payments per loss follow a two point mixture of zero and an Exponential with mean 800, with weights 46.5% and 53.5%.
The moments of a mixture are the mixture of the moments.
This mixed distribution has mean (53.5%)(800) = 428,
second moment $(53.5\%)(2)(800^2) = 684{,}800$,
and thus variance: $684{,}800 - 428^2 = 501{,}616$.
Alternately, the payments per loss follow an aggregate distribution with a Bernoulli frequency with mean 53.5%, and an Exponential severity with mean 800.
This has variance: (mean frequency) (variance of severity) + (mean severity)^2 (variance of freq.)
$= (0.535)(800^2) + (800^2)(0.535)(1 - 0.535) = 501{,}616$.]

Sums of Independent, Identically Distributed Exponentials:[24]

The sum of n independent, identically Exponential Distributions each with mean θ, is a Gamma Distribution with parameters α = n and θ.[25]

Exercise: What is the distribution of the sum of four independent, identically distributed Exponentials each with hazard rate 0.1?
[Solution: A Gamma Distribution with $\alpha = 4$ and $\theta = 1/0.1 = 10$.]

[23] See Example 5.4 in Introduction to Probability Models by Ross.
See 3, 11/00, Q.21 and MAS-1, 5/19, Q.6.
[24] See Section 5..2.3 of Introduction to Probability Models by Ross.
[25] The Gamma Distribution is discussed in a subsequent section.

Comparing Independent Exponentials:[26]

If X_1 and X_2 are independent Exponentials with failure rates λ_1 and λ_2, then the probability that $X_1 < X_2$ is: $\frac{\lambda_1}{\lambda_1 + \lambda_2}$.[27] [28] Prob[$X_1 < X_2$] = $\frac{\theta_2}{\theta_1 + \theta_2}$.

Exercise: Claims from theft are Exponentially Distributed with mean 1000.
Claims from vandalism are Exponentially Distributed with mean 500.
Determine the probability that a random vandalism claim is bigger than a random theft claim.
[Solution: Prob[$X_1 < X_2$] = $\frac{\lambda_1}{\lambda_1 + \lambda_2} = \frac{1/1000}{1/1000 + 1/500} = \frac{500}{500 + 1000}$ = 1/3.

Comment: Vandalism claims having the smaller mean, have a probability of being bigger that is less than a half. Prob[$X_1 < X_2$] = $\frac{\theta_2}{\theta_1 + \theta_2} = \frac{500}{1000 + 500}$ = 1/3.]

Similarly, if one has n independent Exponentials with failure rates λ_i, then the probability that X_i is the smallest one is: $\frac{\lambda_i}{\sum_{i=1}^{n} \lambda_i}$.[29]

[26] See Section 5.2.3 of Introduction to Probability Models by Ross.
[27] See Equation 5.5 in Introduction to Probability Models by Ross.
This is mathematically equivalent to a similar result for comparing Poisson Processes, to be discussed in a subsequent section.
[28] If the two exponentials are identically distributed, then as expected this probability is 1/2.
[29] See page 289-290 of Introduction to Probability Models by Ross.

Minimums of Exponentials:

The minimum of a sample of size n from a single distribution, is an example of an Order Statistic.[30] Prob[Min > x] = Prob[All members of the sample > x] = $S(x)^n$.

Exercise: What is the Distribution of the minimum of a sample of size 4 from an Exponential Distribution with mean 12?
[Solution: Prob[Min > x] = $S(x)^4$ = $\exp[-x/12]^4$ = exp[-x/3].
Thus the distribution of the minimum is an Exponential with mean 3.]

In general, the minimum of a sample of size n from an Exponential Distribution with mean θ, is another Exponential with mean θ/n.

Exercise: You have just reached the head of the line waiting to buy a commuter rail ticket.
There are three clerks currently each dealing with a customer.
The time for a clerk to deal with a customer is Exponentially Distributed with mean 30 seconds.
What is the average time until you are done buying your ticket?
[Solution: You need to wait until the first of the current three customers is done.
The minimum of three independent Exponentials each with mean 30 is an Exponential with mean: 30/3 = 10. So on average you wait 10 seconds to be served and then an additional 30 seconds to buy your ticket, for a total of 40 seconds.]

Exercise: In the previous exercise, what is the probability that of the three customers currently being served and yourself, you are the last one to finish buying a ticket?
[Solution: One of the three customers currently being served will leave first, at which point you start being served by a clerk. Due to the memoryless property, as that point all three of you being served have the same Exponential Distribution. Thus, there is 1/3 chance you will be the last of these three to leave. Thus the probability that of the three customers currently being served and yourself, you are the last one to finish buying a ticket is 1/3.
Comment: See Example 5.3 in Ross.]

Exercise: Claims from theft are Exponentially Distributed with mean 1000.
Claims from vandalism are Exponentially Distributed with mean 500.
Determine the distribution of the minimum of a randomly chosen claim of each type.
[Solution: Prob[Min > x] = Prob[theft > x] Prob[vandalism > x] = $e^{-x/1000}\ e^{-x/500} = e^{-0.003x}$.
Thus the distribution of the minimum is an Exponential with failure rate 0.003 and mean 333.33.]

In general, the minimum of two independent Exponentials is another Exponential with the sum of the hazard rates. For example, in the previous exercise: 0.003 = 1/1000 + 1/500.

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[30] Order Statistics are discussed extensively in "Mahler's Guide to Statistics."

Exercise: We have two independent Exponentials X and Y, with hazard rates λ_X and λ_Y.
Show that Min[X, Y] is independent of whether X < Y or Y < X.
In other words, the distributions of: [Min[X , Y] | X < Y], [Min[X , Y] | Y < X], and Min[X , Y] are the same.
[Solution: The conditional density of X, given X < Y is proportional to:
Prob[X=x] Prob[Y > x] = $f_X(X)\ S_Y(x) = \lambda_X \exp[-\lambda_X x] \exp[-\lambda_Y x] = \lambda_X \exp[-(\lambda_X + \lambda_Y)x]$.
Since this is proportional to the density of an Exponential Distribution with hazard rate $\lambda_X + \lambda_Y$, this must be the conditional density of X, given X < Y.
Thus, X given X < Y follows the same distribution as the minimum of X and Y.
Therefore, the minimum is independent of the rank order of the two Exponentials.
Comment: See Exercise 8 of Chapter 5 of Introduction to Probability Models by Ross.]

We have shown that for two independent Exponentials U and V,
E[U | U < V] = E[V | V < U] = E[Min[U, V]].

If one has n independent Exponentials with failure rates λ_i, then the minimum of a sample consisting of a random draw from each Exponential is another Exponential with the sum of the failure rates.[31]
Also the rank order of the X_i and the minimum are independent each other.

Exercise: You have just reached the head of the line waiting to buy a commuter rail ticket.
There are three clerks currently each dealing with a customer.
The time for a clerk to deal with a customer is Exponentially Distributed.
However the first clerk has a mean service time of 1/3 minute, the second clerk has a mean service time of 1/2 minute, and the final clerk has a mean service time of 2/3 minute.
What is the average time until you are done buying your ticket?
[Solution: You need to wait until the first of the current three customers is done.
The minimum of three independent Exponentials is another Exponential with the sum of the failure rates: 3 + 2 + 1.5 = 6.5. So on average you wait 1/6.5 minutes to be served.
The probability that you are served by each of the three clerks are: 3/6.5, 2/6.5, and 1.5/6.5.
Thus the average time you spend being served is:
(3/6.5)(1/3) + (2/6.5)(1/2) + (1.5/6.5)(2/3) = 3/6.5
Thus the average time until you are done buying your ticket is:
1/6.5 + 3/6.5 = 4/6.5 = 0.6154 minutes.
Comment: See Example 5.8 in Ross. In general with n clerks (servers), the average time until you are done buying your ticket is: $\frac{n+1}{\text{sum of their failure rates}}$.]

[31] See the Proposition at page 290 of Introduction to Probability Models by Ross.

Exercise: In the previous exercise, what is the probability that all three customers currently being served will finish being served before you do?
[Solution: You need to wait until the first of the current three customers is done.
The probability that you are served by each of the three clerks are: 3/6.5, 2/6.5, and 1.5/6.5.
If you are served by the first clerk, then the chance that the second clerk finishes first is: 2/6.5.
In which case, the chance that the third clerk finishes before the first clerk is: 1.5/(3 + 1.5) = 1/3.
If you are served by the first clerk, then the chance that the third clerk finishes first is: 1.5/6.5.
In which case, the chance that the second clerk finishes before the first clerk is: 2/(3 + 2) = 0.4.
Thus if you are served by the first clerk, then the chance that you are last is:
(2/6.5)(1.5/4.5) + (1.5/6.5)(2/5) = 0.1949.
Similarly, if you are served by the second clerk, then the chance you are last is:
(3/6.5)(1.5/3.5) + (1.5/6.5)(3/5) = 0.3363.
Similarly, if you are served by the third clerk, then the chance you are last is:
(3/6.5)(2/3.5) + (2/6.5)(3/4.5) = 0.4689.
The probability you are last is: (3/6.5)(0.1949) + (2/6.5)(0.3363) + (1.5/6.5)(0.4689) = 30.2%.
<u>Comment</u>: Similar to Exercise 5.6 in Ross.]

Greedy Algorithms:[32]

Assume we wish to assign five people to five jobs, with each person assigned to one job.
C_{ij} is the cost when person i is assigned to job j.
We wish to minimize the total cost.
Let us assume the C_{ij} are independent, identically distributed Exponentials; for simplicity assume their means are one.

One possible algorithm would work as follows:[33]
1. Start with person 1.
2. Assign the person to the available job with the lowest cost.
3. Eliminate the job that has been assigned.
4. Go to the next person and unless you have run out of people return to step 2.

The cost for assigning the first person to a job is the minimum of 5 i.i.d. Exponentials, which is an Exponential with rate 5 and thus mean 1/5. Thus the mean cost to assign the first person to a job is 1/5.

Exercise: What is the average cost to assign the second person to a job?
[Solution: Whatever job person one has been assigned to is eliminated.
Thus we choose the minimum of 4 i.i.d. Exponentials; which is an Exponential with mean 1/4.]

Exercise: What is the expected total cost of assigning all 5 people.
[Solution: 1/5 + 1/4 + 1/3 + 1/2 + 1 = 2.283.

Comment: With n people the expected total cost is: $\sum_{i=1}^{n} 1/i$.]

A different possible algorithm would work as follows:
1. Choose the smallest remaining C_{ij}.
2. Assign person i to job j.
3. Eliminate person i and j job that have been assigned.
4. Unless everyone has been assigned return to step 2.

For the very first assignment, we have picked the C_{ij} that is the smallest of the 25.
Thus we have the minimum of 25 independent identically distributed Exponentials each with mean one, which is an Exponential with mean 1/25. The expected cost of the first assignment is 1/25.

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32 See Example 5.7 of Introduction to Probability Models by Ross.
33 Alternately, one could start with a random person in step 1 and pick a random person out of the remaining people in step 4.

Now we eliminate the assigned job and person, leaving us with the task of assigning four people to four jobs; we have 4^2 = 16 remaining costs to choose from. However, remember that we have already selected the minimum of the original 25 times and used it. Thus the expected value of the remaining training costs is higher than the a priori mean of 1.

Thus we have to be a little careful in our analysis. We will use a special property of Exponentials.

Let X and Y are independent, identically distributed Exponential variables with mean 100.
I choose the minimum of X and Y; let us assume it is Y and has a value of 77.
Then X - 77 is positive.
From the memoryless property of the Exponential X - 77 is also an Exponential with 100.
More specifically, for x > 0, Prob[X - 77 > x | X > 77] = Prob[X > 77 + x | X > 77] =
Prob[X > 77 + x] / Prob[X > 77] = exp[-(77+x)/100] / exp[-77/100] = exp[-x/100].

More generally, when one chooses the minimum of a sample of i.i.d. Exponentials, if one subtracts the minimum from each of the remaining variables, then one gets i.i.d. Exponentials with the original mean. This is a special property of Exponentials.

Call the cost of the first assignment A_1. $E[A_1]$ = 1/25.
If we subtract A_1 from each of the remaining 16 training costs, then we get 16 i.i.d. Exponentials with the original mean of one. The minimum of these 16 i.i.d. Exponentials has expected value 1/16. Adding back in A_1 that we subtracted from each training cost, the expected cost of the second assignment is: $E[A_1]$ + 1/16 = 1/25 + 1/16.

Exercise: What is the expected cost of the third assignment?
[Solution: Call the cost of the second assignment A_2. $E[A_2]$ = 1/25 + 1/16.
If we subtract A_2 from the remaining 9 training costs, they follow an Exponential with the original mean of one. The minimum of these 9 i.i.d. Exponentials has expected value 1/9. Thus, the expected cost of the third assignment is: $E[A_2]$ + 1/9 = 1/25 + 1/16 + 1/9.
Comment: We add $1/3^2$ = 1/9 to the expected value of the second assignment.]

Exercise: Using this second algorithm, what is the expected total cost of assigning all 5 people?
[Solution: 1/25 + (1/25 + 1/16) + (1/25 + 1/16 + 1/9) + (1/25 + 1/16 + 1/9 + 1/4) +
(1/25 + 1/16 + 1/9 + 1/4 + 1) = 5/25 + 4/16 + 3/9 + 2/4 + 1
= 1/5 + 1/4 + 1/3 + 1/2 + 1 = 2.283.
Comment: The same expected total cost as for the first algorithm. Ross shows that in general given the assumptions, the two algorithms will have the same expected cost. Note that for any particular set of costs, the two algorithms will usually produce different assignments with different total costs.]

Integrals Involving the Density of the Exponential Distribution:

Let $f(x) = e^{-x/\theta} / \theta$, the density of an Exponential Distribution with mean θ.

Then, $$\int_0^x t^n f(t)\, dt = \theta^n \left(1 - \sum_{i=0}^{i=n} (x/\theta)^i e^{-x/\theta}/i!\right) n!.$$ [34]

$$\int_0^x t\, f(t)\, dt = \theta\, \{1 - e^{-x/\theta} - (x/\theta)e^{-x/\theta}\}\, 1! = \theta - (\theta + x)e^{-x/\theta}.$$

$$\int_0^x t^2 f(t)\, dt = \theta^2 \{1 - e^{-x/\theta} - (x/\theta)e^{-x/\theta} - (x/\theta)^2 e^{-x/\theta}/2\}\, 2! = 2\theta^2 - (2\theta^2 + 2\theta x + x^2)e^{-x/\theta}.$$

$$\int_0^x t^3 f(t)\, dt = \theta^3 \{1 - e^{-x/\theta} - (x/\theta)e^{-x/\theta} - (x/\theta)^2 e^{-x/\theta}/2 - (x/\theta)^3 e^{-x/\theta}/6\}\, 3!$$

$$= 6\theta^3 - (6\theta^3 + 6\theta^2 x + 3\theta x^2 + x^3)e^{-x/\theta}.$$

Exercise: For an exponential distribution with mean 10, what is $\int_7^{15} x\, f(x)\, dx$?

[Solution: $\int_0^{15} x\, f(x)\, dx = (10)\, \{1 - e^{-15/10} - (15/10)e^{-15/10}\} = 4.422.$

$\int_0^{7} x\, f(x)\, dx = (10)\, \{1 - e^{-7/10} - (7/10)e^{-7/10}\} = 1.558.$

$\int_7^{15} x\, f(x)\, dx = 4.422 - 1.558 = 2.864.$]

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[34] This result is based on the Incomplete Gamma Function.
One can derive individual results via integration by parts.

Sums of Exponential Variables:[35]

The sum of n independent Exponential Distributions each with different hazard rate λ_i, has density:[36]

$$f(t) = \sum_{i=1}^{n} \lambda_i \exp[-\lambda_i t] \prod_{j \neq i} \frac{\lambda_j}{\lambda_j - \lambda_i}.$$

Exercise: What is the density of the sum of two independent distributed Exponentials one with hazard rate 0.1 and the other with hazard rate 0.5?

[Solution: $f(t) = 0.1 \exp[-0.1t] \frac{0.5}{0.5 - 0.1} + 0.5 \exp[-0.5t] \frac{0.1}{0.1 - 0.5} = 0.125\, e^{-0.1t} - 0.125\, e^{-0.5t}$.]

Exercise: What is the density of the sum of three independent distributed Exponentials with hazard rates 0.1, 0.3, and 0.5?

[Solution: $f(t) = 0.1 \exp[-0.1t] \frac{0.3}{0.3 - 0.1} \frac{0.5}{0.5 - 0.1} + 0.3 \exp[-0.3t] \frac{0.1}{0.1 - 0.3} \frac{0.5}{0.5 - 0.3}$

$+ 0.5 \exp[-0.5t] \frac{0.1}{0.1 - 0.5} \frac{0.3}{0.3 - 0.5} = 0.1875\, e^{-0.1t} - 0.375\, e^{-0.3t} + 0.1875\, e^{-0.5t}$.]

Sums of a Random Number of Exponential Variables:

One could have the number of Exponential variables summed being a random variable itself.[37]

Exercise: One has a series of three independent Exponential variables, with hazard rates 0.1, 0.5, and 0.3. There is a 60% chance one will have just the first variable, a 30% chance one will sum the first two variables, and a 10% chance one will sum all three variables.
What is the resulting density?

[Solution: From a previous exercise, the density of the sum of the first two Exponentials has density: $f(t) = 0.125\, e^{-0.1t} - 0.125\, e^{-0.5t}$.

From a previous exercise, the density of the sum of the three Exponentials has density:
$f(t) = 0.1875\, e^{-0.1t} - 0.375\, e^{-0.3t} + 0.1875\, e^{-0.5t}$.

Thus the density for the sum of a random number of Exponentials is:

$(0.6)(0.1\, e^{-0.1t}) + (0.3)(0.125\, e^{-0.1t} - 0.125\, e^{-0.5})$

$+ (0.1)(0.1875\, e^{-0.1t} - 0.375\, e^{-0.3t} + 0.1875\, e^{-0.5t})$

$= 0.11625\, e^{-0.1t} - 0.0375\, e^{-0.3t} - 0.01875 e^{-0.5t}$.]

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[35] See Section 5.2.4 of Introduction to Probability Models by Ross.

[36] Ross refers to such a sum as a hypoexponential random variable.

[37] Ross calls this a Coxian random variable.

Problems:

Use the following information for the next eight questions:

Let X be an exponentially distributed random variable, the probability density function of which is: $f(x) = 8\ e^{-8x}$, $x \geq 0$

2.1 (1 point) Which of the following is the mean of X?
A. less than 0.06
B. at least 0.06 but less than 0.08
C. at least 0.08 but less than 0.10
D. at least 0.10 but less than 0.12
E. at least 0.12

2.2 (1 point) Which of the following is the median (50th percentile) of X?
A. less than 0.06
B. at least 0.06 but less than 0.08
C. at least 0.08 but less than 0.10
D. at least 0.10 but less than 0.12
E. at least 0.12

2.3 (1 point) Which of the following is the mode of X?
A. less than 0.06
B. at least 0.06 but less than 0.08
C. at least 0.08 but less than 0.10
D. at least 0.10 but less than 0.12
E. at least 0.12

2.4 (1 point) What is the chance that X is greater than 0.3?
A. less than 0.06
B. at least 0.06 but less than 0.08
C. at least 0.08 but less than 0.10
D. at least 0.10 but less than 0.12
E. at least 0.12

2.5 (1 point) What is the variance of X?
A. less than 0.015
B. at least 0.015 but less than 0.016
C. at least 0.016 but less than 0.017
D. at least 0.017 but less than 0.018
E. at least 0.018

2.6 (1 point) What is the coefficient of variation (standard deviation divided by mean) of X?
A. less than 0.5
B. at least 0.5 but less than 0.7
C. at least 0.7 but less than 0.9
D. at least 0.9 but less than 1.1
E. at least 1.1

2.7 *(2 points) What is the coefficient of skewness of X?*
A. less than 0
B. at least 0 but less than 1
C. at least 1 but less than 2
D. at least 2 but less than 3
E. at least 3

2.8 *(3 points) What is the kurtosis of X?*
A. less than 3
B. at least 3 but less than 5
C. at least 5 but less than 7
D. at least 7 but less than 9
E. at least 9

2.9 (1 point) Prior to the application of any deductible, losses follow an Exponential Distribution with $\theta = 135$. If there is a deductible of 25, what is the density of non-zero payments at 60?
A. less than 0.0045
B. at least 0.0045 but less than 0.0050
C. at least 0.0050 but less than 0.0055
D. at least 0.0055 but less than 0.0060
E. at least 0.0060

2.10 (2 points) You are given the following:
● Claim sizes follow an exponential distribution with density function

$f(x) = 0.1\ e^{-0.1x}$, $0 < x < \infty$.

● You observe 8 claims.
● The number of claims and claim sizes are independent.
Determine the probability that the largest of these claim is less than 17.
A. less than 25%
B. at least 25% but less than 30%
C. at least 30% but less than 35%
D. at least 35% but less than 40%
E. at least 40%

2.11 (2 points) What is the density of the sum of two independent distributed Exponentials one with hazard rate 0.3 and the other with hazard rate 0.8?

2.12 (2 points) You are given:
● Future lifetimes follow an Exponential distribution with a mean of θ.
● The force of interest is δ. In other words the discount factor is $e^{-\delta t}$.
● A whole life insurance policy pays 1 upon death.
What is the actuarial present value of this insurance?
(A) $e^{-\delta\theta}$
(B) $1 / (1 + \delta\theta)$
(C) $e^{-2\delta\theta}$
(D) $1 / (1 + \delta\theta)^2$
(E) None of A, B, C, or D.

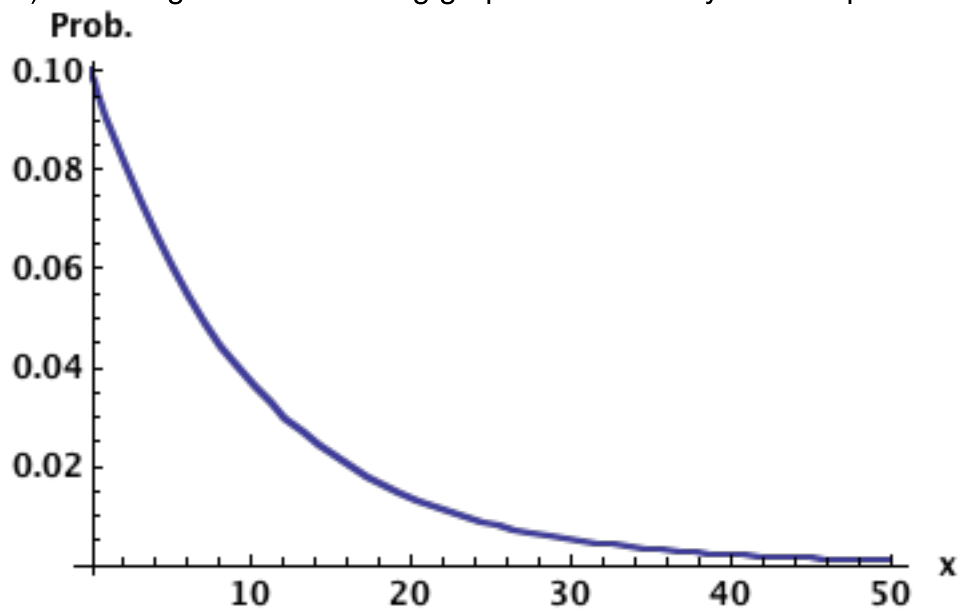
2.13 (1 point) Prior to the application of any deductible, losses follow an Exponential Distribution with $\theta = 25$. If there is a deductible of 5, what is the variance of the non-zero payments?
A. less than 600
B. at least 600 but less than 650
C. at least 650 but less than 700
D. at least 700 but less than 750
E. at least 750

2.14 (2 points) Prior to the application of any deductible, losses follow an Exponential Distribution with $\theta = 31$. There is a deductible of 10. What is the variance of the amount paid by the insurer for one loss, including the possibility that the amount paid is zero?
A. less than 900
B. at least 900 but less than 950
C. at least 950 but less than 1000
D. at least 1000 but less than 1050
E. at least 1050

2.15 (2 points) Size of loss is Exponential with mean θ.
Y is the minimum of N losses.
What is the distribution of Y?

2.16 (2 points) You are given:
● A claimant receives payments at a rate of 1 paid continuously while disabled.
● Payments start immediately.
● The length of disability follows an Exponential distribution with a mean of θ.
● The force of interest is δ. In other words the discount factor is $e^{-\delta t}$.
At the time of disability, what is the actuarial present value of these payments?
(A) $1 / (\delta+\theta)$
(B) $1 / (1 + \delta\theta)$
(C) $\theta / (\delta+\theta)$
(D) $\theta / (1 + \delta\theta)$
(E) None of A, B, C, or D.

2.17 (2 points) You are given the following graph of the density of an Exponential Distribution.



What is the third moment of this Exponential Distribution?
A. 1000 B. 2000 C. 4000 D. 6000 E. 8000

2.18 (3 points) Belle Chimes and Leif Blower are engaged to be married.
The cost of their wedding will be 110,000. They will receive 200 gifts at their wedding.
The size of each gift has distribution: F(x) = 1 - exp[-(x - 100)/500], x > 100.
What is the probability that the total value of the gifts will not exceed the cost of their wedding?
Use the Normal Approximation.
A. 6% B. 8% C. 10% D. 12% E. 14%

2.19 (1 point) Dagwood Bumstead is waiting on a street corner to meet his wife Blondie.
His waiting time is exponentially distributed with mean 6 minutes.
If Dagwood has already waited 20 minutes for Blondie, what is the probability he needs to wait more than an additional 10 minutes?
A. 11% B. 13% C. 15% D. 17% E. 19%

2.20 (3 points) What is the density of the sum of three independent distributed Exponentials with means 2, 5, and 10?

2.21 (1 point) The exponential distribution with a mean of 28,700 hours was used to describe the hours to failure of a fan on diesel engines.
A diesel engine fan has gone 10,000 hours without failing.
Determine the probability of this fan lasting at least an additional 5000 hours.

2.22 (2 points) At a seismically active site, the distribution of the magnitude of earthquakes is given by an Exponential Distribution with hazard rate ln(10).
Over the last 20 years, at that site there have been 150 earthquakes of magnitude at least 3 and less than 5.
Estimate the probability that next year there will be an earthquake of magnitude at least 6.

2.23 (2 points) A set of 4 cities is to be connected via communications links.
The costs to connect each pair of cities are independent exponential random variables with mean of $60 million.
The following algorithm is used to construct the communications links:
- Construct the cheapest link between any two cities.
- Then for the two cities that are not yet linked, construct the cheapest of the links to one of the already linked cities.
- Then for the city that is not yet linked, construct the cheapest of the links to one of the already linked cities.

Determine the expected cost of the second link in millions of dollars.
A. 16 B. 19 C. 22 D. 25 E. 28

Use the following information for the next 5 questions:
- X is a random variable with probability density function: $f(x) = 0.1\ e^{-0.1x}$, $x \geq 0$

2.24 (1 point) Which of the following is the mean of X?
A. 10 B. 12 C. 14 D. 16 E. 18

2.25 (1 point) What is the variance of X?
A. 60 B. 70 C. 80 D. 190 E. 100

2.26 (1 point) What is the chance that X is less than 6?
A. 30% B. 35% C. 40% D. 45% E. 50%

2.27 (1 point) What is the failure rate at 13?
A. 0.06 B. 0.08 C. 0.10 D. 0.12 E. 0.14

2.28 (1 point) What is the median (50th percentile) of X?
A. less than 6
B. at least 6 but less than 8
C. at least 8 but less than 10
D. at least 10 but less than 12
E. at least 12

2.29 (2 points) Monk's Cafe needs to decide how many pounds of pastrami to buy for the upcoming week. Demand for pastrami each week is Exponential with mean 30 pounds.
Monk's pays $10 per pound for pastrami.
Monk's makes a profit of $5 per pound of pastrami sold.
Assume that any pastrami that is not sold is donated to a local food bank.
Assume there is no penalty for failing to meet demand.
How much pastrami should Monk's order so as to maximize its expected profit?
A. less than 15
B. at least 15 but less than 20
C. at least 20 but less than 25
D. at least 25 but less than 30
E. at least 30

Use the following information for the next 5 questions:
- The lifetimes in days of industrial fans are exponential with failure rate 0.0007.
- The lifetimes in days of transformers are exponential with failure rate 0.0028.

2.30 (1 point) Determine the probability that a random transformer will outlast a random industrial fan.

2.31 (1 point) What is the probability that both items are still working after 400 days?

2.32 (1 point) What is the probability that neither item is still working after 800 days?

2.33 (2 points) What is the expected value of the maximum of the two failure times.

2.34 (1 point) You have an industrial fan and a transformer.
One of the two items has just failed.
What is the expected future lifetime of the remaining item?

2.35 (1 point) Losses prior to any deductible follow an Exponential Distribution with $\theta = 2000$.
A policy has a deductible of size 500.
In other words, the insurer pays nothing for a loss of size less than 500,
and pays 500 less than the size of loss for a large loss.
What is the distribution of non-zero payments under that policy?

2.36 (2 point) Two independent variables X and Y have failure rates as a function of time of $r_X(t)$ and $r_Y(t)$. Given that the minimum of X and Y is m, what is the probability that $X < Y$?

2.37 (2 points) Bob has two friends Howard and Jerry, who are stopping by to visit today.
The time until Howard arrives is Exponential with mean 30 minutes.
The length of Howard's visit is Exponential with mean 70 minutes.
The time until Jerry arrives is Exponential with mean 50 minutes.
The length of Jerry's visit is Exponential with mean 80 minutes.
Determine the probability that Howard arrives first and leaves last.
A. 9% B. 11% C. 13% D. 15% E. 17%

Use the following information for the next 4 questions:
- There are two brands of batteries in a drawer.
- The lifetime of each battery is Exponential.
- 30% of the batteries are Neverending, with failure rate of 0.01 / hour.
- 70% of the batteries are Durabell, with failure rate of 0.005 / hour.
- You pick a battery at random

2.38 (1 point) What is the probability that the battery you picked lasts less than 80 hours?
A. 36% B. 38% C. 40% D. 42% E. 44%

2.39 (1 point) What is the average time the battery you picked lasts?
A. 150 B. 170 C. 190 D. 210 E. 230

2.40 (1 point) What is the variance of the time that the battery you picked lasts?
A. 27,000 B. 29,000 C. 31,000 D. 33,000 E. 35,000

2.41 (2 points) What is the failure rate at 120 hours?
A. 0.006 B. 0.008 C. 0.010 D. 0.012 E. 0.014

2.42 (2 points) Thelma and Louise are running a race against each other.
The time it takes Thelma to finish the race is exponentially distributed with mean 5.
The time it takes Thelma to finish is independent of the time it takes Louise to finish the race, which is exponentially distributed with mean 4.
The one who finishes first is awarded \$1000 $e^{-0.2t}$, where t is the time it took to finish.
The loser gets nothing. What is the expected amount that Thelma wins?
A. 310 B. 330 C. 350 D. 370 E. 390

2.43 (2 points) You have just reached the head of a single line waiting to buy tickets.
There are three clerks currently each dealing with a customer.
The time for a clerk to deal with a customer is Exponentially Distributed.
However the first clerk has a mean service time of 1/5 minute, the second clerk has a mean service time of 1/4 minute, and the final clerk has a mean service time of 2/5 minute.
You will be served by whichever clerk finishes with his current customer first.
What is the average time in seconds until you are done buying your ticket?
A. less than 18
B. at least 18 but less than 20
C. at least 20 but less than 22
D. at least 24 but less than 24
E. at least 26

Use the following information for the next 9 questions:
• X and Y are independent Exponential Variables.
• X has mean 5.
• Y has mean 8.
• M is the minimum of X and Y

2.44 (2 points) Determine E[X M | M = X].

2.45 (3 points) Determine E[X M | M = Y].

2.46 (2 points) Determine Cov[X, M].

2.47 (2 points) Determine E[Y M | M = Y].

2.48 (3 points) Determine E[Y M | M = X].

2.49 (2 points) Determine Cov[Y, M].

2.50 (2 points) Determine the covariance of the minimum and the maximum of X and Y.

2.51 (3 points) Determine the variance of the maximum of X and Y.

2.52 (2 points) Determine the correlation of M and the maximum of X and Y.

2.53 (2 points) The lifetime of batteries is Exponential with mean 6.
Batteries are sold for $100 each.
If a battery lasts less than 2 years, the manufacturer will pay the purchaser the pro rata share of the purchase price. For example if the battery lasts only 1.5 years, the manufacturer will pay the purchaser (100)(2 - 1.5)/2 = 25.
What is the expected amount paid by the manufacturer per battery sold?
(A) 11 (B) 13 (C) 15 (D) 17 (E) 19

2.54 (2 points) The random variable X has the density function:

$$f(x) = 0.4 \exp(-x/1128)/1128 + 0.6 \exp(-x/5915)/5915, \quad 0 < x < \infty.$$

Determine the variance of X.
(A) 21 million (B) 23 million (C) 25 million (D) 27 million (E) 29 million

2.55 (4 points) X and Y are two independent Exponential Distributions with failure rates λ_1 and λ_2.
Let V = X - Y. Determine the density of V.

Use the following information for the next 4 questions:
- Robots can fail due to two independent decrements: Internal and External.
  (Internal includes normal wear and tear. External includes accidents.)
- Assuming no external events, a robot's time until failure is given by an Exponential Distribution with $\theta = 20$.
- Assuming no internal events, a robot's time until failure is given by an Exponential Distribution with $\theta = 40$.

2.56 (1 point) What is the probability that a robot does not fail by time 50?

2.57 (1 point) What is the probability that a robot fails first due to an external cause?

2.58 (1 point) What is the distribution of the time until failure assuming the robot fails first due to internal causes rather than external causes.

2.59 (1 point) Robby the robot has been functioning for a period of time 30.
At that point Roderick the robot starts functioning.
What is the probability that Robby is the first one to fail?

2.60 (2 points) You are given the following information about a random variable, X.
- For all r, s ≥ 0, $\Pr(X > r + s) = \Pr(X > r) \Pr(X > s)$.
- $E[X \mid X > 200] = 600$.

Determine $E[X \mid X > 100]$.
A. Less than 420
B. At least 420, but less than 440
C. At least 440, but less than 460
D. At least 460, but less than 480
E. At least 480

2.61 (3 points) X and Y are two independent exponential random variables with hazard rates $\lambda_X = 0.3$ and $\lambda_Y = 0.5$, respectively.
Calculate the expected value of Y, conditional on $2 < Y < X$.
A. Less than 3.1
B. At least 3.1, but less than 3.3
C. At least 3.3, but less than 3.5
D. At least 3.5, but less than 3.7
E. At least 3.7

Use the following information for the next 4 questions:
$F(x) = (0.3)(1 - e^{-x/10}) + (0.5)(1 - e^{-x/25}) + (0.2)(1 - e^{-x/50})$.

2.62 (1 point) What is the probability that x is more than 15?
A. 48% B. 49% C. 50% D. 51% E. 52%

2.63 (1 point) What is the mean?
A. 25 B. 30 C. 35 D. 40 E. 45

2.64 (1 point) What is the variance?
A. Less than 1000
B. At least 1000, but less than 1200
C. At least 1200, but less than 1400
D. At least 1400, but less than 1600
E. At least 1600

2.65 (2 points) What is the hazard rate (failure rate) at 50?
A. Less than 0.020
B. At least 0.020, but less than 0.025
C. At least 0.025, but less than 0.030
D. At least 0.030, but less than 0.035
E. At least 0.035

2.66 (2 points) Matthew, Mark, Luke, and John are four actuaries who work at a consulting firm. They are to each be assigned to one of four different consulting assignments by their boss Paul. The time for each actuary to complete each assignment is an independent exponential random variable with mean of 300 hours.
To minimize the total time spent working on these assignments, Paul uses the following procedure to assign his employees:
- Matthew is assigned to the task which minimizes his time spent.
- Mark is then assigned to the remaining task which minimizes his time spent.
- Luke is then assigned to the remaining task which minimizes his time spent.
- John is then assigned to the remaining task.

Calculate the total expected time for these four actuaries to complete their assignments.
A. Less than 600
B. At least 600, but less than 650
C. At least 650, but less than 700
D. At least 700, but less than 750
E. At least 750

2.67 (1 point) You are given the following:
- The size of loss distribution is given by

$f(x) = 2e^{-2x}, x > 0$
- Under a basic limits policy, individual losses are capped at 1.
- The expected annual claim frequency is 13.

What are the expected annual total loss payments on a basic limits policy?
A. less than 5.0
B. at least 5.0 but less than 5.5
C. at least 5.5 but less than 6.0
D. at least 6.0 but less than 6.5
E. at least 6.5

2.68 (3 points) You are given the following information
- An electronic device has 7 different parts:
- There are 3 resistors with expected lifetimes of 30, 40, and 60.
- There are 4 transistors with expected lifetimes of 20, 50, 70, and 80.
- The lifetimes of all parts are independent and exponentially distributed.

Calculate the probability that a resistor will fail before a transistor.
A. Less than 0.35
B. At least 0.35, but less than 0.40
C. At least 0.40, but less than 0.45
D. At least 0.45, but less than 0.50
E. At least 0.50

2.69 (3 points) John, Paul, George, and Ringo are consultants.
They are to each be assigned to one of four different consulting assignments by their boss "Beetle" Bailey.
The time for each a consultant to complete each assignment is an independent exponential random variable with mean of 576 hours.
To minimize the total time spent working on these assignments, Bailey uses the following procedure to assign her employees:
- She finds the lowest time, and assigns the corresponding consultant to that assignment.
- This leaves only 9 remaining times, 3 employees for each of 3 assignments. Bailey finds the lowest remaining time, and assigns the corresponding consultant to that assignment.
- This leaves only 4 remaining times, 2 employees for each of 2 assignments. Bailey finds the lowest remaining time, and assigns the corresponding consultant to that assignment.
- The remaining consultant is then assigned to the remaining assignment.

Calculate the total expected time for these four consultants to complete their assignments.
A. 1000 B. 1200 C. 1400 D. 1600 E. 1800

2.70 (3 points) A call center currently has 3 representatives and 2 interns who can handle calls. If all representatives including interns are currently on a call, an incoming call will be placed on hold until either a representative or intern is available.
You are given the following information:
• For each representative, the time taken to handle each call is given by an exponential distribution with a mean value equal to 4.
• For each intern, the time taken to handle each call is given by an exponential distribution with a mean value equal to 6.
• Handle times are independent
A customer calls the call center and is placed on hold, and is the first person in line.
Calculate the expected time to complete the call (including both hold time and service).
A. Less than 5.4
B. At least 5.4, but less than 5.5
C. At least 5.5, but less than 5.6
D. At least 5.6, but less than 5.7
E. At least 5.7

2.71 (2 points) Losses follow a memoryless distribution with mean 800.
Each loss is insured and subject to a deductible of 250.
Calculate the average insurance payment made on losses that exceed the deductible.
A. Less than 700
B. At least 700, but less than 800
C. At least 800, but less than 900
D. At least 900, but less than 1000
E. At least 1000

2.72 (2 points) You are given:
• The times until Sodium 24 atoms decay into Magnesium 24 are independent and exponentially distributed with a mean of 1295 minutes.
• Sodium 24 Atom I has been observed for 2000 minutes without decaying.
• Sodium 24 Atom II has just come under observation.
• Calculate the absolute difference between Atom I's failure rate and Atom II's failure rate.
A. Less than 0.01
B. At least 0.01, but less than 0.02
C. At least 0.02, but less than 0.03
D. At least 0.03, but less than 0.04
E. At least 0.04

2.73 (2 points) You are given the following information:
- V, X, Y, and Z are independent random variables.
- V, X, Y, and Z follow exponential distributions with means 8, 10, 12, and 15, respectively.
- A single observation is taken from each of the four random variables.

Calculate the probability that the maximum of the four observed values is more than 16.
A. Less than 0.70
B. At least 0.70, but less than 0.75
C. At least 0.75, but less than 0.80
D. At least 0.80, but less than 0.85
E. At least 0.85

2.74 (2 points) You are given the following information:
- The severity of each theft loss in an insurance policy independently follows an exponential distribution with mean 400.
- The insurance company only pays the amount exceeding the per-loss deductible of 100.

Calculate the standard deviation of the amount the insurance company pays per theft.
A. Less than 200
B. At least 200, but less than 250
C. At least 250, but less than 300
D. At least 300, but less than 350
E. At least 350

2.75 (160, 11/86, Q.9) (2.1 points) X_1 and X_2 are independent random variables each with Exponential distributions. The expected value of X_1 is 9.5. The variance of X_2 is 2.25.
Determine the probability that $X_1 < X_2$.
(A) 2/19 (B) 3/22 (C) 3/19 (D) 3/16 (E) 2/3

2.76 (4, 5/87, Q.32) (1 point) Let X be an exponentially distributed random variable, the probability density function of which is: f(x) = 10 exp(-10x), where $x \geq 0$.
Which of the following statements regarding the mode and median of X is true?
A. The median of X is 0; the mode is 1/2.
B. The median of X is (ln 2) / 10; the mode of X is 0.
C. The median of X is 1/2; the mode of X does not exist.
D. The median of X is 1/2; the mode of X is 0.
E. The median of X is 1/10; and the mode of X is (ln 2) /10.

2.77 (2, 5/90, Q.11) (1.7 points) Let X be a continuous random variable with density function $f(x) = \lambda e^{-\lambda x}$ for $x > 0$. If the median of this distribution is 1/3, then what is λ?
A. (1/3) ln(1/2) B. (1/3) ln(2) C. 2 ln(3/2) D. 3 ln(2) E. 3

2.78 (2, 2/96, Q.40) (1.7 points)
Let $X_1, \ldots, X_{100}$ be a random sample from an exponential distribution with mean 1/2.
Determine the approximate value of $P[\sum_{i=1}^{100} X_i > 57]$ using the Central Limit Theorem.
A. 0.08 B. 0.16 C. 0.31 D. 0.38 E. 0.46

2.79 (2, 2/96, Q.41) (1.7 points) Let X be a continuous random variable with density function $f(x) = e^{-x/2}/2$ for $x > 0$. Determine the 25th percentile of the distribution of X.
A. ln(4/9) B. ln(16/9) C. ln(4) D. 2 E. ln(16)

2.80 (Course 160 Sample Exam #2, 1996, Q.1) (1.9 points) You are given:
(i) Two independent random variables X_1 and X_2 have exponential distributions with means θ_1 and θ_2, respectively.
(ii) $Y = X_1 X_2$.
Determine E[Y].
(A) $\frac{1}{\theta_1} + \frac{1}{\theta_2}$ (B) $\frac{\theta_1 \theta_2}{\theta_1 + \theta_2}$ (C) $\theta_1 + \theta_2$ (D) $\frac{1}{\theta_1 \theta_2}$ (E) $\theta_1\theta_2$

2.81 (4B, 11/99, Q.17) (2 points) Claim sizes follow a distribution with density function $f(x) = e^{-x}$, $0 < x < \infty$. Determine the probability that the second claim observed will be more than twice as large as the first claim observed.
A. e^{-3} B. e^{-2} C. 1/3 D. e^{-1} E. 1/2

2.82 (Course 1 Sample Exam, Q.23) (1.9 points) The value, v, of an appliance is based on the number of years since purchase, t, as follows: $v(t) = e^{(7 - 0.2t)}$.
If the appliance fails within seven years of purchase, a warranty pays the owner the value of the appliance. After seven years, the warranty pays nothing.
The time until failure of the appliance has an exponential distribution with mean 10.
Calculate the expected payment from the warranty.
A. 98.70 B. 109.66 C. 270.43 D. 320.78 E. 352.16

2.83 (1, 5/00, Q.3) (1.9 points) The lifetime of a printer costing 200 is exponentially distributed with mean 2 years. The manufacturer agrees to pay a full refund to a buyer if the printer fails during the first year following its purchase, and a one-half refund if it fails during the second year. If the manufacturer sells 100 printers, how much should it expect to pay in refunds?
(A) 6,321 (B) 7,358 (C) 7,869 (D) 10,256 (E) 12,642

2.84 (1, 5/00, Q.18) (1.9 points) An insurance policy reimburses dental expense, X, up to a maximum benefit of 250. The probability density function for X is: $c\, e^{-0.004x}$ for $x > 0$, where c is a constant. Calculate the median benefit for this policy.
(A) 161 (B) 165 (C) 173 (D) 182 (E) 250

2.85 (1, 11/00, Q.9) (1.9 points) An insurance company sells an auto insurance policy that covers losses incurred by a policyholder, subject to a deductible of 100.
Losses incurred follow an exponential distribution with mean 300.
What is the 95th percentile of actual losses that exceed the deductible?
(A) 600 (B) 700 (C) 800 (D) 900 (E) 1000

2.86 (1, 11/00, Q.14) (1.9 points) A piece of equipment is being insured against early failure. The time from purchase until failure of the equipment is exponentially distributed with mean 10 years. The insurance will pay an amount x if the equipment fails during the first year, and it will pay 0.5x if failure occurs during the second or third year. If failure occurs after the first three years, no payment will be made. At what level must x be set if the expected payment made under this insurance is to be 1000?
(A) 3858 (B) 4449 (C) 5382 (D) 5644 (E) 7235

2.87 (3, 11/00, Q.21) (2.5 points)
A claim severity distribution is exponential with mean 1000.
An insurance company will pay the amount of each claim in excess of a deductible of 100.
Calculate the variance of the amount paid by the insurance company for one claim, including the possibility that the amount paid is 0.
(A) 810,000 (B) 860,000 (C) 900,000 (D) 990,000 (E) 1,000,000

2.88 (1, 5/01, Q.20) (1.9 points) A device that continuously measures and records seismic activity is placed in a remote region. The time, T, to failure of this device is exponentially distributed with mean 3 years. Since the device will not be monitored during its first two years of service, the time to discovery of its failure is X = max(T, 2). Determine E[X] .
(A) $2 + e^{-6}/3$ (B) $2 - 2e^{-2/3} + 5e^{-4/3}$ (C) 3 (D) $2 + 3e^{-2/3}$ (E) 5

2.89 (1, 5/01, Q.32) (1.9 points) A company has two electric generators.
The time until failure for each generator follows an exponential distribution with mean 10.
The company will begin using the second generator immediately after the first one fails.
What is the variance of the total time that the generators produce electricity?
(A) 10 (B) 20 (C) 50 (D) 100 (E) 200

2.90 (1, 5/03, Q.4) (2.5 points) The time to failure of a component in an electronic device has an exponential distribution with a median of four hours. Calculate the probability that the component will work without failing for at least five hours.
(A) 0.07 (B) 0.29 (C) 0.38 (D) 0.42 (E) 0.57

2.91 (SOA M, 11/05, Q.14) (2.5 points) You are given:
(i) T is the future lifetime random variable.
(ii) $h(t) = \mu$, $t \geq 0$, where h(t) is the hazard rate (the failure rate).
(iii) Var[T] = 100.
Calculate $E[T \wedge 10] = E[\text{Min}[T, 10]]$.
(A) 2.6 (B) 5.4 (C) 6.3 (D) 9.5 (E) 10.0

2.92 (CAS LC, 11/14, Q.6) (2.5 points) You are given two independent lives (x) and (y) with constant forces of mortality (failure rates) μ and $k\mu$ respectively, where $k \geq 1$.
You learn that the expected time to the second death is equal to three times the expected time to the first death.
Calculate k.
A. Less than 1. 5
B. At least 1.5, but less than 2.0
C. At least 2.0, but less than 2.5
D. At least 2.5, but less than 3.0
E. At least 3.0

2.93 (CAS LC, 5/15, Q.4) (2.5 points)
You are given the following information:
• T_x and T_y are independent time-to-failure random variables.
• T_x has an exponential distribution with mean 20.
• T_y has an exponential distribution with mean 5.
• T_{xy} is the time until the first one of them fails.
Calculate Var(T_{xy}).
A. Less than 14.5
B. At least 14.5, but less than 15.5
C. At least 15.5, but less than 16.5
D. At least 16.5, but less than 17.5
E. At least 17.5

2.94 (CAS S, 11/15, Q.5) (2.2 points) You are given the following information:
• The amount of damage involved in a home theft loss is an Exponential random variable with mean 2,000.
• The insurance company only pays the amount exceeding the deductible amount of 500.
• The insurance company is considering changing the deductible to 1,000.
Calculate the absolute value of the change in the expected value of the amount the insurance company pays per theft loss by changing the deductible from 500 to 1,000.
A. Less than 330
B. At least 330, but less than 350
C. At least 350, but less than 370
D. At least 370, but less than 390
E. At least 390

2.95 (CAS S, 5/16, Q.1) (2.2 points) Alice, Bob and Chris are hired by a firm to drive three vehicles - a bus, a taxi and a train (only one driver is needed for each). Each employee has different skills and requires different amounts of training on each vehicle. The cost to train each employee i for vehicle j, $C_{i,j}$, is an independent exponential random variable with mean of 200.
To minimize the total cost of training, the firm uses the following procedure to assign the employees to their vehicle:
- Alice is assigned to the vehicle which minimizes her training cost, $C_{Alice,j}$.
- Bob is then assigned one of the two remaining vehicles which minimizes $C_{Bob,j}$.
- Chris is then assigned the remaining vehicle.

Calculate the firm's expected total cost of training these three employees, if it uses this assignment algorithm.
A. Less than 200
B. At least 200, but less than 300
C. At least 300, but less than 400
D. At least 400, but less than 500
E. At least 500

2.96. (3 points) Use the information from the previous question CAS S, 5/16, Q.1.
To minimize the total cost of training, the firm instead uses the following procedure to assign the employees to their vehicle:
- The firm finds the lowest training cost, and assigns the corresponding driver to that vehicle.
- This leaves only 4 remaining training costs, 2 employees for each of 2 vehicles.
  The firm finds the lowest remaining training cost, and assigns the corresponding driver to that vehicle.
- The remaining employee is then assigned to the remaining vehicle.

Calculate the firm's expected total cost of training these three employees, if it uses this assignment algorithm.

2.97 (CAS S, 5/16, Q.6) (2.2 points) You are given the following information:
- The lifetimes of all light bulbs follow the exponential distribution.
- A new incandescent light bulb has a hazard rate of 0.20 per year.
- The expected lifetime of an LED light bulb is twice the expected future lifetime of a 2-year-old incandescent light bulb.

Calculate the hazard rate per year for an LED light bulb.
A. Less than 0.05
B. At least 0.05, but less than 0.07
C. At least 0.07, but less than 0.09
D. At least 0.09, but less than 0.11
E. At least 0.11
<u>Note</u>: I have slightly reworded this past exam question.

2.98 (CAS S, 11/16, Q.3) (2.2 points)
The time X to wait in line is an exponentially distributed random variable with mean 5 minutes.
Calculate the probability that the total waiting time will be longer than 30 minutes from the time that individual arrived in line, given that the wait has already been 20 minutes.
A. Less than 0.1
B. At least 0.1, but less than 0.2
C. At least 0.2, but less than 0.3
D. At least 0.3, but less than 0.4
E. At least 0.4

2.99 (CAS S, 11/16, Q.5) (2.2 points)
A call center currently has 2 representatives and 2 interns who can handle customer calls.
If all representatives including interns are currently on a call, an incoming call will be placed on hold until a representative or intern is available.
You are given the following information:
• For each representative, the time taken to handle each call is given by an exponential distribution with a mean value equal to 1.
• For each intern, the time taken to handle each call is given by an exponential distribution with mean 2.
• Handle times are independent
A customer calls the call center and is placed on hold, and is the first person in line.
Calculate the expected time to complete the call (including both hold time and service).
A. Less than 1.2
B. At least 1.2, but less than 1.4
C. At least 1.4, but less than 1.6
D. At least 1.6, but less than 1.8
E. At least 1.8

2.100 (CAS S, 11/16, Q.7) (2.2 points)
You are given the following information about a watch with 6 different parts:
• There are 3 red wires with expected lifetimes of 50, 75, and 100.
• There are 3 yellow wires with expected lifetimes of 25, 50, and 75.
• The lifetimes of all wires are independent and exponentially distributed.
Calculate the probability that a red wire will break down before a yellow wire.
A. Less than 0.20
B. At least 0.20, but less than 0.25
C. At least 0.25, but less than 0.30
D. At least 0.30, but less than 0.35
E. At least 0.35

2.101 (CAS S, 5/17, Q.1) (2.2 points)
You are given the following information about a random variable, X.
- For all r, s ≥ 0, Pr(X > r + s) = Pr(X > r) Pr(X > s).
- E[X I X > 10] = 30.

Calculate the result for the expression: E[X I X > 20].
A. Less than 30
B. At least 30, but less than 40
C. At least 40, but less than 50
D. At least 50, but less than 60
E. At least 60

2.102 (CAS S, 5/17, Q.6) (2.2 points) X and Y are two independent exponential random variables with hazard rates $\lambda_X = 2$ and $\lambda_Y = 8$, respectively.
Calculate the expected value of X, conditional on 1 < X < Y.
A. Less than 1.20
B. At least 1.20, but less than 1.40
C. At least 1.40, but less than 1.60
D. At least 1.60, but less than 1.80
E. At least 1.80

2.103 (CAS S, 11/17, Q.6) (2.2 points) A customer calls a customer service center with two servers. Both servers are busy at the time of the call, therefore the customer is placed in a queue that is currently empty. The next available server will handle this customer's call.
You are given:
- Service times are exponentially distributed and independent.
- Server 1 is more experienced, so she handles calls in 5 minutes on average.
- Server 2 is less experienced, so she handles calls in 7 minutes on average.

Calculate the expected total waiting plus service time, in minutes, until this customer is finished.
A. Less than 7
B. At least 7, but less than 8
C. At least 8, but less than 9
D. At least 9, but less than 10
E. At least 10

2.104 (CAS S, 11/17, Q.7) (2.2 points) A produce vendor must decide how many oranges to order per week in order to maximize her profit. You are given:
- The cost of oranges to the vendor is 0.50 per orange.
- The vendor is able to sell oranges at 1.00 per orange.
- Quantity demanded follows an exponential distribution with a hazard rate of λ of 0.001 per week.
- Any inventory left over at the end of the week will rot and is worthless.
- There is no penalty to the vendor if she cannot meet all of the demand.

Calculate the optimal amount of oranges the vendor should purchase in a week to maximize profit.
A. Less than 650
B. At least 650 but less than 675
C. At least 675 but less than 700
D. At least 700 but less than 725
E. At least 725
Note: I have rewritten this past exam question.

2.105 (MAS-1, 5/18, Q.1) (2.2 points) You are given the following information:
- The amount of time one spends in an IRS office is exponentially distributed with a mean of 30 minutes.
- P_1 is the probability that John will spend more than an hour of total time in the IRS office, given that he has already been in the office for 20 minutes.
- P_2 is the probability that Lucy will spend more than an hour of total time in the IRS office, given that she has just arrived.

Calculate the difference $(P_1 - P_2)$·
A. Less than 0.00
B. At least 0.00, but less than 0.05
C. At least 0.05, but less than 0.10
D. At least 0.10, but less than 0.15
E. At least 0.15

2.106 (MAS-1, 5/18, Q.5) (2.2 points) You are given:
- Computer lifetimes are independent and exponentially distributed with a mean of 24 months.
- Computer I has been functioning properly for 36 months.
- Computer II is a brand new and functioning computer.
- Calculate the absolute difference between Computer I's failure rate and Computer II's failure rate.

A. Less than 0.01
B. At least 0.01, but less than 0.02
C. At least 0.02, but less than 0.03
D. At least 0.03, but less than 0.04
E. At least 0.04

2.107 (MAS-1, 5/18, Q.23) (2.2 points) You are given the following information:
- X, Y, and Z are independent random variables.
- X, Y, and Z follow exponential distributions with means 2, 4, and 5, respectively.
- A single observation is taken from each of the three random variables.

Calculate the probability that the maximum of the three observed values is less than 6.
A. Less than 0.20
B. At least 0.20, but less than 0.30
C. At least 0.30, but less than 0.40
D. At least 0.40, but less than 0.50
E. At least 0.50

2.108 (MAS-1, 5/19, Q.4) (2.2 points) You are given the following information:
- Computer lifetimes are exponentially distributed with mean of 10 months.
- Computer A has been functioning properly for 12 months.

Calculate the probability that Computer A will function properly for at least 4 more months.
A. Less than 0.64
B. At least 0.64, but less than 0.66
C. At least 0.66, but less than 0.68
D. At least 0.68, but less than 0.70
E. At least 0.70

2.109 (MAS-1, 5/19, Q.6) (2.2 points) You are given the following information:
- The severity of each theft loss in a Homeowners insurance policy independently follows an exponential distribution with mean 2,000.
- The insurance company only pays the amount exceeding the per-loss deductible.
- σ_j is the standard deviation of the amount the insurance company pays per theft with a deductible of j.

Calculate the absolute difference between σ_{1000} and σ_{500}.
A. Less than 90
B. At least 90, but less than 100
C. At least 100, but less than 110
D. At least 110, but less than 120
E. At least 120

2.110 (MAS-1, 11/19, Q.1) (2.2 points)
Losses follow a memoryless distribution with mean 1,000.
Each loss is insured and subject to a deductible of 500.
Calculate the average insurance payment made on losses that exceed the deductible.
A. Less than 500
B. At least 500, but less than 700
C. At least 700, but less than 900
D. At least 900, but less than 1100
E. At least 1100

Solutions to Problems:

2.1. E. An exponential with θ =1/8; mean = θ = **0.125**.

2.2. C. An exponential with θ = 1/8; $F(x) = 1 - e^{-x/\theta} = 1 - e^{-8x}$.
At the median: F(0.5) = 0.5 = $1 - e^{-8x}$. $\Rightarrow$ x = -ln(0.5)/8 = **0.0866**.

2.3. A. The mode of the exponential is always **zero**.
(The density $8e^{-8x}$ decreases for x > 0 and thus attains its maximum at x=0.)

2.4. C. An exponential with $1/\theta = 8$; $F(x) = 1 - e^{-x/\theta} = 1 - e^{-8x}$.
$1 - F(0.3) = e^{-(8)(0.3)} = e^{-2.4}$ = **0.0907**.

2.5. B. An exponential with $1/\theta = 8$; variance = θ^2 = **0.015625**.

2.6. D. An exponential always has a coefficient of variation of **1**.
The C.V. = standard deviation / mean = $(0.015625)^{0.5}$ / 0.125 = 1.

2.7. D. An exponential always has skewness of **2**. Specifically the moments are:
$E[X] = (1!)\ \theta^1 = 1/8 = 0.125$. $E[X^2] = (2!)\ \theta^2 = 2 / 8^2 = 0.03125$.
$E[X^3] = (3!)\ \theta^3 = 6 / 8^3 = 0.01172$.
Standard Deviation = $(0.03125 - 0.125^2)^{0.5}$ = 0.125.
Skewness = $\{E[X^3] - (3\ E[X]\ E[X^2]) + (2\ E[X]^3)\}$ / $STDDEV^3$ =
$\{0.01172 - (3)(0.125)(0.03125)) + (2)(0.125)^3)\} / 0.125^3$ = 0.0039075 / 0.001953 = **2.00**.

2.8. E. An exponential always has kurtosis of **9**. Specifically the moments are:
$E[X] = (1!)\ \theta^1 = \theta$. $E[X^2] = (2!)\ \theta^2 = 2\theta^2$. $E[X^3] = (3!)\ \theta^3 = 6\theta^3$. $E[X^4] = (4!)\ \theta^4 = 24\ \theta^4$.
Standard Deviation = $(2\theta^2 - \theta^2)^{0.5} = \theta$.
fourth central moment = $E[X^4] - (4\ E[X]\ E[X^3]) + (6\ E[X]^2 E[X^2]) - 3\,E[X]^4$

$$= 24\ \theta^4 - (4)(\theta)(6\theta^3) + (6)(\theta^2)(2\theta^2) - (3)\theta^4 = 9\theta^4.$$

kurtosis = (fourth central moment) / $STDDEV^4 = 9\theta^4 / \theta^4$ = **9**.

2.9. B. After truncating and shifting from below, one gets the same Exponential Distribution with θ = 135, due to its memoryless property.
The density is: $e^{-x/135}/135$, which at x = 60 is: $e^{-60/135}/135$ = **0.00475**.

2.10. A. For this exponential distribution, $F(x) = 1 - e^{-0.1x}$. $F(17) = 1- e^{-(0.1)(17)} = 0.817$.
The chance that all eight claims will be less than or equal to 17, is: $F(17)^8 = 0.817^8$ = **19.9**%.
Comment: This is an example of an order statistic. The maximum of the 8 claims is less than or equal to 17 if and only if each of the 8 claims is less than or equal to 17.

2.11. The sum of n independent Exponential Distributions each with <u>different</u> hazard rate λ_i, has density: $f(t) = \sum_{i=1}^{n} \lambda_i \exp[-\lambda_i t] \prod_{j \neq i} \frac{\lambda_j}{\lambda_j - \lambda_i}$.

$f(t) = 0.3 \exp[-0.3t] \frac{0.8}{0.8 - 0.3} + 0.8 \exp[-0.8t] \frac{0.3}{0.3 - 0.8}$ = $\mathbf{0.48\ e^{-0.3t} - 0.48\ e^{-0.8t}}$.

<u>Comment</u>: See Section 5.2.4 of Ross.

2.12. B. The probability of death at time t, is the density of the Exponential Distribution:
$f(t) = e^{-t/\theta}/\theta$. The present value of a payment of one at time t is $e^{-\delta t}$.
Therefore, the actuarial present value of this insurance is:

$$\int_0^\infty e^{-\delta t}\ e^{-t/\theta}/\theta \ dt = (1/\theta) \int_0^\infty e^{-(\delta+1/\theta)t} \ dt = (1/\theta) / (\delta + 1/\theta) = 1 / (1 + \delta\theta).$$

2.13. B. After truncating and shifting from below, one gets the same Exponential Distribution with $\theta = 25$, due to its memoryless property. The variance is $\theta^2 = 25^2 = 625$.

2.14. A. Due to its memoryless property, the nonzero payments are Exponential with $\theta = 31$, with mean 31 and variance 31^2. The probability of a nonzero payment is the probability that a loss is greater than the deductible of 10; $S(10) = e^{-10/31} = 0.7243$.
Let I = 0 if there is a zero payment, and I = 1 if there is a positive payment.
I is Bernoullis with q = 0.7243.
Let Y be the payment variable.
E[Y | I] = 31 I. Var[E[Y | i]] = 31^2 Var[I] = $(31^2)(0.7243)(1 - 0.7243) = 191.9$.
Y = 0 if I = 0 and an Exponential with $\theta = 31$ and variance 31^2 if I = 1.
Therefore, Var[Y | I] = 31^2 I. E[Var[Y | I]] = (31^2) E[I] = $(31^2)(0.7243) = 696.1$.
By the conditional variance formula:
Var[Y] = E[Var[Y| I]] + Var[E[Y | i]] = 696.1 + 191.9 = **888**.
Alternately, the payments of the insurer can be thought of as an aggregate distribution, with Bernoulli frequency with mean 0.7243 and Exponential severity with mean 31.
The variance of this aggregate distribution is:
(Mean Freq.)(Var. Sev.) + (Mean Sev.)^2(Var. Freq.) =
$(0.7243)(31^2) + (31)^2 \{(0.7243)(1 - 0.7243)\}$ = **888**.
<u>Comment</u>: See Example 5.4 in Ross. Similar to 3, 11/00, Q.21.
I is 31 times a Bernoulli variable. When we multiply a variable by a constant, we multiply the second moment and the variance by that constant squared.
I prefer the second solution which uses the formula for the variance of an aggregate distribution, which is worthwhile learning.

2.15. The survival function of Y is: Prob[all N losses > y] = $S(y)^N = (e^{-x/\theta})^N = e^{-xN/\theta}$.
The distribution of Y is Exponential with mean θ/N.

2.16. D. Given a disability of length t, the present value of an annuity certain is:
$(1-e^{-\delta t})/\delta$. The expected present value is the average of this over all t:

$$\int_0^\infty \{(1-e^{-\delta t})/\delta\} f(t)\,dt = \int_0^\infty \{(1-e^{-\delta t})/\delta\} (e^{-t/\theta}/\theta)\,dt = (1/\delta)\int_0^\infty e^{-t/\theta}/\theta - e^{-t(\delta + 1/\theta)}/\theta\,dt =$$

$(1/\delta)\{1 - (1/(\delta + 1/\theta))/\theta\} = (1/\delta)\{1 - (1/(1 + \delta\theta)\} = \theta / (1 + \delta\theta)$.

2.17. D. For an Exponential, $f(x) = e^{-x/\theta}/\theta$. $f(0) = 1/\theta$. Thus $1/10 = 1/\theta$. $\Rightarrow \theta = 10$.
Third moment is: $6\theta^3 = 6000$.

2.18. B. Let Y = X - 100. Then Y is Exponential with mean $\theta = 500$.
E[X] = E[Y] + 100 = 500 + 100 = 600. Var[X] = Var[Y] = 500^2 = 250,000.
The mean total value of gifts is: (200)(600) = 120,000.
The variance of the total value of gifts is: (200)(250,000) = 50,000,000.
Prob[gifts ≤ 110,000] $\cong \Phi[(110{,}000 - 120{,}000)/\sqrt{50{,}000{,}000}\,] = \Phi[-1.41]$ = **7.9%**.
Comment: The distribution of the size of gifts is a Shifted Exponential.

2.19. E. Due to the memoryless property, if he has already waited 20 minutes, that does not affect his future waiting time. Probability he waits more than 10 additional minutes is:
$e^{-10/6}$ = **18.9%**.

2.20. The sum of n independent Exponential Distributions each with different hazard rate λ_i, has density: $f(t) = \sum_{i=1}^{n} \lambda_i \exp[-\lambda_i t] \prod_{j \neq i} \frac{\lambda_j}{\lambda_j - \lambda_i}$.

$$f(t) = 0.5 \exp[-0.5t]\,\frac{0.2}{0.2 - 0.5}\,\frac{0.1}{0.1 - 0.5} + 0.2 \exp[-0.2t]\,\frac{0.5}{0.5 - 0.2}\,\frac{0.1}{0.1 - 0.2}$$
$$+ 0.1 \exp[-0.1t]\,\frac{0.5}{0.5 - 0.1}\,\frac{0.2}{0.2 - 0.1} = \mathbf{0.0833\, e^{-0.5t} - 0.3333\, e^{-0.2t} + 0.25\, e^{-0.1t}}.$$

Comment: See Section 5.2.4 of Ross.

2.21. Due to its memoryless property, the future lifetime follows the original Exponential.
$S(5000) = e^{-5000/28,700}$ = **84.0%**.

2.22. $F(3) = 1 - \exp[-3 \ln(10)] = 1 - 10^{-3} = 0.999$.
$F(5) = 1 - \exp[-5 \ln(10)] = 1 - 10^{-5} = 0.99999$.
$F(5) - F(3) = 0.99999 - 0.999 = 0.00099$. $S(6) = \exp[-6 \ln(10)] = 10^{-6}$.
The annual rate of earthquakes of magnitude at least 3 and less than 5 is: 150/20 = 7.5.
Thus the inferred annual rate of all earthquakes is: 7.5/0.00099.
Thus the expected number of large earthquakes next year is:
$(7.5/0.00099)(10^{-6})$ = **0.76%**.
<u>Comment</u>: Based on the Gutenberg-Richter law.
I have ignored the small probability of two large earthquakes in one year.

2.23. D. At the first stage there are: $\binom{4}{2}$ = 6 possible links.

E[Cost of first link] = E[minimum of 6 exponentials] = $\theta/6$.
At the next stage, each of two unlinked cities could be connected to each of two linked cities, for a total of 4 possible links. However, we have already eliminated the minimum of the original 6. If we subtract this overall minimum from each of these 4 Exponentials, the remaining variables are Exponential with mean θ.
E[Cost of second link - cost of first link] = E[minimum of 4 exponentials] = $\theta/4$.
$\Rightarrow$ E[Cost of second link] = E[cost of first link] + $\theta/4 = \theta/6 + \theta/4 = \theta\, 5/12 = (60)(5/12)$ = **25**.
<u>Comment</u>: When one chooses the minimum of a sample of i.i.d. Exponentials, if one subtracts the minimum from each of the remaining variables, then one gets i.i.d. Exponentials with the original mean. This is a special property of Exponentials.
Label the cities W, X, Y, Z. Then if the first link is W↔X, then the second link can not be Y↔Z.

Thus the second link may not be the second cheapest of the original six links.
As discussed in "Mahler's Guide to Statistics", the expected value of second cheapest of the original six links is: $(1/6 + 1/5)\theta = (60)(11/30) = 22 < 25$.
Getting the expected value of the third rather than the second link would be much more difficult.

2.24. A. This is an Exponential Distribution with mean **10**.

2.25. E. This is an Exponential Distribution with variance: $\theta^2 = 10^2$ = **100**.

2.26. D. $F(x) = 1 - e^{-0.1x}$. $F(6) = 1 - e^{-0.6}$ = **45.1%**.

2.27. C. This is an Exponential Distribution with constant failure rate of **0.1**.

2.28. B. $0.5 = F(x) = 1 - e^{-0.1x}$. $\Rightarrow$ x = **6.93**.

2.29. A. Let y be the amount of pastrami ordered by Monk's.
Let X be the demand for pastrami by the people who eat at Monk's.
Then Monk's sells the minimum of X and y.
Profit is: 15 Min[X, y] - 10y.
Expected profit is: $15\ E[X \wedge y] - 10\ y = (15)\ (30)\ (1 - e^{-y/30}) - 10\ y$.
Setting the partial derivative with respect to y equal to zero: $15e^{-y/30} - 10 = 0$.
$\Rightarrow y = 30 \ln(1.5) =$ **12.2** pounds.
Comment: Similar to Exam S, 11/17, Q.7.
See Example 5.5 in Introduction to Probability Models by Ross.
Limited Expected Value = $E[X \wedge x] = E[\text{Min}[X, x]]$.
For the Exponential Distribution, $E[X \wedge x] = \theta\ (1 - e^{-x/\theta})$.

2.30. $\text{Prob}[X_1 < X_2] = \dfrac{\lambda_1}{\lambda_1 + \lambda_2} = \dfrac{0.0007}{0.0007 + 0.0028} = 7/35 =$ **1/5**.

Comment: Mathematically equivalent to the probability that the first event from a Poisson Process with $\lambda = 0.0007$ will appear before the first event from a Poisson Process with $\lambda = 0.0028$.

2.31. The minimum is Exponential with failure rate: 0.0007 + 0.0028 = 0.0035.
Prob[Minimum > 400] = exp[-(0.0035)(400)] = **24.7%.**
Comment: Therefore the expected value of the minimum is: 1/0.0035 = 286 days.

2.32. Prob[Neither item working at time 800] =
$F_1(800)\ F_2(800) = \{1 - e^{-(0.0007)(800)}\}\{1 - e^{-(0.0028)(800)}\} =$ **38.3%.**

2.33. Prob[Maximum ≤ t] = Prob[fan failed by time t] Prob[transformer failed by time t] =
$\{1 - e^{-0.0007t}\}\{1 - e^{-0.0028t}\} = 1 - e^{-0.0007t} - e^{-0.0028t} + e^{-0.0035t}$.
Thus $\text{Prob}[\text{Maximum} > t] = e^{-0.0007t} + e^{-0.0028t} - e^{-0.0035t}$.
Integrating this survival function from zero to infinity gives the mean of the maximum:
1/0.0007 + 1/0.0028 - 1/0.0035 = **1500 days**.
Alternately, given only two items: X + Y = Min[X, Y] + Max[X, Y].
Taking expected values: 1/0.0007 + 1/0.0028 = 1/0.0035 + E[Max[X, Y]].
⇒ E[Max[X, Y]] = 1/0.0007 + 1/0.0028 - 1/0.0035 = **1500 days**.
Comment: For two independent Exponentials:
$E[\text{Min}[X, Y]] = 1 / (\lambda_1 + \lambda_2)$. $E[\text{Max}[X, Y]] = 1 / \lambda_1 + 1 / \lambda_2 - 1 / (\lambda_1 + \lambda_2)$.
If the two Exponentials are identical, then $E[\text{Min}] = \theta/2$ and $E[\text{Max}] = 3\theta/2$.

2.34. The probability the industrial fan fails first is: $\frac{\lambda_1}{\lambda_1 + \lambda_2} = \frac{0.0007}{0.0007 + 0.0028}$ = 7/35 = 1/5.

In which case, the expected future lifetime of the transformer is: 1/0.0028 = 357.1.

The probability the transformer fails first is: $\frac{\lambda_2}{\lambda_1 + \lambda_2} = \frac{0.0028}{0.0007 + 0.0028}$ = 28/35 = 4/5.

In which case, the expected future lifetime of the fan is: 1/0.0007 = 1428.6.
The expected future lifetime of the remaining item is: (1/5)(357.1) + (4/5)(1428.6) = **1214.3**.
Comment: See Exercise 5.30 in Ross.

2.35. By the memoryless property, this is also an Exponential Distribution with θ = 2000.

2.36. If the minimum is m and Y > X, then X = m and Y > m; this has probability: $f_X(m)\ S_Y(m)$.
If the minimum is m and Y < X, then Y = m and X > m; this has probability: $f_Y(m)\ S_X(m)$.
Thus the desired probability is:

$$\frac{f_X(m)\ S_Y(m)}{f_X(m)\ S_Y(m) + f_Y(m)\ S_X(m)} = \frac{f_X(m) / S_X(m)}{f_X(m) / S_X(m) + f_Y(m) / S_Y(m)} = \frac{r_X(m)}{r_X(m) + r_Y(m)}.$$

Comment: See Exercise 5.7 in Introduction to Probability Models by Ross.

If both X and Y are Exponential, then $\frac{r_X(m)}{r_X(m) + r_Y(m)} = \frac{\lambda_X}{\lambda_X + \lambda_Y}$.

2.37. E. The probability that Howard arrives first is: 50/(30 + 50) = 5/8.
Given that Howard shows up first, the probability that Jerry shows up before Howard leaves is: 70 / (50 + 70) = 7/12.
Given that both friends are there, the probability that Jerry leaves first is: 70 / (70 + 80) = 7/15.
Thus the probability that Howard arrives first and leaves last is: (5/8)(7/12)(7/15) = **17.0%**.
Comment: See Exercise 5.14 in Ross.
λ_1 = 1/30. λ_2 = 1/50. Prob[#1 before #2] = $\lambda_1 / (\lambda_1 + \lambda_2)$ = (1/30) / (1/30 + 1/50) = 50/(50 + 30).

2.38. C. The distribution function of the mixture is the mixture of the distribution functions.
30% {1 - exp[-(0.01)(80)]} + 70% {1 - exp[-(0.005)(80)]} = **39.6%.**

2.39. B The mean of the mixture is the mixture of the means.
(30%)(1/0.01) + (70%)(1/0.005) = **170 hours**.

2.40. D. The second moment of the mixture is: (30%)($2/0.01^2$) + (70%)($2/0.005^2$) = 62,000.
Thus the variance of the mixture is: 62,000 - 170^2 = **33,100**.

2.41. A. f(120) = (30%)(0.01)exp[-(0.01)(120)] + (70%)(0.005)exp[-(0.005)(120)] = 0.002824.
S(120) = (30%) exp[-(0.01)(120)] + (70%) exp[-(0.005)(120)] = 0.4745.
r(120) = f(120) / S(120) = 0.002824 / 0.4745 = **0.00595**.
Comment: The failure rate of the mixture is not the mixture of the failure rates.

2.42. A. The time of the minimum is Exponential with failure rate: 1/5 + 1/4 = 0.45.
The distribution of the minimum is independent of who finishes first.

The expected reward is: $\int_0^\infty 1000\ e^{-0.2t}\ 0.45\ e^{-0.45t}\ dt = 450 \int_0^\infty e^{-0.65t}\ dt$ = 450/0.65 = 692.3.

The probability that Thelma wins is: 4 / (5 + 4) = 4/9.
The expected amount that Thelma wins is: (4/9)(692.3) = **$308**.
Comment: See Exercise 5.19 in Ross.

2.43. C. You need to wait until the first of the current three customers is done.
The minimum of three independent Exponentials is another Exponential with the sum of the failure rates: 5 + 4 + 2.5 = 11.5. So on average you wait 1/11.5 minutes to be served.
The probability that you are served by each of the three clerks are:
5/11.5, 4/11.5, and 2.5/11.5.
Thus the average time you spend being served is:
(5/11.5)(1/5) + (4/11.5)(1/4) + (2.5/11.5)(2/5) = 3/11.5
Thus the average time until you are done buying your ticket is:
1/11.5 + 3/11.5 = 4/11.5 = 0.348 minutes = **20.9 seconds**.
Comment: Similar to Exam S, 11/17, Q.6. See Example 5.8 in Ross. In general with n clerks (servers), the average time until you are done buying your ticket is: $\frac{n+1}{\text{sum of their failure rates}}$.

2.44. The minimum is an Experiential Distribution with failure rate: 1/5 + 1/8 = 0.325, regardless of whether X or Y is bigger.
E[X M | M = X] = E[M^2 | M = X] = E[M^2] = 2/0.325^2 = **18.935**.

2.45. If M = y, then X > y. E[X | X > y] = E[X - y + y | X > y] = y + E[X - y | X > y] = y + 5.
Where I have applied to X the memoryless property of the Exponential Distribution.
Thus given M = y, E[X M] = (y + 5) y = y^2 + 5y.
Thus E[X M | M = Y] = E[y^2 + 5y | M = Y] = E[M^2 | M = Y] + 5 E[M | M = Y] = E[M^2] + 5 E[M] = 2/0.325^2 + 5/0.325 = **34.320.**

2.46. Prob[X < Y] = 8 / (8 + 5) = 8/13. E[X M] = (8/13)(18.935) + (5/13)(34.320) = 24.852.
Cov[X, M] = E[X M] - E[X] E[M] = 24.852 - (5)(1/0.325) = **9.467**.

2.47. The minimum is an Experiential Distribution with failure rate: 1/5 + 1/8 = 0.325, regardless of whether X or Y is bigger.
E[Y M | M = Y] = E[M^2 | M = Y] = E[M^2] = 2/0.325^2 = **18.935**.

2.48. If M = x, then Y > x. E[Y | Y > x] = E[Y - x + x | Y > x] = x + E[Y - x | Y > x] = x + 8.
Where I have applied to Y the memoryless property of the Exponential Distribution.
Thus given M = x, E[Y M] = (x + 8) x = x^2 + 8x.
Thus E[Y M | M = X] = E[x^2 + 8x | M = X] = E[M^2 | M = X] + 8 E[M | M = X] = E[M^2] + 8 E[M] = 2/0.325^2 + 8/0.325 = **43.550.**

2.49. Prob[Y < X] = 5 / (8 + 5) = 5/13.
E[Y M] = (5/13)(18.935) + (8/13)(43.550) = 34.083.
Cov[Y, M] = E[Y M] - E[Y] E[M] = 34.083 - (8)(1/0.325) = **9.467**.
Comment: See Exercise 5.10 in Ross.
In general, Cov[X, M] = $\frac{1}{(\lambda_X + \lambda_Y)^2}$ = Cov[Y, M].

2.50. Min + Max = X + Y. ⇒ Max = X + Y - Min.
Cov[Min , Max] = Cov[M , X + Y - M] = Cov[X, M] + Cov[Y, M] - Cov[M , M] =
9.467 + 9.467 - Var[M] = (2)(9.467) - $1/0.325^2$ = **9.467**.
Comment: In general, Cov[Min, Max] = $\frac{1}{(\lambda_X + \lambda_Y)^2}$.

2.51. Prob[Max ≤ t] = Prob[X ≤ t] Prob[Y ≤ t] = $(1 - e^{-t/5})(1 - e^{-t/8}) = 1 - e^{-t/5} - e^{-t/8} + e^{-0.325t}$.
Differentiating, the density of the maximum is: $e^{-t/5}/5 + e^{-t/8}/8 - 0.325e^{-0.325t}$.
Multiplying by t, and integrating from zero to infinity, and using the fact that each term is the density of an Exponential distribution, the mean of the maximum is: 5 + 8 - 1/0.325 = 9.923.
Multiplying by t^2, and integrating from zero to infinity, and using the fact that each term is the density of an Exponential distribution, the second moment of the maximum is:
$(2)(5^2) + (2)(8^2) - 2/0.325^2 = 159.065$.
Thus the variance of the maximum is: $159.065 - 9.923^2$ = **60.60**.
Comment: Beyond what you are likely to be asked on your exam.
In general, the variance of the maximum is: $\frac{1}{\lambda_X^2} + \frac{1}{\lambda_Y^2} - \frac{3}{(\lambda_X + \lambda_Y)^2}$,
which in this case is: $5^2 + 8^2 - 3/0.325^2 = 60.60$.

2.52. Correlation of the minimum and maximum is: $\frac{9.467}{\sqrt{(9.467)(60.60)}}$ = **0.395**

Comment: In general, the correlation of the minimum and maximum is:

$$\sqrt{\frac{\frac{1}{(\lambda_X + \lambda_Y)^2}}{\frac{1}{\lambda_X^2} + \frac{1}{\lambda_Y^2} - \frac{3}{(\lambda_X + \lambda_Y)^2}}} = \frac{1}{\sqrt{(1 + \frac{\lambda_Y}{\lambda_X})^2 + (1 + \frac{\lambda_X}{\lambda_Y})^2 - 3}}.$$

2.53. C. For t < 2, the amount paid is: (100)(2 - t)/2 = 100 - 50t.
The density of the Exponential is: $f(t) = e^{-t/6}/6$.
Thus the expected amount paid by the manufacturer is:

$$\int_0^2 (100 - 50t)\, e^{-t/6}/6 \; dt = 100 \int_0^2 e^{-t/6}/6 \; dt - 50 \int_0^2 t\, e^{-t/6}/6 \; dt =$$

$(100)(1 - e^{-2/6}) - (50)(6)\{1 - e^{-2/6} - (2/6)e^{-2/6}\} =$ **14.96**.
Alternately, the expected amount by which lifetimes are less than 2 is:
$2 - E[X \wedge 2] = 2 - (6)(1 - e^{-2/6}) = 0.2992$.
The expected amount paid per battery is: (100)(0.2992/2) = **14.96**.

Comment: For an Exponential Distribution with mean θ: $\int_0^x t\, f(t)\; dt = \theta\,\{1 - e^{-x/\theta} - (x/\theta)e^{-x/\theta}\}$.

2.54. D. The mean of each Exponential is: θ. The second moment of each Exponential is: $2\theta^2$.
The mean and second moment of the mixed distribution are the weighted average of those of the individual distributions. Therefore, the mixed distribution has mean:
$0.4\theta_1 + 0.6\theta_2 = (0.4)(1128) + (0.6)(5915) = 4000$ and
second moment: $2(0.4\theta_1^2 + 0.6\theta_2^2) = 2\{(0.4)(1128^2) + (0.6)(5915^2)\} = 43{,}002{,}577$.
Variance = $43{,}002{,}577 - 4000^2$ = **27.0 million**.

2.55. The joint density of X and Y is: $\lambda_1 \exp[-\lambda_1 x]\ \lambda_2 \exp[-\lambda_2 y]$.
X = Y + V.

Thus for v ≥ 0, $f(v) = \int_0^\infty \lambda_1 \exp[-\lambda_1 (y+v)]\ \lambda_2 \exp[-\lambda_2 y]\; dy =$

$$\lambda_1 \lambda_2 \exp[-\lambda_1 v] \int_0^\infty \exp[-(\lambda_1 + \lambda_2)y]\; dy = \lambda_1 \lambda_2 \exp[-\lambda_1 v] / (\lambda_1 + \lambda_2).$$

If v < 0, then since X > 0, y > -v, and $\int_{-v}^\infty \lambda_1 \exp[-\lambda_1 (y+v)]\ \lambda_2 \exp[-\lambda_2 y]\; dy =$

$$\lambda_1 \lambda_2 \exp[-\lambda_1 v] \int_{-v}^\infty \exp[-(\lambda_1 + \lambda_2)y]\; dy = \lambda_1 \lambda_2 \exp[-\lambda_1 v] \exp[-(\lambda_1 + \lambda_2)(-v)] / (\lambda_1 + \lambda_2) =$$

$\lambda_1 \lambda_2 \exp[l_2 v] / (\lambda_1 + \lambda_2)$.

$$f(v) = \frac{\lambda_1 \lambda_2}{\lambda_1 + \lambda_2} \begin{cases} \exp[\lambda_2 v], & v \le 0 \\ \exp[-\lambda_1 v], & v \ge 0 \end{cases}.$$

Comment: Beyond what you are likely to be asked on your exam.

2.56. Since the decrements are independent, the failure rates add: 1/20 + 1/40 = 0.075.
For an Exponential with failure rate 0.075, S(50) = exp[-(0.075)(50)] = **2.35%**.
Alternately, the probability of surviving past time t is the product of the probabilities of surviving both of the independent decrements:
$S(t) = S_1(t)\ S_2(t)$ = exp[-t/20] exp[-t/40] = exp[-0.075t].
S(50) = exp[-(0.075)(50)] = **2.35%**.

2.57. (1/40) / (1/20 + 1/40) = **1/3**.

2.58. Assume the robot fails at time t due to internal causes, and has survived to time t as far as external causes go. The probability is proportional to: $f_1(t)\ S_2(t)$ = (exp[-t/20] / 20) exp[-t/40].
This is proportional to an Exponential Distribution with failure rate: 1/20 + 1/40 = 0.075
or mean 13.33.
Alternately, the minimum is independent of the rank order of the two Exponentials.
The minimum is an Exponential Distribution with failure rate: 1/20 + 1/40 = 0.075
or mean 13.33.
Comment: Similar to Exercise 5.8 in Ross.

2.59.
Due to the memoryless property, it does not matter that Robby has been functioning for 30.
Starting at time 30, each robot is equally likely to be the first to fail. Probability is **1/2**.
Comment: Similar to Exercise 5.9 in Ross.

2.60. E. Pr(X > r + s) = Pr(X > r) Pr(X > s) is one way to write the memoryless property, which means we have an Exponential Distribution.
600 = E[X | X > 200] = E[X - 200 | X > 200] + 200 = e(200) + 200 = θ + 200. ⇒ θ = 400.
E[X | X > 100] = e(100) + 100 = θ + 100 = 400 + 100 = **500**.
Comment: Similar to Exam S, 5/17, Q.1.
See Equation 5.3 in Introduction to Probability Models, by Ross.

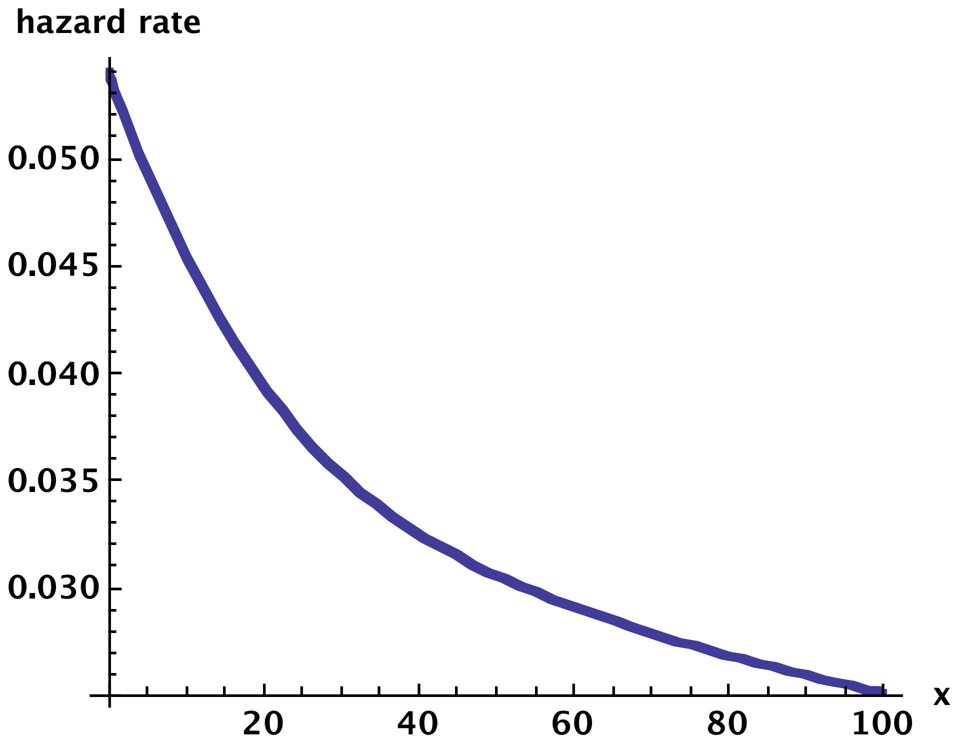
2.61. B. Truncate and shift each of X and Y at 2; subtract 2 from each value and only keep the positive results. Call these new variables X* and Y*.
Then due to memoryless property of the Exponential, X* and Y* are two independent exponential random variables with hazard rates 0.3 and 0.5, respectively.
$E[Y \mid 2 < Y < X] = 2 + E[Y^* \mid 0 < Y^* < X^*] = 2 + E[Y^* \mid Y^* < X^*]$.
It can be shown that for two independent Exponentials U and V,
$E[U \mid U < V] = E[V \mid V < U] = E[\text{Min}[U, V]]$.
In other words, the minimum is independent of the rank order of the two Exponentials.
In this case, Min[X*, Y*] is Exponential with hazard rate 0.3 + 0.5 = 0.8, and thus mean 1/0.8.
Thus, $E[Y \mid 2 < Y < X] = 2 + 1/0.8 =$ **3.25**.
Comment: Similar to Exam S, 5/17, Q.6. Difficult!

2.62. B. $S(x) = 1 - F(x) = 0.3\ e^{-x/10} + 0.5\ e^{-x/25} + 0.2\ e^{-x/50}$.
$S(15) = 0.3\ e^{-15/10} + 0.5\ e^{-15/25} + 0.2\ e^{-15/50} =$ **48.95%**.

2.63. A. The mean of the mixed distribution is a weighted average of the mean of each Exponential Distribution: (0.3)(10) + (0.5)(25) + (0.2)(50) = **25.5**.
Comment: This is a 3-point mixture of Exponential Distributions, with means of 10, 25, and 50 respectively.

2.64. B. Each Exponential Distribution has a second moment of $2\theta^2$.
The second moment of the mixture is a weighted average of the individual second moments:
$(0.3)(2)(10^2) + (0.5)(2)(25^2) + (0.2)(2)(50^2) = 1685$.
Variance = $1685 - 25.5^2 =$ **1034.75**.
Comment: Coefficient of variation = $\sqrt{1034.75}$ / 25.5 = **1.26**. Note that while the CV of every Exponential is 1, the CV of a mixed exponential is always greater than one.

2.65. D. $f(x) = (0.3)(e^{-x/10}/10) + (0.5)(e^{-x/25}/25) + (0.2)(e^{-x/50}/50)$. $f(50) = 0.00438036$.
$S(x) = 0.3e^{-x/10} + 0.5e^{-x/25} + 0.2e^{-x/50}$. $S(50) = 0.143265$.
$h(50) = f(50)/S(50) = 0.00438036 / 0.143265 =$ **0.0306**.
Comment: Note that while the hazard rate of each Exponential Distribution is independent of x, that is not true for the Mixed Exponential Distribution.



2.66. B. E[Time for Matthew] = E[minimum of 4 exponentials] = $\theta/4$.
E[Time for Mark] = E[minimum of 3 exponentials] = $\theta/3$.
E[Time for Luke] = E[minimum of 2 exponentials] = $\theta/2$.
E[Time for John] = E[exponential] = θ.
Expected total time: $\theta(1/4 + 1/3 + 1/2 + 1) = (300)(2.08333) =$ **625**.
Comment: Similar to CAS Exam S, 5/16, Q.1.

2.67. C. The distribution is an Exponential Distribution with $\theta = 1/2$.
For the Exponential Distribution $E[X \wedge x] = \theta (1 - e^{-x/\theta})$.
The average size of the capped losses is: $E[X \wedge 1] = (1/2)(1 - e^{-2}) = 0.432$.
Thus the expected annual total loss payments on a basic limits policy are: (13)(0.432) = **5.62**.
Alternately, one can use the relation between the mean excess loss and the Limited Expected Value: $e(x) = \{ \text{mean} - E[X \wedge x] \} / \{1 - F(x)\}$, therefore $E[X \wedge x] = \text{mean} - e(x)\{1 - F(x)\}$.
For the Exponential Distribution, the mean excess loss is a constant = θ = mean.
Therefore, $E[X \wedge x] = \text{mean} - e(x)\{1 - F(x)\} = \theta - \theta(e^{-x/\theta})$. Proceed as before.

2.68. C. Minimum of the failure times for resistors is Exponential with hazard rate:
1/30 + 1/40 + 1/60 = 0.075.
Minimum of the failure times for transistors is Exponential with hazard rate:
1/20 + 1/50 + 1/70 + 1/80 = 0.09679.

Prob[resistor before transistor] = $\frac{0.075}{0.075 + 0.09679}$ = **43.66%**.

Comment: Similar to CAS S, 11/16, Q.7.

2.69. B. For the very first assignment, we have picked the minimum of 16 independent, identically distributed Exponentials, which is an Exponential with mean 576/16 = 36.
Call the time for the first assignment A_1. $E[A_1]$ = 36.
At this point we eliminate the person and assignment that has already been assigned, leaving us with 9 times. However, in the first step we have chosen the minimum of the original sample of 16.
Thus, the expected value of the remaining times is higher than the a priori mean of 576.
When one chooses the minimum of a sample of i.i.d. Exponentials, if one subtracts the minimum from each of the remaining variables, then one gets i.i.d. Exponentials with the original mean.
(A special property of the Exponential Distribution.)
Therefore, if we subtract A_1 from each of the remaining 9 times, then we get 9 i.i.d. Exponentials with the original mean of 576. The minimum of these 9 i.i.d. Exponentials has expected value: 576/9 = 64. Adding back in A_1 that we subtracted from each training cost, the expected cost of the second assignment is: 64 + $E[A_1]$ = 64 + 36 = 100.
Now we have 4 times left. Call the time for the second assignment A_2. $E[A_2]$ = 100.
If we subtract A_2 from these times, they follow an Exponential with the original mean of 576.
The minimum of these 4 i.i.d. Exponentials has expected value: 576/4 = 144.
Adding back in A_2 that we subtracted from each training cost, the expected cost of the second assignment is: 144 + $E[A_2]$ = 144 + 100 = 244.
Now we have one time left. Call the cost for the third assignment A_3. $E[A_3]$ = 244.
If we subtract A_3 from this time, it follows an Exponential with the original mean of 576. Thus, the expected value for the final assignment is: 576 + $E[A_3]$ = 576 + 244 = 820.
Thus the expected total training cost is: 36 + 100 + 244 + 820 = **1200**.
Comment: This uses the second greedy algorithm in Ross. We would get the same expected value if we had instead uses the first greedy algorithm in Ross:
(1/4 + 1/3 + 1/2 + 1) (576) = 1200.

2.70. C. The time until he is served, is the minimum of 5 independent Exponentials with hazard rates of: 1/4, 1/4, 1/4, 1/6, 1/6.
This is an Exponential with hazard rate: 1/4 + 1/4 + 1/4 + 1/6 + 1/6 = 1.0833.
⇒ The mean time waiting is: 1/1.0833 = 0.9231.

The probability that the first person available is a representative is: $\frac{(3)(1/4)}{(3)(1/4) + (2)(1/6)}$ = 0.6923.

⇒ The mean service time is: (0.6923)(4) + (1 - 0.6923)(6) = 4.6154.
⇒ Expected total time is: 0.9231 + 4.6154 = **5.5385**.
Comment: Similar to CAS S, 11/16, Q.5.

2.71. C.
Since the distribution is memoryless, the average payment per payment is the mean of **800**.
Comment: Similar to MAS-1, 11/19, Q.1.

2.72. A. Due to the memoryless property of the Exponential, the failure rate is constant.
The failure rates are the same; the absolute difference in failure rates is **zero**.
Comment: Similar to MAS-1, 5/18, Q.5.
The Exponential Distribution is a very good model of radioactive decay.
The half-life of a radioactive isotope is the median, the time when on average half of the atoms have decayed. For an Exponential with mean θ, the median is $\theta \ln(2)$.

2.73. A. Max[V, X, Y, Z] < 16. $\Leftrightarrow$ V < 16, and X < 16, and Y < 16, and Z < 16.
$\Rightarrow$ Prob[Max[V, X, Y, Z] < 16] = $(1 - e^{-16/8})(1 - e^{-16/10})(1 - e^{-16/12})(1 - e^{-16/15}) = 0.3333$.
$\Rightarrow$ The desired probability is: 1 - 0.3333 = **0.6667**.
Comment: Similar to MAS-1, 5/18, Q.23.

2.74. E. Due to the memoryless property of the Exponential, with a deductible, the amount the insurance company pays per non-zero claim is independent of the size of the deductible.
$\Rightarrow$ With a deductible of d, the payments per theft follow a two point mixture of zero and an Exponential with mean 400, with weights F(d) and S(d).
This mixed distribution has mean S(d) 400, second moment $(2)(400^2)$ S(d),
and thus variance: (320,000) S(d) - (160,000) $S(d)^2$.
$S(100) = e^{-100/400} = 0.77880$.
Thus $\sigma = \sqrt{(320{,}000)(0.77880) - (160{,}000)(0.77880^2)}$ = **390**.

Alternately, with a deductible of d, the payments per theft follow an aggregate distribution with a Bernoulli frequency with mean S(d), and an Exponential Severity with mean 400.
This has variance: (mean frequency) (variance of severity) + (mean severity)^2 (variance of freq.)
= S(d) (400^2) + (400^2) S(d){1 - S(d)} = (320,000) S(d) - (160,000) $S(d)^2$.
Proceed as before.
Comment: Similar to MAS-1, 5/19, Q.6.
See Example 5.4 in Introduction to Probability Models by Ross.
The moments of a mixture are the mixture of the moments.
Variance of an aggregate distribution is:
(mean frequency) (variance of severity) + (mean severity)^2 (variance of frequency).

2.75. B. $\theta_2^2 = 2.25$. $\Rightarrow \theta_2 = 1.5$. Given $X_1 = t$, Prob[$X_2 > t$] = $e^{-t/1.5}$.

$$\text{Prob}[X_2 > X_1] = \int_0^\infty e^{-t/1.5}\, e^{-t/9.5}/9.5 \, dt = (1/9.5)/(1/1.5 + 1/9.5) = \mathbf{3/22}.$$

Comment: This is mathematically equivalent to two independent Poisson Processes, with $\lambda_1 = 1/9.5$ and $\lambda_2 = 1/1.5$. The probability of observing an event from the first process before the second process is: $\lambda_1 / (\lambda_1 + \lambda_2)$ = (1/9.5) / (1/9.5 + 1/1.5) = 3/22.

2.76. B. The median is where F(x) = 0.5. $F(x) = 1 - e^{-10x}$. Therefore solving for x, the median = -ln(0.5) / 10 = ln(2) / 10. The mode is that point where f(x) is largest. Since f(x) declines for x ≥ 0, f(x) is at its maximum at x = 0. Therefore, the mode is zero.

2.77. D. $F(x) = 1 - e^{-\lambda x}$. F(1/3) = 0.5. $\Rightarrow 0.5 = e^{-\lambda/3}$. $\Rightarrow \lambda =$ **3 ln(2)**.

2.78. A. The sum of 100 Exponential distributions has mean (100)(1/2) = 50,

and variance $(100)(1/2^2) = 25$. $P[\sum_{i=1}^{100} X_i > 57] \cong 1 - \Phi[(57 - 50)/5] = 1 - \Phi[1.4] =$ **0.0808**.

2.79. B. $0.25 = F(x) = 1 - e^{-x/2}$. $\Rightarrow$ x = -2ln(0.75) = 2ln(4/3) = **ln(16/9)**.

<u>Comment</u>: $F(\ln(16/9)) = 1 - \exp[-\ln(16/9)/2] = 1 - \sqrt{9/16} = 1 - 3/4 = 1/4$.

2.80. E. $E[X_1X_2] = E[X_1]E[X_2] = \theta_1\theta_2$.

2.81. C. Given that the first claim is of size x, the probability that the second will be more than twice as large is $1 - F(2x) = S(2x) = e^{-2x}$. The overall average probability is:

$$\int_0^\infty (\text{Probability given x})\ f(x)\ dx = \int_0^\infty e^{-2x}\ e^{-x}\ dx = \int_0^\infty e^{-3x}\ dx = \mathbf{1/3}.$$

2.82. D. The density of the time of failure is: $f(t) = e^{-t/10}/10$.

Expected payment is: $\int_0^7 v(t)\ f(t)\ dt = \int_0^7 e^{(7-0.2t)}\ e^{-t/10}/10\ dt = 0.1\ e^7 \int_0^7 e^{-0.3t}\ dt =$

$0.1\ e^7(1 - e^{-2.1})\ 0.3 =$ **320.78**.

2.83. D. Prob[fails during the first year] = $F(1) = 1 - e^{-1/2} = 0.3935$.
Prob[fails during the second year] = $F(2) - F(1) = e^{-1/2} - e^{-2/2} = 0.2386$.
Expected Cost = 100{(200)(0.3935) + (100)(0.2386)} = **10,256**.

2.84. C. This is an Exponential with 1/θ = 0.004. θ = 250. The median of this Exponential is: 250 ln(2) = **173.3**, which since it is less than 250 is also the median benefit.

2.85. E. By the memoryless property of the Exponential Distribution, the non-zero payments excess of a deductible are also an Exponential Distribution with mean 300. Thus the 95th percentile of the nonzero payments is: -300 ln(1 - 0.95) = 899. Adding back the 100 deductible, the 95th percentile of the losses that exceed the deductible is: **999**.

2.86. D. Prob[fails during the first year] = $F(1) = 1 - e^{-1/10} = 0.09516$.
Prob[fails during the second or third year] = $F(3) - F(1) = e^{-1/10} - e^{-3/10} = 0.16402$.
Expected Cost = 0.09516x + 0.16402x/2 = 1000. $\Rightarrow$ x = **5644**.

2.87. D. Due to its memoryless property, the nonzero payments are Exponential with $\theta = 1000$, with mean 1000 and variance 1000^2. The probability of a nonzero payment is the probability that a loss is greater than the deductible of 100; $S(100) = e^{-100/1000} = 0.90484$.
Let I = 0 if there is a zero payment, and I = 1 if there is a positive payment.
I is Bernoullis with q = 0.90484.
Let Y be the payment variable.
E[Y | I] = 1000 I. Var[E[Y | i]] = 1000^2 Var[I] = $(1000^2)(0.90484)(1 - 0.90484) = 86{,}105$.
Y = 0 if I = 0 and an Exponential with $\theta = 1000$ and variance 1000^2 if I = 1.
Therefore, Var[Y | I] =1000^2 I. E[Var[Y | I]] = (1000^2) E[I] = $(1000^2)(0.90484) = 904{,}840$.
By the conditional variance formula:
Var[Y] = E[Var[Y| I]] + Var[E[Y | i]] = 904,840 + 86,105 = **990,945**.
Alternately the payments of the insurer can be thought of as an aggregate distribution, with Bernoulli frequency with mean 0.90484 and Exponential severity with mean 1000.
The variance of this aggregate distribution is:
(Mean Freq.)(Var. Sev.) + (Mean Sev.)^2(Var. Freq.) =
$(0.90484)(1000^2) + (1000)^2 \{(0.90484)(0.09516)\}$ = **990,945**.
<u>Comment</u>: See Example 5.4 in <u>Introduction to Probability Models</u> by Ross.

2.88. D. max(T, 2) + min(T, 2) = T + 2.
E[max(T, 2)] = E[T+ 2] - E[min(T, 2)] = E[T] + 2 - E[T $\wedge$ 2] = $3 + 2 - 3(1 - e^{-2/3})$ = $\mathbf{2 + 3e^{-2/3}}$.

2.89. E. Each Exponential has variance $10^2 = 100$.
The variances of independent variables add: 100 + 100 = **200**.
<u>Comment</u>: The total time is Gamma with $\alpha = 2$, $\theta = 10$, and variance $(2)(10^2) = 200$.

2.90. D. Median = 4 $\Rightarrow 0.5 = 1 - e^{-4/\theta}$. $\Rightarrow \theta = 5.771$. $S(5) = e^{-5/5.771}$ = **0.421**.

2.91. C. A constant force of mortality is an Exponential Distribution.
Variance = $\theta^2 = 100$. $\Rightarrow \theta = 10$.
E[T $\wedge$ 10] = $\theta(1 - e^{-10/\theta}) = (10)(1 - e^{-1})$ = **6.32**.

2.92. A. The time of the first death is the minimum of two independent Exponentials, and thus is also Exponential with the sum of the failure rates: $\mu + k\mu = (1+k)\mu$.
Let the two variables be X and Y.
Then X + Y = Min[X, Y] + Max[X, Y]. $\Rightarrow$ E[X] + E[Y] = E[Min] + E[Max].
$\Rightarrow$ E[Max] = $1/\mu + 1/(k\mu) - 1/\{(1+k)\mu\}$.
We are given E[Max] = 3 E[Min]. $\Rightarrow 1/\mu + 1/(k\mu) - 1/\{(1+k)\mu\} = 3/\{(1+k)\mu\}$.
$\Rightarrow k(k+1) + k+1 - k = 3k$. $\Rightarrow k^2 - 2k + 1 = 0$. $\Rightarrow$ k = **1**.
<u>Comment</u>: As discussed in "Mahler's Guide to Statistics," for two independent, identically distributed Exponentials with mean θ, E[Min] = $\theta/2$, and E[Max] = $3\theta/2$ = 3 E[Min].

2.93. C. T_{xy} is the minimum of X and Y, and is an Exponential with hazard rate:
1/20 + 1/5 = 0.25.
$Var(T_{xy}) = 1/0.25^2 =$ **16**.

2.94. B. Due to the memoryless property of the Exponential, in each case the average payment per (non-zero) payment is the same 2000.
With the 500 deducible, the probability of a payment is: $e^{-500/2000} = 0.7788$, and thus the average payment per theft loss is: (0.7788)(2000) = 1557.6.
With the 1000 deducible, the probability of a payment is: $e^{-1000/2000} = 0.6065$, and thus the average payment per theft loss is: (0.6065)(2000) = 1213.0.
|1557.6 - 1213.0| = **344.6**.

2.95. C. E[Training costs of Alice] = E[minimum of 3 exponentials] = θ/3.
E[Training costs of Bob] = E[minimum of 2 exponentials] = θ/2.
E[Training costs of Chris] = E[exponential] = θ.
Expected total training costs: θ(1/3 + 1/2 + 1) = (200)(11/6) = **366.67**.
Comment: An example of a “Greedy Algorithm for Assignment” as discussed in Example 5.7 of Introduction to Probability Models, by Ross.

2.96. For the very first assignment, we have picked the minimum of 9 independent, identically distributed Exponentials, which is an Exponential with mean 200/9. The expected cost of the first assignment is: 200/9 = 22.222. Call the cost of the first assignment A_1. $E[A_1] = 22.222$.
At this point we eliminate the person and vehicle that has already been assigned, leaving us with 4 training costs. However, in the first step we have chosen the minimum of the original sample of 9.
Thus, the expected value of the remaining training costs is higher than the a priori mean of 200.
When one chooses the minimum of a sample of i.i.d. Exponentials, if one subtracts the minimum from each of the remaining variables, then one gets i.i.d. Exponentials with the original mean.
(A special property of the Exponential Distribution.)
Therefore, if we subtract A_1 from each of the remaining 4 training costs, then we get 4 i.i.d. Exponentials with the original mean of 200. The minimum of these 4 i.i.d. Exponentials has expected value 200/4 = 50. Adding back in A_1 that we subtracted from each training cost, the expected cost of the second assignment is: $50 + E[A_1] = 50 + 22.222 = 72.222$.
Now we have one training cost left. Call the cost of the second assignment A_2. $E[A_2] = 72.222$.
If we subtract A_2 from this last training cost, it follows an Exponential with the original mean of 200. Thus, the expected value of the final training cost is: $200 + E[A_2] = 200 + 72.222 =$ 272.222.
Thus the expected total training cost is: 22.222 + 72.222 + 272.222 = **366.666**.
Comment: This uses the second greedy algorithm in Ross. We get the same expected value as in CAS S, 5/16, Q.1, which instead uses the first greedy algorithm in Ross.

2.97. D. The hazard rate for an Exponential Distribution is constant and equal to 1/mean.
The expected lifetime of a 2-year old incandescent light bulb is: 1/0.2 = 5,
same as that of a new one.
Thus LED light bulbs have mean: (2)(5) = 10, and hazard rate: **1/10**.

2.98. B. Since the process is memoryless, we want the probability of waiting at least 10 more minutes: $e^{-10/5}$ = **0.135**.

2.99. D. The time until he is served, is the minimum of 4 independent Exponentials with hazard rates: 1, 1, 1/2, 1/2. This is an Exponential with hazard rate: 1 + 1 + 1/2 + 1/2 = 3.
⇒ The mean time waiting is 1/3.

The probability that the first person available is a representative is: $\frac{(2)(1)}{(2)(1) + (2)(1/2)}$ = 2/3.

⇒ The mean service time is: (2/3)(1) + (1 - 2/3)(2) = 4/3.
⇒ Expected total time is: 1/3 + 4/3 = **5/3**.

2.100. E. Minimum of the failure times for red wires is Exponential with hazard rate:
1/50 + 1/75 + 1/100 = 0.0433.
Minimum of the failure times for yellow wires is Exponential with hazard rate:
1/25 + 1/50 + 1/75 = 0.0733.

Prob[red before yellow] = $\frac{0.0433}{0.0433 + 0.0733}$ = **37.1%**.

2.101. C. Pr(X > r + s) = Pr(X > r) Pr(X > s) is one way to write the memoryless property, which means we have an Exponential Distribution, with mean θ.
30 = E[X | X > 10] = E[X - 10 | X > 10] + 10 = e(10) + 10 = θ + 10. ⇒ θ = 20.
E[X | X > 20] = e(20) + 20 = θ + 20 = 20 + 20 = **40**.
Comment: See Equation 5.3 in Introduction to Probability Models, by Ross.
For the Exponential: Pr(X > r + s) = $e^{-\lambda(r+s)} = e^{-\lambda r} e^{-\lambda s}$ = Pr(X > r) Pr(X > s).
For the Exponential, due the memoryless property: Prob[X > r + s | X > r] = Prob[X > s].
⇔ Pr(X > r + s) / Pr(X > r) = Prob[X > s]. ⇔ Pr(X > r + s) = Pr(X > r) Prob[X > s].

2.102. A. Truncate and shift each of X and Y at 1; subtract 1 from each value and only keep the positive results. Call these new variables X* and Y*.
Then due to memoryless property of the Exponential, X* and Y* are two independent exponential random variables with hazard rates 2 and 8, respectively.
E[X | 1 < X < Y] = 1 + E[X* | 0 < X* < Y*] = 1 + E[X* | X* < Y*].
It can be shown that for two independent Exponentials U and V,
E[U | U < V] = E[V | V < U] = E[Min[U, V]].
In other words, the minimum is independent of the rank order of the two Exponentials; this is another special property of the Exponential Distribution.
In this case, Min[X*, Y*] is Exponential with hazard rate 2 + 8 = 10, and thus mean 1/10.
Thus, E[X | 1 < X < Y] = 1 + 1/10 = **1.1**.
Alternately, the joint density of X and Y is: $f(x, y) = 2\ e^{-2x}\ 8\ e^{-8y}$.

The numerator of E[X | 1 < X < Y] is: $\int_1^\infty \int_1^y x\ 2\ e^{-2x}\ 8\ e^{-8y}\ dx\ dy = \int_1^\infty 8\ e^{-8y} (-x\ e^{-2x} - e^{-2x}/2) \Big]_{x=1}^{x=y} dy =$

$\int_1^\infty 8\ e^{-8y} (-y\ e^{-2y} - e^{-2y}/2 + 1.5e^{-2})\ dy = \int_1^\infty -8\ y\ e^{-10y} - 4\ e^{-10y} + e^{-2}\ 12\ e^{-8y}\ dy =$

$(0.8\ y\ e^{-10y} + 0.08\ e^{-10y})\Big]_{y=1}^{y=\infty} - (4/10)e^{-10} + e^{-2}\ (12/8)\ e^{-8} =$

$-0.8\ e^{-10} - 0.08\ e^{-10} - 0.4\ e^{-10} + 1.5\ e^{-10} = 0.22\ e^{-10}$.

The denominator of E[X | 1 < X < Y] is: $\int_1^\infty \int_1^y 2\ e^{-2x}\ 8\ e^{-8y}\ dx\ dy = \int_1^\infty 8\ e^{-8y} (e^{-2} - e^{-2y})\ dy =$

$\int_1^\infty e^{-2}\ 8\ e^{-8y} - 8\ e^{-10y}\ dy = e^{-2}\ e^{-8} - (8/10)\ e^{-10} = 0.2\ e^{-10}$.

Thus E[X | 1 < X < Y] = $(0.22\ e^{-10}) / (0.2\ e^{-10})$ = **1.1**.

Comment: Difficult! For an Exponential with mean θ, $\int_0^x t\ f(t)\ dt = \theta\{1 - e^{-x/\theta} - (x/\theta)e^{-x/\theta}\}$.

Thus, $\int_1^y x\ 2\ e^{-2x}\ dx = \int_0^y x\ 2\ e^{-2x}\ dx - \int_0^1 x\ 2\ e^{-2x}\ dx =$

$(1/2)(1 - e^{-2y} - 2y\ e^{-2y}) - (1/2)(1 - e^{-2} - 2e^{-2}) = -ye^{-2y} - e^{-2y}/2 + 1.5e^{-2}$.

2.103. C. The combined service process has a rate of: 1/5 + 1/7 = 12/35.
Thus the customer's average wait to be served is: 35/12 = 2.917.
7/(5 + 7) = 7/12 chance that the customer gets served by the first server,
and 5/(5+7) = 5/12 chance the customer gets served by the second server.
Thus the average time the customer is being served is: (5)(7/12) + (7)(5/12) = 70/12 = 5.833.
Thus the expected total waiting plus service time is: 2.917 + 5.833 = **8.750** minutes.
Comment: See Example 5.8 in Introduction to Probability Models by Ross.
In general with n servers, the average time until the customer is done being served is:
(n + 1) / (sum of their failure rates). In this case: (2 + 1) / (1/5 + 1/7) = 8.75.

2.104. C. If the vendor buys y oranges and the demand is x, then the profit is: Min[x. y] - 0.5 y.
Thus the expected profit is: $E[X \wedge y] - 0.5 y$.
For an Exponential with mean θ, the limited expected value is: $E[X \wedge y] = \theta (1 - e^{-y/\theta})$.
Assume the exam question intended to say a rate of 0.001 and thus a mean of 1000.
Then the expected profit is: $1000 (1 - e^{-y/1000}) - 0.5y$.
In order to maximize the expected profit, set the partial derivative with respect to q equal to zero:
$0 = e^{-y/1000} - 0.5. \Rightarrow y = 1000 \ln(2) =$ **693**.
Comment: This past exam question originally read "θ of 0.001 per week."
See Example 5.5 in Introduction to Probability Models by Ross.

2.105. D. Due to the memoryless property of the Exponential Distribution, the future time John spends is also Exponential with mean 30.
P_1 = Prob[John waits more than 40 additional minutes] = $e^{-40/30}$.
$P_2 = e^{-60/30}. \Rightarrow P_1 - P_2 = e^{-4/3} - e^{-2} =$ **0.1283**.

Comment: IRS ⇔ United States Internal Revenue Service.

2.106. A. Due to the memoryless property of the Exponential, the failure rate is constant.
The failure rates are the same; the absolute difference in failure rates is **zero**.
Comment: Both computers have a failure rate of 1/24.
A constant hazard rate is not a realistic model for computers.

2.107. E. Max[X, Y, Z] < 6. ⇔ X < 6 and Y < 6 and Z < 6.

⇒ The desired probability is: $(1 - e^{-6/2}) (1 - e^{-6/4}) (1 - e^{-6/5}) =$ **0.5159**.

Comment: Prob[maximum < t] = $(1 - e^{-t/2}) (1 - e^{-t/4}) (1 - e^{-t/5})$.
⇒ The survival function of the maximum is: $S(t) = 1 - (1 - e^{-t/2}) (1 - e^{-t/4}) (1 - e^{-t/5})$

$= e^{-t/2} + e^{-t/4} + e^{-t/5} - e^{-t/2} e^{-t/4} - e^{-t/2} e^{-t/5} - e^{-t/4} e^{-t/5} + e^{-t/2}\ e\text{-}t/4\ e^{-t/5}$
$= e^{-t/2} + e^{-t/4} + e^{-t/5} - e^{-0.75t} - e^{-0.7t} - e^{-0.45t} + e^{-0.95t}$.
Integrating the survival function from zero to infinity, the expected value of the maximum is:
2 + 4 + 5 - 1/0.75 - 1/0.7 - 1/0.45 + 1/0.95 = 7.0685.

2.108. C. Due to the memoryless property of the Exponential Distribution, it does not matter how long the computer has already been functioning.
S(4) = exp[-4/10] = **0.670**.

2.109. D. Due to the memoryless property of the Exponential, with a deductible, the amount the insurance company pays per non-zero claim is independent of the size of the deductible.
⇒ With a deductible of d, the payments per theft follow a two point mixture of zero and an Exponential with mean 2000, with weights F(d) and S(d).
This mixed distribution has mean S(d) 2000, second moment $(2)(2000^2)$ S(d),
and thus variance: (8 million) S(d) - (4 million) $S(d)^2$.
$S(500) = e^{-500/2000} = 0.77880$.
Thus $\sigma_{500} = \sqrt{(8 \text{ million}) (0.77880) - (4 \text{ million}) (0.77880^2)} = 1950.46$.

$S(1000) = e^{-1000/2000} = 0.60653$.
Thus $\sigma_{1000} = \sqrt{(8 \text{ million}) (0.60653) - (4 \text{ million}) (0.60653^2)} = 1838.67$.

The absolute difference between σ_{1000} and σ_{500} is: |1950.46 - 1838.67| = **111.8**.
Alternately, with a deductible of d, the payments per theft follow an aggregate distribution with a Bernoulli frequency with mean S(d), and an Exponential Severity with mean 2000.
This has variance: (mean frequency) (variance of severity) + $(\text{mean severity})^2$ (variance of freq.)
$= S(d) (2000^2) + (2000^2) S(d)\{1 - S(d)\} = (8 \text{ million}) S(d) - (4 \text{ million}) S(d)^2$.
Proceed as before.
Comment: Difficult.
See Example 5.4 in Introduction to Probability Models by Ross.
The moments of a mixture are the mixture of the moments.
Variance of an aggregate distribution is:
(mean frequency) (variance of severity) + $(\text{mean severity})^2$ (variance of frequency).

2.110. D.
Since the distribution is memoryless, the average payment per payment is the mean of **1000**.
Comment:
"Calculate the average insurance payment made on losses that exceed the deductible."
This language indicates to me the average payment per large loss, in other words per non-zero payment.

Section 3, Homogeneous Poisson Processes

A (homogeneous) **Poisson Process has a constant claims rate (intensity) λ**, **and what happens on disjoint intervals (touching at an endpoint is okay) is independent**.
The number of claims observed in time interval of width h is given by the Poisson Distribution with mean hλ.[38] [39] [40]

For example, assume we have a Poisson Process on (0, 5) with $\lambda = 0.03$.
Then the total number of claims is given by a Poisson Distribution with mean: (5)(0.03) = 0.15.

Exercise: For a Poisson Process with $\lambda = 0.7$, what is the chance of exactly 3 claims by time 2?
[Solution: Poisson Distribution with mean: (2)(0.7) = 1.4. $f(3) = 1.4^3 e^{-1.4}/3! = 11.3\%$.]

Thus if you understand the Poisson frequency distribution, you understand the (homogeneous) Poisson Process. However, in the case of a (homogeneous) Poisson Process one normally keeps track of both the total number of claims and the times they each occurred. If one had three claims, one would also want to know the three times at which they occurred.

Counting Processes:

A stochastic process N(t) on $t \geq 0$ is a counting process if N(t) represents the total number of events by time t. N(t) must satisfy:
1. $N(t) \geq 0$.
2. N(t) is integer valued.
3. If $s < t$, then $N(s) \leq N(t)$.
4. For $s < t$, $N(t) - N(s)$ equals the number of events that have occurred in the interval (s, t].

Thus the state space of a counting process is the nonnegative integers.
A counting process is nondecreasing with time.

A **counting process is what an actuary would call a claims frequency process.** A Poisson Process is an example of a counting process. N(t) is the number of claims that have occurred by time t. $N(t) - N(s)$ is the number of claims that have occurred in the interval (s, t], is the increment to the process.

A claims frequency process has independent increments if the number of claims in two disjoint periods of time are independent of each other.

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[38] See Theorem 5.1 of Introduction to Probability Models by Sheldon M. Ross.
[39] Ross and Daniel refer to "events."
I will usually refer to "claims", which is the most common application of these ideas for actuaries.
[40] *Note that by changing the time scale and therefore the claims intensity, one can always reduce to a mathematically equivalent situation where the interval is (0,1).*

For example, for practical purposes the number of dental claims a large insurer gets this week are probably independent of the number of dental claims it gets next week. Thus we could model this frequency process with independent increments. However, the number of claims for the flu this week might be correlated with the number of claims for the flu last week.

A claims frequency has stationary increments if the distribution of the increment is a function of the width of the interval t - s, rather than s or t individually.

Stationary Increments ⇔ the distribution of N(s + Δ) - N(s) is independent of s.

A (homogeneous) Poisson Process is a counting process with stationary and independent increments.

The assumption of stationary and independent increments is basically equivalent to asserting that at any point in time the process probabilistically restarts itself. In other words, the process has no memory.

If exposure levels are changing, then we would be unlikely to have stationary increments and therefore unlikely to have a (homogeneous) Poisson Process. Rather one would have a Nonhomogeneous Poisson Process, to be discussed subsequently.

Definition:

For a homogeneous Poisson Process with claims rate (intensity) λ:[41]

1. It is a counting process that starts at time = 0 with zero events; N(0) = 0.

2. It has independent increments.[42]

3. The increment N[t + h] - N[t] is Poisson with mean λh.[43]

[41] See Definition 1.2 in "Poisson Processes" by Daniel.
[42] $N[t_1 + h_1] - N[t_1]$ and $N[t_2 + h_2] - N[t_2]$ are independent if $(t_1, t_1 + h_1]$ does not overlap with $(t_2, t_2 + h_2]$. Touching at endpoints is okay.
[43] Increments are stationary, with claims rate (intensity) lambda.
As discussed subsequently, for a nonhomogeneous Poisson Process, lambda is a function of time, and the increment is Poisson with mean equal to the integral of λ(t) over the time interval.

o(h)

o(h) is a mathematical property a function may have.
A function f(x) is said to be o(h), if the limit as h approaches zero of f(h)/h is zero.[44]

In other words as h approaches zero, f(h) goes to zero more quickly than h.
For example, $f(x) = x^{1.5}$ is o(h). Thus x is of lower order of magnitude than $x^{1.5}$.
On the other hand, $\sqrt{x}$ is not o(h).

For a Poisson Process, the chance of having more than 1 claim in an interval of length h, is o(h). In other words, as the length of an interval approaches zero, the chance of having more than one claim in that interval also approaches zero, faster than the length of that interval approaches zero. Prob[more than one claim in an interval of length h]/h, goes to zero as h goes to zero.

For a Poisson Process, the chance of having 1 claim in an interval of length h, is
λh + o(h). In other words, as h goes to zero,
Prob[one claim in interval of length h]/h, goes to λ as h goes to zero.

A function f(x) is equivalent to x as x goes to zero, f(x) ~ x, if the limit as h approaches zero of f(h)/h is one. For example, sin(x) ~ x, as x approaches zero.

A function f(x) is said to be O(h), if there is a positive constant K, such that for all sufficiently small h, $| f(h)/h | \le K$. For example, $f(x) = x(1 + a/x)^x$ is O(h), since the limit as x goes to zero of $(1 + a/x)^x$ is e^a. If a function is either o(h) or equivalent to h as h goes zero, then it is also O(h).

Another Definition of a Poisson Process:[45]

A Poisson Process is a counting process, N(t), such that:
N(0) = 0.
Stationary and independents increments.
Prob[N(t+h) - N(t) = 1] = λh + o(h).
Prob[N(t+h) ≥ 2] = o(h).

[44] See Definition 5.1 of Introduction to Probability Models by Sheldon M. Ross.
[45] See Definition 5.2 of Introduction to Probability Models by Sheldon M. Ross

Problems:

3.1 (1 point) A Poisson Process has a claims intensity of 0.3.
What is the probability of exactly 2 claims from time 0 to time 10?
A. 22% B. 24% C. 26% D. 28% E. 30%

3.2 (1 point) A Poisson Process has a claims intensity of 0.3.
Given that there have been 2 claims from time 0 to time 10, what is the probability of exactly 2 claims from time 10 to time 20?
A. 22% B. 24% C. 26% D. 28% E. 30%

3.3 (3 points) A Poisson Process has a claims intensity of 0.3.
Given that there have been 2 claims from time 0 to time 10, what is the probability of exactly 2 claims from time 6 to time 16?
A. 19% B. 21% C. 23% D. 25% E. 27%

3.4 (1 point) A Poisson Process has $\lambda = 0.6$.
What is the probability of exactly 3 claims from time 5 to time 9?
A. 19% B. 21% C. 23% D. 25% E. 27%

3.5 (3 points) A Poisson Process has $\lambda = 0.6$.
If there are three claims from time 2 to time 8, what is the probability of exactly 3 claims from time 5 to time 9?
A. 19% B. 21% C. 23% D. 25% E. 27%

3.6 (2 points) A Poisson Process has $\lambda = 0.6$.
What is the probability of 2 claims from time 0 to time 4 and 5 claims from time 0 to time 10?
A. 4.0% B. 4.5% C. 5.0% D. 5.5% E. 6.0%

3.7 (3 points) A Poisson Process has $\lambda = 0.6$. What is the probability of at least 1 claim from time 0 to time 4 and at least 3 claims from time 0 to time 6?
A. 69% B. 71% C. 73% D. 75% E. 77%

3.8 (1 point) A Poisson Process has $\lambda = 0.6$.
We observe 3 claims from time 0 to 1.
What is the probability of 4 claims from time 0 to 2?
A. 33% B. 35% C. 37% D. 39% E. 41%

3.9 (2 points) Claims are given by a Poisson Process with $\lambda = 7$. What is probability that the number of claims between time 5 and 15 is at least 60 but no more than 80?
Use the Normal Approximation.
(A) 73% (B) 75% (C) 77% (D) 79% (E) 81%

3.10 (3 points) Claims are given by a Poisson Process with $\lambda = 7$. What is probability that the number of claims between time 5 and 11 is greater than the number of claims from time 11 to 15? Use the Normal Approximation.
(A) 87% (B) 89% (C) 91% (D) 93% (E) 95%

Use the following information for the next two questions:
The android Data is stranded on the planet Erehwon.
- Data uses energy uniformly at a rate of 10 gigajoules per year.
- If Data's stored energy reach 0, he ceases to function.
- Data gets his energy from dilithium crystals.
- Data gets 6 gigajoules of energy from each dilithium crystal.
- Data finds dilithium crystals at a Poisson rate of 2 per year.
- Data can store dilithium crystals without limit until needed.
- Data currently has 8 gigajoules of energy stored.

3.11 (4 points) What is the probability that Data ceases to function within the next 2.5 years?
(A) 30% (B) 33% (C) 36% (D) 39% (E) 42%

3.12 (3 points) What is the expected number of gigajoules of energy found by Data in the next 2.5 years?
(A) 20 (B) 22 (C) 24 (D) 26 (E) 28

3.13 (3 points) The number of phone calls to a hotline are given by a Poisson Process with $\lambda = 5$. What is probability that the number of calls between time 2 and 10 is greater than the number of calls from time 7 to 16? Use the Normal Approximation.
(A) 17% (B) 19% (C) 21% (D) 23% (E) 25%

3.14 (1 point) Claims follow a homogeneous Poisson Process.
The average time between claims is 2.5 days.
How many whole number of days do we need to observe, in order to have at least a 95% probability of seeing at least one claim?
(A) 8 (B) 9 (C) 10 (D) 11 (E) 12

3.15 (2 points) A Poisson Process has a claims intensity of 0.4 per day.
How many whole number of days do we need to observe, in order to have at least a 95% probability of seeing at least two claims?
(A) 11 (B) 12 (C) 13 (D) 14 (E) 15

3.16 (1 point) A Poisson Process has a claims intensity of 0.04 per day. If there is at least 1 claim during a week, what is the probability that there are at least 2 claims during that week?
(A) 7% (B) 9% (C) 11% (D) 13% (E) 15%

3.17 (3 points) The number of atomic particles registering on a counter follows a Poisson Process with intensity of 0.5.
What is the probability that the third particle registers between time 5 and time 8?
(A) 10% (B) 15% (C) 20% (D) 25% (E) 30%

3.18 *(3 points) Data the Android gets new Emails at a Poisson rate of 15 per hour.*
Data checks for new Emails every x hours, where x has distribution $F(x) = 1 - 1/(27x^3)$, $x > 1/3$.
What is the variance of the number of new Emails Data finds when he checks?
(A) 18 (B)20 (C) 22 (D) 24 (E) 26

3.19 (3 points) Joe has four homework assignments: Math, English, History, and Chemistry.
He randomly chooses one of his assignments, and works on it until he completes it.
When Joe completes an assignment, he randomly chooses one of the remaining assignments, and works on it until completed.
Joe completes assignments at a Poisson rate of 1 assignment per 40 minutes.
Calculate the probability that Joe has completed his English assignment within 1 hour of starting his homework.
A. Less than 25%
B. At least 25%, but less than 30%
C. At least 30%, but less than 35%
D. At least 35%, but less than 40%
E. At least 40%

3.20 (3 points) For a certain company, losses follow a Poisson process with $\lambda = 3$ per year.
The amount of every loss is \$100.
An insurance policy covers all losses in a year, subject to an annual aggregate deductible of \$100.
(The insured pays the first \$100 of loss during a year. The insurer pays any remaining losses.)
There have been no payments for this insurance policy during the first half of the year.
Calculate the expected claim payments for this insurance policy during the second half of the year.
(A) 90 (B) 100 (C) 110 (D) 120 (E) 130

3.21 (3 points) One has a Poisson Process with $\lambda = 25$.
Let A = the number of events that occur from time 1 to time 6.
Let B = the number of events that occur from time 3 to time 10.
Determine the correlation of A and B.
A. 30% B. 40% C. 50% D. 60% E. 70%

Use the following information for the next three questions:
One has a Poisson Process with $\lambda = 3$.
There is one event between time 0 to time 2.

3.22 (1 point) What is the expected number of events from time 0 to 4?
A. 5 B. 6 C. 7 D. 8 E. 9

3.23 (2 points) What is the variance of the number of events from time 0 to 4?
A. 5 B. 6 C. 7 D. 8 E. 9

3.24 (2 points) What is the probability that there are 5 events from time 0 to 4?
A. 5% B. 7% C. 9% D. 11% E. 13%

3.25 (2 points)
Policyholder calls to a call center follow a homogenous Poisson process with $\lambda = 250$ per day.
Using the normal approximation with continuity correction, calculate the probability of receiving at least 260 calls in a day.
A. Less than 27%
B. At least 27%, but less than 29%
C. At least 29%, but less than 31%
D. At least 31%, but less than 33%
E. At least 33%

3.26 (2 points) One has a Poisson Process with $\lambda = 10$ per hour.
You observe the process for 2 hours.
What is the probability that the total number of events during the first 20 minutes and the last 10 minutes combined is 6?
A. 10% B. 15% C. 20% D. 25% E. 30%

3.27 (2 points) Often a manufacturer of military hardware must demonstrate its reliability.
For example, an electronic system was to be tested for 10,000 hours.
It was agreed that the hardware would pass the test if there were no more than 2 failures.
The manufacturer wants to know the probability that the hardware will fail the test, which is called the producer's risk.
Assuming the number of failures follows a Poisson Process with $\lambda = 0.000115$ failures per hour, determine the producer's risk.
A. 11% B. 12% C. 13% D. 14% E. 15%

3.28 (2 points)
Insurance claims are made according to a Poisson process $\{N(t), t \geq 0\}$ with rate $\lambda = 0.2$.
Calculate $E[N(5)\ N(12)]$.
A. Less than 3
B. At least 3, but less than 4
C. At least 4, but less than 5
D. At least 5 but less than 6
E. At least 6

3.29 (2 points) Cars arrive according to a Poisson process at a rate of three per hour.
Calculate the probability that in a given hour, exactly one car will arrive during the first twenty five minutes and no other cars will arrive during that hour.
A. Less than 0.03
B. At least 0.03 but less than 0.06
C. At least 0.06 but less than 0.09
D. At least 0.09 but less than 0.12
E. At least 0.12

3.30 (2 points) You are given the following information about Doug Graves:
● Doug's annual claim frequency follows.a Poisson Distribution with standard deviation of 1.2.
● Doug has a claim on January 1, 2025.
● Doug has at least one more claim during 2025.
Determine the probability that Doug has a total of exactly 2 claims during 2025.

3.31 (CAS Part 2 Exam, 1965, Q. 30) (1.5 points) If, on the average, 400 cubic centimeters of water will have 8 bacteria in the solution, what is the probability that, if 20 cubic centimeters are drawn off, there will be not more than 1 bacterium present in the 20 cubic centimeters?

Use the following information for 3, 11/01 questions 10 and 11:
For a tyrannosaur with 10,000 calories stored:
(i) The tyrannosaur uses calories uniformly at a rate of 10,000 per day.
If his stored calories reach 0, he dies.
(ii) The tyrannosaur eats scientists (10,000 calories each) at a Poisson rate of 1 per day.
(iii) The tyrannosaur eats only scientists.
(iv) The tyrannosaur can store calories without limit until needed.

3.32 (3, 11/01, Q.10) (2.5 points)
Calculate the probability that the tyrannosaur dies within the next 2.5 days.
(A) 0.30 (B) 0.40 (C) 0.50 (D) 0.60 (E) 0.70

3.33 (3, 11/01, Q.11) (2.5 points)
Calculate the expected calories eaten in the next 2.5 days.
(A) 17,800 (B) 18,800 (C) 19,800 (D) 20,800 (E) 21,800

3.34 (CAS3, 11/03, Q.32) (2.5 points) Daniel, in "Poisson Processes (and mixture distributions)", identifies four requirements that a counting process N(t) must satisfy.
Which of the following is NOT one of them?
A. N(t) must be greater than or equal to zero.
B. N(t) must be an integer.
C. If $s<t$, then N(s) must be less than or equal to N(t).
D. The number of events that occur in disjoint time intervals must be independent.
E. For $s<t$, N(t)-N(s) must equal the number of events that have occurred in the interval (s, t].

3.35 (SOA3, 11/03, Q.26) (2.5 points) A member of a high school math team is practicing for a contest. Her advisor has given her three practice problems: #1, #2, and #3.
She randomly chooses one of the problems, and works on it until she solves it. Then she randomly chooses one of the remaining unsolved problems, and works on it until solved.
Then she works on the last unsolved problem.
She solves problems at a Poisson rate of 1 problem per 5 minutes.
Calculate the probability that she has solved problem #3 within 10 minutes of starting the problems.
(A) 0.18 (B) 0.34 (C) 0.45 (D) 0.51 (E) 0.59

3.36 (CAS3, 11/04, Q.18) (2.5 points) Justin takes the train to work each day.
It takes 10 minutes for Justin to walk from home to the train station.
In order to get to work on time, Justin must board the train by 7:50 a.m.
Trains arrive at the station at a Poisson rate of 1 every 8 minutes.
What is the latest time he must leave home each morning so that he is on time for work at least 90% of the time?
A. 7:21 a.m. B. 7:22 a.m. C. 7:31 a.m. D. 7:32 a.m. E. 7:41 a.m.

3.37 (CAS3, 11/04, Q.19) (2.5 points) XYZ Insurance introduces a new policy and starts a sales contest for 1,000 of its agents. Each agent makes a sale of the new product at a Poisson rate of 1 per week. Once an agent has made 4 sales, he gets paid a bonus of $1,000. The contest ends after three weeks. Assuming 0% interest, what is the expected cost of the contest?
A. $18,988 B. $57,681 C. $168,031 D. $184,737 E. $352,768

3.38 (SOA3, 11/04, Q.16) (2.5 points) For a water reservoir:
(i) The present level is 4999 units.
(ii) 1000 units are used uniformly daily.
(iii) The only source of replenishment is rainfall.
(iv) The number of rainfalls follows a Poisson process with $\lambda = 0.2$ per day.
(v) The distribution of the amount of a rainfall is as follows:

Amount	Probability
8000	0.2
5000	0.8

(vi) The numbers and amounts of rainfalls are independent.
Calculate the probability that the reservoir will be empty sometime within the next 10 days.
(A) 0.27 (B) 0.37 (C) 0.39 (D) 0.48 (E) 0.50

3.39 (CAS3, 5/05, Q.39) (2.5 points) Longterm Insurance Company insures 100,000 drivers who have each been driving for at least five years.
Each driver gets "violations" at a Poisson rate of 0.5/year.
Currently, drivers with 1 or more violations in the past three years pay a premium of 1000.
Drivers with 0 violations in the past three years pay 850.
Your marketing department wants to change the pricing so that drivers with 2 or more violations in the past five years pay 1,000 and drivers with zero or one violations in the past five years pay X.
Find X so that the total premium revenue for your firm remains constant when this change is made.
A. Less than 900
B. At least 900, but less than 925
C. At least 925, but less than 950
D. At least 950, but less than 975
E. 975 or more
Note: I have slightly reworded this past exam question.

3.40 (CAS3, 11/06, Q.26) (2.5 points) Which of the following is/are true?
1. A counting process is said to possess independent increments if the number of events that occur between time s and t is independent of the number of events that occur between time s and t+u for all u > 0.
2. All Poisson processes have stationary and independent increments.
3. The assumption of stationary and independent increments is essentially equivalent to asserting that at any point in time the process probabilistically restarts itself.
A. 1 only B. 2 only C. 3 only D. 1 and 2 only E. 2 and 3 only

3.41 (CAS3L, 11/08, Q.2) (2.5 points) You are given the following:
- Hurricanes occur at a Poisson rate of 1/4 per week during the hurricane season.
- The hurricane season lasts for exactly 15 weeks.

Prior to the next hurricane season, a weather forecaster makes the statement,
"There will be at least three and no more than five hurricanes in the upcoming hurricane season."
Calculate the probability that this statement will be correct.
A. Less than 54%
B. At least 54%, but less than 56%
C. At least 56%, but less than 58%
D. At least 58%, but less than 60%
E. At least 60%

3.42 (CAS3L, 5/10, Q.13) (2.5 points)
You are given the following information regarding bank collapses:
● Bank collapses occur Monday through Thursday and are classified into two categories: severe and mild.
● Severe bank collapses occur only on Mondays and Tuesdays, and follow a Poisson distribution with $\lambda = 1$ on each of those days.
● Mild bank collapses can occur only on Wednesdays and Thursdays.
If there was no more than one severe bank collapse earlier in the week, then mild bank collapses follow a Poisson distribution with $\lambda = 1$ for Wednesdays and Thursdays.
If there was more than one severe bank collapse earlier in the week, then mild bank collapses follow a Poisson distribution with $\lambda = 2$ for Wednesdays and Thursdays.
Calculate the probability that no mild bank collapses occur during one week.
A. Less than 6.5%
B. At least 6.5% but less than 7.0%
C. At least 7.0% but less than 7.5%
D. At least 7.5% but less than 8.0%
E. At least 8.0%

3.43 (MAS-1, 11/18, Q.1) (2.2 points)
Insurance claims are made according to a Poisson process $\{N(t), t \geq 0\}$ with rate $\lambda = 1$.
Calculate $E[N(1) * N(2)]$.
A. Less than 1.5
B. At least 1.5, but less than 2.5
C. At least 2.5, but less than 3.5
D. At least 3.5, but less than 4.5
E. At least 4.5

3.44 (MAS-1, 5/19, Q.1) (2.2 points)
Cars arrive according to a Poisson process at a rate of two per hour.
Calculate the probability that in a given hour, exactly one car will arrive during the first ten minutes and no other cars will arrive during that hour.
A. Less than 0.03
B. At least 0.03 but less than 0.06
C. At least 0.06 but less than 0.09
D. At least 0.09 but less than 0.12
E. At least 0.12

Solutions to Problems:

3.1. A. Number of claims is Poisson with mean: (0.3)(10) = 3. Prob[2 claims] = $3^2e^{-3}/2$ = **0.224**.

3.2. A. What happens from time 10 to 20 is independent of what happened from time 0 to 10.
Number of claims is Poisson with mean: (0.3)(10) = 3. Prob[2 claims] = $3^2e^{-3}/2$ = **0.224**.

3.3. E. Due to the constant independent claims intensity, each of the 2 claims in (0, 10) has a 4/10 = 0.4 chance of being in (6, 10), independent of the other claim.
Therefore, there is 0.6^2 = 0.36 chance of zero claims in (6, 10), (2)(0.6)(0.4) = 0.48 chance of one claim in (6, 10), and 0.4^2 = 0.16 chance of two claims in (6, 10).
What happens from time 10 to 16 is independent of what happened from time 6 to 10.
Number of claims from 10 to 16 is Poisson with mean: (.03)(6) = 1.8.
Prob[2 claims in (6, 16)] = Prob[0 claims in (6,10)] Prob[2 claims in (10, 16)] +
Prob[1 claim in (6, 10)] Prob[1 claim in (10, 16)] +
Prob[2 claims in (6, 10)] Prob[0 claims in (10, 16)] =
$(0.36)(1.8^2e^{-1.8}/2) + (0.48)(1.8e^{-1.8}) + (0.16)(e^{-1.8})$ = **0.266**.
Comment: Difficult. We are given that there are 2 claims from 0 to 10. Therefore, the number of claims from 6 to 10 is conditional on there being exactly 2 claims in 0 to 10. For example, there can not be 3 claims in 6 to 10. In contrast, we know nothing about what happened beyond time 10. Therefore, for the interval from 10 to 16 we use the unconditional (Poisson) probabilities.

3.4. B. Over the interval 5 to 9, the number of claims is Poisson with mean:
(9 - 5)(0.6) = 2.4. f(3) = $e^{-2.4}2.4^3/3!$ = **20.9%**.

3.5. C. What happens from time 2 to 8 is independent of what happens from time 8 to 9.
Number of claims from 8 to 9 is Poisson with mean: (0.6)(1) = 0.6.
The 3 claims from time 2 to 8 are independently uniformly distributed from 2 to 8.
Each claim has a 3/6 = 1/2 chance of being in (5, 8).
Therefore, there is 1/8 chance of zero claims in (5, 8), 3/8 chance of one claim, and 3/8 chance of two claims, and 1/8 chance of 3 claims.
(These probabilities are Binomial with q = 1/2 and m = 3.
Total of 3 claims with each one having an independent 50% chance of being in the interval.)
Prob[3 claims in (5, 9)] = Prob[0 claims in (5, 8)] Prob[3 claims in (8, 9)] +
Prob[1 claim in (5, 8)] Prob[2 claims in (8, 9)] + Prob[2 claims in (5, 8)] Prob[1 claim in (8, 9)] +
Prob[3 claims in (5, 8)] Prob[0 claims in (8, 9)] =
$(1/8)(0.6^3e^{-0.6}/6) + (3/8)(0.6^2e^{-0.6}/2) + (3/8)(0.6e^{-0.6}) + (1/8)(e^{-0.6})$ = **23.2%**.

3.6. D. Over the interval 0 to 4, the number of claims is Poisson with mean: (4)(0.6) = 2.4.
f(2) = $e^{-2.4}2.4^2/2!$ = 0.261. What happens over the interval 4 to 10 is independent of what happens over the interval 0 to 4. Over the interval 4 to 10, the number of claims is Poisson with mean: (6)(0.6) = 3.6. f(3) = $e^{-3.6}3.6^3/3!$ = 0.212.
Prob[2 claims from time 0 to time 4 and 5 claims from time 0 to time 10] =
Prob[2 claims from time 0 to time 4 and 3 claims from time 4 to time 10] =
Prob[2 claims from time 0 to 4] Prob[3 claims from time 4 to 10] **=** (0.261)(0.212) **= 0.055**.

3.7. A. Over the interval 0 to 4, the number of claims is Poisson with mean: (4)(0.6) = 2.4.
$f(0) = e^{-2.4} = 0.091$. $f(1) = e^{-2.4}2.4 = 0.218$. $f(2) = e^{-2.4}2.4^2/2! = 0.261$.
What happens over the interval 4 to 6 is independent of what happens over the interval 0 to 4.
Over the interval 4 to 6, the number of claims is Poisson with mean: (2)(0.6) = 1.2.
$f(0) = e^{-1.2} = 0.301$. $f(1) = e^{-1.2}1.2 = 0.361$.
Prob[at least 1 claim from time 0 to time 4 and at least 3 claims from time 0 to time 6] =
Prob[1 claim from time 0 to 4] Prob[at least 2 claims from time 4 to 6] +
Prob[2 claims from time 0 to 4] Prob[at least 1 claim from time 4 to 6] +
Prob[at least 3 claims from time 0 to 4] Prob[at least 0 claims from time 4 to 6]
= (0.218)(1 - 0.301 - 0.361) + (0.261)(1 - 0.301) + (1 - 0.091 - 0.218 - 0.261)(1) **= 0.686.**
Alternately, one can calculate the probability of the desired condition failing:
Prob[0 claim from time 0 to 4]
+ Prob[1 claims from time 0 to 4] Prob[0 or 1 claim from time 4 to 6]
+ Prob[2 claims from time 0 to 4] Prob[0 claims from time 4 to 6]
= (0.091) + (0.218)(0.301 + 0.361) + (0.261)(0.301) = 0.314.
Probability of the condition being met is: 1 - 0.314 = **0.686**.

3.8. A. What happens from 1 to 2 is independent of what happened from 1 to 2.
We need the probability of one claim from time 1 to 2: $0.6\ e^{-0.6}$ = **32.9%**.

3.9. D. The number of claims from time 5 to 15 is Poisson with mean (10)(7) = 70.
It has variance 70. Prob(60 ≤ # claims ≤ 80) $\cong \Phi[(80.5 - 70)/\sqrt{70}] - \Phi[(59.5 - 70)/\sqrt{70}]$ =
$\Phi(1.255) - \Phi(-1.255)$ = 0.8953 - (1 - 0.8953) = **79.1%**.

3.10. E. The number of claims from time 5 to 11 is Poisson with mean (6)(7) = 42.
This can be approximated by a Normal with mean 42 and variance 42.
The number of claims from time 11 to 15 is Poisson with mean: (4)(7) = 28.
This can be approximated by a Normal with mean 28 and variance 28.
The number of claims from 5 to 11 is independent of the number of claims from 11 to 15.
Therefore, the difference in the number of claims can be approximated by a Normal Distribution with mean: 42 - 28 = 14 and variance: 42 + 28 = 70.
Prob(More claims in (5, 11) than (11, 15)) $\cong 1 - \Phi[(0.5 - 14)/\sqrt{70}] = \Phi(1.61)$ = **94.6%.**

Comment: Here we want to see if one variable is greater than another.
This is true if their difference, in the correct order, is positive.
When one subtracts two variables, one subtracts the means.
When one subtracts two independent variables, one adds the variances.

3.11. C. If Data does not find a crystal within the first 0.8 years, he exhausts his initial energy store and ceases to function.
Prob[cease function @ t = 0.8] = Prob[0 crystals by time = 0.8] = $e^{-(0.8)(2)}$ = .2019.
If Data functions at time = 0.8, he ceases to function at t = 0.8 + 0.6 = 1.4 if he has not found at least a total of 2 crystals by time 1.4. Prob[cease function @ t = 1.4] =
Prob[1 crystal for $0 < t \le 0.8$]Prob[0 crystals for $0.8 < t \le 1.4$ | 1 crystal for $0 < t \le 0.8$] =
Prob[1 crystal for $0 < t \le 0.8$]Prob[0 crystals for $0.8 < t \le 1.4$] =
$1.6e^{-1.6}e^{-1.2}$ = 0.0973. If Data functions at time = 1.4, he ceases to function at t = 1.4 + 0.6 = 2 if he has not found at least a total of 3 crystals by time 2. Prob[cease function @ t = 2] =
Prob[2 crystals, $0 < t \le 1.4$ & at least one crystal, $0 < t \le 0.8$] Prob[0 crystals, $1.4 < t \le 2$] =
Prob[1 crystal for $0 < t \le 0.8$] Prob[1 crystal for $0.8 < t \le 1.4$]Prob[0 crystals, $1.4 < t \le 2$] +
Prob[2 crystals for $0 < t \le 0.8$] Prob[0 crystals for $0.8 < t \le 1.4$] Prob[0 crystals, $1.4 < t \le 2$] =
$1.6e^{-1.6}\ 1.2e^{-1.2}\ e^{-1.2} + (1.6^2e^{-1.6}/2)\ e^{-1.2}\ e^{-1.2}$ = 0.0586.
If Data functions at time 2, then he also functions at t = 2.5 < 2 + 0.6 .
The probability that Data ceases to function within the next 2.5 years:
0.2019 + 0.0973 + 0.0586 = **0.358**.
Comment: Similar to 3, 11/01, Q.10.

3.12. C. A functioning Data is expected to find 2 crystals or 12 gigajoules per year.
time [0, 0.8]: (0.8)(12) = 9.6. time [.8, 1.4]: (1 - 0.2019)(0.6)(12) = 5.746.
time [1.4, 2]: {1 - (0.2019 + 0.0973)}(0.6)(12) = 5.046.
time [2, 2.5]: {1 - (0.2019 + 0.0973 + 0.0586)}(0.5)(12) = 3.853.
Expected energy found: 9.6 + 5.746 + 5.046 + 3.853 = **24.2**.

3.13. D. The subinterval (7, 10) is in both intervals. Therefore,
More calls in (2, 10) than (7, 16). ⇔ More calls in (2, 7) than (10, 16).
The number of calls from time 2 to 7 is Poisson with mean (5)(5) = 25.
This can be approximated by a Normal with mean 25 and variance 25.
The number of calls from time 10 to 16 is Poisson with mean (5)(6) = 30.
This can be approximated by a Normal with mean 30 and variance 30.
The number of calls from 2 to 7 is independent of the number of calls from 10 to 16.
Therefore, the difference in the number of calls can be approximated by a Normal Distribution with mean: 25 - 30 = -5 and variance: 25 + 30 = 55.
Prob(More calls in (2, 7) than (10, 16)) ≅ $1 - \Phi[(0.5 + 5)/\sqrt{55}] = 1 - \Phi(0.74)$ = **23.0%**.

3.14. A. average time between claims = $1/\lambda$. ⇒ λ = 1/2.5 = 0.4
We want Prob[0 claims] ≤ 0.05 ⇒ $e^{-0.4t} \le 0.05$ ⇒ t ≥ -ln(0.05)/0.4 = 7.49.
Need to observe at least **8** days.

3.15. B. Over t days, the number of claims is Poisson with mean 0.4t.
We want Prob[0 or 1 claim] ≤ 0.05 ⇒ $e^{-0.4t} + 0.4te^{-0.4t} \le 0.05$ ⇒
ln(1+ 0.4t) - 0.4t ≤ ln(0.05) = -2.996 ⇒ 0.4t - ln(1+0.4t) ≥ 2.996.

t	ln(1+0.4t)	0.4t - ln(1+0.4t)
11	1.6864	2.7136
12	1.7579	3.0421
13	1.8245	3.3755
14	1.8871	3.7129
15	1.9459	4.0541

Need to observe at least **12** days.

3.16. D. Mean number of claims during 7 days is 0.28.
Prob[at least 1 claim] = $1 - e^{-0.28}$ = 0.2442.
Prob[at least 2 claims] = $1 - e^{-0.28} - 0.28\ e^{-0.28}$ = 0.0326.
Prob[at least 2 claims | at least 1 claim] =
Prob[at least 2 claims]/Prob[at least 1 claim] = 0.0326/0.2442 = **0.133**.

3.17. E. Number of particles from time 0 to 5 is Poisson with mean (5)(0.5) = 2.5.
Number of particles from time 5 to 8 is Poisson with mean (3)(0.5) = 1.5.
Number of particles from time 0 to 5 is independent of the number from time 5 to 8.
Prob[3rd particle is from 5 to 8] =
Prob[2 particles from 0 to 5] Prob[At least one particle from 5 to 8]
+ Prob[1 particle from 0 to 5] Prob[At least two particles from 5 to 8]
+ Prob[0 particles from 0 to 5] Prob[At least three particles from 5 to 8] =
$(2.5^2e^{-2.5}/2)(1 - e^{-1.5}) + (2.5e^{-2.5})(1 - e^{-1.5} - 1.5e^{-1.5}) + (e^{-2.5})(1 - e^{-1.5} - 1.5e^{-1.5} - 1.5^2e^{-1.5}/2)$ =
(0.2565)(0.7769) + (0.2052)(0.4422) + (0.0821)(0.1912) = **0.306**.
Alternately, Prob[3rd particle is from 5 to 8] =
Prob[3rd particle by time 8] - Prob[3rd particle by time 5] =
Prob[at least 3 particles by time 8] - Prob[at least 3 particles by time 5]
$(1 - e^{-4} - 4e^{-4} - 4^2e^{-4}/2) - (1 - e^{-2.5} - 2.5e^{-2.5} - 2.5^2e^{-2.5}/2)$ = 0.7619 - 0.4562 = **0.306**.

3.18. E. Given he waits x hours to check his Emails, the number of Emails he finds, N, is Poisson with mean 15x. Therefore, E[N | x] = 15x and Var[N | x] = 15x.
By analysis of variance, Var[N] = E[Var[N | x]] + Var[E[N | x]] = E[15x] + Var[15x] =
15E[X] + 225Var[X].
X follows a Single Parameter Pareto Distribution with $\alpha = 3$ and $\theta = 1/3$.
$E[X^n] = \{\alpha / (\alpha - n)\}\theta^n$. $E[X] = \{\alpha / (\alpha - 1)\}\theta = 1/2$. $E[X^2] = \{\alpha / (\alpha - 2)\}\theta^2 = 1/3$.
$Var[X] = 1/3 - 1/2^2 = 1/12$. Var[N] = (15)(1/2) + (225)(1/12) = **26.25**.
Comment: Difficult!

3.19. D. $\lambda = 1/40$. It is equally likely that he does English first, second, third, or fourth.
If he does English first, then Prob[completed English within 60 minutes of starting] =
Prob[completed at least one assignment during 60 minutes] = $1 - e^{-60/40} = 1 - e^{-1.5}$ = 77.7%.
If he does English second, then Prob[completed English within 60 minutes of starting] =
Prob[completed at least 2 assignments during 60 minutes] = $1 - e^{-1.5} - 1.5e^{-1.5}$ = 44.2%.
If he does English third, then Prob[completed English within 60 minutes of starting] =
Prob[completed at least 3 assignments during 60 minutes] = $1 - e^{-1.5} - 1.5e^{-1.5} - 1.5^2e^{-1.5}/2$ = 19.1%. If he does English fourth, then Prob[completed English within 60 minutes of starting] =
Prob[completed at least 4 assignments during 60 minutes] =
$1 - e^{-1.5} - 1.5e^{-1.5} - 1.5^2e^{-1.5}/2 - 1.5^3e^{-1.5}/6$ = 6.6%.
Prob[completed English within 60 minutes of starting] =
(77.7% + 44.2% + 19.1% + 6.6%)/4 = **36.9%**.
Alternately, if he does English first, then Prob[complete English within 60 minutes of starting] =
Prob[complete at least one assignment during 60 minutes] = $1 - e^{-60/40}$ = 0.77689.
If he does English second, then Prob[complete English within 60 minutes of starting] =
Prob[complete at least two assignment during 60 minutes] = $1 - e^{-60/40} - 1.5e^{-1.5}$ = 0.44217.
If he does English third, then Prob[complete English within 60 minutes of starting] =
Prob[complete at least three assignment during 60 minutes] = $1 - e^{-1.5} - 1.5e^{-1.5} - 1.5^2e^{-1.5}/2$ = 0.19115.
If he does English fourth, then Prob[complete English within 60 minutes of starting] =
Prob[complete at least four assignment during 60 minutes]
= $1 - e^{-1.5} - 1.5e^{-1.5} - 1.5^2e^{-1.5}/2 - 1.5^3e^{-1.5}/6$ = 0.06564.
It is equally likely that he does English first, second, third, or fourth.
Therefore, Prob[complete English within 60 minutes of starting] =
(0.77689 + 0.44217 + 0.19115 + 0.06564) / 4 = **36.9%**.
Comment: In the first solution, we are assuming we have a Poisson Process which continues through time 1 hour. We pretend that if he completes the 4 given assignments, he works on some additional assignments. Then the probability he completes the 4 given assignments within 1 hour is the same as the probability of 4 or more events in the Poisson Process.

3.20. D. If there had been 1 loss during the first half of the year, then due to the annual aggregate deductible, no insurance payment would have been made. If there had been 2 losses during the first half of the year, then an insurance payment would have been made.
Therefore, no payments for this insurance policy during the first half of the year ⇒
0 or 1 loss during the first half of the year.
Number of losses during first half of the year is Poisson with mean 3/2.
$\text{Prob}[N = 0 \mid N \le 1] = e^{-\lambda}/(e^{-\lambda} + \lambda e^{-\lambda}) = 1/(1+\lambda) = 1/(1 + 1.5) = 0.4.$
$\text{Prob}[N = 1 \mid N \le 1] = \lambda e^{-\lambda}/(e^{-\lambda} + \lambda e^{-\lambda}) = \lambda/(1+\lambda) = 1.5/(1 + 1.5) = 0.6.$
If there is one loss during the first half of the year, then any losses during the second half of the year result in payments. We expect 3/2 = 1.5 losses, costing (1.5)(100) = 150.
If there is no loss during the first half of the year, then any losses beyond the first during the second half of the year result in payments.
Let N_2 be the number of losses in the second half of the year.
$E[N_2 \mid N_2 > 0]\text{Prob}[N_2 > 0] + 0\text{Prob}[N_2 = 0] = E[N_2] = 1.5. \Rightarrow E[N_2 \mid N_2 > 0]\text{Prob}[N_2 > 0] = 1.5.$
If there is no loss during the first half of the year, then the expected number of losses on which there are payments is: $E[N_2 - 1 \mid N_2 \ge 1] = E[N_2 \mid N_2 > 0] - 1.$
If there is no loss during the first half of the year, then the expected number of payments is:
$E[N_2 - 1 \mid N_2 \ge 1]\ \text{Prob}[N_2 \ge 1] = (E[N_2 \mid N_2 > 0] - 1)\text{Prob}[N_2 > 0] = 1.5 - \text{Prob}[N > 0]$
$= 1.5 - (1 - e^{-1.5}) = 0.723.$
Thus, if there is no loss during the first half of the year, we expect to pay: (100)(0.723) = 72.3 in the second half of the year.
Thus, the expected payments in the second half of the year are: (0.4)(72.3) + (0.6)(150) = **119**.
<u>Comment</u>: Very difficult!
Let $Y_+ = Y$ if $Y > 0$ and $Y_+ = 0$ if $Y \le 0$.
Let $N \wedge 1$ be the minimum of N and 1.
If there is no loss during the first half of the year, then the expected number of payments is:
$E[(N-1)_+] = E[N] - E[N \wedge 1] = 1.5 - \text{Prob}[N \ge 1] = 1.5 - (1 - e^{-1.5}) = 0.723.$

3.21. C. A is Poisson with mean 5λ and variance 5λ.
B is Poisson with mean 7λ and variance 7λ.
Let N(t) = the number of events by time t.
Cov[A, B] = Cov[N(6) - N(1), N(10) - N(3)]
= Cov[{N(6) - N(3)} + {N(3) - N(1)}, {N(10) - N(6)} + {N(6) - N(3)}] =
= Cov[N(6) - N(3), N(10) - N(6)] + Cov[N(6) - N(3), N(6) - N(3)] +
Cov[N(3) - N(1), N(10) - N(6)] + Cov[N(3) - N(1), N(6) - N(3)] =
0 + Var[N(6) - N(3)] + 0 + 0 = 3λ.
$\text{Corr}[A, B] = \text{Cov}[A, B] / \sqrt{\text{Var}[A]\ \text{Var}[B]} = 3\lambda / \sqrt{5\lambda\ 7\lambda} = 3 / \sqrt{35} = \mathbf{0.507}.$
<u>Comment</u>: What happens over disjoint intervals is independent.
[1, 6] and [3, 10] overlap, and therefore what happens over them is positively correlated.

3.22. C. What happens from 0 to 2 is independent of what will happen from time 2 to 4.
Expected number of events from time 2 to 4 is: (2)(3) = 6.
There is one event between time 0 to time 2.
Therefore, the expected number of events from time 0 to 4 is: 1 + 6 = **7**.

3.23. B. What happens from 0 to 2 is independent of what will happen from time 2 to 4.
Number of events from time 2 to 4 is Poisson with mean: (2)(3) = 6.
Var[number of events from time 0 to 4] =
Var[number of events from time 0 to 2] + Var[number of events from time 2 to 4] = 0 + 6 = **6**.

3.24. E. What happens from 0 to 2 is independent of what will happen from time 2 to 4.
The number of events from time 2 to 4 is Poisson with mean: (2)(3) = 6.
We need the probability of 4 events from time 2 to 4: $6^4 e^{-6} /4!$ = **13.4%**.
Comment: See page 5 of "Poisson Processes", by Daniel.

3.25. B. Applying the normal approximation with mean and variance equal to 250:
Prob[at least 260] $\cong 1 - \Phi[(259.5 - 250)/\sqrt{250}\,] = 1 - \Phi[0.60]$ = **27.4%**.
Comment: Using the continuity correction, 259 is out and 260 is in:
259 259.5 260
$|\!\longrightarrow$

3.26. B. The number of events over a period of length 30 minutes is Poisson with mean 5.
$f(6) = e^{-5}\ 5^6 / 6!$ = **14.6%**.
Comment: The fact that the two periods of time are not consecutive does not matter.

3.27. A. The number of failures is Poisson, with mean (10,000)(0.000115) = 1.15.
Prob[2 or fewer failures] = $e^{-1.15}\ (1 + 1.15 + 1.15^2/2) = 0.890$.
Thus the producer's risk is: 1 - 0.890 = **11.0%**.

3.28. B. N(12) - N(5) is independent of N(5).
$\Rightarrow$ E[{N(12) - N(5)} N(5)] = E[N(12) - N(5)] E[N(5)] = $7\lambda 5\lambda = 35\lambda^2$.
$\Rightarrow$ E[N(12) N(5)] - E[N(5)^2] = $35\lambda^2$.
N(5) is Poisson with mean $5\lambda$and variance 5λ. $\Rightarrow$ E[N(5)^2] = $5\lambda + (5\lambda)^2$.
$\Rightarrow$ E[N(12) N(5)] = E[N(5)^2] + $35\lambda^2 = 5\lambda + 60\lambda^2 = (5)(0.2) + (60)(0.2^2)$ = **3.4**.
Comment: Similar to MAS-1, 11/18, Q.1.

3.29. C. Number of cars over 25 minutes is Poisson with mean: (3)(25/60) = 1.25.
Prob[one car first 25 minutes] = $1.25\ e^{-1.25} = 0.3581$.
Number of cars over 35 minutes is Poisson with mean: (3)(35/60) = 1.75.
Prob[no car next 35 minutes] = $e^{-1.75} = 0.1738$.
What happens over the first 25 minutes is independent of what happens over the last 35 minutes.
Prob[one car first 25 minutes and no cars over the next 35 minutes] = (0.3581)(0.1738) = **0.0622**.
Comment: Similar to MAS-1, 5/19, Q.1.

3.30. The number of claims in a year follows a Poisson Process with $\lambda = 1.2^2 = 1.44$.
The number of claims during the remainder of the year is independent of Doug having a claim at the very start of 2024. Let N be the number of claims during the remainder of the year.
We want Prob[N = 1 | N ≥ 1] = Prob[N=1]/Prob[N≥1] = $1.44e^{-1.44} / (1 - e^{-1.44})$ = **44.7%.**
Alternately, for the remaining 364 days of the year the number of claims is Poisson with $\lambda = (1.44)(364/365) = 1.436$.
Prob[N = 1 | N ≥ 1] = Prob[N=1]/Prob[N≥1] = $1.436e^{-1.436} / (1 - e^{-1.436})$ = **44.8%.**

3.31. A reasonable assumption is that number of bacterium is a Poisson Process with intensity: 8/400 = 1/50.
Thus the number of bacterium in 20 cubic centimeters is Poisson with mean: 20/50 = 0.4.
$f(0) + f(1) = e^{-0.4} + 0.4\ e^{-0.4}$ = **0.9384**.

3.32. C. If the tyrannosaur does not eat a scientist within the first day it exhausts its initial store of calories and dies. Prob[death @ t = 1] = Prob[0 scientists by time = 1] = e^{-1}.
If the tyrannosaur survives at time = 1, it dies at t = 2 if it has not eaten at least a total of 2 scientists by time 2. This requires that the tyrannosaur eats 1 scientist from time 0 to 1, and no scientists from time 1 to 2.
Prob[death @ t = 2] = Prob[1 scientist for $0 < t \le 1$] Prob[0 scientists for $1 < t \le 2$] = $e^{-1}\ e^{-1} = e^{-2}$.
If the tyrannosaur survives to t = 2 it will survive to t = 2.5.
The probability that the tyrannosaur dies within the next 2.5 days is: $e^{-1} + e^{-2}$ = **0.503**.
<u>Comment</u>: Due to the particular inputs in this case, the tyrannosaur can only die at the end of a day. Unlike a ruin theory model, the good events (for the tyrannosaur) are random, while the bad events are nonrandom.

3.33. B. A <u>living</u> tyrannosaur is expected to eat 1 scientist or 10,000 calories per day.
Day 1: 10,000. Day 2: $(10{,}000)(1 - e^{-1})$. First half of Day 3: $(10{,}000/2)(1 - e^{-1} - e^{-2})$.
Expected calories: $10{,}000\{1 + 1 - e^{-1} + (1 - e^{-1} - e^{-2})/2\}$ = **18,805**.
<u>Comment</u>: Uses the solution to the previous question.
Over the first 2.5 days, a tyrannosaur survives an average of 1.8805 days.

3.34. D. If the number of events that occur in disjoint time intervals is independent, then one has independent increments. This is a feature of the Poisson Process, but not of all counting processes.

3.35. E. λ = 0.2 per minute. It is equally likely that she does #3 first, second, or third.
If she does #3 first, then Prob[solved #3 within 10 minutes of starting] =
Prob[solve at least one problem during 10 minutes] = $1 - e^{-(0.2)(10)} = 1 - e^{-2}$.
If she does #3 second, then Prob[solved #3 within 10 minutes of starting] =
Prob[solve at least two problems during 10 minutes] = $1 - e^{-2} - 2e^{-2}$.
If she does #3 third, then Prob[solved #3 within 10 minutes of starting] =
Prob[solve more than 2 problems during 10 minutes] = $1 - e^{-2} - 2e^{-2} - 2^2e^{-2}/2$.
Prob[solved #3 within 10 minutes of starting] =
(1/3)Prob[solved #3 within 10 minutes of starting | did #3 first] +
(1/3)Prob[solved #3 within 10 minutes of starting | did #3 second] +
(1/3)Prob[solved #3 within 10 minutes of starting | did #3 third] =
$(1/3)(1 - e^{-2}) + (1/3)(1 - e^{-2} - 2e^{-2}) + (1/3)(1 - e^{-2} - 2e^{-2} - 2^2e^{-2}/2) = 1 - 3e^{-2}$ = **0.594**.
Alternately, if she solves 1 problem within 10 minutes, there is 1/3 chance she solved #3.
If she solves 2 problems within 10 minutes, there is 2/3 chance she solved #3.
If she solves at least 3 problems within 10 minutes, then she solved #3.
Prob[solved #3 within 10 minutes of starting] =
(1/3)Prob[solved 1 problem within 10 minutes of starting] +
(2/3)Prob[solved 2 problems within 10 minutes of starting] +
(3/3)Prob[solved at least 3 problems within 10 minutes of starting]=
$(1/3)(2e^{-2}) + (2/3)(2^2e^{-2}/2) + (3/3)(1 - e^{-2} - 2e^{-2} - 2^2e^{-2}/2) = 1 - 3e^{-2}$ = **0.594**.
Comment: In the alternate solution, we are assuming we have a Poisson Process which continues through time 10. We pretend that if she completes the 3 given problems, she works on some additional problems. Then the probability she completes the 3 given problems within 10 minutes is the same as the probability of 3 or more events in the Poisson Process. If you prefer this method than use it, while if you prefer the first technique use it; if used properly then they produce the same answer.

3.36. A. Assuming Justin waits at the station t minutes, the chance of no train arriving is $e^{-t/8}$. Setting that equal to 10%: $0.1 = e^{-t/8}$. $\Rightarrow$ t = 18.4.
7:50 - 18.4 minutes - 10 minutes for walk = **7:21 a.m**.
Comment: Leaving at 7:22 a.m. would result in a $e^{-18/8}$ = 10.5% chance of not getting a train in time. Leaving at 7:21 a.m., this probability is: $e^{-19/8}$ = 9.3%.

3.37. E. The number of sales an agent makes over 3 weeks is Poisson with mean 3.
Prob[4 or more sales] = 1 - f(0) - f(1) - f(2) - f(3) = $1 - e^{-3} - 3e^{-3} - 3^2e^{-3}/2 - 3^3e^{-3}/6$ = 0.352768.
($1000)(1000 agents)(0.352768) = **$352,768**.

3.38. D. If there is no rain, then the reservoir will be empty within 5 days.
If there is not more than 5000 rain, then the reservoir will be empty within 10 days.
Thus there are two ways that the reservoir can be empty sometime within the next 10 days:
1) There is no rainfall within the next 5 day.
2) There is exactly 1 rainfall in the next 5 days, it is of amount 5000, and there are no further rainfalls.
Prob[empty sometime within the next 10 days] =
Prob[0 in 5 days] + Prob[1 in 5 days]Prob[amount = 5000] Prob[0 in next 5 days]
= exp[-(0.2)(5)] + (5)(.2)exp[-(0.2)(5)](.8)exp[-(0.2)(5)] = 0.3679 + 0.1083 = **0.476**.

3.39. A. The number of violations over three years is Poisson with mean 1.5.
Prob[0 in 3 years] = $e^{-1.5}$ = 0.2231.
⇒ Current Average Premium is: (0.2231)(850) + (1 - 0.2231)(1000) = 966.54.
The number of violations over five years is Poisson with mean 2.5.
Prob[0 or 1 in 5 years] = $e^{-2.5} + 2.5e^{-2.5}$ = 0.2873.
0.2873X + (1 - 0.2873)(1000) = 966.54. ⇒ X = **883.5**.
Comment: The total number of drivers is not used.
The exam questions said ”with 2 or more accidents” rather than ”with 2 or more violations.”

3.40. E. A counting process is said to possess independent increments if the number of events that occur in disjoint time intervals are independent. Statement #1 is false.
Statement #1 would have been true if it had said
“if the number of events that occur between time s and t is independent of the number of events that occur between time t and t+u for all u > 0.”
Statement #2 is true of homogeneous Poisson Processes but not nonhomogeneous Poisson Processes. Nonhomogeneous Poisson Processes do not have stationary increments, (except for the special case where they are homogeneous.) Statement #3 is true.
Comment: The intended answer was C, corresponding to Statement #2 not being true, since Ross in Introduction to Probability Models uses “Poisson process” when he means the homogeneous case. However, Daniel uses the term Poisson Process to include both the homogeneous and nonhomogeneous case.

3.41. B. The number of hurricanes is Poisson with mean: 15/4 = 3.75.
$f(3) + f(4) + f(5) = \lambda^3 e^{-\lambda}/6 + \lambda^4 e^{-\lambda}/24 + \lambda^5 e^{-\lambda}/120 = (e^{-3.75}\ 3.75^3/6)(1 + 3.75/4 + 3.75^2/20)$
= **54.6%**.

3.42. B. Severe bank collapses are Poisson with mean: (2)(1) = 2.
Probability of at most one severe collapse is: $e^{-2} + 2e^{-2}$ = 40.60%.
Probability of more than one severe collapse is: 1 - 40.60% = 59.40%.
If there is at most one severe collapse, then mild collapses are Poisson with mean: (2)(1) = 2.
If there is more than one severe collapse, then mild collapses are Poisson with mean: (2)(2) = 4.
The probability that no mild bank collapses occur during one week:
Prob[At most 1 severe] Prob[no mild | at most one severe] +
Prob[More than 1 severe] Prob[no mild | more than one severe] =
(40.60%)(e^{-2}) + (59.40%)(e^{-4}) = **6.58%**.

3.43. C. N(2) - N(1) is independent of N(1).
$\Rightarrow$ E[{N(2) - N(1)} N(1)] = E[N(2) - N(1)] E[N(1)] = $\lambda\lambda = \lambda^2$.
$\Rightarrow$ E[N(2) N(1)] - E[N(1)^2] = λ^2.
N(1) is Poisson with mean λ and variance λ. $\Rightarrow$ E[N(1)^2] = $\lambda + \lambda^2$.
$\Rightarrow$ E[N(1) N(2)] = E[N(1)^2] + $\lambda^2 = \lambda + 2\lambda^2 = 1 + (2)(1^2)$ = **3**.
Alternately, let N(1) = X and N(2) = Y.
Since X and Y - X are independent: E[X(Y-X)] = E[X] E[Y-X].
$\Rightarrow$ E[XY] - E[X^2] = E[X] (E[Y] - E[X]). $\Rightarrow$
E[XY] = E[X^2] + E[X]E[Y] - E[X^2] = Var[X] + E[X] E[Y] = $\lambda + \lambda(2\lambda) = \lambda + 2\lambda^2 = 1 + (2)(1^2)$ = **3**.
Comment: I waited to the end to substitute in the value of lambda, to illustrate the general ideas.

3.44. B. Number of cars over 10 minutes is Poisson with mean: (2)(10/60) = 1/3.
Prob[one car first ten minutes] = (1/3) $e^{-1/3}$ = 0.2388.
Number of cars over 50 minutes is Poisson with mean: (2)(50/60) = 5/3.
Prob[no car next fifty minutes] = $e^{-5/3}$ = 0.1889.
What happens over the first 10 minutes is independent of what happens over the last 50 minutes.
Prob[one car first ten minutes and no cars over the next 50 minutes] = (0.2388)(0.1889) = **0.0451**.

Section 4, Interevent Times, Poisson Processes

Exercise: What is the probability that the waiting time until the first claim is less than or equal to 10 for a Poisson Process with $\lambda = 0.03$?
[Solution: The number of claims by time 10 is Poisson with mean: (10)(0.03) = 0.3.
Waiting time is ≤ 10 ⇔ At least one claim by t = 10. Prob[0 claims by time 10] = $e^{-0.3}$.
Prob[1 or more claims by time 10] = 1 - $e^{-0.3}$ = 25.9%.]

In this case, the distribution function of the waiting time until the first claim is: $F(t) = 1 - e^{-0.03t}$, an Exponential Distribution with mean 1/0.03.
In general, for a Poisson Process with claims intensity λ,
the waiting time until the first claim has an Exponential Distribution with mean 1/λ.
$F(t) = 1 - e^{-\lambda t}$.[46]

Interevent Times:

The interevent times (interarrival times) **are the times between events**.
V_1 is the waiting time until the first event.
V_2 is the time from the first event until the second event.
V_j is the time from the j-1th event until the jth event.[47]

Exercise: What is the probability that the waiting time from the third claim to the fourth claim, V_4, is less than or equal to 10 for a Poisson Process with $\lambda = 0.03$?
[Solution: Interevent time from 3rd to 4th claim ≤ 10 ⇔ # claims ≥ 1 in interval of length 10.
The number of claims over a time interval of length 10 is Poisson with mean: (10)(0.03) = 0.3.
Prob[0 claims] = $e^{-0.3}$. Prob[1 or more claims] = 1 - $e^{-0.3}$ = 25.9%.]

Due to the constant, independent claims intensity, the interevent time between the 1st and 2nd claim has the same distribution as the waiting time until the first claim. We can start a new Poisson Process, with claims intensity λ, when the 1st claim occurs; therefore, the wait until the next claim is Exponential with mean $1/\lambda$. Similarly, we can start a new Poisson Process, with claims intensity λ, when the n^{th} claim occurs; therefore, the wait until the next claim is Exponential with mean $1/\lambda$.

This interevent time, V_n, is independent of what happened before the n^{th} claim occurs.

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[46] The survival function is $\exp[-\lambda t]$, the same as that for a constant force of mortality λ. As long as we wait for the first claim, the mathematics is the same as the time until death for a constant force of mortality λ.
[47] While Daniel uses the letter V, Ross uses the letter T for the interarrival times.

For a Poisson Process with claims intensity λ, the interevent times are independent Exponential Distributions each with mean 1/λ.[48] [49]

For example, assume we have a Poisson Process with $\lambda = 0.03$, with an unrestricted time horizon. Then the waiting time to the first claim is Exponential, with $\theta = 1/0.03 = 33.33$.

The second interevent time is an independent Exponential, with $\theta = 33.33$.
Each interevent time is an independent Exponential, with $\theta = 33.33$.

The interevent times for a Poisson Distribution are those of the corresponding Poisson Process. If claims follow a Poisson Distribution with mean annual frequency of 0.02, then each interevent time is an independent Exponential, with $\theta = 1/0.02 = 50$.

Event Times:

T_j is the time of the jth event (or claim).[50] **$V_j = T_j - T_{j-1}$.** $T_n = V_1 + V_2 + ... + V_n$.

The Poisson Process is a random process. Sometimes you have to wait a long time for the next claim, and sometimes the next claim shows up right away. Similarly, T_3, how long you have to wait for the third claim to occur is random.

Fewer claims showing up. ⇔ Event time is larger.

More claims showing up. ⇔ Event time is smaller.

A key idea relates event times to the number of events by a given time.
For example, T_2, the time of the second event, is less than or equal to 10, if and only if there have been at least 2 events by time 10. $T_2 \le 10$. ⇔ $N(10) \ge 2$.

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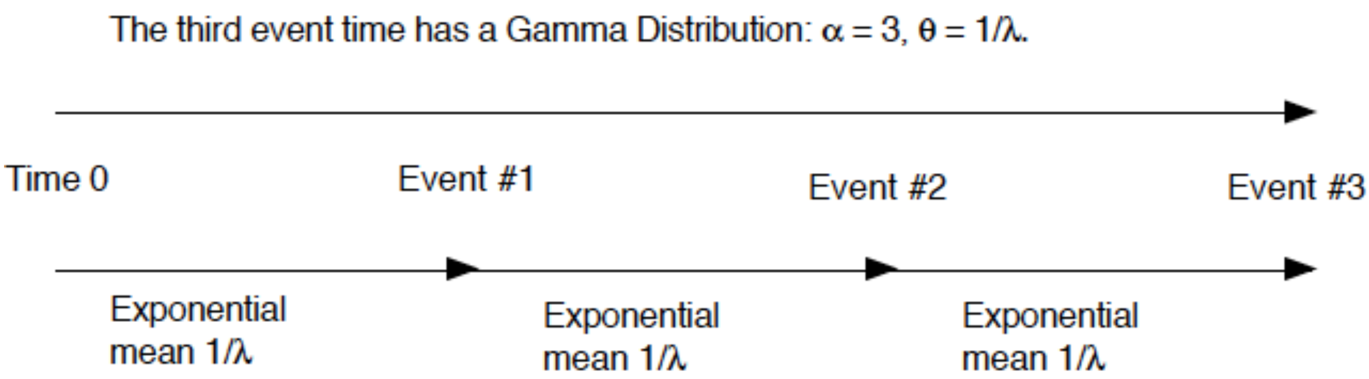
[48] See Proposition 5,1 in Introduction to Probability Models by Ross.
[49] Conversely, if the interevent times are independent, identically distributed Exponentials with mean θ, then the counting process is Poisson with hazard rate 1/θ.
[50] While Daniel uses the letter T, Ross uses the letter S for the arrival times.

Exercise: Assume we have a Poisson Process with $\lambda = 0.4$.
What is the probability that $T_2 \le 10$?
[Solution: This is the same as the probability that we have observed at least 2 events by time 10. The number of events in the time period from 0 to 10 is Poisson Distributed with mean: $(10)(0.4) = 4$.
Therefore, the chance of zero events in this interval is e^{-4}.
The chance of one event in this time interval is $4e^{-4}$.
Thus the chance of at least 2 events in the time interval is: $1 - e^{-4} - 4e^{-4} = 90.8\%$.
Comment: $\text{Prob}[T_2 \le 10] = \text{Prob}[N(10) \ge 2]$.]

In general, $\mathbf{T_k \le t. \Leftrightarrow N(t) \ge k.}$ $\mathbf{Prob[T_k \le t] = Prob[N(t) \ge k].}$

Exercise: Assume we have a Poisson Process with $\lambda = 0.03$.
What is the probability that we have observed at least 4 claims by time 100?
[Solution: The number of claims in the time period from 0 to 100 is Poisson Distributed with mean $(0.03)(100) = 3$. Therefore, the chance of zero claims in this interval is e^{-3}. The chance of one claim in this time interval is $3e^{-3}$. The chance of two claims in this time interval is $3^2e^{-3}/2$. The chance of three claims in this time interval is $3^3e^{-3}/6$. Thus the chance of at least 4 claims in the time interval is:
$1 - \{e^{-3} + 3e^{-3} + 3^2e^{-3}/2 + 3^3e^{-3}/6\} = 1 - 0.647232 = 0.352768$.
Comment: $\text{Prob}[T_4 \le 100] = \text{Prob}[N(100) \ge 4] = 0.352768$.]

Relationship to the Gamma Distribution:



The interevent times are independent, identically distributed Exponential Distributions with mean $1/\lambda$. The n^{th} event time, T_n, is the sum on n independent Exponential Variables, each with mean $1/\lambda$. Therefore, **the n^{th} event time, T_n, follows a Gamma Distribution with parameters α = n and $\theta = 1/\lambda$.**[51]

The mean time until the n^{th} claim is n/λ. Note that if we restrict the Poisson Process to (0,T), then there may be fewer than n claims; in other words the time until the n^{th} claim could be greater than T.

For example, assume we have a Poisson Process with $\lambda = 0.03$, with an unrestricted time horizon. Then the 4th event time is given by a Gamma Distribution with parameters $\alpha = 4$ and $\theta = 33.33$. The mean time of the 4th claim is the mean of this Gamma Distribution:
$\alpha\theta = (4)(33.33) = 133.33$.
T_4 follows a Gamma Distribution with $\alpha = 4$ and $\theta = 33.33$.
Thus $F(100) = \Gamma[4; 100/33.33] = \Gamma[4; 3]$.

However, in the previous exercise, we have computed that $\text{Prob}[T_4 \le 100] = 0.352768$.
Thus we have shown that $\Gamma[4; 3] = 0.352768$, without the use of a computer!

[51] The Gamma Distribution is discussed in a subsequent section.
See Definition and Properties 1.20 in Daniel.
The Gamma Distribution is parameterized as per Appendix A of Loss Models.

A Formula for the Incomplete Gamma Function:

In general, assume claims are given by a Poisson Process with claims intensity λ. Then the claims in the interval from (0, t) are Poisson Distributed with mean $t\lambda$. One can calculate the chance that there are least n claims in two different ways. First, the chance of at least n claims is a sum of Poisson densities:

$$1 - F(n-1) = 1 - \sum_{i=0}^{n-1} e^{-\lambda t} (\lambda t)^i / i! = \sum_{i=n}^{\infty} e^{-\lambda t} (\lambda t)^i / i!.$$

On the other hand, the n^{th} event time is a Gamma Distribution with $\alpha = n$ and $\theta = 1/\lambda$.
Thus the n^{th} claim has distribution function at time t of: $\Gamma[\alpha; t/\theta] = \Gamma[n ; \lambda t]$.
The chance of at least n claims by time t is the probability that the n^{th} claim occurs by time t, which is: $\Gamma[n ; \lambda t]$.

Comparing the two results for t = 1: $\Gamma[n ; \lambda] = 1 - \sum_{i=0}^{n-1} e^{-\lambda} \lambda^i / i! = \sum_{i=n}^{\infty} e^{-\lambda} \lambda^i / i!$.

Thus, the Incomplete Gamma Function with positive integer shape parameter α can be written in terms of a sum of Poisson densities:[52]

$$\Gamma(\alpha ; x) = 1 - \sum_{i=0}^{\alpha-1} \frac{x^i e^{-x}}{i!} = \sum_{i=\alpha}^{\infty} \frac{x^i e^{-x}}{i!}.$$

For example, $\Gamma[3; 3.5] = 1 - \{e^{-3.5} + 3.5e^{-3.5} + 3.5^2 e^{-3.5}/2\} = 1 - 0.320847 = 0.679153$.

We have also established a formula for the Distribution Function of a Poisson with mean λ, in terms of the Incomplete Gamma Function:

$$F(x) = \sum_{i=0}^{x} e^{-\lambda} \lambda^i / i! = 1 - \Gamma[x+1 ; \lambda].$$

[52] See Theorem A.1 in Appendix A of Loss Models, not on the syllabus.

Relationship to the Chi-Square Distribution:[53]

Since the Chi-Square Distribution is a special case of the Gamma Distribution, one can use a Chi-Square Table in order to estimate the Distribution Function of a Poisson.

A Chi-Square Distribution with ν degrees of freedom is a Gamma Distribution, as per Loss Models, with parameters $\alpha = \nu/2$ and $\theta = 2$.

Exercise: Use the following Chi-Square Table in order to estimate F(5) for a Poisson with mean 10.5.

	Significance Levels				
Degrees of Freedom	0.100	0.050	0.025	0.010	0.005
12	18.55	21.03	23.34	26.22	28.30

[Solution: $F(5) = 1 - \Gamma(5+1; 10.5) = 1 - \Gamma(6; 21/2) =$
1 - Chi-Square Distribution for 12 degrees of freedom at 21 $\cong$ 5%.]

In general, for a Poisson with mean λ, F(x)
= 1 - Chi-Square Distribution with (2x+2) degrees of freedom at 2λ
= significance level of 2λ for a Chi-Square with (2x+2) degrees of freedom.

Time Until the Next Event:

Let T(x) = the time from x until the next event.

Since what happened before time x is independent of what happens after time x, we can start a new process at time x. Therefore, T(x) is Exponential with mean $1/\lambda$. $\Pr[T(x) > t] = e^{-\lambda t}$.

Exercise: Assume we have a Poisson Process with $\lambda = 0.03$.
What is the probability that the first event to occur after time 10 occurs by time 15?
[Solution: $\Pr[T(10) \le 5] = 1 - \exp[-(0.03)(5)] = 13.9\%$.]

[53] The Chi-Square Distribution is discussed in “Mahler’s Guide to Statistics.”

Problems:

Use the following information for the next seven questions:
The number of births at a small hospital follows a Poisson Process
with an intensity of 0.05 per hour.

4.1 (1 point) What is the mean time until the first birth?
A. 5 B. 10 C. 15 D. 20 E. 25

4.2 (1 point) What is the mean time until the tenth birth?
A. 50 B. 100 C. 200 D. 300 E. 400

4.3 (1 point) What is the probability that the time until the first birth is greater than 35?
A. Less than 14%
B. At least 14%, but less than 15%
C. At least 15%, but less than 16%
D. At least 16%, but less than 17%
E. At least 17%

4.4 (1 point) What is the probability that the time from the fifth birth to the sixth birth is less than 10?
A. Less than 36%
B. At least 36%, but less than 38%
C. At least 38%, but less than 40%
D. At least 40%, but less than 42%
E. At least 42%

4.5 (2 points) What is the probability that the time from the eighth birth to the tenth birth is greater than 50?
A. Less than 28%
B. At least 28%, but less than 30%
C. At least 30%, but less than 32%
D. At least 32%, but less than 34%
E. At least 34%

4.6 (1 point) What is the median time until the first birth?
A. 12 B. 14 C. 16 D. 18 E. 20

4.7 (1 point) What is the variance of the time from the fourth to the ninth birth?
A. 2000 B. 2500 C. 3000 D. 3500 E. 4000

4.8 (2 points) For a claim number process you are given that the elapsed times between successive claims are mutually independent and identically distributed with distribution function:
$F(t) = 1 - e^{-t/2}$, $t \geq 0$.
Determine the probability of exactly 3 claims in an interval of length 7.
A. Less than 23%
B. At least 23%, but less than 25%
C. At least 25%, but less than 27%
D. At least 27%, but less than 29%
E. At least 29%

Use the following information for the next two questions:
The number of failures on a certain stretch of power line follows a Poisson Process.
The average time between failures is 500.

4.9 (2 points) What is the probability that we have observed at least 2 failures by time 900?
A. Less than 53%
B. At least 53%, but less than 55%
C. At least 55%, but less than 57%
D. At least 57%, but less than 59%
E. At least 59%

4.10 (2 points) What is the probability that we have observed exactly 2 failures by time 900?
A. Less than 23%
B. At least 23%, but less than 25%
C. At least 25%, but less than 27%
D. At least 27%, but less than 29%
E. At least 29%

4.11 (2 points) Use the following information:
- Customers arrive at a subway token booth in accordance with a Poisson process with mean 4 per minute.
- There is one clerk and his service time is exponentially distributed with a mean of 10 seconds.

When Joe arrives at the token booth, no other customers are there.
What is the probability that Joe is done being served before another customer arrives?
(A) 40% (B) 45% (C) 50% (D) 55% (E) 60%

4.12 (2 points) Lucky Tom finds coins on his way to work at a Poisson rate of 0.5 coins/minute.
What is the probability that during his one-hour walk tomorrow Tom finds his third coin during the second five minutes of his walk?
A. 40% B. 42% C. 44% D. 46% E. 48%

Use the following information for the next seven questions:

- You and your friend George go together to the subway platform.
- George is waiting for the uptown train and you are waiting for the downtown train.
- Uptown trains arrive via a Poisson Process at a rate of 5 per hour.
- Downtown trains arrive via a Poisson Process at a rate of 5 per hour.
- The arrival of downtown and uptown trains are independent.

4.13 (2 points) What is the average time until the first one of you catches your train?
(A) 4 minutes (B) 6 minutes (C) 8 minutes (D) 10 minutes (E) 12 minutes

4.14 (2 points) What is the median time until the first one of you catches your train?
(A) 4 minutes (B) 6 minutes (C) 8 minutes (D) 10 minutes (E) 12 minutes

4.15 (2 points) What is the average time until the last one of you catches your train?
(A) 15 minutes (B) 18 minutes (C) 21 minutes (D) 24 minutes (E) 27 minutes

4.16 (2 points) What is the 60^{th} percentile of the time until the last one of you catches your train?
(A) 15 minutes (B) 18 minutes (C) 21 minutes (D) 24 minutes (E) 27 minutes

4.17 (2 points) What is the average time you wait on the platform without George?
(A) 6 minutes (B) 8 minutes (C) 10 minutes (D) 12 minutes (E) 14 minutes

4.18 (1 point) On average how many uptown trains pass while you wait on the platform?
(A) 1/2 (B) 3/4 (C) 1 (D) 3/2 (E) 2

4.19 (2 points) What is the probability that exactly three downtown trains pass while George waits on the platform?
(A) 2% (B) 3% (C) 4% (D) 5% (E) 6%

4.20 (3 points) A Poisson Process has $\lambda = 0.9$.
What is the probability that the fourth event occurs between time 2 and time 5?
A. 40% B. 45% C. 50% D. 55% E. 60%

4.21 (1 point) Claims occur via a homogeneous Poisson Process. The expected waiting time until the first claim is 770 hours. If the claims intensity had been 5 times as large, what would have been the expected waiting time until the first claim?
A. 154 hours B. 765 hours C. 770 hours D. 3850 hours E. None of the above.

4.22 (2 points) Assume a driver's claim frequency is given by a Poisson distribution, with an average annual claim frequency of 8%.
What is the probability that it will be more than 25 years until this driver's first claim?
A. less than 10%
B. at least 10% but less than 11%
C. at least 11% but less than 12%
D. at least 12% but less than 13%
E. at least 13%

4.23 (2 points) For a Poisson Process, the average time between claims is 0.2.
What is the probability that we have observed at least 3 claims by time 1.2?
A. Less than 93%
B. At least 93%, but less than 95%
C. At least 95%, but less than 97%
D. At least 97%, but less than 99%
E. At least 99%

4.24 (3 points) Use the following information:
- In the late afternoon, customers arrive at a bagel shop via a Poisson Process at a rate of 20 per hour.
- Each customer first has his order filled and then pays for that order.
- The time to fill an order is exponentially distributed with a mean of 90 seconds.
- The time to pay for an order is exponentially distributed with a mean of 10 seconds.
- The times to fill an order and to pay for that order are independent.

When Mary arrives at the bagel shop, no other customers are there.
What is the probability that Mary is done getting her order and paying for it before another customer arrives?
(A) 63% (B) 65% (C) 67% (D) 69% (E) 71%

4.25 (2 points) Buses arrive via a Poisson Process with $\lambda = 6$ per hour. Sandy arrives at 9:02.
Sandy is still waiting for a bus at 9:05.
What is the probability that the next bus arrives by 9:10?
A. 35% B. 40% C. 45% D. 50% E. 55%

4.26 (1 point) Debbie receives phone calls at work via a Poisson Process at a rate of 5 per hour.
Right after finishing a phone call, Debbie leaves her desk for 10 minutes in order to get a snack.
Determine the probability that Debbie received at least one call while away from her desk.
(A) 57% (B) 59% (C) 61% (D) 63% (E) 65%

4.27 *(2 points) One has a counting process N(t) with stationary and independent increments.*
$N(0) = 0$. $Prob[N(x) = 1] = 8x + o(x)$. $Prob[N(x) \geq 2] = o(x)$.
What is the variance of the time between the fourth and fifth events?
A. 1/64 B. 1/8 C. 1/4 D. 1/2 E. 1

Use the following information for the next five questions:
Bonnie and Clyde arrive at the bus terminal and each wait for their bus to leave.
Bonnie is going to Brighton and Clyde is going to Chelsea.
Brighton buses leave the terminal via a Poisson Process at a rate of 5 per hour.
Chelsea buses leave the terminal via a Poisson Process at a rate of 15 per hour.
Brighton buses and Chelsea buses are independent.

4.28 (2 points) On average how many Chelsea buses leave while Bonnie is at the terminal?
A. 1 B. 2 C. 3 D. 4 E. 5

4.29 (3 points) Determine the average time Bonnie waits if Clyde's bus leaves first.
A. 12 minutes B. 13 minutes C. 14 minutes D. 15 minutes E. 16 minutes

4.30 (2 points) What is the average time until the first one of them catches his or her bus?
A. 3 minutes B. 4 minutes C. 5 minutes D. 6 minutes E. 7 minutes

4.31 (2 points) What is the average time until the last one of them catches his or her bus?
A. 12 minutes B. 13 minutes C. 14 minutes D. 15 minutes E. 16 minutes

4.32 (2 points) What is the probability that exactly five Chelsea busses leave the terminal while Bonnie waits?
A. 5% B. 6% C. 7% D. 8% E. 9%

4.33 (2 points) Lucky Tom finds coins on his way to work at a Poisson rate of 0.5 coins/minute. What is the probability that the first two coins Lucky Tom finds during his one-hour walk today are found within one minute of each other?
A. 30% B. 35% C. 40% D. 45% E. 50%

4.34 (2 points) Trains arrive via a Poisson Process. The average time between trains is 10. Conditional on the first train arriving before time 15, what is the expected waiting time until the first train?
A. 5.7 B. 5.9 C. 6.1 D. 6.3 E. 6.5

4.35 *(3 points) A bus leaves at 10 minute intervals.*
Passengers arrive via a Poisson Process with $\lambda = 3$ per minute.
Let X be the total time in minutes spent waiting by all of the passengers who board a single bus.
Determine the variance of X.
A. 250 B. 500 C. 750 D. 1000 E. 1250

Use the following information for the next two questions:
- Cautious Clarence wants to cross a road.
- Vehicles pass the spot where Clarence is waiting via a Poisson Process at a rate of one every 4 seconds.
- Clarence will wait until he can see that no vehicle will come by in the next 10 seconds.

4.36 (1 point) What is the probability that Clarence does not have to wait?
A. 6% B. 8% C. 10% D. 12% E. 14%

4.37 *(3 points) Calculate Clarence's average waiting time.*
A. 20 B. 25 C. 30 D. 35 E. 40

Use the following information for the next two questions:
- Mistakes in cell division occur via a Poisson Process with $\lambda = 2$ per year.
- An individual dies when 150 such mistakes have occurred.

4.38 (2 points) What is the variance of the lifetime of an individual?
A. Less than 30
B. At least 30, but less than 35
C. At least 35, but less than 40
D. At least 40, but less than 45
E. At least 45

4.39 (2 points) Using the Normal Approximation, estimate the probability that an individual survives to age 85.
A. 3% B. 5% C. 7% D. 9% E. 11%

4.40 *(3 points) Flip Wilson flips a penny once per minute.*
The chance of a head is p for each flip, independent of the other flips.
What is the distribution of the interevent times between heads?

Use the following information for the next three questions:
People visit an ATM via a Poisson process at a rate of 20 per hour.
There have been 10 people between 6:00 PM and 7:00 PM.
It is now 7:00 PM.

4.41 (1 point) At what time do you expect the 13^{th} person?
A. Before 7:05 PM
B. On or after 7:05 PM, but before 7:10 PM
C. On or after 7:10 PM, but before 7:15 PM
D. On or after 7:15 PM, but before 7:20 PM
E. On or after 7:20 PM

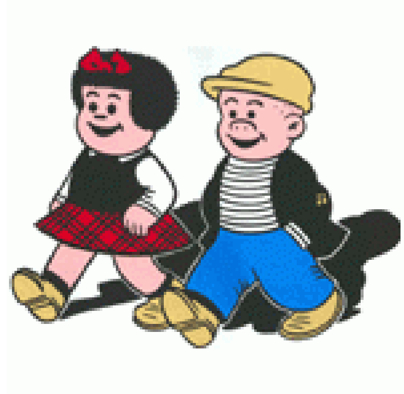
4.42 (1 point) What is the probability that the 11^{th} person arrives by 7:05 PM?
A. Less than 70%
B. At least 70%, but less than 75%
C. At least 75%, but less than 80%
D. At least 80%, but less than 85%
E. At least 85%

4.43 (2 points) What is the probability that the 13^{th} person arrives by 7:12 PM?
A. Less than 70%
B. At least 70%, but less than 75%
C. At least 75%, but less than 80%
D. At least 80%, but less than 85%
E. At least 85%

4.44 (2 points) Customers arrive at Madame Zelda the fortune teller one at a time.
The number of customers during non-overlapping time intervals are independent of one another.
Customers arrive at the rate of 3 per hour.
Determine the probability that her second customer arrives after at least one hour has passed.
A. 14% B. 16% C. 18% D. 20% E. 22%

<u>**4.45**</u> (2 points) The Seville Barber Shop has 5 barbers working.
Customers wait to be served by the next available barber on a first come first served basis.
All barbers are currently cutting the hair of customers.
Each barber services customers at a Poisson rate of 3 customers per hour.
When Bartolo arrives, there are 4 customers waiting ahead of him.
Calculate the probability that Bartolo must wait more than 15 minutes for the next available barber.
A. Less than 30%
B. At least 30%, but less than 40%
C. At least 40%, but less than 50%
D. At least 50%, but less than 60%
E. At least 60%

4.46 (3 points) Emails arrive at a Poisson rate of one per 2 minutes.



Nancy determines the exact probability that 4 or more emails will arrive in the next 6 minutes. Sluggo estimates the same probability by applying the Normal Approximation to the distribution of the fourth event time. What is the absolute difference in their estimates?
A. Less than 2%
B. At least 2%, but less than 3%
C. At least 3%, but less than 4%
D. At least 4%, but less than 5%
E. At least 5%

4.47 (1 point) For a homogeneous Poisson Process, the probability that more than 5 days elapses between events is 0.30.
Calculate the expected number of events in the next 100 days.
A. Less than 10
B. At least 10, but less than 15
C. At least 15, but less than 20
D. At least 20, but less than 25
E. At least 25

4.48 (2 points) Very large asteroids hit the moon at a Poisson rate of one every 10 million years. What is the probability that more than 110 very large asteroids will hit the moon in the next 1000 million years? Use the Normal Approximation.
A. 15% B. 17% C. 19% D. 21% E. 23%

4.49 (2 points) Major earthquakes (of magnitude 8 and higher) occur according to a homogeneous Poisson process, with a rate of 1.36 per year.
Calculate the variance of the waiting time until the tenth major earthquake occurs.
A. Less than 4.8
B. At least 4.8, but less than 5.0
C. At least 5.0, but less than 5.2
D. At least 5.2, but less than 5.4
E. At least 5.4

4.50 (3 points) You are given the following information about the interarrival times between goals in a particular soccer match.
(Soccer is what is called football in most of the world other than the U.S.)
- The waiting times in minutes between goals follows independent, identically distributed exponential distributions.
- The waiting time for the first goal follows this same exponential distribution, and is independent of the other exponential distributions.
- The probability that more than 10 minutes elapses between goals is 0.30.

Calculate the standard deviation (in minutes) of the waiting time until the third goal.
A. Less than 14
B. At least 14, but less than 15
C. At least 15, but less than 16
D. At least 16, but less than 17
E. At least 17

4.51 (2 points) You are given the following information:
- Klem Handler processes insurance claims.
- Klem works on one claim until he is done processing it, and then starts on another.
- The time it takes Klem to process different claims are independent and identically distributed.
- The time it takes Klem to process a group of claims follows the Gamma distribution.
- The expected time for Klem to process 10 claims received at the same time is 80 minutes.

Calculate the probability that it will take Klem more than 20 minutes to process a claim from the time he starts working on it.
A. Less than 8%
B. At least 8%, but less than 10%
C. At least 10%, but less than 12%
D. At least 12%, but less than 14%
E. At least 14%

4.52 (4 points) Fifty-eight independent industrial fans are put in service.
Each has a lifetime that comes from an exponential distribution with a mean of 28,700 hours.
Predict the number of such fans that will fail in the next 2000 hours of service on each fan.
(a) Assume that failed fans are replaced by a fan with a new design that does not fail.
(b) For part (a), calculate a value that is above the observed number of failures with 90% probability.
(c) Assume instead that each failed fan is immediately replaced by a fan of the old design.
(d) For part (c), calculate a value that is above the observed number of failures with 90% probability.

Use the following information for the next two questions:
- The gamma rays detected from a small amount of Cesium 137 follow a Poisson Process with a rate of 0.01 per second.

4.53 (2 points) What is the probability that the next three gamma rays to be detected arrive within a minute of each other?
A. 10% B. 12% C. 14% D. 16% E. 18%

4.54 (2 points) A gamma ray from the sample has just been detected.
Determine the probability that the next gamma ray from the sample is detected between 30 and 60 seconds from now, and the one after that is detected between 100 and 160 seconds from now.
A. 4% B. 4% C. 5% D. 6% E. 7%

4.55 (2 points) You are given the following information about the waiting time until a certain number of events occur:
- The underlying events follow a homogenous Poisson process
- T_n is the time until the n^{th} event occurs
- $E[T_3] = 5$

Calculate the variance of T_7.
A. Less than 4
B. At least 4, but less than 8
C. At least 8, but less than 12
D. At least 12, but less than 16
E. At least 16

4.56 (2, 5/90, Q.43) (1.7 points) Customers arrive randomly and independently at a service window, and the time between arrivals has an exponential distribution with a mean of 12 minutes. Let X equal the number of arrivals per hour. What is P[X = 10]?
A. $10e^{-12}/10!$ B. $10^{12}e^{-10}/10!$ C. $12^{10}e^{-10}/10!$ D. $12^{10}e^{-12}/10!$ E. $5^{10}e^{-5}/10!$

4.57 (Course 151 Sample Exam #2, Q.7) (0.8 points)
For a claim number process $\{N(t), t \geq 0\}$ you are given that the elapsed times between successive claims are mutually independent and identically distributed with distribution function

$F(t) = 1 - e^{-3t}, t \geq 0.$

Determine the probability of exactly 4 claims in an interval of length 2.
(A) 0.11 (B) 0.13 (C) 0.15 (D) 0.17 (E) 0.19

4.58 (1, 11/00, Q.34) (1.9 points) The number of days that elapse between the beginning of a calendar year and the moment a high-risk driver is involved in an accident is exponentially distributed. An insurance company expects that 30% of high-risk drivers will be involved in an accident during the first 50 days of a calendar year.
What portion of high-risk drivers are expected to be involved in an accident during the first 80 days of a calendar year?
(A) 0.15 (B) 0.34 (C) 0.43 (D) 0.57 (E) 0.66

4.59 (IOA 101, 4/01, Q.3) (2.25 points) Suppose that the occurrence of events which give rise to claims in a portfolio of automobile insurance policies can be modeled as follows: the events occur through time at random, at rate μ per hour. Then the number of events which occur in a given period of time has a Poisson distribution (you are given this).
Show that the time between two consecutive events occurring has an exponential distribution with mean $1/\mu$ hours.

4.60 (1, 5/01, Q.22) (1.9 points) The waiting time for the first claim from a good driver and the waiting time for the first claim from a bad driver are independent and follow exponential distributions with means 6 years and 3 years, respectively.
What is the probability that the first claim from a good driver will be filed within 3 years and the first claim from a bad driver will be filed within 2 years?
(A) $(1 - e^{-2/3} - e^{-1/2} + e^{-7/6})/18$
(B) $e^{-7/6}/18$
(C) $1 - e^{-2/3} - e^{-1/2} + e^{-7/6}$
(D) $1 - e^{-2/3} - e^{-1/2} + e^{-1/3}$
(E) $1 - e^{-2/3}/3 - e^{-1/2}/6 + e^{-7/6}/18$

4.61 (IOA 101, 9/01, Q.6) (3.75 points)
(i) (1.5 points) The occurrence of claims in a group of 200 policies is modeled such that the probability of a claim arising in the next year is 0.015 independently for each policy.
Each policy can give rise to a maximum of one claim.
Calculate an approximate value for the probability that more than 10 claims arise from this group of policies in the next year by approximating via a Poisson.
Leave your answer in terms of an Incomplete Gamma Function.
(ii) (2.25 points) The occurrence of claims in a group of 2000 policies is modeled such that the probability of a claim arising in the next year is 0.015 independently for each policy.
Each policy can give rise to a maximum of one claim.
Using the Normal Approximation, calculate an approximate value for the probability that more than 40 claims arise from this group of policies in the next year.

4.62 (CAS3, 11/05, Q.28) (2.5 points) Big National Bank has 3 teller windows open for customer service. Each teller services customers at a Poisson rate of 6 customers per hour.
There is a single line to wait for the next available teller and all tellers are currently serving customers.
If there are 2 people in line when the next customer arrives, calculate the probability that he must wait more than 10 minutes for the next available teller.
A. Less than 30%
B. At least 30%, but less than 40%
C. At least 40%, but less than 50%
D. At least 50%, but less than 60%
E. At least 60%

4.63 (CAS3, 11/06, Q.27) (2.5 points)
A customer service operator accepts calls continuously throughout the work day.
The length of each call is exponentially distributed with an average of 3 minutes.
Calculate the probability that at least one call will be completed in the next 2 minutes.
A. Less than 0.50
B. At least 0.50, but less than 0.55
C. At least 0.55, but less than 0.60
D. At least 0.60, but less than 0.65
E. At least 0.65

4.64 (SOA M, 11/06, Q.8) (2.5 points)
The time elapsed between claims processed is modeled such that V_k represents the time elapsed between processing the k-1th and kth claim.
(V_1 = time until the first claim is processed).
You are given:
(i) V_1, V_2,... are mutually independent.
(ii) The pdf of each V_k is $f(t) = 0.2\ e^{-0.2t}$, $t > 0$, where t is measured in minutes.
Calculate the probability of at least two claims being processed in a ten minute period.
(A) 0.2 (B) 0.3 (C) 0.4 (D) 0.5 (E) 0.6

4.65 (CAS3, 5/07, Q.1) (2.5 points) You are given the following information:
- The number of wild fires per day in a state follows a Poisson distribution.
- The expected number of wild fires in a thirty-day time period is 15.

Calculate the probability that the time between the eighth and ninth fire will be greater than three days.
A. Less than 20%
B. At least 20%, but less than 25%
C. At least 25%, but less than 30%
D. At least 30%, but less than 35%
E. At least 35%

4.66 (SOA MLC, 5/07, Q.5) (2.5 points)
Heart/Lung transplant claims in 2007 have interevent times that are independent with a common distribution which is exponential with mean one month.
As of the end of January, 2007 no transplant claims have arrived.
Calculate the probability that at least three Heart/Lung transplant claims will have arrived by the end of March, 2007.
(A) 0.18 (B) 0.25 (C) 0.32 (D) 0.39 (E) 0.45

4.67 (SOA MLC, 5/07, Q.26) (2.5 points)
A certain scientific theory supposes that mistakes in cell division occur according to a Poisson process with rate 4 per day, and that a specimen fails at the time of the 289^{th} such mistake.
This theory explains the only cause of failure.
T is the time-of-failure random variable in days for a newborn specimen.
Using the normal approximation, calculate the probability that $T > 68$.
(A) 0.84 (B) 0.86 (C) 0.88 (D) 0.90 (E) 0.92

4.68 (CAS3, 11/07, Q.1) (2.5 points)
You are given the following information about the interarrival times for tornadoes in county XYZ.

- The waiting time in days between tornadoes follows an exponential distribution and remains constant throughout the year.
- The probability that more than 30 days elapses between tornadoes is 0.60.

Calculate the expected number of tornadoes in the next 90 days.
A. Less than 1.0
B. At least 1.0, but less than 1.5
C. At least 1.5, but less than 2.0
D. At least 2.0, but less than 2.5
E. At least 2.5

4.69 (CAS3, 11/07, Q.2) (2.5 points)
Car crashes occur according to a Poisson process at a rate of 2 per hour.
There have been 12 crashes between 9:00 AM and 10:00 AM.
Given that it is now 10:00 AM, at what time do you expect the 13th crash?
A. Before 10:05 AM
B. On or after 10:05 AM, but before 10:15 AM
C. On or after 10:15 AM, but before 10:25 AM
D. On or after 10:25 AM, but before 10:35 AM
E. On or after 10:35 AM

4.70 (CAS3L, 11/10, Q.10) (2.5 points) Hurricanes make landfall according to a homogeneous Poisson process, with a rate of 1.5 per month.
Calculate the variance (in months squared) of the waiting time until the third hurricane makes landfall.
A. Less than 1.35
B. At least 1.35, but less than 1.45
C. At least 1.45, but less than 1.55
D. At least 1.55, but less than 1.65
E. At least 1.65

4.71 (CAS3L, 5/11, Q.9) (2.5 points) You are given the following information:
- Buses depart from the bus stop at a Poisson rate of six per hour.
- Today, Jim arrives at the bus stop just as a bus is departing,
  and will have to wait for the next bus to depart.
- Yesterday, Jim arrived at the bus stop six minutes after the prior bus departed
  and had to wait for the subsequent bus to depart.

Calculate how much longer Jim's expected wait time is today compared to yesterday.
A. Less than 1 minute
B. At least 1 minute, but less than 2 minutes
C. At least 2 minutes, but less than 3 minutes
D. At least 3 minutes, but less than 4 minutes
E. At least 4 minutes

4.72 (CAS3L, 11/11, Q.11) (2.5 points) You are given the following information:
Claims are reported according to a homogeneous Poisson process.
Starting from time zero, the expected waiting time until the second claim is three hours.
Calculate the standard deviation of the waiting time until the second claim.
A. Less than 1.75
B. At least 1.75, but less than 2.00
C. At least 2.00, but less than 2.25
D. At least 2.25, but less than 2.50
E. At least 2.50

4.73 (CAS3L, 11/12, Q.10) (2.5 points)
You are given the following information about processing time for wildfire claims:
- The time to process a group of claims follows the Gamma distribution.
- The expected time to process 100 claims received at the same time is 50 days.

Calculate the probability that an individual claim will not be processed on the first day it is received.
A. Less than 8%
B. At least 8%, but less than 10%
C. At least 10%, but less than 12%
D. At least 12%, but less than 14%
E. At least 14%

4.74 (CAS3L, 5/13, Q.10) (2.5 points) You are given the following information:
- The number of gold nuggets found per day on a property follows a Poisson process.
- The expected number of nuggets found in a 30 day time period is 15.

Calculate the probability that the time between finding the 8^{th} and 9^{th} nuggets will be greater than 3 days.
A. Less than 20%
B. At least 20%, but less than 25%
C. At least 25%, but less than 30%
D. At least 30%, but less than 35%
E. At least 35%.

4.75 (CAS3L, 11/13, Q.23) (2.5 points) You are given the following information:
- Accidents follow a Poisson Process with a rate of 0.05 accidents per day.
- Two people have accidents independently of one another.

Calculate the expected number of days until both people have had at least one accident.
A. Less than 29.5
B. At least 29.5, but less than 30.5
C. At least 30.5, but less than 31.5
D. At least 31.5, but less than 32.5
E. At least 32.5

4.76 (CAS ST, 11/15, Q.2) (2.5 points) For a health insurance policy, the annual number of claims follows the Poisson process with the mean, λ, equal to 5.
Calculate the probability that the third claim occurs after one year, that is $\Pr(T_3 > 1)$.
A. Less than 0.10
B. At least 0.10, but less than 0.11
C. At least 0.11, but less than 0.12
D. At least 0.12, but less than 0.13
E. At least 0.13

4.77 (CAS S, 11/15, Q.1) (2.2 points) You are given the following information:
- Buses depart from the bus stop at a Poisson rate of 5 per hour.
- Today, Joe arrives at the bus stop just as a bus is leaving and will have to wait for the next bus to depart.
- Yesterday, Joe arrived at the bus stop 5 minutes after the prior bus departed and had to wait for the subsequent bus to depart.

Calculate how much longer Joe's expected wait time is today compared to yesterday.
A. Less than 1 minute
B. At least 1 minute, but less than 2 minutes
C. At least 2 minutes, but less than 3 minutes
D. At least 3 minutes, but less than 4 minutes
E. At least 4 minutes

4.78 (CAS ST, 5/16, Q.2) (2.5 points) For a health insurance policy the annual number of claims follows the Poisson process with mean $\lambda = 5$.
Calculate the probability that the time interval between the second and third claims exceeds 0.6 years.
A. Less than 0.01
B. At least 0.01, but less than 0.04
C. At least 0.04, but less than 0.07
D. At least 0.07, but less than 0.10
E. At least 0.10

4.79 (MAS-1, 5/19, Q.2) (2.2 points) You are given the following information about the waiting time until a certain number of events occur:
- The underlying events follow a homogenous Poisson process
- T_n is the time until the n^{th} event occurs
- $E[T_2] = 2$

Calculate the variance of T_{10}.
A. Less than 4
B. At least 4, but less than 8
C. At least 8, but less than 12
D. At least 12, but less than 16
E. At least 16

Solutions to Problems:

4.1. D. Mean time until the first birth is $1/\lambda = 1/0.05 =$ **20**.

4.2. C. Mean time until the tenth birth is $10/\lambda = 10/0.05 =$ **200**.

4.3. E. Time until the first birth is Exponentially Distributed with mean $1/\lambda = 1/0.05 = 20$.
$S(35) = e^{-35/20} = e^{-1.75} =$ **0.174**.
Alternately, the number of births by time 35 is Poisson with mean (35)(0.05) = 1.75.
Prob[0 births by time 35] = $e^{-1.75} =$ **0.174**.

4.4. C. Time between any two consecutive births is Exponentially Distributed with mean:
$1/\lambda = 1/0.05 = 20$. $F(10) = 1 - e^{-10/20} = 1 - e^{-0.5} =$ **0.393**.

4.5. B. Prob[time from the eighth birth to the tenth birth is > 50] =
Prob[0 or 1 birth in time period of length 50] =
Prob[0 births from Poisson with mean (0.05)(50) = 2.5] +
Prob[1 birth from Poisson with mean (0.05)(50) = 2.5] = $e^{-2.5} + 2.5e^{-2.5} =$ **0.287**.
Alternately, the time between any two consecutive births is Exponentially Distributed with mean $\theta = 1/\lambda = 1/0.05 = 20$. Interevent times are independent.
Therefore, the time between the 8th and 10th births is Gamma with $\alpha = 2$ and $\theta = 20$.
$F(t) = \Gamma[\alpha; t/\theta] = \Gamma[2; t/20]$. Prob[time from the eighth birth to the tenth birth is > 50] =
$1 - \Gamma[2; 50/20] = 1 - \Gamma[2; 2.5] = e^{-2.5} + 2.5e^{-2.5} =$ **0.287**.
Alternately, the density of this Gamma is: $t\, e^{-t/20}/400$.
The desired probability is the integral of this density from 50 to infinity:

$$\int_{50}^{\infty} t\, e^{-t/20}/400 \; dt = (1/400)(-20\, t\, e^{-t/20} - 400\, e^{-t/20}) \Big]_{t=50}^{t=\infty} = 1400e^{-2.5}/400 = \mathbf{0.287}.$$

Comment: For example, assume the 8th occurs at time 200.

Number of births between times 200 and 250	Does the 10th birth occur by time 250?
0	No
1	No
2 or more	Yes

Due to the memoryless property, one can just start a new Poisson Process whenever the 8^{th} birth shows up. Then the given question is equivalent to:
A Poisson Process has a births intensity of 0.05.
What is the probability that the time until the second birth is greater than 50?

4.6. B. The time until the first event is exponential with mean: 1/0.05 = 20.
The median is where the distribution function is 50%.
$0.5 = 1 - e^{-t/20}$. $\Rightarrow t = 20 \ln(2) =$ **13.86**.

4.7. A. The Interevent times are independent, identically distributed exponentials with mean 20.
Each one has variance: $20^2 = 400$.
The sum of 5 of independent variables has variance: (5)(400) = **2000**.
Alternately, the time from the fourth to the ninth birth is Gamma with $\alpha = 5$ and $\theta = 20$,
and variance: $\alpha\theta^2 = (5)(20^2) =$ **2000**.

4.8. A. A claims process with interevent times independent and identically distributed with an exponential distribution is a Poisson Process. If the intensity is λ, then the interevent times have distribution $F(t) = 1 - e^{-\lambda t}$, so in this case $\lambda = 0.5$.
Thus the mean number of claims over a period of length 7 is: $7\lambda = 3.5$.
The density at 3 of a Poisson distribution with mean 3.5 is: $e^{-3.5}\ 3.5^3 / 3! =$ **0.216**.
<u>Comment</u>: Similar to Course 151 Sample Exam #2, Q.7.

4.9. B. $\lambda = 1/500$. The number of claims by time 900 is Poisson Distributed with mean 9/5 = 1.8.
Therefore, the chance of zero claims in this interval is: $e^{-1.8}$.
The chance of one claim in this time interval is: $1.8e^{-1.8}$.
Thus the chance of at least 2 claims is: $1 - \{e^{-1.8} + 1.8e^{-1.8}\} =$ **53.7%**.

4.10. C. The number of claims by time 900 is Poisson Distributed with mean 9/5 = 1.8.
$f(2) = 1.8^2 e^{-1.8}/2 =$ **26.8%**.

4.11. E. Service time is Exponential with mean 1/6 minute; the probability that Joe finishes his service exactly at time t in minutes is: $6e^{-6t}$, the density of this Exponential Distribution.
The time to the next customer is exponential with mean 1/4 minute; the probability that no customer arrives by time t is: e^{-4t}.
Prob[Joe done being served before another customer arrives] =

$$\int \text{Prob[Joe done being served at time t] Prob[no customer arrives by time t]}\ dt =$$

$$\int_0^{\infty} 6e^{-6t}\ e^{-4t}\ dt = -6e^{-10t}/10 \Big]_{t=0}^{t=\infty} = 6/10 = \textbf{60\%}.$$

4.12. B. Prob[3rd coin by time 5] = Prob[At least 3 coins by time 5]
$= 1 - e^{-2.5}(1 + 2.5 + 2.5^2/2) = 45.62\%$.
Prob[3rd coin by time 10] = Prob[At least 3 coins by time 10]
$= 1 - e^{-5}(1 + 5 + 5^2/2) = 87.53\%$.
Prob[3rd coin found between time 5 and 10] = 87.53% - 45.62% = **41.91%**.

4.13. B. Summing the two Poisson processes, we get a Poisson process with rate:
5 + 5 = 10 per hour. The average wait for a train (of either type) is 1/10 hour = **6 minutes**.
Alternately, Prob[neither train arrived by time t] =
Prob[uptown has not arrived by time t] Prob[downtown has not arrived by time t]
= $e^{-5t}e^{-5t} = e^{-10t}$. Mean time until the first train is the integral of its survival function:
Integral from 0 to ∞ of e^{-10t} is: 1/10 hour = **6 minutes**.

4.14. A. Summing the two Poisson processes, we get a Poisson process with rate:
5 + 5 = 10 per hour.
Thus the wait until the first train arrives is Exponential with mean: 60/10 = 6 minutes.
The median is where the distribution function is 50%:
$1 - e^{-t/6} = 0.5. \Rightarrow t =$ **4.15** minutes.

4.15. B. Prob[both trains arrived by time t] =
Prob[uptown has arrived by time t] Prob[downtown has arrived by time t]
= $(1 - e^{-5t})(1 - e^{-5t}) = 1 - 2e^{-5t} + e^{-10t}$.
Mean time until the both of your trains arrive the integral of its survival function:
Integral from 0 to ∞ of $2e^{-5t} - e^{-10t}$ is: 2/5 - 1/10 = 3/10 hour = **18 minutes**.

4.16. B. Prob[both trains arrived by time t] =
Prob[uptown has arrived by time t] Prob[downtown has arrived by time t]
= $(1 - e^{-5t})(1 - e^{-5t})$.
The 60^{th} percentile is where the distribution function is 60%:
$(1 - e^{-5t})^2 = 0.6. \Rightarrow 1 - e^{-5t} = 0.7746. \Rightarrow t = 0.2980$ hours = **17.88** minutes.

4.17. A. By symmetry, there is a 50% chance the uptown train arrives first.
Alternately, let s be the time of arrival of the first uptown train. $f(s) = 5e^{-5s}$.
Prob[downtown train has not arrived by time s] = e^{-5s}.
⇒ Density that the uptown train arrives first at time s is: $5e^{-5s}e^{-5s} = 5e^{-10s}$.
Probability that the uptown train arrives first is:

$$\int_0^\infty 5e^{-10s}\, ds = 5/10 = 0.5.$$

When an uptown train arrives, the average wait until the next downtown train is 1/5 hours
= 12 minutes. (In fact, this is true whenever we start observing.)
⇒ Average time you wait without George = (0.5)(12 minutes) = **6 minutes**.
<u>Comment</u>: By symmetry, the average time George waits without you is also 6 minutes.
E[arrival of first train] + E[you wait alone] + E[George waits alone] = 6 + 6 + 6 = 18 =
E[arrival of both trains].
We have two possibilities, George's train shows up first or your train shows up first.
The answer is:
Prob(George's train first) (time from George's train to your train) + Prob(your train first) (0).

4.18. C. By symmetry, you expect to see on average an equal number of uptown and downtown trains while you wait on the platform. Since you will take the first downtown train that comes, you will see exactly one downtown train. Therefore, on average you see **one** uptown train.
Alternately, since there are on average 5 downtown trains per hour, your average wait for your downtown train is: (60 minutes) / 5 = 12 minutes.
The rate at which uptown trains arrive is: 5 per hour. ⇔ 1 per 12 minutes.
⇒ λ = 1/ (12 minutes).
During those 12 minutes, the average number of uptown trains to pass is: (12)(1/12) = **1**.
Comment: By symmetry, the average number of downtown trains that pass while George waits on the platform is also 1.
When solving a question it is always important to pick unit of time and use it consistently.
In this case, I have chosen to work in minutes, but one could have worked in hours instead.

4.19. E. By symmetry, there is an equal 1/2 chance that an uptown or downtown train arrives first. Since the Poisson processes have no memory, this is also true of the second train, and each subsequent train. Prob[3 downtown and then a uptown] = $(1/2)^3$ (1/2) = **1/16**.
Comment: The average number of downtown trains that pass while George waits is:
$(1/2)(0) + (1/2)^2(1) + (1/2)^3(2) + (1/2)^4(3) + (1/2)^5(4) + \ldots = 1/4 + 2/8 + 3/16 + 4/32 + \ldots$,
which does in fact equal one, although it is not obvious.
The number of downtown trains that pass while George waits on the platform follows a Geometric Distribution with $\beta/(1+\beta) = 1/2$ or $\beta = 1$.
By symmetry, 1/16 is also the probability that exactly three uptown trains pass while you wait on the platform.

4.20. D. Prob[4th event has occurred by time 2] = Prob[At least 4 claims by time 2]
$= 1 - e^{-1.8}(1 + 1.8 + 1.8^2/2 + 1.8^3/6) = 10.87\%$.
Prob[4th event has occurred by time 5] = Prob[At least 4 claims by time 5]
$= 1 - e^{-4.5}(1 + 4.5 + 4.5^2/2 + 4.5^3/6) = 65.77\%$.
Prob[fourth event occurs between time 2 and 5] = 65.77% - 10.87% = **54.90%**.
Comment: One could instead add up the probabilities of:
(three events between 0 and 2) and (at least one event between 2 and 5)
(two events between 0 and 2) and (at least two events between 2 and 5)
(one event between 0 and 2) and (at least three events between 2 and 5)
(zero events between 0 and 2) and (at least four events between 2 and 5).

4.21. A. When the claims intensity is multiplied by 5, the expected waiting time until the first claim is divided by 5: 770/5 = **154**.
Comment: For a (homogeneous) Poisson Process, the waiting time until the first claim is the inverse of the claims intensity.

4.22. E. The time until the next accident is exponentially distributed with the mean equal to $1/\lambda$:
$F(t) = 1 - e^{-0.08t}$. $S(25) = 1 - F(25) = e^{-2}$ = **13.5%**.
Alternately, the number of accidents over the next 25 years is given by a Poisson with mean (25)(8%) = 2. Therefore, the chance of zero accidents over the next 25 years is e^{-2}. This e^{-2} is just the desired chance that it will be more than 25 years until this driver's next accident.

4.23. B. $\lambda = 1/0.2 = 5$.
The number of claims in the time period from 0 to 1.2 is Poisson Distributed with mean (5)(1.2) = 6. Therefore, the chance of zero claims in this interval is e^{-6}.
The chance of one claim in this time interval is $6e^{-6}$.
The chance of two claims in this time interval is $6^2e^{-6}/2$.
Thus the chance of at least 3 claims in the time interval is :
$1 - \{e^{-6} + 6e^{-6} + 6^2e^{-6}/2\} = 1 - 0.062 =$ **93.8%**.

4.24. A. Let X be the time in hours to fill Mary's order and Y be the time in hours for Mary to pay.
Then X is Exponential with mean 1/40 hour and has density $40e^{-40x}$.
Y is Exponential with mean 1/360 hour and has density $360e^{-360y}$.
The time to the next customer is exponential with mean 1/20 hours; the probability that no customer arrives by time x + y is the survival function of this Exponential at x+y: $e^{-20(x+y)}$.
Note that this also the density at zero of a Poisson with mean 20(x+y).
Prob[Mary done before another customer arrives] =

$$\int \text{Prob}[X = x]\ \text{Prob}[Y = y]\ \text{Prob}[\text{no customer arrives by time } x + y]\ dx\ dy =$$

$$\int 40e^{-40x}\, 360e^{-360y}\, e^{-20(x+y)}\, dx\ dy = 14{,}400 \int_0^\infty e^{-60x}\, dx \int_0^\infty e^{-380y}\, dy = \frac{14{,}400}{(60)\ (380)} = \textbf{63.2\%}.$$

Alternately, let Z = X + Y = total time Mary spends. Then
Prob[Mary done before another customer arrives] = $\int \text{Prob}[Z = z]\ \text{Prob}[\text{no customer by } z]\ dz =$

$$\int \text{Prob}[Z = z]\ e^{-20z}\ dz = \text{Moment Generating Function of Z at -20.}$$

Since X and Y are independent, m.g.f. of Z = X + Y is the product of the m.g.f.s of X and Y.
X is Exponential with $\theta = 1/40$ hour and m.g.f. $1/(1 - \theta t) = 40/(40 - t)$.
Y is Exponential with $\theta = 1/360$ hour and m.g.f. $1/(1 - \theta t) = 360/(360 - t)$.
m.g.f. of Z is: $\{40/(40 - t)\}\{360/(360 - t)\} = 14{,}400 / \{(40 - t)(360 - t)\}$.
Moment Generating Function of Z at t = -20 is: (14,400) / {(60)(380)} = **63.2%**.

4.25. B. Start a new Poisson Process at 9:05.
The time until the next bus is Exponential with mean: 60/6 = 10 minutes.
Prob[Waiting time from 9:05 is less than or equal to 5] = $1 - e^{-5/10}$ = **39.3%**.
Comment: The Poisson process is memoryless. The arrival of buses after 9:05 is independent of the arrival of buses from 9:02 to 9:05. However, the fact that no bus arrived between 9:02 to 9:05 is why Sandy is still waiting for a bus at 9:05.

4.26. A. Start a new Poisson Process when Debbie leaves her desk. λ = 1/12 minutes.
Prob[no calls in 10 minutes] = $e^{-10l} = e^{-10/12} = 0.4346$. 1 - 0.4346 = **56.54%**.
Comment: See Example 1.7.2 in Hogg, McKean, and Craig.]

4.27. A. This is a definition of a Poisson Process, with $\lambda = 8$.
Each interevent time is Exponential with mean 1/8, and variance $1/8^2$ = **1/64**.

4.28. C. Bonnie waits on average 1/5 of an hour, during which time we expect:
(15)(1/5) = **3** Chelsea buses to leave.
Alternately, assume Bonnie's bus leaves at time t. $S(t) = e^{-5t}$. $f(t) = 5e^{-5t}$.
Between time 0 and t, on average 15t of Clyde's buses leave.

Average number of Chelsea buses: $\int_0^\infty 15t\ 5e^{-5t}\ dt = 15/5 = \mathbf{3}$.

Comment: The integral is the mean of an Exponential Distribution.

4.29. D. Assume Bonnie's bus leaves at time t. $S(t) = e^{-5t}$. $f(t) = 5e^{-5t}$.
The probability Clyde's bus has left by time t is: $1 - e^{-15t}$.
⇒ Prob[Bonnie's bus leaves at time t and Clyde's bus has already left] = $(1 - e^{-15t})\ 5e^{-5t}$.

⇒ Prob[Clyde's bus left first] = $\int_0^\infty (1 - e^{-15t})\ 5e^{-5t}\ dt = \int_0^\infty (5e^{-5t} - 5e^{-20t})\ dt = 1 - 1/4 = 3/4$.

Now via integration we take a weighted average of the time Bonnie's bus left, t, using as weights: Prob[Bonnie's bus leaves at time t and Clyde's bus has already left] = $(1 - e^{-15t})\ 5e^{-5t}$.
Thus the average total time Bonnie waits conditional on Clyde's bus leaving first is:

$$\frac{\int_0^\infty (1 - e^{-15t})\ 5e^{-5t}\ t\ dt}{\int_0^\infty (1 - e^{-15t})\ 5e^{-5t}\ dt} = \frac{\int_0^\infty (5e^{-5t}\ t - 5e^{-20t}\ t)\ dt}{3/4} = (4/3)(1/5 - 5/20^2) = \mathbf{1/4\ hour}.$$

Comment: The probability that Poisson Process 2 occurs before Poisson Process 1 is:
$\lambda_2/(\lambda_1+\lambda_2) = 15/(5 + 15) = 3/4$.

$$\int_0^\infty t\ e^{-\lambda t}\ dt = \int_0^\infty t\ \lambda e^{-\lambda t}\ dt\ /\ \lambda = (\text{mean of Exponential with hazard rate } \lambda)\ /\ \lambda = 1/\lambda^2.$$

Conceptually, if Clyde's bus takes a short time to arrive it is more likely to be first.
If Bonnie's bus takes a long time to arrive, then Clyde's bus is more likely to be first.
So conditional on Clyde's bus being first, Bonnie's wait is longer than average.

4.30. A. Summing the two Poisson processes, we get a Poisson process with rate:
5 + 15 = 20 per hour. The average wait for a bus (of either type) is 1/20 hour = **3 minutes**.
Alternately, Prob[neither bus arrived by time t] =
Prob[Brighton has not arrived by time t] Prob[Chelsea has not arrived by time t]
$= e^{-5t}\ e^{-15t} = e^{-20t}$. Mean time until the first bus is the integral of its survival function:
Integral from 0 to ∞ of e^{-20t} is: 1/20 hour = **3 minutes**.

4.31. B. Prob[both buses arrived by time t] =
Prob[Brighton has arrived by time t] Prob[Chelsea has arrived by time t]
= $(1 - e^{-5t})(1 - e^{-15t}) = 1 - e^{-5t} - e^{-15t} + e^{-20t}$.
Mean time until the both of their buses have arrived is the integral of its survival function:

$$\int_0^\infty e^{-5t} + e^{-15t} - e^{-20t}\ dt = 1/5 + 1/15 - 1/20 = 13/60 \text{ hour} = \mathbf{13\ minutes}.$$

4.32. B. The probability that the first bus to leave is a Chelsea bus is: 15/(5 + 15) = 3/4.
Since the Poisson processes have no memory, this is also true of the second bus, and each subsequent bus.
Prob[5 Chelsea and then a Brighton] = $(3/4)^5$ (1/4) = **5.93%.**
Comment: The number of Chelsea busses that leave the terminal while Bonnie waits follows a Geometric Distribution with $\beta/(1+\beta) = 3/4$ or $\beta = 3$.
The number of Brighton busses that leave the terminal while Clyde waits follows a Geometric Distribution with $\beta/(1+\beta) = 1/4$ or $\beta = 1/3$.

4.33. C. Start a new Poisson Process, whenever Lucky Tom finds his first coin.
Then the remaining time until the second coin is found is Exponential with mean: 1/0.5 = 2 minutes.
The probability this time is less than or equal to one is: $1 - e^{-1/2}$ = **39.3%**.
Comment: I have ignored the extremely small possibility that Lucky Tom fails to find at least two coins today, since the number of coins he finds is Poisson with mean: (60)(0.5) = 30.

4.34. A. $\lambda = 1/10$. The conditional average time waiting is:

$$\int_0^{15} t\ e^{-t/10}/10\ dt \ \Big/ \int_0^{15} e^{-t/10}/10\ dt = -t\ e^{-t/10} - 10e^{-t/10} \Big]_{t=0}^{t=15} / (1 - e^{-1.5}) =$$

$(10 - 25e^{-1.5}) / (1 - e^{-1.5})$ = **5.69**.
Alternately, if no train has arrived by t = 15, by the memoryless property we expect to wait an additional 10 minutes for a train. E[wait I wait > 15] = 15 + 10 = 25.
E[wait] = E[wait I wait ≤ 15]Prob[wait ≤ 15] + E[wait I wait > 15]Prob[wait > 15] ⇒
10 = E[wait I wait ≤ 15]$(1 - e^{-1.5}) + 25e^{-1.5}$. ⇒
E[wait I wait ≤ 15] = $(10 - 25e^{-1.5})/(1 - e^{-1.5})$ = **5.69**.
Alternately, the time waiting is Exponential with mean 10. We are looking for the average size of loss for losses of size 0 to 15: {E[X∧ 15] - 15S(15)} / F(15) =
*$\{(10)(1 - e^{-15/10}) - (15)(e^{-15/10})\}/(1 - e^{-15/10}) = (10 - 25e^{-1.5}) / (1 - e^{-1.5})$ = **5.69**.*

4.35. D. The number of passengers N, is Poisson with mean: (10)(3) = 30.
Each passenger waits on average: 10/2 = 5 minutes.
E[X | N] = 5N. Var[E[X | N]] = 5^2 Var[N] = (25)(30) = 750.
Due to constant intensity, the wait for each passenger is uniform from 0 to 10, with variance:
$10^2/12$ = 8.333. Var[X | N] = 8.333N. E[Var[X | N]] = 8.333E[N] = (8.333)(30) = 250.
Var[X] = E[Var[X | N]] + Var[E[X | N]] = 250 + 750 = **1000**.
Alternately, one can think of this as a Compound Poisson, with a uniform severity.
Each passenger is like one claim.
The number of passengers is like the number of claims.
The time each passenger waited for the bus is like the size of each claim.
The second moment of the uniform distribution from 0 to 10 is: $10^2/3$ = 100/3.
The number of passengers is Poisson with mean: (10)(3) = 30.
Thus the variance of the compound Poisson is: (30) (100/3) = 1000.
Comment: Difficult. Similar to Q. 5.64 in Introduction to Probability Models by Ross.

4.36. B. Prob[no vehicle in next 10 seconds] = $e^{-10/4}$ = **8.21%**.

4.37. D. Let x be Clarence's average waiting time.
Condition on the time of the first vehicle t.
If t > 10, then Clarence's waiting time is zero.
If t ≤ 10, then Clarence waits time t, plus an additional time after the time of the first vehicle.
We can start a new Poisson Process when the first vehicle arrives, and therefore this additional time has an average of x.
t is Exponential with mean 4. E[t | t ≤10] = {E[t ∧ 10] - 10S(10)} / F(10) =
{4(1 - $e^{-10/4}$) - 10$e^{-10/4}$} / (1 - $e^{-10/4}$) = 3.106.
Therefore, x = (0)Prob[t > 10] + (3.106 + x)Prob[t ≤ 10] = (3.106 + x)(1 - $e^{-10/4}$).
⇒ x = 3.106(1 - $e^{-10/4}$)/$e^{-10/4}$ = **34.7**.
Comment: x = 4$e^{2.5}$ - 14 = 34.7. Difficult.
Similar to Q. 5.44 in Introduction to Probability Models by Ross.

4.38. C. The 150th event time is Gamma with α = 150 and $\theta = 1/\lambda$ = 1/2.
Variance is: $\alpha\theta^2$ = 150/4 = **37.5**.
Comment: Similar to Q. 5.39 in Introduction to Probability Models by Ross.

4.39. B. Lifetime has mean: $\alpha\theta$ = 150/2 = 75, and variance: $\alpha\theta^2$ = 150/4 = 37.5.
Prob[Lifetime ≥ 85] ≅ 1 - $\Phi[(85 - 75)/\sqrt{37.5}]$ = 1 - Φ[1.633] = **5.1%**.
Comment: Using a computer, the exact answer is 0.0556344.

4.40. Assume Flip has just flipped a head.
The probability that the next flip is a head is p.
Therefore, the probability that the interevent time is 1 is p.
The probability that the next flip is a tail, followed by a head is (1-p)p.
Therefore, the probability that the interevent time is 2 is (1-p)p.
The probability that the next flips are tail, tail, head is $(1-p)^2p$.
Therefore, the probability that the interevent time is 3 is $(1-p)^2p$.
The probability that the next flips are tail, tail, tail, head is $(1-p)^3p$.
Therefore, the probability that the interevent time is 4 is $(1-p)^3p$.
The probability that the interevent time is n is: $(1-p)^{n-1}p$, n = 1, 2, 3, ...
Comment: The distribution of the interevent times is a zero-truncated Geometric with $\beta = (1 - p)/p$, not on the syllabus of this exam.
This is a series of Bernoulli trials. The number tails prior to getting the next head is Geometric with $\beta = (1 - p)/p$.

4.41. B. Due to the memoryless property, we can start a new process at 7:00 PM.
The expected wait for three more people is 3/20 hours = 9 minutes.
The expected time of the 13th person is **7:09 PM**.
Comment: Similar to CAS3, 11/07, Q.2.

4.42. D. Since what happened before 7:00 PM is independent of what happens after 7:00 PM, we can start a new process at 7:00 PM.
Therefore, Prob[11th person arrives by 7:05 PM] = Prob[at least one person by 7:05 PM]
= 1 - exp[-(20)(1/12)] = **81.1%**.

4.43. C. Since what happened before 7:00 PM is independent of what happens after 7:00 PM, we can start a new process at 7:00 PM.
The number of people between 7:00 PM and 7:12 PM is Poisson with mean: (20)(12/60) = 4.
Therefore, Prob[13th person arrives by 7:12 PM] = Prob[at least three people by 7:12 PM]
= $1 - e^{-4} - 4 e^{-4} - 4^2 e^{-4} /2$ = **76.2%**.

4.44. D. This is a homogeneous Poisson process with $\lambda = 3$.
Thus the number of customers by time 1 is Poisson with mean 3.
Prob[second customer arrives after one hour has passed]
= Prob[fewer than 2 customers by time 1]
= Prob[0 customer by time 1] + Prob[1 customer by time 1] = $e^{-3} + 3 e^{-3}$ = **19.9%**.

4.45. E. The number of people serviced in 15 minutes by one barber is Poisson with mean: (15)(3/60) = 3/4. Thus the sum of the number of people serviced in 15 minutes by 5 barbers is Poisson with mean: (5)(3/4) = 3.75.
If 5 or more customers are served in the next 15 minutes, then a barber is ready to start cutting Bartolo's hair within 15 minutes.
Prob[must wait more than 15 minutes] = Prob[fewer than 5 people serviced in 15 minutes] =
$f(0) + f(1) + f(2) + f(3) + f(4) = e^{-3.75}(1 + 3.75 + 3.75^2/2 + 3.75^3/6 + 3.75^4/24)$ = **67.8%**.
Comment: Similar to CAS3, 11/05, Q.28.

4.46. D. The number of emails over 6 minutes is Poisson with mean 3.
Prob[4 or more] = $1 - e^{-3}(1 + 3 + 3^2/2 + 3^3/6) = 0.3528$.
The waiting time until the first email is Exponential with mean 2, and variance: $2^2 = 4$.
Therefore, the fourth event time has a mean of: (4)(2) = 8, and variance of: (4)(4) = 16.
Therefore, using the Normal Approximation:
Prob[4 or more emails by time 6] = Prob[4th event time ≤ 6] $\cong \Phi[(6 - 8) / \sqrt{16}] = \Phi[-0.50]$ =
0.3085. |0.3528 - 0.3085| = **0.0443**.
Comment: If Sluggo had instead applied the Normal Approximation to the Poisson Distribution:
Prob[4 or more] $\cong 1 - \Phi[(3.5 - 3) / \sqrt{3}] = 1 - \Phi[0.29]$ = 38.6%.

4.47. D. Prob[more than 5 days between events] = $e^{-5\lambda}$.
We are given that $e^{-5\lambda} = 0.3$. $\Rightarrow \lambda = \ln(0.3)/(-5) = 0.2408$. $\Rightarrow 100\lambda$ = **24.08**.
Comment: Similar to CAS3, 11/07, Q.1.

4.48. A. The number of large asteroids in 1000 million years is Poisson with mean 100.
Therefore, using the Normal Approximation with continuity correction:
Prob[more than 110] $\cong 1 - \Phi[(110.5 - 100)/\sqrt{100}] = 1 - \Phi[1.05]$ = **14.7%**.
Alternately, the waiting time until the first large asteroid is exponential with mean 10 (million), and variance 10^2.
Thus the waiting time until the 111th large asteroid has mean: (111)(10) = 1110,
and variance $(111)(10^2)$ = 11,100.
Prob[more than 110 large asteroids by time 1000] =
Prob[waiting time until 111th large asteroid ≤ 1000] $\cong \Phi[(1000 - 1110) / \sqrt{11{,}100}]$
$= \Phi[-1.04]$ = **14.9%**.

4.49. E. The waiting time until the first major earthquake is Exponential with $\theta = 1/1.36$, and variance: $\theta^2 = 1/1.36^2$.
The waiting time until the tenth major earthquake is the sum of 10 independent, identically distributed waiting times, each with variance $1/1.36^2$.
Thus the variance of the waiting time until the tenth major earthquake occurs is: $10/1.36^2$ = **5.41**.
Alternately, the waiting time until the tenth major earthquake is Gamma with $\alpha = 10$ and $\theta = 1/1.36$.
The variance of this Gamma is: $\alpha\theta^2 = (10)(1/1.36^2)$ = **5.41**.
Comment: Similar to CAS3L, 11/10, Q.10.

4.50. B. 0.3 = Exp[-10λ]. $\Rightarrow \lambda$ = -ln[0.3]/10 = 0.1204.
Thus, the waiting time until the first goal is Exponential with mean: 1/0.1204 = 8.306.
Therefore, the time from the first to the second goal as well as the time from the second to third goal are also Exponential with mean 8.306.
These three Exponentials are independent, so their variance add:
$8.306^2 + 8.306^2 + 8.306^2 = 207.0$.
The standard deviation of the waiting time until the third goal is: $\sqrt{207.0}$ = **14.4**.

Alternately, waiting time until the third goal follows a Gamma Distribution, with $\alpha = 3$ and $\theta = 8.306$.
Variance of this Gamma Distribution is: $\alpha\,\theta^2 = (3)\,(8.306)^2 = 207.0$.
The standard deviation of the waiting time until the third goal is: $\sqrt{207.0}$ = **14.4**.
<u>Comment</u>: Similar to CAS3, 11/07, Q.1 and CAS3L, 11/11, Q.11.

4.51. B. Assume that the time to process a single claim is Exponential with mean: 80/10 = 8.
(If 10 claims are processed sequentially, and the time to process each claim follows an independent Exponential with mean 8, then the time to process the group of 10 claims follows a Gamma with $\alpha = 10$, $\theta = 8$, and mean 80.)
Then the probability that it takes more than 20 minutes to process a claim from the time he starts working on it is: exp[-20/8] = **8.2%**.
<u>Comment</u>: Similar to CAS 3L, 11/12, Q.10.

4.52 (a) The probability that each fan fails by time 2000 is: $1 - e^{-2000/28700} = 0.067314$.
Thus, the expected number of fans that fail is: (58)(0.067314) = **3.9042**.
(b) The number of fans that fail is Binomial with m = 58 and q = 0.067314.
The variance is: (58)(0.067314)(1- 0.067314) = 3.6414.
$\Phi[1.282] = 0.90$. Thus the desired value is: $3.9042 + 1.282\sqrt{3.6414}$ = **6.351**.
(c) With each failed fan replaced by a new fan of the old design, we have a (homogeneous) Poisson Process, with λ = 58/28,700.
Thus the expected number of failures over 2000 hours is: (2000)(58/28,700) = **4.0418**.
(d) The number of fans that fail is Poisson with mean 4.0418. The variance is 4.0418.
$\Phi[1.282] = 0.90$. Thus the desired value is: $4.0418 + 1.282\sqrt{4.0418}$ = **6.619**.

4.53. B. Start a new Poisson Process, whenever the first gamma ray arrives.
Then we want the probability of at least 2 additional gamma rays in the next 60 seconds.
The number of gamma rays within the next 60 seconds is Poisson with mean: (60)(0.01) = 0.6.
Prob[at least 2 gamma rays in the next 60 seconds] = $1 - e^{-0.6} - 0.6\,e^{-0.6}$ = **12.2%**.

4.54. C. We need no events from 0 to 30, one event from 30 to 60,
no events from 60 to 100, and at least one event from 100 to 160.
The probability is: $e^{-(30)(0.01)}\,\{(30)(0.01)\,e^{-(30)(0.01)}\}\,e^{-(40)(0.01)}\,\{1 - e^{-(60)(0.01)}\}$ =
(0.7408) (0.2222) (0.6703) (0.4512) = **4.98%**.

4.55. E. $5 = E[T_3] = 3/\lambda. \Rightarrow \lambda = 3/5.$
T_7 is the sum of 7 independent Exponential Distributions each with mean $1/\lambda = 5/3$.
Thus $Var[T_7] = (7)(5/3)^2 =$ **19.44**.
Alternately, T_3 is Gamma with $\alpha = 3$ and θ. $\Rightarrow \alpha\theta = E[T_3] = 5. \Rightarrow \theta = 5/3$.
Thus T_7 is Gamma with $\alpha = 7$ and $\theta = 5/3$. $\Rightarrow Var[T_7] = \alpha\theta = (7)(5/3)^2 =$ **19.44**.
Comment: Similar to MAS-1, 5/19, Q.2.

4.56. E. A Poisson Process with mean $\lambda = 1/12$ minutes. Number of arrivals over 60 minutes is Poisson with mean 60/12 = 5. $P[X = 10] = \mathbf{5^{10}e^{-5}/10!}$.

4.57. B. A claims process with interevent times independent and identically distributed with an exponential distribution is a Poisson Process. If the intensity is λ, then the interevent times have distribution $F(t) = 1 - e^{-\lambda t}$, so in this case $\lambda = 3$.
Thus the mean number of claims over a period of length 2 is: $2\lambda = 6$.
The density at 4 of a Poisson distribution with mean 6 is: $e^{-6}\ 6^4 / 4! =$ **0.134**.

4.58. C. This is mathematically the same as a Poisson Process with intensity λ.
70% = Prob[0 claims in 50 days] = (Density at 0 of Poisson with mean 50λ) = $e^{-50\lambda}$.
$\Rightarrow \lambda = -\ln(0.7)/50 = 0.00713$.
Prob[0 claims in 80 days] = exp[-(80)(0.00713)] = 0.565.
Prob[1 or more claims in 80 days] = 1 - 0.565 = **0.435**.

4.59. Let the event occur at time s. Mean number of events by time s + t is $t\mu$.
Prob[no event by time s + t] = density at zero of a Poisson with mean $t\mu$: $\exp[-t\mu]$.
This is the Survival Function of an Exponential Distribution with mean $1/\mu$.

4.60. C. This is mathematically equivalent to two independent Poisson Processes with $\lambda = 1/6$ and $1/3$. Prob[waiting time ≤ 3 | good] Prob[waiting time ≤ 2 | bad] = $(1 - e^{-3/6})(1 - e^{-2/3})$
= $\mathbf{1 - e^{-2/3} - e^{-1/2} + e^{-7/6}}$.

4.61. (i) Approximate via a Poisson with the same mean, $\lambda = (200)(0.015) = 3$.
Prob[10 or fewer claim] = Prob[11th event time > 1] = 1 - $\Gamma[11 ; 3]$.
Prob[more than 10 claims] = $\mathbf{\Gamma[11 ; 3]}$.
(ii) $1 - \Phi[(40.5 - 30)/\sqrt{29.55}] = 1 - \Phi[1.93] =$ **2.68%**.
Comment: $\Gamma[11 ; 3] = 0.000292337$. The exact probability in part (i) is 0.000251737.
Using the Normal Approximation would give an estimate of: $1 - \Phi[(10.5 - 3)/\sqrt{2.955}]$
$= 1 - \Phi[4.363] = 6.4 \times 10^{-6}$. The Normal Approximation is poor in this case, since this Binomial Distribution is highly skewed. In part (ii) the exact probability is 0.0312644.

4.62. C. The number of people serviced in 10 minutes by one teller is Poisson with mean: (10)(6/60) = 1. Thus the sum of the number of people serviced in 10 minutes by 3 tellers is Poisson with mean 3.
If 3 or more customers are served in the next 10 minutes, then a teller is ready to start serving this customer within 10 minutes.
Prob[must wait more than 10 minutes] = Prob[fewer than 3 people serviced in 10 minutes] =
$f(0) + f(1) + f(2) = e^{-3}(1 + 3 + 3^2/2)$ = **42.32%**.
Comment: Due to the independent and stationary increments, it does not matter that the customers currently being serviced did not just come up to their tellers.
If you start observing a Poisson Process at any point in time, the time until the next event has the same Exponential Distribution.
We can start a new Poisson Process whenever this customer arrives.

4.63. A. The ends of calls have independent, identically distributed exponential interevent times.
This is mathematically equivalent to a Poisson Process, with $\lambda = 1/3$.
Over 2 minutes, the number of calls completed is Poisson with mean (2)(1/3).
$1 - f(0) = 1 - e^{-2/3}$ = **0.487**.
Alternately, when we first start observing, the remaining length of the current call is Exponential, also with mean 3 minutes, due to the memoryless property of the Exponential Distribution.
Prob[at least one call will be completed in the next 2 minutes] =
Prob[current call is completed in the next 2 minutes] =
Distribution Function at 2 for an Exponential Distribution with mean 3 = $1 - e^{-2/3}$ = **0.487**.
Comment: "continuously throughout the work day" is intended to mean that when one call ends, another begins right away. They did not specify that the lengths of calls were independent; while this assumption is needed to have a Poisson Process, it is not needed to determine the probability of at least one call.

4.64. E. A Poisson Process with $\lambda = 0.2$.
Number of claims processed in 10 minutes is Poisson with mean: (10)(0.2) = 2.
Prob[at least 2] = $1 - f(0) - f(1) = 1 - e^{-2} - 2e^{-2}$ = **59.4%**.

4.65. B. $\lambda = 15/30 = 0.5$ per day.
The interarrival time between any two fires is Exponential, $S(t) = e^{-0.5t}$.
$S(3) = e^{-1.5}$ = **22.3%**.

4.66. C. This is a Poisson process with λ = 1/(mean of exponential interarrival times) = 1 per month.
The number of claims over two months is Poisson with $\lambda = 2$.
Prob[at least 3 claims] = $1 - f(0) - f(1) - f(2) = 1 - e^{-2} - 2e^{-2} - 2^2e^{-2}/2!$ = **0.323**.

4.67. A. Let N(t) be the number of mistakes by time t. Prob[T > 68] = Prob[N(68) < 289].
N(68) is Poisson with mean (4)(68) = 272.
Prob[N(68) < 289] $\cong \Phi[(288.5 - 272)/\sqrt{272}\,] = \Phi[1.00]$ = **84.13%**.
Alternately, the time of the 289th failure is Gamma with α = 289 and θ = 1/4.
This Gamma has mean 289/4 and variance 289/16.
Prob[T > 68] $\cong 1 - \Phi[(68 - 289/4)/\sqrt{289/16}\,] = 1 - \Phi[-1.00]$ = **84.13%**.
Comment: The time of the 289th failure is the sum of 289 independent Exponentials each with mean 1/4 and variance $(1/4)^2$ = 1/16. Therefore, the time of the 289th failure has a mean of 289/4 and a variance of 289/16.

4.68. C. Prob[more than 30 days between events] = $e^{-30\lambda}$.
We are given that $e^{-30\lambda}$ = 0.6. $\Rightarrow \lambda$ = ln(0.6)/(-30) = 0.01703. $\Rightarrow 90\lambda$ = **1.533**.

4.69. D. Due to the memoryless property, we can start a new process at 10:00 A.M.
The time until the next event, the first event after 10:00 A.M., is Exponential with mean 1/2 hour.
The expected time of the next event is: 10 + 1/2 = **10:30 A.M.**
Comment: What happens after time 10 is independent of what happened between time 9 and 10.

4.70. A. The waiting time until the first hurricane is Exponential with θ = 1/1.5 = 2/3, and variance: θ^2 = 4/9.
The waiting time until the third hurricane is the sum of three independent, identically distributed waiting times, each with variance 4/9.
Thus the variance of the waiting time until the third hurricane makes landfall is: (3)(4/9) = **4/3**.
Alternately, the waiting time until the third hurricane is Gamma with α = 3 and θ = 1/1.5 = 2/3.
The variance of this Gamma is: $\alpha\theta^2 = (3)(2/3)^2$ = **4/3**.

4.71. A. Yesterday, due to the memoryless property of the Poisson Process, we can start a new Poisson when Jim arrives; his average wait is the usual 1/6 of an hour.
Today Jim's average wait is also 1/6 of an hour.
His expected wait today is the same as his expected wait yesterday; it is **0 minutes** longer.
Comment: For a (homogeneous) Poisson Process, his average wait is independent of when Jim arrives. Real buses do not follow a Poisson process, so using your intuition may be harmful.

4.72. C. Since the interevent times are independent and identically distributed, the expected waiting time until the first claim is 3/2 hours.
Thus, the waiting time until the first claim is Exponential with mean 1.5.
Therefore, the time from the first to the second claim is also Exponential with mean 1.5.
These two Exponentials are independent, so their variance add: $1.5^2 + 1.5^2 = 4.5$.
The standard deviation of the waiting time until the second claim is: $\sqrt{4.5}$ = **2.12**.

Alternately, waiting time until the second claim follows a Gamma Distribution, with $\alpha = 2$ and $\theta = 1/\lambda$.
$3 = \text{Mean} = \alpha\theta = 2/\lambda. \Rightarrow \lambda = 2/3$.
$\Rightarrow$ Waiting time until the second claim follows a Gamma Distribution, with $\alpha = 2$ and $\theta = 3/2$.
Variance of this Gamma Distribution is: $\alpha\,\theta^2 = (2)\,(3/2)^2 = 9/2$.
The standard deviation of the waiting time until the second claim is: $\sqrt{9/2}$ = **2.12**.
Comment: In general, the waiting time until event n follows a Gamma Distribution, with $\alpha = n$ and $\theta = 1/\lambda$.
This Gamma Distribution has a mean of: $\alpha\theta = n/\lambda$, and a variance of: $\alpha\,\theta^2 = n/(\lambda^2) = (\text{mean})^2 / n$.

4.73. D. Assume that the time to process a single claim is Exponential with mean: 50/100 = 1/2 day.
(Assuming 100 claims are processed sequentially, and the time to process each claim follows an independent Exponential with mean 1/2, then the time to process the group of 100 claims follows a Gamma with $\alpha = 100$, $\theta = 1/2$, and mean 50 days.)
Then the probability that it takes more than one day to process a single claim is:
exp[-1/(1/2)] = **13.53%**.
Comment: A very poorly worded question!
"The expected time to process 100 claims received at the same time is 50 days."
The writer of the question seems to assume that the claims are processed sequentially; in other words the person or office works on one claim to the exclusion of all others until it is processed.
Once the first claim is processed, we move on to processing another claim, etc.
In that case, the average time to process a claim once we start to work on it is 1/2 day.
"The time to process a group of claims follows the Gamma distribution."
The writer of the question seems to assume that this implies that the time to process each claim in the group follows the same independent Exponential Distribution.
The sum of Independent, identically distributed Exponentials follows a Gamma Distribution; however, a Gamma Distribution can result from other mathematical situations.

4.74. B. $\lambda = 15/30 = 1/2$.
Thus the time between consecutive events is Exponential with mean 2.
$S(3) = e^{-3/2}$ = **22.3%**.

4.75. B. Prob[a person has at least one accident by time t] = 1 - exp[-0.05t].
Prob[both have at least one accident by time t] = $(1 - \exp[-0.05t])^2$ =
1 - 2 exp[-0.05t] + exp[-0.1t].
Thus the corresponding survival function is one minus this probability: 2 exp[-0.05t] - exp[-0.1t].
Mean time until both people have had at least one accident is the integral of this survival function:

$$\int_0^{\infty} 2 \exp[-0.05t] - \exp[-0.1t] \, dt = 2/0.05 - 1/0.1 = \mathbf{30}.$$

4.76. D. The number of claims by time 1 is Poisson with mean 5.
Prob[third claim occurs after one year] = Prob[at most 2 claims by time 1] =
$f(0) + f(1) + f(2) = e^{-5} (1 + 5 + 5^2/2)$ = **12.47%.**

4.77. A. The expected waiting time is independent of when the last bus left, since a Poisson Process is memoryless. **Joe's expected wait time today is equal to yesterday**.
<u>Comment</u>: Similar to CAS3L, 5/11, Q.9.
Both expected waiting times are: 1/5 hour = 12 minutes.

4.78. C. The time between successive claims is Exponential with mean: 1/5 = 0.2.
$S(0.6) = \exp[-0.6/0.2] = e^{-3}$ = **4.98%.**

4.79. C. $2 = E[T_2] = 2/\lambda. \Rightarrow \lambda = 1$.
T_{10} is the sum of 10 independent Exponential Distributions each with mean $1/\lambda = 1$.
Thus $\text{Var}[T_{10}] = (10)(1^2)$ = **10**.
Alternately, T_2 is Gamma with $\alpha = 2$ and θ. $\Rightarrow \alpha\theta = E[T_2] = 2. \Rightarrow \theta = 1$.
Thus T_{10} is Gamma with $\alpha = 10$ and $\theta = 1$. $\Rightarrow \text{Var}[T_{10}] = \alpha\theta^2 = (10)(1^2)$ = **10**.

Section 5, Thinning and Adding Poisson Processes

One can thin or add Poisson Processes.[54]

Thinning a Poisson Process:[55]

If we select at random a fraction of the claims from a Poisson Process, we get a new Poisson Process, with smaller claims intensity. This is called **thinning a Poisson Process**.

For example, assume we have a Poisson Process on (0,5) with $\lambda = 0.03$. If one accepts at random 1/3 of the claims from this first Poisson Process, then one has a new Poisson Process, with $\lambda = 0.03/3 = 0.01$.[56] The remaining 2/3rds of the original claims are also a Poisson Process, with $\lambda = (0.03)(2/3) = 0.02$. Also, these two Poisson Processes are independent.[57]

If claims are from a Poisson Process, and one divides these claims into subsets in a manner independent of the frequency process, then the claims in each subset are <u>independent</u> Poisson Processes.

Exercise: Assume claims are given by a Poisson Process with claims intensity $\lambda = 10$. Assume frequency and severity are independent. 30% of claims are of size less than $10,000, 50% of claims are of size between $10,000 and $25,000, and 20% of the claims are of size greater than $25,000. What are the frequency processes for the claims of different sizes?
[Solution: There are three independent Poisson Processes. Claims of a size less than $10,000, have a claims intensity of 3. Claims of size between $10,000 and $25,000 have a claims intensity of 5. Claims of size greater than $25,000 have a claims intensity of 2.]

Exercise: In the prior exercise, what is the probability that by time 0.7 we will have had at least 2 claims of size less than $10,000?

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[54] Similarly, one can thin or add Poisson frequency distributions.
See "Mahler's Guide to Frequency and Loss Distributions."
[55] See Proposition 5.2 in Introduction to Probability Models by Sheldon M. Ross.
[56] The key thing is that the selection process can not depend in any way on the frequency process. For example, if severity is independent of frequency, we may select only the large losses. As another example, we may select only those claims from insureds with middle initial C.
[57] The two (or more) smaller Poisson Processes that result from thinning can be added up to recover the original Poisson Process.

[Solution: Since claims of a size less than $10,000 are given by a Poisson Process with a claims intensity of 3, the number of such claims in the time period from 0 to 0.7 is Poisson Distributed with mean (3)(0.7) = 2.1. Therefore, the chance of zero such claims in this interval is $e^{-2.1}$. The chance of one such claims in this time interval is $2.1e^{-2.1}$. Thus the chance of at least 2 claims in the time interval is: $1 - \{e^{-2.1} + 2.1e^{-2.1}\} = 1 - 0.379615 = 0.620385$.
Alternately, since claims of a size less than $10,000, are given by a Poisson Process with a claims intensity of 3, the 2nd event time for a claim of a size less than $10,000 is Gamma Distributed with $\alpha = 2$ and $\theta = 1/3$. Thus the chance that the time until the 2nd such claim is less than or equal to than 0.7 is: $F(0.7) = \Gamma[2; 0.7/(1/3)] = \Gamma[2; 2.1]$.
Comment: $\Gamma[2; 2.1] = 0.620385$.
We have previously calculated this value of the incomplete Gamma Function for integer shape parameter as per Theorem A.1 in Appendix A of Loss Models.]

Exercise: In the prior exercise, if by time 0.7 we see 3 claims of size greater than $25,000, what is the probability that by time 0.7 we will have had at least 2 claims of size less than $10,000?
[Solution: These are independent processes, so the number of large claims gives us no information about the number of small claims. Therefore the solution is the same as that of the previous exercise: 0.620385.]

Exercise: Assume claims are given by a Poisson Process with claims intensity $\lambda = 10$.
Assume frequency and severity are independent. 30% of claims are of size less than $10,000, 50% of claims are of size between $10,000 and $25,000, and 20% of the claims are of size greater than $25,000. What is the probability that by time 0.7 we will have had at least 2 claims of size less than $10,000, at least 3 claims of size between $10,000 and $25,000, and at least 1 claim of size greater than $25,000?
[Solution: There are three independent Poisson Processes. Claims of a size less than $10,000, have a claims intensity of 3. Claims of size between $10,000 and $25,000 have a claims intensity of 5. Claims of size greater than $25,000 have a claims intensity of 2. The chance of at least 2 small claims in the time interval is: 0.620385, from a previous solution. The number of medium claims is Poisson with mean (0.7)(5) = 3.5. The chance of at least 3 medium claims in the time interval is:
$1 - \{e^{-3.5} + 3.5e^{-3.5} + 3.5^2e^{-3.5}/2\} = 0.679153$. The waiting time until the first claim of size greater than $25,000 is Exponentially Distributed with $\theta = 1/2$. Thus $F(0.7) = 1 - e^{-0.7/(1/2)} = 0.753403$.
Since the three Poisson Processes are independent, the probability that by time 0.7 we will have had at least 2 claims of size < $10,000, at least 3 claims of size between $10,000 and $25,000, and at least 1 claim of size > $25,000 is: (0.753403)(0.679153)(0.620385) = 0.317436.]

If the thinning factor varies over time, then thinning a homogeneous Poisson process produces a nonhomogeneous Poisson Process, to be discussed subsequently.

For example, let us assume the survival function of the size of losses at time t is:

$S(x, t) = \left(\frac{100}{100 + x/1.05^t}\right)^3$.[58] Then if all losses have a Poisson rate of 20, losses of size greater than 50, are a nonhomogeneous Poisson Process with $\lambda(t) = 20 \left(\frac{100}{100 + 50/1.05^t}\right)^3$.

<u>One Event of Each Kind</u>:[59]

Sometimes one is interested in the time until one has at least one event or claim of each type. For example, assume you are sitting by the side of a country road watching cars pass in one direction. Assume cars pass with a Poisson Process with intensity $\lambda = 0.10$. Assume 20% of the cars that pass are red, 30% of the cars that pass are blue, and 50% of the cars that pass are green. Then the passing of red cars is a Poisson process with intensity $\lambda = 0.02$. The Poisson process for red cars is independent of the Poisson Process for blue cars with intensity 0.03, and of the Poisson Process for green cars with intensity 0.05.

The waiting time until the first red car is Exponential with mean 1/0.02 = 50. Thus the chance that at least one red car has passed by time t, is: $1 - e^{-t/50} = 1 - e^{-0.02t}$.
Since the processes are independent, the chance that at least one car of each color (red, blue, and green) has passed by time t is the product of the three individual probabilities:
$(1 - e^{-0.02t})\ (1 - e^{-0.03t})\ (1 - e^{-0.05t}) = (1 - e^{-0.02t})\ (1 - e^{-0.03t} - e^{-0.05t} + e^{-0.08t})$
$= 1 - e^{-0.02t} - e^{-0.03t} - e^{-0.05t} + e^{-0.08t} + e^{-0.05t} + e^{-0.07t} - e^{-0.10t}$.

Exercise: What is the chance that at least 1 car of each color has passed by time 10?
[Solution: $(1 - e^{-0.02t})(1 - e^{-0.03t})(1 - e^{-0.05t}) = (1 - e^{-0.2})(1 - e^{-0.3})(1 - e^{-0.5})$
$= (0.1813)(0.2592)(0.3935) = 0.01849$. Alternately, $1 - e^{-0.02t} - e^{-0.03t} + e^{-0.08t} + e^{-0.07t} - e^{-0.10t}$
$= 1 - e^{-0.2} - e^{-0.3} + e^{-0.8} + e^{-0.7} - e^{-1.0} = 0.01849$.]

The chance that it takes more than time t for at least one car of each color to pass is:
$S(t) = 1 - F(t) = 1 - (1 - e^{-0.02t})(1 - e^{-0.03t})(1 - e^{-0.05t}) = e^{-0.02t} + e^{-0.03t} - e^{-0.08t} - e^{-0.07t} + e^{-0.10t}$.
In general, one can calculate the expected value of x by taking an integral of the survival function S(x). Thus the expected time until at least one car of each color has passed is:

$$\int_0^\infty S(t)\ dt = \int_0^\infty (e^{-0.02t} + e^{-0.03t} - e^{-0.08t} - e^{-0.07t} + e^{-0.10t})\ dt =$$

1/0.02 + 1/0.03 - 1/0.08 - 1/0.07 + 1/0.10 = 50 + 33.33 - 12.5 - 14.29 + 10 = 66.54.

[58] *This is a Pareto Distribution with $\alpha = 3$, $\theta = 100$, and 5% inflation per year.*
[59] See Example 5.17 in <u>Introduction to Probability Models</u> by Sheldon M. Ross.

Time Dependent Classification of Poisson Processes:[60]

Assume claims come from a Poisson Process with claims intensity 50.

Assume that the time it takes to report a claim to the insurer is distributed as per an Exponential Distribution with θ = 4.
Then assuming claims started occurring at time equals 0, what is the expected number of claims that are still unreported at time 10?

For claims that occurred at time t, by time 10 the percent reported is F(10-t).
The portion unreported is: 1 - F(10-t) = S(10-t) = exp[-(10-t)/4].
Since the claims intensity is a constant 50, the expected unreported claims are:

$$50 \int_0^{10} \exp[-(10-t)/4] \, dt = 200 \exp[-(10-t)/4] \Big]_{t=0}^{t=10} = 200 (1 - e^{-2.5}) = 183.6.$$

Thus we expect 183.6 unreported claims at time 10. By time 10 we expect (10)(50) = 500 claims in total. So we expect 183.6/500 = 36.7% of the claims to be unreported.

In general, in order to get the expected number of unreported claims one takes an integral of the survival function of the time to reporting distribution.

$$\int_0^T \lambda \, S(T-t) \, dt = \lambda \int_0^T S(T-t) \, dt .$$

The expected number of reported claims is λT - expected number of unreported claims:

$$\int_0^T \lambda \, F(T-t) \, dt = \lambda \int_0^T F(T-t) \, dt .$$

As will be discussed in subsequent sections, these are examples of nonhomogeneous Poisson processes. For a nonhomogeneous Poisson process, the claims intensity is a function of time, λ(t).

The intensity of claims that occur at time t but are unreported claims by time 10 is a function of t rather than constant: 50 S(10-t).
The intensity of claims that occur at time t but are reported claims by time 10 is: 50 F(10-t).

We have thinned a homogeneous Poisson process and obtained two nonhomogeneous Poisson processes; the thinned processes are nonhomogeneous since the tinning factor depends on t.

[60] See Proposition 5.3 in Introduction to Probability Models by Sheldon M. Ross.

Therefore, the number of reported and unreported claims are independent Poisson random variables, whose means can be calculated as above. In the above example, the number of reported claims is Poisson with mean 500 - 183.6 = 316.4, and the number of unreported claims is Poisson with 183.6, and the number of reported and unreported claims are independent.

Exercise: In the above example, what is the probability of more than 300 reported claims and fewer than 200 unreported claims?
[Solution: Prob[more than 300 reported claims] $\cong 1 - \Phi[(300.5 - 316.4) / \sqrt{316.4}\,] =$
$1 - \Phi[-0.8939] = 81.4\%$.
Prob[less than 200 unreported claims] $\cong \Phi[(199.5 - 183.6) / \sqrt{183.6}\,] = \Phi[1.1734] = 88.0\%$.
Independent processes, so the probability of both events is: (81.4%)(88.0%) = 71.6%.]

Assume instead that the Poisson process only goes from (0, 1), but we see how many claims are reported by time R > 1.[61]

Then the expected number of reported claims by time R is: $\int_0^1 \lambda\, F(R\text{-}t)\, dt$.

More, generally, assume we divide the claims from a Poisson Process into subsets; then provided the manner of dividing does not depend on the number of claims, the number of claims in each subset by a given time are independent Poisson random variables. The mean number of type i at time t is: $\lambda \int_0^t \text{Prob[event is of type i at time t | given event occurred at time s]}\, ds$.[62]

Examples include:
1. In a queue, how many customers have completed service by a given time?
2. In a queue, how many customers are being served at a given time?
3. The number of cars that will pass your car on the road by a given time.
4. The number of persons infected by a disease by a certain time.
5. The number of infected persons who have shown symptoms of a given disease by a certain time.
6. The number of infected persons who have not shown symptoms of a given disease by a certain time.

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[61] *An accident year consists of those losses from events that occur during a given year.*
[62] See Proposition 5.3 in Introduction to Probability Models by Sheldon M. Ross, not on the syllabus.

Adding Poisson Processes:[63]

If one adds two independent Poisson Processes, one gets a new Poisson Process, with claims intensity the sum of the two individual claims intensities.
This is the case, because when we add two independent claims intensities, each of which is constant with independent increments, so is their sum.

Exercise: Claims from illness are a Poisson Process with claims intensity 13.
Claims from accident are a Poisson Process with claims intensity 7.
The two processes are independent.
What is the probability of at least 3 claims by time 0.1?
[Solution: Claims are a Poisson Process with $\lambda = 13 + 7 = 20$.
Thus the number of claims by time 0.1 is Poisson with mean 2.
The probability of at least 3 claims is: $1 - \{e^{-2} + 2e^{-2} + 2^2e^{-2}/2\} = 0.323$.]

[63] See Section 1.3.2 of “Poisson Processes” by Daniel.

Problems:

5.1 (3 points) Claims are reported according to a Poisson process with intensity 25 per month.
The number of claims reported and the claim amounts are independently distributed.
Claim amounts are distributed via a Weibull Distribution, $F(x) = 1 - \exp[-(x/\theta)^\tau]$, $x > 0$,
with $\tau = 0.7$ and $\theta = 1000$.
Calculate the number of complete months of data that must be gathered to have at least a 99.5% chance of observing at least 3 claims each exceeding 3000.

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Use the following information for the next 5 questions:
As he walks, Clumsy Klem loses coins at a Poisson rate of 0.2 coins/minute.
The denominations are randomly distributed:

(i) 50% of the coins are worth 5;
(ii) 30% of the coins are worth 10; and
(iii) 20% of the coins are worth 25.

5.2 (1 point) Calculate the expected value of the coins Klem loses during his one-hour walk today.

(A) 120 (B) 122 (C) 124 (D) 126 (E) 128

5.3 (2 points) Calculate the conditional expected value of the coins Klem loses during his one-hour walk today, given that among the coins he lost exactly three were worth 5 each.

(A) 103 (B) 105 (C) 107 (D) 109 (E) 111

5.4 (2 points) Calculate the conditional variance of the value of the coins Klem loses during his one-hour walk today, given that among the coins he lost exactly three were worth 5 each.

(A) 1800 (B) 1850 (C) 1900 (D) 1950 (E) 2000

5.5 (2 points) Klem goes on a four hour walk for charity. Determine the probability that Klem loses a coin worth 10 within seven minutes subsequent to the time of losing his first coin worth 5.

A. 30% B. 35% C. 40% D. 45% E. 50%

5.6 (3 points) If Klem loses a least 3 coins worth 10, what is the expected amount of money he loses during a one-hour walk?

A. 131 B. 132 C. 133 D. 134 E. 135

Use the following information for the next 2 questions:
Taxicabs leave a hotel with a group of passengers at a Poisson rate λ = 10 per hour. The number of people in each group taking a cab is independent and has the following probabilities:

Number of People	Probability
1	0.60
2	0.30
3	0.10

5.7 (2 points) What is the probability that at least one taxicab of each type has left during the first half hour?
A. 21% B. 23% C. 25% D. 27% E. 29%

5.8 (3 points) What is the average number of minutes until at least one taxicab of each type has left?
A. 60 B. 65 C. 70 D. 75 E. 80

Use the following information for the next two questions:
- Number of claims follows a Poisson Process with intensity 5.
- Claim severity is independent of the number of claims and has the following probability density function: $f(x) = 3.5\ x^{-4.5}$, $x > 1$.

5.9 (2 points) What is the average time until the 20th claim of size greater than 3?
A. Less than 150
B. At least 150, but less than 60
C. At least 160, but less than 70
D. At least 170, but less than 80
E. At least 180

5.10 (2 points) What is the probability that the 20th claim of size greater than 3 has occurred by time 200?
Use the Normal Approximation.
A. Less than 50%
B. At least 50%, but less than 60%
C. At least 60%, but less than 70%
D. At least 70%, but less than 80%
E. At least 80%

<u>5.11</u> (3 points) Individuals contract the HIV virus via a Poisson Process with unknown constant intensity λ. Symptoms of AIDS appear after a lag.
The lag from contracting the disease to the appearance of symptoms is given by an Exponential Distribution with a mean of 8 years.
A given population was first exposed to the disease 20 years ago.
In this population, 3410 cases of AIDS have been reported.
Estimate how many members of this population have contracted HIV, but have yet to show symptoms of AIDS.
A. 1900 B. 2000 C. 2100 D. 2200 E. 2300

<u>5.12</u> (2 points) You are given the following:
● Number of customers follows a Poisson Process with intensity 0.0125 per minute.
● The amount that a single customer spends has a uniform distribution on [0, 5000].
● Number of customers and the amount each customer spends are independent.
Calculate the probability that it will take more than 1000 minutes for a single customer to spend more than 4000.
(A) 4% (B) 6% (C) 8% (D) 10% (E) 12%

5.13 (3 points) Claims occur via a Poisson Process with unknown constant intensity λ.
Claims are reported after a lag.
The lag from occurrence to report is given by an Exponential Distribution with a mean of 0.6.
480 claims that occurred between time 0 and time 1 have been reported by time 1.25.
Estimate λ.
A. Less than 680
B. At least 680, but less than 690
C. At least 690, but less than 700
D. At least 700, but less than 710
E. At least 710

5.14 (2 points) For Broward County, Florida, hurricane season is 24 weeks long. It is assumed that the time between hurricanes is exponentially distributed with a mean of 6 weeks. It is also assumed that 30% of all hurricanes will hit Broward County.
Calculate the probability that in any given hurricane season, there will be three hurricanes of which exactly one hits Broward County.
A. Less than 5%
B. At least 5%, but less than 6%
C. At least 6%, but less than 7%
D. At least 7%, but less than 8%
E. 8% or more

Use the following information for the next 5 questions:

● The number of families who immigrate to Vancougar from Honwan is given by a Poisson Process with intensity per day of 0.8.

● The sizes of these families are distributed as follows:

1	2	3	4	5	6	7	8	9	10
10%	15%	22%	20%	12%	8%	6%	4%	2%	1%

● The number of families and the sizes of the families are independent.

5.15 (1 point) What is the mean number of days until a family of size 5 immigrates?
A. 8 B. 10 C. 12 D. 14 E. 16

5.16 (2 points) How many days must one observe to be 95% certain of observing a family of size 8 or more?
A. Less than 35
B. At least 35, but less than 40
C. At least 40, but less than 45
D. At least 45, but less than 50
E. At least 50

5.17 (1 point) Last year there were 66 families of size 4 who immigrated.
Estimate the number of families of size 3 or less that immigrated last year.
A. Less than 135
B. At least 135, but less than 140
C. At least 140, but less than 145
D. At least 145, but less than 150
E. At least 150

5.18 (3 points) What is the probability that in the next 100 days at least one family of each size will immigrate?
A. Less than 30%
B. At least 30%, but less than 35%
C. At least 35%, but less than 40%
D. At least 40%, but less than 45%
E. At least 45%

5.19 (3 points) What is the probability that in the next 100 days there are at least three families of size 8, at least two families of size 9, and at least one family of size 10?
A. 12% B. 14% C. 16% D. 18% E. 20%

Use the following information for the next 6 questions:
- Theft losses follow a Poisson Process with $\lambda = 10$.
- The size of Theft losses follows an Exponential Distribution with mean 400.
- Fire losses follow a Poisson Process with $\lambda = 3$.
- The size of Fire losses follows a Pareto Distribution, $F(x) = 1 - \left(\frac{\theta}{\theta+x}\right)^{\alpha}$, $x > 0$,

  with $\alpha = 3$ and $\theta = 4000$.
- Theft losses and Fire losses are independent of each other.
- An insurance policy that covers Fire losses and Theft losses has a 1000 deductible.

5.20 (1 point) What is the probability of exactly 8 theft losses by time 1?
A. 11% B. 13% C. 15% D. 17% E. 19%

5.21 (1 point) What is the probability of exactly 8 fire losses by time 1?
A. Less than 1%
B. At least 1%, but less than 2%
C. At least 2%, but less than 3%
D. At least 3%, but less than 4%
E. At least 4%

5.22 (1 point) What is the probability of exactly 8 losses by time 1?
A. Less than 1%
B. At least 1%, but less than 2%
C. At least 2%, but less than 3%
D. At least 3%, but less than 4%
E. At least 4%

5.23 (2 points) What is the probability of exactly 3 non-zero payments due to theft losses by time 1?
A. 0.5% B. 1% C. 2% D. 3% E. 4%

5.24 (2 points) What is the probability of exactly 3 non-zero payments due to fire losses by time 1?
A. 7% B. 9% C. 11% D. 13% E. 15%

5.25 (2 points) What is the probability of exactly 3 non-zero payments by time 1?
A. 19% B. 21% C. 23% D. 25% E. 27%

Use the following information for the next 8 questions:
As he walks, Clumsy Klem loses coins at a Poisson rate of 0.2 coins/minute.
The denominations are randomly distributed:

(i) 50% of the coins are worth 5;
(ii) 30% of the coins are worth 10; and
(iii) 20% of the coins are worth 25.

5.26 (2 points) Calculate the probability that the third coin of value 25 has been lost within the first 30 minutes.
(A) 0.12 (B) 0.14 (C) 0.16 (D) 0.18 (E) 0.20

5.27 (2 points) Calculate the probability that in the first ten minutes of his walk he loses 2 coins worth <u>10</u> each, and in the first twenty minutes he loses 3 coins worth <u>5</u> each.
(A) 1.8% (B) 2.0% (C) 2.2% (D) 2.4% (E) 2.6%

5.28 (2 points) Calculate the probability that in the first ten minutes of his walk he loses 2 coins worth <u>10</u> each, and in the first fifteen minutes he loses 3 coins worth <u>10</u> each.
(A) 1.8% (B) 2.0% (C) 2.2% (D) 2.4% (E) 2.6%

5.29 (2 points) Calculate the probability that in the first ten minutes of his walk he loses at least 2 coins worth <u>5</u> each, and in the next twenty minutes he loses at least 4 coins worth <u>5</u> each.
(A) 0.03 (B) 0.04 (C) 0.05 (D) 0.06 (E) 0.07

<u>**5.30**</u> (3 points) Calculate the probability that in the first ten minutes of his walk he loses at least 2 coins worth <u>10</u> each, and in the first thirty minutes he loses at least 4 coins worth <u>10</u> each.
(A) 0.05 (B) 0.06 (C) 0.07 (D) 0.08 (E) 0.09

5.31 (3 points) Calculate the probability that in the first ten minutes of his walk he loses at least 2 coins worth <u>5</u> each, and in the first 25 minutes he loses at least 5 coins worth <u>5</u> each.
(A) 0.05 (B) 0.06 (C) 0.07 (D) 0.08 (E) 0.09

5.32 (2 points) Clumsy Klem takes a 30 minute walk. Let A be the number of coins lost during the first ten minutes of that walk. Let B be the number of coins lost during the last ten minutes of that walk. What is the probability that A + B = 5?
(A) 0.12 (B) 0.14 (C) 0.16 (D) 0.18 (E) 0.20

5.33 (3 points) Clumsy Klem takes a 30 minute walk. Let C be the number of coins lost during the first ten minutes of that walk. Let D be the number of coins lost during the first twenty minutes of that walk. What is the probability that C + D = 4?
(A) 0.12 (B) 0.14 (C) 0.16 (D) 0.18 (E) 0.20

5.34 (3 points) Trucks arrive at the Sheriff Buford T. Justice Memorial Truckstop at a Poisson rate of 15 per hour. The time a truck spends at this truckstop follows a Single Parameter Pareto Distribution, $F(x) = 1 - \left(\frac{\theta}{x}\right)^{\alpha}$, $\theta > 0$, with $\alpha = 1.2$ and $\theta = 0.1$ hour.

What is the probability that there are currently more than 3 trucks that have each been at the Sheriff Buford T. Justice Memorial Truckstop for more than 2 hours?
(A) 45% (B) 50% (C) 55% (D) 60% (E) 65%

Use the following information for the next 3 questions:
Taxicabs leave a hotel with a group of passengers at a Poisson rate $\lambda = 10$ per hour.
The number of people in each group taking a cab is independent and has the following probabilities:

Number of People	Probability
1	0.60
2	0.30
3	0.10

5.35 (2 points) If during an hour no taxicabs leave with 3 passengers each and 8 taxicabs leave with 1 passenger each, estimate how many total people leave via taxicabs during that hour.
(A) 12 or less (B) 13 (C) 14 (D) 15 (E) 16 or more

5.36 (2 points) What is the probability that during the next hour exactly 1 taxicab leaves with 3 passengers, exactly 3 taxicabs leave with 2 passengers each, and exactly 6 taxicabs leave with 1 passenger each?
(A) 0.7% (B) 1.0% (C) 1.3% (D) 1.6% (E) 1.9%

5.37 (2 points) What is the probability that the first taxicab to leave has 2 people and that it leaves within 7 minutes?
A. 7% B. 8% C. 9% D. 10% E. 11%

5.38 (3 points) Assume claims follow a Poisson Process with unknown constant intensity λ.
Assume that the time it takes to report a claim is distributed as per a Exponential Distribution with mean 3. Claims started occurring at time equals 0.
By time equal to 5, there have been 402 claims reported.
Estimate how many claims have occurred but remain unreported by time 5.
A. 340 B. 360 C. 380 D. 400 E. 420

5.39 (2 points) At time zero, severity is Uniform from 0 to 10,000. Inflation is 10% per year.
Losses occur at a Poisson rate of 200 per year.
Over the first 9 years one observes 1100 losses of size greater than or equal to 6000.
What is the conditional expected number of losses observed over the first 9 years of size less than 6000?
(A) 700 (B) 725 (C) 750 (D) 775 (E) 800

5.40 (3 points) A communications network was first put in operation 50 days ago.
Flaws in the network occur via a Poisson Process with $\lambda = 0.03$ per day.
Each flaw will eventually, independently cause the network to fail.
When this happens, that flaw is detected and fixed immediately.
The time from when a flaw occurs to when it results in failure follows a Pareto Distribution,
$F(x) = 1 - \left(\frac{\theta}{\theta + x}\right)^{\alpha}$, $x > 0$, with $\alpha = 5$ and $\theta = 80$.
Determine the probability that there are currently no undetected flaws in the network.
(A) 50% (B) 55% (C) 60% (D) 65% (E) 70%

Use the following information for the next two questions:
Arthur the art dealer is trying to sell a valuable painting at his art gallery.
Unfortunately, Arthur is leaving on an extended buying trip to Europe and has to leave his inexperienced niece Cecilia in charge of his art gallery.
Arthur needs to leave Cecilia with very specific instructions that leave no room for judgment.
Offers for the painting will arrive at a Poisson rate of 2 per week.
The size of an offer is independent of the size of any other offer, and follows a Single Parameter Pareto Distribution with $\alpha = 3$ and $\theta = 10{,}000$, $F(x) = 1 - (\theta/x)^{\alpha}$, $x > \theta$.
Cecilia can either accept an offer or reject it and wait for the next offer.
Arthur will instruct Cecilia to accept the first offer greater than a certain amount x.

5.41 (2 points) If Arthur wants there to be only a 5% chance that the painting will be unsold after 10 weeks, what value of x should he select?
A. 15,000 B. 16,000 C. 17,000 D. 18,000 E. 19,000

5.42 (3 points) Arthur incurs 200 in expense each week the painting remains unsold.
If Arthur wants to maximize the expected value of the difference between the price for which the painting is sold and the expenses incurred, what value of x should he select?
A. 50,000 B. 60,000 C. 70,000 D. 80,000 E. 90,000

5.43 (3 points) Claims occur at a Poisson rate of 10 per month.
The time it takes to report a claim is distributed as per a Exponential Distribution with mean 20 months. Claims started occurring at time equals 0.
Calculate the expected number of claims that will be reported by time 24 months.
A. 85 B. 90 C. 95 D. 100 E. 105

5.44 (2 points) In the previous question, what is the probability that fewer than 90 claims will be reported by time 24 months?
(A) 12% (B) 14% (C) 16% (D) 18% (E) 20%

Use the following information for the next four questions:
Subway trains arrive at a station at a Poisson rate of 15 per hour.
60% of the trains are express and 40% are local.
The type of each train is independent of the types of preceding trains.
Ezra and Fiona work in the same office.
They are waiting at the same station.
An express gets them to work in 20 minutes and a local gets them there in 35 minutes.
Ezra always takes the first express to arrive, while Fiona always take the first train to arrive.
Let E be the time it takes Ezra to arrive at work.
Let F be the time it takes Fiona to arrive at work

5.45 (2 points) Calculate the probability that F < E.
(A) 1% (B) 2% (C) 3% (D) 4% (E) 5%

5.46 (2 points) Calculate the expected value of F - E, in minutes.
(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

5.47 (2 points) Calculate the standard deviation of E, in minutes.
A. Less than 7.0
B. At least 7.0, but less than 7.5
C. At least 7.5, but less than 8.0
D. At least 8.0, but less than 8.5
E. At least 8.5

5.48 (2 points) Calculate the standard deviation of F, in minutes.
A. Less than 7.0
B. At least 7.0, but less than 7.5
C. At least 7.5, but less than 8.0
D. At least 8.0, but less than 8.5
E. At least 8.5

5.49 (2 points) Claims occur via a Poisson Process with intensity 50 per year.
Claims are reported after a lag, which is uniformly distributed from 0 to 0.25.
Determine the probability that fewer than 40 claims are reported by the end of year.
(A) 18% (B) 20% (C) 22% (D) 24% (E) 26%

5.50 *(5 points) Hurricanes hitting the State of Windiana follow a Poisson Process, with λ = 82% on an annual basis. In 1998, the losses from such a hurricane are given by a Pareto Distribution with $\alpha = 2.5$ and $\theta = 400$ million. $F(x) = 1 - \{\theta/(\theta+x)\}^{\alpha}$, $x > 0$.*
Inflation is 5% per year. Frequency and severity are independent.
Starting in the year 2000, how many complete years must be observed in order to have at least a 90% chance of seeing at least one hurricane with more than $250 million of loss hitting the State of Windiana?
A. 7 B. 8 C. 9 D. 10 E. 11

Use the following information for the next two questions:
Subway trains leave the Main Street station at a Poisson rate of 10 per hour.
2/3 of the trains are express and 1/3 are local.
The type of each train is independent of the types of preceding trains.
Ethel and Fred work in the same office.
An express gets them to work in 13 minutes and a local gets them there in 25 minutes.
Ethel always takes the first express to arrive, while Fred always take the first train to arrive.
Ethel arrives at the Main Street station at 7:48.
Fred arrives at the Main Street station at 7:51, and finds Ethel waiting.
Fred takes the local train that leaves at 7:55, leaving Ethel still waiting at the station.
Let E be the time Ethel arrives at work.
Let F be the time Fred arrives at work.

5.51 (2 points) Calculate the expected value of F - E, in minutes.
(A) 0 (B) 1 (C) 3 (D) 7 (E) 10

5.52 (2 points) Calculate the probability that E < F.
A. Less than 40%
B. At least 40%, but less than 50%
C. At least 50%, but less than 60%
D. At least 60%, but less than 70%
E. 70% or more

Use the following information for the next 10 questions:
You are given the following:

- A loss occurrence may be caused by wind, earthquake or theft.
- Wind, earthquake and theft losses occur independently of one another.
- Wind losses follow a Poisson Process.
  The expected amount of time between wind losses is 10 years.
- Earthquake losses follow a Poisson Process.
  The expected amount of time between earthquake losses is 25 years.
- Theft losses follow a Poisson Process.
  The expected amount of time between theft losses is 5 years.
- The size of wind losses follows a LogNormal Distribution, $F(x) = \Phi\left[\frac{\ln(x) - \mu}{\sigma}\right]$, $x > 0$,
  with parameters $\mu = 2$ and $\sigma = 3$.
- The size of earthquake losses follows a Pareto Distribution, $F(x) = 1 - \left(\frac{\theta}{\theta + x}\right)^{\alpha}$, $x > 0$,
  with parameters $\alpha = 2$ and $\theta = 1500$.
- The size of theft losses follows a Weibull Distribution, $F(x) = 1 - \exp\left[-\left(\frac{x}{\theta}\right)^{\tau}\right]$, $x > 0$,
  with parameters $\tau = 2$ and $\theta = 600$.

5.53 (2 points) Determine the expected amount of time between loss occurrences.
A. Less than 2.0 years
B. At least 2.0 years, but less than 2.5 years
C. At least 2.5 years, but less than 3.0 years
D. At least 3.0 years, but less than 3.5 years
E. At least 3.5 years

5.54 (1 point) What is the chance that the fifth loss has occurred by time 11.76?
A. $\Gamma[4 ; 4]$ B. $\Gamma[5 ; 4]$ C. $\Gamma[4 ; 5]$ D. $\Gamma[5 ; 5]$ E. None of A, B, C, or D

5.55 (1 point)
What is the chance that is it is more than 4 years between the seventh and eighth loss?
A. Less than 0.25
B. At least 0.25, but less than 0.26
C. At least 0.26, but less than 0.27
D. At least 0.27, but less than 0.28
E. At least 0.28

5.56 (3 points) What is the chance there has been at least one loss of each type by the end of the 8th year?
A. Less than 11%
B. At least 11%, but less than 11%
C. At least 12%, but less than 13%
D. At least 13%, but less than 14%
E. At least 14%

5.57 (3 points) What is the average time until there has been at least one loss of each type?
A. Less than 27 years
B. At least 27 years, but less than 28 years
C. At least 28 years, but less than 29 years
D. At least 29 years, but less than 30 years
E. At least 30 years

<u>5.58</u> (2 points) What is the chance that the 4th loss is a wind loss of size greater than 1000?
A. Less than 1%
B. At least 1%, but less than 2%
C. At least 2%, but less than 3%
D. At least 3%, but less than 4%
E. At least 4%

5.59 (1 point) What is the expected time between earthquakes of size greater than 1000?
A. Less than 55 years
B. At least 55 years, but less than 60 years
C. At least 60 years, but less than 65 years
D. At least 65 years, but less than 70 years
E. At least 70 years

5.60 (1 point) What is the expected wait until the first theft loss greater than 1000?
A. Less than 75 years
B. At least 75 years, but less than 80 years
C. At least 80 years, but less than 85 years
D. At least 85 years, but less than 90 years
E. At least 90 years

5.61 (2 points) What is the expected wait until the first loss of size greater than 1000 is observed?
A. Less than 15 years
B. At least 15 years, but less than 20 years
C. At least 20 years, but less than 25 years
D. At least 25 years, but less than 30 years
E. At least 30 years

5.62 (3 points) What is the chance there has been at least four theft losses, at least three wind losses and at least two earthquake losses by time 20?
A. $\Gamma[4 ; 2]\ \Gamma[3 ; 4]\ \Gamma[2 ; 0.8]$ B. $\Gamma[4 ; 4]\ \Gamma[3 ; 3]\ \Gamma[2 ; 2]$ C. $\Gamma[4 ; 0.25]\ \Gamma[3; 0.5]\ \Gamma[2 ; 1.25]$
D. $\Gamma[3 ; 4]\ \Gamma[2 ; 3]\ \Gamma[2 ; 1]$ E. None of the above

5.63 (3 points) Tyrannosaurs eat only scientists.
Tyrannosaurs encounter scientists at a Poisson rate of 0.8 per day.
Of the scientists who a tyrannosaur encounters, one quarter of the scientists kill the tyrannosaur with their ray guns, one quarter escape from the tyrannosaur using their invisibility devices, and the remaining one half are eaten by the tyrannosaur.
What is the probability that Terry the tyrannosaur over the next week (7 days) eats at least three scientists and is alive at the end of the week?
(A) 9% (B) 10% (C) 11% (D) 12% (E) 13%

5.64 (3 points) You are given the following information about the Kwik-E-Mart:
- Customers arrive at a Poisson rate of 20 per hour.
- At random, 40% are men, 30% are women, and 30% are children.
- Among the items the Kwik-E-Mart sells are Squishees, large frozen soft drinks.
- No customer buys more than one Squishee.
- 20% of men customers buy a Squishee.
- 10% of women customers buy a Squishee.
- 60% of children customers buy a Squishee.

What is the expected time until the sale of the 50^{th} Squishee?
A. Less than 7.0 hours
B. At least 7.0 hours, but less than 7.5 hours
C. At least 7.5 hours, but less than 8.0 hours
D. At least 8.0 hours, but less than 8.5 hours
E. At least 8.5 hours

Use the following information for the next 9 questions:
As he walks, Clumsy Klem loses coins at a Poisson rate of 0.2 coins/minute.
The denominations are randomly distributed:
(i) 50% of the coins are worth 5 (nickels);
(ii) 30% of the coins are worth 10 (dimes); and
(iii) 20% of the coins are worth 25 (quarters).

5.65 (2 points) If Klem has lost 8 nickels and 5 dimes in the first 20 minutes, what is the probability he loses at least 3 quarters in the first 40 minutes?
(A) 14% (B) 16% (C) 18% (D) 20% (E) 22%

5.66 (1 point) What is the expected time until Klem loses his first coin?
(A) 5 minutes (B) 6 minutes (C) 7 minutes (D) 8 minutes (E) 9 minutes

5.67 (1 point) What is the expected time until Klem loses his first quarter?
(A) 10 minutes (B) 15 minutes (C) 20 minutes (D) 25 minutes (E) 30 minutes

5.68 (1 point) What is the expected time until Klem loses his first coin that is not a quarter?
(A) 5 minutes (B) 6 minutes (C) 7 minutes (D) 8 minutes (E) 9 minutes

5.69 (1 point) What is the expected time until Klem loses his tenth coin?
(A) 20 minutes (B) 30 minutes (C) 40 minutes (D) 50 minutes (E) 60 minutes

5.70 (1 point) What is the expected time until Klem loses his tenth nickel?
(A) 80 minutes (B) 90 minutes (C) 100 minutes (D) 110 minutes (E) 120 minutes

5.71 (3 points) What is the expected time until Klem has lost at least one coin of each type?
(A) 27 minutes (B) 30 minutes (C) 33 minutes (D) 36 minutes (E) 39 minutes

5.72 (4 points) What is the expected time until Klem has lost at least two nickels and at least two dimes?
(A) 38 minutes (B) 40 minutes (C) 42 minutes (D) 44 minutes (E) 46 minutes

5.73 (9 points) What is the expected time until Klem has lost at least two coins of each type?
(A) 57 minutes (B) 60 minutes (C) 63 minutes (D) 66 minutes (E) 69 minutes

5.74 (2 points) Artificial satellites are launched at a Poisson rate of 10 per year.
The time until a satellite falls from the sky is Exponentially distributed with mean 5 years.
Determine the probability that none of the satellites currently in the air was launched more than 20 years ago.
(A) 20% (B) 25% (C) 30% (D) 35% (E) 40%

Use the following information for the next two questions:
- Doris is the nurse at Springfield Elementary School.
- Doris treats minor injuries to students.
- Injured students arrive at Doris's office at a Poison rate of 3 per hour.
- It takes Doris 10 minutes to treat 70% of the students.
- It takes Doris 20 minutes to treat the other 30% of the students.
- If a student arrives while Doris is treating another student, they wait until Doris is free.

5.75 (3 points) At 10:00 Doris is not treating a student.
What is the probability that Doris is not treating a student at 10:20?
(A) 40% (B) 45% (C) 50% (D) 55% (E) 60%

5.76 (5 points) At 10:00 Doris is not treating a student.
What is the probability that Doris is not treating a student at 10:30?
(A) 40% (B) 45% (C) 50% (D) 55% (E) 60%

5.77 (3 points) During the evening rush hour at the Lincoln Circle subway station:
- Uptown trains arrive via a Poisson Process at a rate of 1 per 4 minutes.
- Downtown trains arrive via a Poisson Process at a rate of 1 per 6 minutes.
- The arrival of downtown and uptown trains are independent.
- 80% of uptown trains are expresses.
- 60% of downtown trains are expresses.

What is the average time until the third express train passes through the station?
(A) 4 minutes (B) 6 minutes (C) 8 minutes (D) 10 minutes (E) 12 minutes

5.78 (4 points) Use the following information:
- Employees at a company suffer work related injuries and are temporarily disabled.
- Worker disabilities occur via a Poisson Process with rate 9 per year.
- 70% of the time, the period a worker remains disabled follows an Exponential Distribution with mean 3 months.
- 30% of the time, the period a worker remains disabled follows an Exponential Distribution with mean 9 months.

Determine the probability that as of April 1, 2014 there are exactly three workers whose injuries occurred during 2013 and who are still disabled.
A. 10% B. 12% C. 14% D. 16% E. 18%

Use the following information for the next 7 questions:
As he walks, Clumsy Klem loses coins at a Poisson rate of 0.2 coins/minute.
The denominations are randomly distributed:
(i) 50% of the coins are worth 5 (nickels);
(ii) 30% of the coins are worth 10 (dimes); and
(iii) 20% of the coins are worth 25 (quarters).

5.79 (2 points) During his hour walk, Klem loses 8 nickels.
Calculate the probability that he loses exactly 7 coins that are not nickels.
(A) 14% (B) 15% (C) 16% (D) 17% (E) 18%

5.80 (2 points) Calculate the probability that during his hour walk, Klem loses 12 coins, of which exactly 3 are quarters.
(A) 0.5% (B) 1% (C) 2% (D) 3% (E) 4%

5.81 (3 points) Calculate the probability that in the first ten minutes of his walk Klem has lost coins worth in total exactly 25.
(A) 7% (B) 8% (C) 9% (D) 10% (E) 11%

5.82 (4 points) Calculate the probability that conditional on Klem losing in the first ten minutes of his walk coins worth in total exactly 25, he loses in the first thirty minutes of his walk a total of exactly two dimes.
(A) 24% (B) 25% (C) 26% (D) 27% (E) 28%

5.83 (2 points) Calculate the probability that conditional on Klem losing in the first ten minutes of his walk coins worth in total exactly 25, he loses in the first twenty minutes of his walk coins worth in total exactly 35.
(A) 14% (B) 15% (C) 16% (D) 17% (E) 18%

5.84 (1 point) Calculate the variance of the number of coins not worth 10 that Klem loses during a one hour walk.
(A) 7.0 (B) 7.5 (C) 8.0 (D) 8.5 (E) 9.0

5.85 (2 points) Calculate the probability that in a one hour walk, Klem loses exactly 5 nickels, exactly 4 dimes, and exactly 2 quarters.
(A) 0.5% (B) 0.6% (C) 0.7% (D) 0.8% (E) 0.9%

5.86 (3 points) Assume claims follow a Poisson Process with unknown constant intensity λ.
Assume that the time it takes to report a claim is distributed as per a Pareto Distribution,
$F(x) = 1 - \left(\frac{\theta}{\theta + x}\right)^{\alpha}$, $x > 0$ with $\alpha = 3$ and $\theta = 10$. Claims started occurring at time equals 0.

By time equal to 5, there have been 402 claims reported.
Estimate how many claims have occurred but remain unreported by time 5.
A. 300 B. 350 C. 400 D. 450 E. 500

Use the following information for the next two questions:
- Supernovas occur in a particular galaxy via a Poisson Process with rate one per 50 years.
- 10% of supernovas in this galaxy are Type I.

5.87 (2 points) What is the 90th percentile of the time until the next Type I supernova?
A. 1000 B. 1050 C. 1100 D. 1150 E. 1200

5.88 (2 points) What is standard deviation of the time until the third Type I supernova?
A. 700 B. 900 C. 1100 D. 1300 E. 1500

5.89 (2 points) You are given the following information:
- Elizabeth is interrupted at work by either a phone call or a coworker coming to her cubicle.
- The arrival of phone calls and coworkers are independent.
- Phone calls arrive according to a Poisson process at a rate of 2 per hour.
- Coworkers arrive at her cubicle according to a Poisson process at a rate of 1 per hour.

Calculate the probability it is more than 30 minutes before Elizabeth is next interrupted.
A. Less than 0.15
B. At least 0.15, but less than 0.20
C. At least 0.20, but less than 0.25
D. At least 0.25 but less than 0.30
E. At least 0.30

5.90 *(3 points) Assume claims come from a Poisson Process with claims intensity 40.*
Assume that the time it takes to report a claim is distributed as per a Weibull Distribution,
$F(x) = 1 - \exp[-(x/\theta)^\tau]$, $x > 0$, with parameters $\theta = 2$ and $\tau = 0.6$.
Then assuming claims started occurring at time equals 0, what is the expected number of claims that are still unreported at time 10?
You may leave your answer in terms of Complete and Incomplete Gamma Functions.

5.91 (2 points) Claims follow a Poisson Process with rate of 9 per year.
There are two type of claims, with 70% of Type 1.
For Type 1 claims, Prob[size > 5000] = 4%.
For Type 2 claims, Prob[size > 5000] = 13%.
Determine the probability of more than one claim of size greater than 5000 next year.
A. 6% B. 8% C. 10% D. 12% E. 14%

5.92 (3 points) Use the following information:
- Pat is a senior about to graduate from college.
- Pat receives job offers according to a Poisson process with a rate of 0.8 per week.
- A job offer is acceptable to Pat if the salary offered is at least 70,000 per year.
- The salaries offered are mutually independent and follow a Single Parameter Pareto distribution with $\alpha = 4$ and $\theta = 40,000$.
- For the Single Parameter Pareto Distribution: $F(x) = 1 - \left(\frac{\theta}{x}\right)^{\alpha}$, $x > \theta$.

Calculate the probability that to receive the first acceptable job offer it will take Pat more than 10 weeks but no more than 15 weeks.

A. 11% B. 13% C. 15% D. 17% E. 19%

5.93 (2 points) You are given the following information:
- A claims office receives new claims on Auto and Homeowners Insurance.
- Each type of claim follows a homogeneous Poisson process with the following rates of occurrence:

Auto	1.5 per day.
Homeowners	1 per two days

Calculate the probability that at least 4 claims occur in a 3 day period.

A. Less than 0.81
B. At least 0.81, but less than 0.82
C. At least 0.82, but less than 0.83
D. At least 0.83, but less than 0.84
E. At least 0.84

5.94 (4, 5/90, Q.42) (2 points) Suppose the amount of losses due to a single windstorm follows a Weibull distribution, $F(x) = 1 - \exp\left[-\left(\frac{x}{\theta}\right)^{\tau}\right]$, $x > 0$, with parameters θ = \$156.25 million and $\tau = 0.5$.

Suppose also that the number of windstorms follows a Poisson distribution with constant parameter and that 12 windstorms have exceeded \$5 million (in 1990 dollars) in the past 30 years.

If n is the expected number of years until the next windstorm that exceeds \$1 billion (in 1990 dollars), in what range does n fall?

A. $n < 22$ B. $22 \le n < 24$ C. $24 \le n < 26$ D. $26 \le n < 28$ E. $28 \le n$

5.95 (4B, 11/92, Q.30) (2 points) You are given the following:
- The size of loss distribution for damages from a single earthquake follows
  a Pareto distribution, $F(x) = 1 - \left(\frac{\theta}{\theta + x}\right)^{\alpha}$, $x > 0$,

  with parameters $\alpha = 4$ and θ = \$150,000,000 in constant 1992 dollars,
- The number of earthquakes in a single year has a Poisson distribution with constant mean.
- In the past 50 years, 10 earthquakes have occurred in which damages exceeded
  \$30 million in constant 1992 dollars.

Determine the expected number of years between earthquakes having damage in excess of \$100 million in 1992 dollars.
A. Less than 13
B. At least 13 but less than 15
C. At least 15 but less than 17
D. At least 17 but less than 19
E. At least 19

5.96 (4B, 11/95, Q.27) (2 points) You are given the following:
- The number of storms causing losses in excess of \$500,000 (in constant 1995 dollars)
  in a one-year period follows a Poisson distribution with mean 1.3.
- In constant 1995 dollars, the size of loss for each storm follows a Pareto distribution,
  $F(x) = 1 - \left(\frac{\theta}{\theta + x}\right)^{\alpha}$, $x > 0$, with parameters $\theta = 7000$ and $\alpha = 1.05$.

Determine the expected number of years between storms causing losses in excess of \$5,000,000 (in constant 1995 dollars).
A. Less than 5.0
B. At least 5.0, but less than 7.5
C. At least 7.5, but less than 10.0
D. At least 10.0, but less than 12.5
E. At least 12.5

5.97 (4B, 5/96, Q.20 & Course 3 Sample Exam, Q.23) (2 points)
You are given the following:
- A loss occurrence in excess of $1 billion (in constant 1996 dollars) may be caused by a hurricane, an earthquake or a fire.
- Hurricanes, earthquakes. and fires occur independently of one another.
- The number of hurricanes causing a loss occurrence in excess of $1 billion (in constant 1996 dollars) in a one-year period follows a Poisson distribution. The expected amount of time between such hurricanes is 2.0 years.
- The number of earthquakes causing a loss occurrence in excess of $1 billion (in constant 1996 dollars) in a one-year period follows a Poisson distribution. The expected amount of time between such earthquakes is 5.0 years.
- The number of fires causing a loss occurrence in excess of $1 billion (in constant 1996 dollars) in a one-year period follows a Poisson distribution. The expected amount of time between such fires is 10.0 years.

Determine the expected amount of time between loss occurrences in excess of $1 billion (in constant 1996 dollars).
A. Less than 1.2 years
B. At least 1.2 years, but less than 1.4 years
C. At least 1.4 years, but less than 1.6 years
D. At least 1.6 years, but less than 1.8 years
E. At least 1.8 years

5.98 (3, 5/00, Q.2) (2.5 points) Lucky Tom finds coins on his way to work at a Poisson rate of 0.5 coins/minute. The denominations are randomly distributed:
(i) 60% of the coins are worth 1;
(ii) 20% of the coins are worth 5; and
(iii) 20% of the coins are worth 10.
Calculate the conditional expected value of the coins Tom found during his one-hour walk today, given that among the coins he found exactly ten were worth 5 each.
(A) 108 (B) 115 (C) 128 (D) 165 (E) 180

5.99 (3 points) In the previous question, 3, 5/00, Q.2, what is the probability that Lucky Tom finds his second coin worth 5 prior to finding his first coin worth 10?
(A) 1/6 (B) 1/5 (C) 1/4 (D) 1/3 (E) 2/5

5.100 (3, 11/00, Q.23) (2.5 points) Workers' compensation claims are reported according to a Poisson process with mean 100 per month. The number of claims reported and the claim amounts are independently distributed. 2% of the claims exceed 30,000.
Calculate the number of complete months of data that must be gathered to have at least a 90% chance of observing at least 3 claims each exceeding 30,000.
(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

5.101 (3, 11/00, Q.29) (2.5 points) Job offers for a college graduate arrive according to a Poisson process with mean 2 per month. A job offer is acceptable if the wages are at least 28,000.
Wages offered are mutually independent and follow a lognormal distribution,
$F(x) = \Phi\left[\frac{\ln(x) - \mu}{\sigma}\right]$, $x > 0$, with $\mu = 10.12$ and $\sigma = 0.12$.
Calculate the probability that it will take a college graduate more than 3 months to receive an acceptable job offer.
(A) 0.27 (B) 0.39 (C) 0.45 (D) 0.58 (E) 0.61

5.102 (3, 11/02, Q.9) (2.5 points) Lucky Tom finds coins on his way to work at a Poisson rate of 0.5 coins/minute. The denominations are randomly distributed:
(i) 60% of the coins are worth 1 each
(ii) 20% of the coins are worth 5 each
(iii) 20% of the coins are worth 10 each.
Calculate the probability that in the first ten minutes of his walk he finds at least 2 coins worth 10 each, and in the first twenty minutes finds at least 3 coins worth 10 each.
(A) 0.08 (B) 0.12 (C) 0.16 (D) 0.20 (E) 0.24

5.103 (3, 11/02, Q.20) (2.5 points) Subway trains arrive at a station at a Poisson rate of 20 per hour. 25% of the trains are express and 75% are local. The types of each train are independent.
An express gets you to work in 16 minutes and a local gets you there in 28 minutes.
You always take the first train to arrive. Your co-worker always takes the first express.
You both are waiting at the same station.
Which of the following is true?
(A) Your expected arrival time is 6 minutes earlier than your co-worker's.
(B) Your expected arrival time is 4.5 minutes earlier than your co-worker's.
(C) Your expected arrival times are the same.
(D) Your expected arrival time is 4.5 minutes later than your co-worker's.
(E) Your expected arrival time is 6 minutes later than your co-worker's.

5.104 (10 points) In the previous question 3, 11/02, Q.20:
(a) (2 points) Determine the standard deviation of the amount of time it takes you to get to work.
(b) (2 points) Determine the standard deviation of the amount of time it takes your coworker to get to work.
(c) (1 point) Determine the probability that your coworker takes the fifth train to arrive.
(d) (2 points) Conditional on the first train to show up being a local, determine the conditional distribution of its time of arrival.
(e) (3 points) Conditional on your coworker taking the fourth train to arrive, determine the probability that your coworker waited more than 30 minutes.

5.105 (CAS3, 11/03, Q.31) (2.5 points) Vehicles arrive at the Bun-and-Run drive-thru at a Poisson rate of 20 per hour. On average, 30% of these vehicles are trucks.
Calculate the probability that at least 3 trucks arrive between noon and 1:00 PM.
A. Less than 0.80
B. At least 0.80, but less than 0.85
C. At least 0.85, but less than 0.90
D. At least 0.90, but less than 0.95
E. At least 0.95

5.106 (SOA3, 11/03, Q.11) (2.5 points) Subway trains arrive at a station at a Poisson rate of 20 per hour. 25% of the trains are express and 75% are local. The type of each train is independent of the types of preceding trains. An express gets you to the stop for work in 16 minutes and a local gets you there in 28 minutes. You always take the first train to arrive. Your co-worker always takes the first express. You both are waiting at the same station.
Calculate the probability that the train you take will arrive at the stop for work before the train your co-worker takes.
(A) 0.28 (B) 0.37 (C) 0.50 (D) 0.56 (E) 0.75

5.107 (CAS3, 5/04, Q.31) (2.5 points) Coins are tossed into a fountain according to a Poisson process with a rate of one every three minutes.
The coin denominations are independently distributed as follows:

Coin Denomination	Probability
Penny	0.5
Nickel	0.2
Dime	0.2
Quarter	0.1

Calculate the probability that the fourth dime is tossed into the fountain in the first two hours.
A. Less than 0.89
B. At least 0.89, but less than 0.92
C. At least 0.92, but less than 0.95
D. At least 0.95, but less than 0.98
E. At least 0.98

5.108 (CAS3, 11/04, Q.17) (2.5 points) You are given:
- Claims are reported at a Poisson rate of 5 per year.
- The probability that a claim will settle for less than $100,000 is 0.9.

What is the probability that no claim of $100,000 or more is reported during the next 3 years?
A. 20.59% B. 22.31% C. 59.06% D. 60.65% E. 74.08%

5.109 (CAS3, 5/05, Q.7) (2.5 points) An insurance company pays claims at a Poisson rate of 2,000 per year. Claims are divided into three categories: "minor," "major," and "severe," with payment amounts of $1,000, $5,000, and $10,000, respectively. The proportion of "minor" claims is 50%. The total expected claim payments per year is $7,000,000.
What proportion of claims are "severe"?
A. Less than 11%
B. At least 11%, but less than 12%
C. At least 12%, but less than 13%
D. At least 13%, but less than 14%
E. 14% or more

5.110 (CAS3, 5/05, Q.11) (2.5 points) For Broward County, Florida, hurricane season is 24 weeks long. It is assumed that the time between hurricanes is exponentially distributed with a mean of 6 weeks. It is also assumed that 30% of all hurricanes will hit Broward County.
Calculate the probability that in any given hurricane season, Broward County will be hit by more than 1 hurricane.
A. Less than 15%
B. At least 15%, but less than 20%
C. At least 20%, but less than 25%
D. At least 25%, but less than 30%
E. 30% or more

5.111 (SOA M, 5/05, Q.5) (2.5 points)
Kings of Fredonia drink glasses of wine at a Poisson rate of 2 glasses per day.
Assassins attempt to poison the king's wine glasses. There is a 0.01 probability that any given glass is poisoned. Drinking poisoned wine is always fatal instantly and is the only cause of death. The occurrences of poison in the glasses and the number of glasses drunk are independent events.
Calculate the probability that the current king survives at least 30 days.
(A) 0.40 (B) 0.45 (C) 0.50 (D) 0.55 (E) 0.60

5.112 (SOA M, 5/05, Q.24) (2.5 points)
Subway trains arrive at your station at a Poisson rate of 20 per hour.
25% of the trains are express and 75% are local.
The types and number of trains arriving are independent.
An express gets you to work in 16 minutes and a local gets you there in 28 minutes.
You always take the first train to arrive.
Your co-worker always takes the first express.
You are both waiting at the same station.
Calculate the conditional probability that you arrive at work before your co-worker, given that a local arrives first.
(A) 37% (B) 40% (C) 43% (D) 46% (E) 49%

5.113 (SOA M, 5/05, Q.25) (2.5 points) Beginning with the first full moon in October deer are hit by cars at a Poisson rate of 20 per day. The time between when a deer is hit and when it is discovered by highway maintenance has an exponential distribution with a mean of 7 days. The number hit and the times until they are discovered are independent. Calculate the expected number of deer that will be discovered in the first 10 days following the first full moon in October.
(A) 78 (B) 82 (C) 86 (D) 90 (E) 94

5.114 (2 points) In the previous question, SOA M, 5/05, Q.25, what is the probability that more than 110 deer will be discovered in the first 10 days following the first full moon in October?
(A) 4% (B) 5% (C) 6% (D) 7% (E) 8%

5.115 (CAS3, 11/05, Q.29) (2.5 points) ABC Insurance Company estimates that the time between reported claims is exponentially distributed with mean 0.50 years. Times between claims are independent. Each time a claim is reported, a payment is made with probability 0.70. Calculate the probability that no payment will be made on claims reported during the next two years.
A. 0.06 B. 0.14 C. 0.25 D. 0.30 E. 0.50

5.116 (CAS3, 11/05, Q.31) (2.5 points) The Toronto Bay Leaves attempt shots in a hockey game according to a Poisson process with mean 30. Each shot is independent.
For each attempted shot, the probability of scoring a goal is 0.10.
Calculate the standard deviation of the number of goals scored by the Bay Leaves in a game.
A. Less than 1.4
B. At least 1.4, but less than 1.6
C. At least 1.6, but less than 1.8
D. At least 1.8, but less than 2.0
E. At least 2.0

5.117 (SOA M, 11/05, Q.8) (2.5 points) A Mars probe has two batteries. Once a battery is activated, its future lifetime is exponential with mean 1 year. The first battery is activated when the probe lands on Mars. The second battery is activated when the first fails. Battery lifetimes after activation are independent. The probe transmits data until both batteries have failed. Calculate the probability that the probe is transmitting data three years after landing.
(A) 0.05 (B) 0.10 (C) 0.15 (D) 0.20 (E) 0.25

5.118 (SOA M, 11/06, Q.9) (2.5 points) A casino has a game that makes payouts at a Poisson rate of 5 per hour and the payout amounts are 1, 2, 3,… without limit.
The probability that any given payout is equal to i is $1/ 2^i$. Payouts are independent.
Calculate the probability that there are no payouts of 1, 2, or 3 in a given 20 minute period.
(A) 0.08 (B) 0.13 (C) 0.18 (D) 0.23 (E) 0.28

5.119 (CAS3L, 5/09, Q.8) (2.5 points) Bill receives mail at a Poisson rate of 10 items per day.
The contents of the items are randomly distributed:

- 50% of the items are credit card applications.
- 30% of the items are catalogs.
- 20% of the items are letters from friends.

Bill has received 20 credit card applications in two days.
Calculate the probability that for those same two days, he receives at least 3 letters from friends and exactly 5 catalogs.
A. Less than 6%
B. At least 6%, but less than 10%
C. At least 10%, but less than 14%
D. At least 14%, but less than 18%
E. At least 18%

5.120 (CAS3L, 5/09, Q.9) (2.5 points) You are given the following information:

- Policyholder calls to a call center follow a homogenous Poisson process with λ = 250 per day.
- Policyholders may call for 3 reasons: Endorsement, Cancellation, or Payment.
- The distribution of calls is as follows:

Call Type	Percent of Calls
Endorsement	50%
Cancellation	10%
Payment	40%

Using the normal approximation with continuity correction, calculate the probability of receiving more than 156 calls in a day that are either endorsements or cancellations.
A. Less than 27%
B. At least 27%, but less than 29%
C. At least 29%, but less than 31%
D. At least 31%, but less than 33%
E. At least 33%

5.121 (CAS3L, 11/09, Q.10) (2.5 points) You are given the following information:

- An insurance company processes policy endorsements at a Poisson rate of λ = 200 per day.
- 45% of these endorsements result in a premium increase.
- Policy endorsements are observed for one day.

Using the normal approximation with no continuity correction, calculate the probability that there are more endorsements resulting in a premium increase than endorsements not resulting in a premium increase.
A. Less than 5.0%
B. At least 5.0%, but less than 7.5%
C. At least 7.5%, but less than 10.0%
D. At least 10.0%, but less than 12.5%
E. At least 12.5%

5.122 (CAS3L, 11/09, Q.11) (2.5 points) You are given the following information:
- Claims follow a compound Poisson process.
- Claims occur at the rate of $\lambda = 10$ per day.
- Claim severity follows an exponential distribution with $\theta = 15{,}000$.
- A claim is considered a large loss if its severity is greater than 50,000.

What is the probability that there are exactly 9 large losses in a 30-day period?
A. Less than 5%
B. At least 5%, but less than 7.5%
C. At least 7.5%, but less than 10%
D. At least 10%, but less than 12.5%
E. At least 12.5%

5.123 (CAS3L, 5/11, Q.10) (2.5 points)
You are given the following information about an insurance policy:
- Claim frequency follows a homogeneous Poisson process.
- The average number of claims reported each month is 10.
- Claim severities are independent and follow an exponential distribution with $\theta = 10{,}000$.
- To monitor the impact of large individual claims on the policy aggregate losses, management receives a large loss report any time a claim occurs that is greater than 30,000.

Calculate the standard deviation of the waiting time (in months) until the second large loss report.
A. Less than 1
B. At least 1, but less than 3
C. At least 3, but less than 6
D. At least 6, but less than 12
E. At least 12

5.124 (CAS3L, 5/12, Q.10) (2.5 points) You are given the following information:
- An insurance policy has aggregate losses according to a compound Poisson distribution.
- Claim frequency follows a Poisson process.
- The average number of claims reported each year is 200.
- Claim severities are independent and follow an Exponential distribution with $\theta = 160{,}000$.

Management considers any claim that exceeds 1 million to be a catastrophe.
Calculate the median waiting time (in years) until the first catastrophe claim.
A. Less than 1
B. At least 1, but less than 2
C. At least 2, but less than 3
D. At least 3, but less than 4
E. At least 4

5.125 (CAS3L, 11/12, Q.9) (2.5 points) You are given the following information:
• Claims are given by a Poisson Process with claims intensity $\lambda = 8$.
• Frequency and severity of chums are independent.
• Claim severity follows a discrete distribution that is given in the table below

Claim Amount Interval	Probability
Less than \$7,000	0.40
At least \$7,000 but Less than \$20,000	0.50
At least \$20,000	0.10

What is the probability that by time 0.6 there will be at least two claims with severity less than \$7,000?
A. Less than 0.2
B. At least 0.2, but less than 0.4
C. At least 0.4, but less than 0.6
D. At least 0.6, but less than 0.8
E. At least 0.8

5.126 (CAS3L, 11/13, Q.10) (2.5 points) You are given the following information:
• Cars arrive according to a Poisson Process at a rate of 20 cars per hour.
• 75% of cars are red and 25% of cars are blue.
• 28 red cars and 32 blue cars have arrived after three hours have passed.
Calculate the total expected number of red cars that will have arrived after eight hours have passed.
A. Less than 80
B. At least 80, but less than 100
C. At least 100, but less than 120
D. At least 120, but less than 140
E. At least 140

5.127 (CAS ST, 5/14, Q.2) (2.5 points) You are given the following information:
• Buses arrive according to a Poisson process at a rate of 5 per hour.
• Taxis arrive according to a Poisson process at a rate of 10 per hour.
• The arrival of buses and taxis are independent.
• You get a ride to work from either a bus or a taxi, whichever arrives first.
Calculate the probability you will have to wait more than 10 minutes for a ride to work.
A. Less than 0.05
B. At least 0.05, but less than 0.10
C. At least 0.10, but less than 0.15
D. At least 0.15 but less than 0.20
E. At least 0.20

5.128 (CAS ST, 11/14, Q.1) (2.5 points)
You are trying to get downtown at rush hour and will take the first vehicle to arrive. You are given:

- Taxis arrivals follow the Poisson process with rate λ = 1 per 10 minutes.
- Bus arrivals follow the Poisson process with rate λ = 4 per 30 minutes.
- Streetcar arrivals follow the Poisson process with rate λ = 2 per hour.

Let W denote your waiting time, in minutes.
Calculate the variance of W.
A. Less than 5
B. At least 5, but less than 10
C. At least 10, but less than 15
D. At least 15, but less than 20
E. At least 20

5.129 (CAS ST, 5/15, Q.2) (2.5 points) You arrive at a bus station at exactly 8:30 am and you have the option of either taking Line 1 or Line 2 to bring you to your destination.
Buses along Line 1 and Line 2 arrive independently, both according to a Poisson process.
On the average, one Line 1 bus arrives every 15 minutes and one Line 2 bus arrives every 10 minutes.
If you board Line 1, it will take you 8 minutes to reach your destination. If you board Line 2, it will take you 20 minutes to reach your destination.
You decide to take the first bus that arrives.
Calculate the expected length of time, to the nearest minute, that it will take you to reach your destination.
A. Less than 22 minutes
B. At least 22 minutes, but less than 24 minutes
C. At least 24 minutes, but less than 26 minutes
D. At least 26 minutes, but less than 28 minutes
E. At least 28 minutes

5.130 (CAS S, 11/15, Q.3) (2.2 points) You are given the following information:

- Lucy finds coins at a Poisson rate of 1 coin per 10 minutes.
- The denominations are randomly distributed as follows:
    - 65% of the coins are worth 1 each;
    - 20% of the coins are worth 5 each;
    - 15% of the coins are worth 10 each.

Calculate the probability that in the first 30 minutes she finds at least 1 coin worth 10 each and in the first hour finds at least 2 coins worth 10 each.
A. Less than 0.165
B. At least 0.165, but less than 0.175
C. At least 0.175, but less than 0.185
D. At least 0.185, but less than 0.195
E. At least 0.195

5.131 (CAS S, 5/16, Q.2) (2.2 points) You are given the following information:
- Vehicles arrive at the drive-through bank at a Poisson rate of 30 per hour.
- On average, 10% of these vehicles require cash transactions.

Calculate the probability that at least 3 vehicles require cash transactions between 10 AM and noon.
A. Less than 0.915
B. At least 0.915, but less than 0.925
C. At least 0.925, but less than 0.935
D. At least 0.935, but less than 0.945
E. At least 0.945

5.132 (CAS S, 5/16, Q.4) (2.2 points) You are given the following information about the rate at which calls come into a customer service center:
- The customer service center is open 24 hours a day.
- Calls are received from four regions.
- The rate at which calls arrive is 10 calls per hour and follows a Poisson distribution.
- The probability that a call will be from a given region is independent of other calls and is shown in the table below.

Region	Probability
1	0.2
2	0.3
3	0.1
4	0.4

Calculate the probability that in a one-hour time period at least one call will have been received from each region.
A. Less than 0.25
B. At least 0.25, but less than 0.30
C. At least 0.30, but less than 0.35
D. At least 0.35, but less than 0.40
E. At least 0.40

5.133 (CAS S, 11/16, Q.2) (2.2 points)
For an auto policy, claims occur according to a Poisson process with rate $\lambda = 10$ per day.
Once reported, claims are independently classified as Property Damage (PD) only, Bodily Injury (BI) only, or both PD and BI.
You are given that 30 percent of claims are PD only, 20 percent are BI only, and the remaining 50 percent are both PD and BI.
Calculate the probability that in a given day there will be exactly 4 PD only claims, 2 BI only claims, and 6 both PD and BI claims.
A. Less than 0.63 percent
B. At least 0.63 percent, but less than 0.68 percent
C. At least 0.68 percent, but less than 0.73 percent
D. At least 0.73 percent, but less than 0.78 percent
E. At least 0.78 percent

5.134 (CAS S, 5/17, Q.2) (2.2 points)
You are given the following information about a senior-year college student from an actuarial science program:
- Job offers arrive according to a Poisson process with a rate of three per month.
- A job offer is acceptable if the salary offered is at least 50,000 per year.
- The salaries offered are mutually independent and follow a lognormal distribution with $\mu = 10.5$ and $\sigma = 0.2$.

Calculate the probability that it will take this senior-year college student more than four months to receive an acceptable job offer.
A. Less than 0.505
B. At least 0.505, but less than 0.515
C. At least 0.515, but less than 0.525
D. At least 0.525, but less than 0.535
E. At least 0.535

5.135 (CAS S, 11/17, Q.2) (2.2 points) You are given the following information:
- A reinsurer classifies three types of events as catastrophes: hurricanes, earthquakes and wildfires.
- Each type of catastrophe follows a homogeneous Poisson process with the following rates of occurrence:

Hurricanes:	5 per year
Earthquakes:	1 per 5 years
Wildfires:	1 per year

Calculate the probability that 2 or more catastrophes occur in a six-month period.
A. Less than 0.80
B. At least 0.80, but less than 0.81
C. At least 0.81, but less than 0.82
D. At least 0.82, but less than 0.83
E. At least 0.83

5.136 (MAS-1, 11/19, Q.2)
You are given the following information about waiting times at a subway station:
- Subway trains arrive at a Poisson rate of 20 per hour
- 30% of the trains are Express and 70% are Local
- The arrival times of each train are independent
- An Express train gets you to work in 18 minutes, and a Local train gets you there in 30 minutes
- You always take the first train to arrive and you get to the office in X_1 minutes from the time you arrive at the subway station
- Your coworker always takes the first Express train to arrive and he gets to the office in X_2 minutes from the time he arrives at the subway station

Calculate the expected value of $X_1 - X_2$.

A. Less than -2.0
B. At least -2.0, but less than -1.0
C. At least -1.0, but less than 0.0
D. At least 0.0, but less than 1.0
E. At least 1.0

Solutions to Problems:

5.1. D. For the Weibull, $S(x) = \exp[-(x/\theta)^\tau]$. $S(3000) = \exp[-3^{0.7}] = 0.1156$.
Large claims are a Poisson Process with mean (0.1156)(25) = 2.89 per month.
Over m months, the number of large claims is Poisson distributed with mean 2.89m.
We require that Prob(# large claims < 3) ≤ 0.5%.
Thus we require that: $e^{-2.89m} + 2.89me^{-2.89m} + (2.89m)^2e^{-2.89m}/2 \le 0.005$.
Plugging in m = 1, 2, etc., this inequality is first satisfied when m = **4**.

# months	Exp[-2.89m] (1 + 2.89m + 0.5(2.89m)^2)
1	0.4483
2	0.0725
3	0.0081
4	0.0008
5	0.0001

Comment: Similar to 3, 11/00, Q.23.

5.2. D. Expected number of coins is: (60)(0.2) = 12.
Mean value of a coin is: (50%)(5) + (30%)(10) + (20%)(25) = 10.5.
Expected value of coins lost on a one-hour walk is: (12)(10.5) = **126**.

5.3. E. The losing of the three different types of coins are independent Poisson processes.
Over the course of 60 minutes, Klem expects to lose (0.3)(0.2)(60) = 3.6 coins worth 10 each and (0.2)(0.2)(60) = 2.4 coins worth 25 each. If Klem loses 3 coins worth 5 each, then the expected worth of the coins he loses is: (3)(5) + (3.6)(10) + (2.4)(25) = **111**.
Comment: Similar to 3, 5/00, Q.2**.** A priori, Klem expects to lose on average (0.2)(60) = 12 coins per hour, worth on average: (6)(5) + (3.6)(10) + (2.4)(25) = 126.

5.4. B. The losing of the three different types of coins are independent Poisson processes.
Over the course of 60 minutes, Klem expects to lose (0.3)(0.2)(60) = 3.6 coins worth 10 each.
This Compound Poisson process has variance: $(3.6)(10^2) = 360$.
Over the course of 60 minutes, Klem expects to lose (0.2)(0.2)(60) = 2.4 coins worth 25 each.
This Compound Poisson process has variance: $(2.4)(25^2) = 1500$.
If we know Klem loses 3 coins worth 5 each, then this process contributes nothing to the variance. The conditional variance of the value of the coins Klem loses: 360 + 1500 = **1860**.

5.5. B. Coins worth 10 are a Poisson Process with λ = (0.3)(0.2) = 0.06 per minute, and this process is independent of the Poisson Process of coins worth 5.
Start a new Poisson Process, whenever Klem loses his first coin worth 5.
Then the time until the next coin worth 10 is lost is Exponential with mean: 1/0.06.
The probability this time is less than or equal to 7 is: $1 - e^{-(7)(0.06)}$ = **34.3%**.
Comment: I have ignored the extremely small possibility that Klem fails to lose at least one coin worth 5 on his four hour walk.
Prob[Klem does not lose at least one nickel] = exp[(240)(0.1)] = exp[-24] = 2.6×10^{-10}.
Prob[Klem loses at least one nickel] ×
Prob[Klem loses a dime within 7 minutes of losing the first nickel | Klem lost at least nickel] = {1 - exp[-24]} (0.343) = 0.343.

5.6. E. The nickels, dimes, and quarters are three independent Poisson processes.
The losing of nickels is a Poisson process with mean: (12)(50%) = 6.
The losing of quarters is a Poisson process with mean: (12)(20%) = 2.4.
The losing of dimes is a Poisson process with mean: (12)(30%) = 3.6.
Let m = mean number of dimes given at least 3 dimes.
We know that: 0 f(0) + 1 f(1) + 2 f(2) + m{1 - f(0) - f(1) - f(2)} = 3.6.
Therefore, $m = (3.6 - 3.6\ e^{-3.6} - 2\ 3.6^2\ e^{-3.6}/2)/(1 - e^{-3.6} - 3.6\ e^{-3.6} - 3.6^2\ e^{-3.6}/2)$
$= (3.6e^{3.6} - 3.6 - 12.96)/(e^{3.6} - 1 - 3.6 - 6.48) = 4.514$.
Therefore, the conditional expected amount of money he loses during a one-hour walk is:
(6)(5) + (2.4)(25) + (4.514)(10) = **135.1**.

5.7. E. Prob[at least 1 taxicab @1, first 1/2 hour] = $1 - e^{-(5)(0.6)} = 1 - e^{-3}$ = 0.950.
Prob[at least 1 taxicab @2, first 1/2 hour] = $1 - e^{-(5)(0.3)} = 1 - e^{-1.5}$ = 0.777.
Prob[at least 1 taxicab @3, first 1/2 hour] = $1 - e^{-(5)(0.1)} = 1 - e^{-0.5}$ = 0.393.
Since the Poisson Processes are independent, the probability that at least one taxicab of each type has left during the first half hour is: (0.950)(0.777)(0.393) = **29.0%**.

5.8. B. The probability of at least one taxicab of each type by time t is:
$(1 - e^{-6t})(1 - e^{-3t})(1 - e^{-t}) = 1 - e^{-6} - e^{-3t} - e^{-t} + e^{-9t} + e^{-7t} + e^{-4t} - e^{-10t}$.

$$\text{mean time} = \int_0^\infty S(t)\ dt = \int_0^\infty e^{-6t} + e^{-3t} + e^{-t} - e^{-9t} - e^{-7t} - e^{-4t} + e^{-10t}\ dt =$$

1/6 + 1/3 + 1/1 - 1/9 - 1/7 - 1/4 + 1/10 = 1.096 hours = **65.8 minutes**.

5.9. E. Severity is a Single Parameter Pareto with $S(x) = 1 - F(x) = x^{-3.5}$. S(3) = 0.0214.
Such large claims follow a Poisson Process with intensity: (.0214)(5) = 0.107.
The average time until the first such claim is: 1/0.107 = 9.35.
The average time until the 20th such claim is: (20)(9.35) = **187**.

5.10. C. The waiting time until the first large claim is Exponential with mean 9.35.
The time until the 20th large claim is a sum of 20 independent, identically distributed Exponentials each with $\theta = 9.35$. ⇔ a Gamma Distribution with $\alpha = 20$ and $\theta = 9.35$.
It has mean: (20)(9.35) = 187, and variance $(20)(9.35^2) = 1748.5$.
Prob[20th large event time ≤ 200] $\cong \Phi[(200 - 187)/\sqrt{1748.5}\,] = \Phi(0.31) =$ **62.2%**.
Alternately, the number of large claims by time 200 is Poisson with mean: (0.107)(200) = 21.4.
Prob[# large claims ≥ 20] $\cong 1 - \Phi[(19.5 - 21.4)/\sqrt{21.4}\,] = 1 - \Phi(-0.41) =$ **65.9%**.
Comment: The exact answer is: $\Gamma[20{:}\ 200/9.35] = \Gamma[20;\ 21.4] = 0.647$.
The two approximations give different numerical answers, but the same letter solution.

5.11. B. Someone who contacted HIV at time t, has a probability of showing symptoms at time 20 of 1 - exp[-(20-t)/8]. Expected number with symptoms:

$$\lambda \int_0^{20} (1 - \exp[-(20-t)/8])\ dt = \lambda(20 + 8e^{-20/8} - 8) = \lambda(12 + 8e^{-20/8}).$$

Expected number without symptoms:

$$\lambda \int_0^{20} \exp[-(20-t)/8]\ dt = \lambda 8(1 - e^{-20/8}).$$

Set $3410 = \lambda(12 + 8e^{-20/8})$. $\Rightarrow \lambda = 3410/(12 + 8e^{-20/8})$.
Expected number without symptoms: $8(1 - e^{-20/8})3410/(12 + 8e^{-20/8}) =$ **1978**.
Comment: Similar to Example 5.20 in Introduction to Probability Models by Ross.

5.12. C. 1/5 of the time a customer spends more than 4000. Such customers are a Poisson Process with $\lambda = 0.0125/5 = 0.0025$. The probability of 0 such customers over 1000 minutes is: $e^{-(1000)(.0025)} = e^{-2.5} =$ **8.2%**.

5.13. D. The probability that a claim that occurred at time t, will be reported by time 1.25 is: $1 - e^{-(1.25-t)/0.6}$. Claims occur from t = 0 to t = 1, at an intensity λ.
The number of expected number of claims reported by time 1.25 is:

$$\int_0^1 \lambda\ (1 - e^{-(1.25-t)/0.6})\ dt = \lambda\ \{t - (0.6)e^{-(1.25-t)/0.6}\}\Big]_{t=0}^{t=1} = \lambda\{1 - (0.6)(e^{-0.4166} - e^{-2.0833})\} =$$

$\lambda\{1 - (0.6)(0.6593 - 0.1245)\} = 0.6791\lambda$.
Setting the observed number of reported claims equal to the expected number:
$480 = 0.6791\ \lambda$. $\Rightarrow \lambda = 480/0.6791 =$ **707**.
Comment: A simple example of Claim Count Development, as discussed on the syllabus of CAS Exam on Basic Reserving. We have estimated that there are 707 - 480 = 227 claims that have occurred during this year, but have yet to be reported.

5.14. E. Since the interevent times are Exponential, the number of hurricanes is Poisson.
The number of hurricanes over the season is Poisson with mean: 24/6 = 4.
The number of hurricanes over the season that hit Broward County is Poisson with mean: (30%)(4) = 1.2.
The number of hurricanes over the season that do not hit Broward County is Poisson with mean: (70%)(4) = 2.8. The two Poisson processes are independent.
Prob[two not hitting Broward] Prob[one hitting Broward] = $(2.8^2 e^{-2.8}/2)1.2e^{-1.2}$ = **8.62%**.

5.15. B. Families of size 5 are a Poisson Process, with λ = (0.12)(0.8) = 0.096.
Mean time until next family of size 5 is: 1/0.096 = **10.4 days**.

5.16. E. For n days, the mean number of families of size 8 or more is:
n(0.8)(4% + 2% + 1%) = 0.056n. Prob[no such families] = $e^{-0.056n} \le 0.05$. n ≥ **53.5**.

5.17. B. The number of families of size 3 or less is independent of the number of families of size 4. Expected number of families of size 3 or less is: (0.1 + 0.15 + 0.22)(0.8)(365) = **137**.

5.18. D. The number of families of size 1 over eighty days is Poisson with mean (0.1)(100)(0.8) = 8. The probability of at least one family of size 1 is: $1 - e^{-(80)(0.1)} = 1 - e^{-8}$.
The different size families are independent Poisson Processes.
The probability of at least one family of each size is:
$(1 - e^{-8})(1 - e^{-12})(1 - e^{-17.6})(1 - e^{-16})(1 - e^{-9.6})(1 - e^{-6.4})(1 - e^{-4.8})(1 - e^{-3.2})(1 - e^{-1.6})(1 - e^{-0.8})$
= **0.417**.
Comment: Only about the last 4 terms are significantly less than 1.

5.19. C. Families of size 8 are a Poisson Process with mean over 80 days of: (0.04)(80) = 3.2.
Probability of at least three family of size 8 is: $1 - e^{-3.2} - 3.2e^{-3.2} - 3.2^2 e^{-3.2}/2$ = .620.
Probability of at least two family of size 9 is: $1 - e^{-1.6} - 1.6e^{-1.6}$ = .475.
Probability of at least one family of size 10 is: $1 - e^{-0.8}$ = .551.
The different size families are independent Poisson Processes.
Therefore, the desired probability is: (0.620)(0.475)(0.551) = **0.162**.

5.20. A. $f(8) = 10^8 e^{-10}/8!$ = **11.3%**.

5.21. A. $f(8) = 3^8 e^{-3}/8!$ = **0.8%**.

5.22. E. The sum of the two independent processes is another Poisson Process with λ = 10 + 3 = 13. $f(8) = 13^8 e^{-13}/8!$ = **4.6%**.

5.23. E. $S(1000) = e^{-1000/400}$ = 8.21%. Non-zero payments due to theft losses are a Poisson Process with λ = (8.21%)(10) = 0.821. $f(3) = 0.821^3 e^{-0.821}/3!$ = **4.1%**.

5.24. D. $S(1000) = (4000/(4000 + 1000))^3$ = 51.2%. Non-zero payments due to fire losses are a Poisson Process with λ = (51.2%)(3) = 1.536. $f(3) = 1.536^3 e^{-1.536}/3!$ = **13.0%**.

5.25. B. The sum of the two independent processes is another Poisson Process with
$\lambda = 0.821 + 1.536 = 2.357$. $f(3) = 2.357^3 e^{-2.357}/3! =$ **20.7%**.

5.26. A. Coins of worth 25 are lost via a Poisson Process with mean: (20%)(0.2) = 0.04 per minute. The number of coins worth 25 over the first 30 minutes is Poisson with mean 1.2.
Prob[at least 3 worth 25 in the first 30 minutes] = $1 - (e^{-1.2} + 1.2e^{-1.2} + (1.2^2/2)\, e^{-1.2})$ = **0.1205**.
Comment: Similar to CAS3, 5/04, Q.31.

5.27. A. Coins worth 5 and 10 are independent Poisson Processes. Coins of worth 10 are lost via a Poisson Process with mean: (30%)(0.2) = 0.06 per minute. The number of coins worth 10 over the first 10 minutes is Poisson with mean 0.6. Coins of worth 5 are lost via a Poisson Process with mean: (50%)(0.2) = 0.1 per minute. The number of coins worth 5 over the first 20 minutes is Poisson with mean 2.
Prob[2 worth 10 in first 10 minutes] Prob[3 worth 5 in the first 20 minutes] =
$(0.6^2 e^{-0.6}/2)(2^3 e^{-2}/6) = (0.0988)(0.1804) =$ **0.0178**.

5.28. C. Coins of worth 10 are lost via a Poisson Process with mean: (30%)(0.2) =
0.06 per minute. The number of coins over the first 10 minutes is Poisson with mean 0.6.
The number of coins over the next 5 minutes is Poisson with mean 0.3. What happens over the first 10 minutes is independent of what happens over the next 5 minutes.
Prob[2 in first 10 minutes and 3 in first 15 minutes] =
Prob[2 in first 10 minutes] Prob[1 in next 5 minutes] = $(0.6^2 e^{-0.6}/2)(0.3e^{-0.3}) = (0.0988)(0.2222) =$
0.0220.

5.29. B. Coins of worth 5 are lost via a Poisson Process with mean:
(50%)(0.2) = 0.1 per minute. What happens in the first 10 minutes is independent of what happens during the next twenty minutes.
Prob[at least 2 in first 10 minutes and at least 4 in next 20 minutes] =
Prob[at least 2 in first 10 minutes]Prob[at least 4 in next 20 minutes] =
$(1 - e^{-1} - e^{-1})\,(1 - e^{-2} - 2e^{-2} - 2^2 e^{-2}/2 - 2^3 e^{-2}/6) = (0.2642)(0.1429) =$ **0.0378**.

5.30. A. Coins of worth 10 are lost via a Poisson Process with mean: (30%)(0.2) =
0.06 per minute or 0.6 per 10 minutes. The number of such coins over the first 10 minutes is Poisson with mean 0.6. The number of such coins over the next 20 minutes is Poisson with mean 1.2. The number of coins over the first 10 minutes is independent of the number of coins over the next 20 minutes.
Prob[at least 2 in first 10 minutes and at least 4 in first 30 minutes] =
Prob[2 in first 10 minutes] Prob[at least 2 in next 20 minutes] +
Prob[3 in first 10 minutes] Prob[at least 1 in next 20 minutes] +
Prob[at least 4 in the first 10 minutes] =
$(0.6^2 e^{-0.6}/2)(1 - e^{-1.2} - 1.2e^{-1.2}) + (0.6^3 e^{-0.6}/6)(1 - e^{-1.2})$
$+ (1 - e^{-0.6} - 0.6e^{-0.6} - 0.6^2 e^{-0.6}/2 - 0.6^3 e^{-.6}/6)$
$= (0.0988)(0.337) + (0.0198)(0.699) + 0.0034 =$ **0.0505**.
Comment: Similar to 3, 11/02, Q.9.

5.31. D. Coins of worth 5 are lost via a Poisson Process with mean:
(50%)(0.2) = 0.1 per minute.
Prob[at least 2 in first 10 minutes and at least 5 in first 25 minutes] =
Prob[2 in first 10 minutes] Prob[at least 3 in next 15 minutes] +
Prob[3 in first 10 minutes] Prob[at least 2 in next 15 minutes] +
Prob[4 in first 10 minutes] Prob[at least 1 in next 15 minutes] +
Prob[at least 5 in the first 10 minutes] =
$(1^2e^{-1}/2)(1 - e^{-1.5} - 1.5e^{-1.5} - 1.5^2e^{-1.5}/2) + (1^3e^{-1}/6)(1 - e^{-1.5} - 1.5e^{-1.5}) + (1^4e^{-1}/24)(1 - e^{-1.5})$
$+ (1 - e^{-1} - 1e^{-1} - 1^2e^{-1}/2 - 1^3e^{-1}/6 - 1^4e^{-1}/24) =$
(0.1839)(0.1912) + (0.0613)(0.4422) + (0.0153)(0.7769) + 0.0037 = **0.0778**.

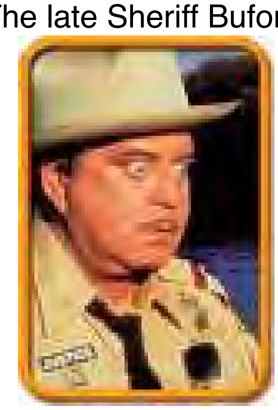
5.32. C. What happens during the first ten and last ten minutes are independent, since the two time periods are disjoint. A is Poisson with mean 2. B is Poisson with mean 2.
A + B is Poisson with mean 4. Prob[A + B = 5] = $e^{-4}4^5/5!$ = **15.6%.**

5.33. A. What happens during the first ten and next ten minutes are independent, since the two time periods are disjoint. C is Poisson with mean 2. D - C is Poisson with mean 2.
Prob[C + D = 4] = Prob[C = 0 and D = 4] + Prob[C = 1 and D = 3] + Prob[C = 2 and D = 2] =
Prob[C = 0]Prob[D - C = 4] + Prob[C = 1]Prob[D - C = 2] + Prob[C = 2]Prob[D - C = 0] =
$(e^{-2})(e^{-2}2^4/4!) + (2e^{-2})(e^{-2}2^2/2!) + (e^{-2}2^2/2!)(e^{-2})$ = **12.2%**.

5.34. D. For this Single Parameter Pareto Distribution, $S(x) = 1 - F(x) = (\theta/x)^\alpha = 0.1^{1.2}/x^{1.2}$.
Take the current time as 0.
If a truck arrived at time t, then the probability that it is still at the truckstop is: 1 - F(-t) = S(-t).
The mean number of trucks that have been there for more than 2 hours is:

$$\int_{-\infty}^{-2} 15\, S(-t)\, dt = \int_2^{\infty} 15\, S(t)\, dt = \int_2^{\infty} \frac{0.1^{1.2}}{t^{1.2}}\, dt = (15)\, \frac{0.1^{1.2}}{(0.2)\,(2^{0.2})} = (15)(0.2746) = 4.12.$$

The number of trucks that have been there for more than 2 hours is Poisson with mean 4.12.
Probability of more than 3 such trucks is: $1 - e^{-4.12}(1 + 4.12 + 4.12^2/2 + 4.12^3/6)$ = **59.0%**.
Comment: The late Sheriff Buford T. Justice:



5.35. C. The taxicabs with 1 person, 2 people, and 3 people are three independent Poisson Processes. Therefore, knowing how many cabs left with 1 or 3 people does not tell you anything about the number with 2 people.
Over an hour we expect (10)(0.3) = 3 cabs with 2 people.
Expected total number of people is: (8)(1) + (3)(2) + (0)(3) = **14**.

5.36. C. We thin the overall lambda of 10, by multiplying by 60%, 30% and 10%.
Thus the taxicabs with 3 people are Poisson with lambda = (10%)(10) = 1.
The taxicabs with 2 people are Poisson with lambda = (30%)(10) = 3.
The taxicabs with 1 person are Poisson with lambda = (60%)(10) = 6.
The taxicabs with 1 person, 2 people, and 3 people are three independent Poisson Processes, with respectively λ = 6, 3, and 1.
Now we can use each Poisson to calculate probabilities for the first hour.
Prob[1 taxicab with 3 people] = $f(1 \mid \lambda = 1) = e^{-1} = 0.368$.
Prob[3 taxicabs with 2 people] = $f(3 \mid \lambda = 3) = 3^3 e^{-3}/3! = 0.224$.
Prob[6 taxicabs 1 person] = $f(6 \mid \lambda = 6) = 6^6 e^{-6}/6! = 0.161$.
Since the Poisson Processes are independent, we can multiply these probabilities.
Prob[1 taxicab @3, 3 taxicabs @2, and 6 taxicabs @1] = (0.368)(0.224)(0.161) = **1.3%**.
Comment: The question would have been the same if instead:
Taxicabs arrive at a hotel at a Poisson rate λ = 10 per hour.
The color of cabs has the following probabilities:

Color of Cabs	Probability
Yellow	0.60
Black	0.30
Blue	0.10

What is the probability that during the next hour exactly 1 blue taxicab arrives, exactly 3 black taxicabs arrive, and exactly 6 yellow taxicabs arrive?

5.37. C. Prob[1st cab has two people] = 30%.
Taxicabs with 2 people are Poisson with rate: (0.3)(10) = 3 per hour = 0.05 per minute.
Prob[1st cab leaves within 10 minutes | cab has two people] = 1 - exp[-(7)(0.05)] = 29.5%.
Prob[1st cab has two people and leaves within 10 minutes] = (30%)(29.5%) = **8.9%**.

5.38. C. A claim that occurred at time $t \le 5$, has a probability of being reported by time 5 of: $F(5 - t) = 1 - \exp[-(5-t)/3]$. Let $s = 5 - t$.
Then the expected number of reported claims:

$$\lambda \int_0^5 (1 - \exp[-(5-t)/3])\ dt = \lambda \int_5^0 (1 - \exp[-s/3])\ (-ds) = \lambda \int_0^5 (1 - \exp[-s/3])\ ds$$

$= \lambda(5 + 3e^{-5/3} - 3) = \lambda(2 + 3e^{-5/3})$.

Expected number of unreported claims: $\lambda \int_0^5 \exp[-(5-t)/3]\ dt = \lambda 3(1 - e^{-5/3})$.

Set $402 = \lambda(2 + 3e^{-5/3})$. $\Rightarrow \lambda = 402/(2 + 3e^{-5/3})$.
Expected number of unreported claims: $3(1 - e^{-5/3})402/(2 + 3e^{-5/3})$ = **381**.

5.39. B. The number of large and small losses reported by time 9 are independent Poisson Processes. Therefore, the number of large losses observed does not affect the expected number of small losses.
At time t, losses are uniform 0 to $10{,}000\ 1.1^t$.
Prob[size $\le$ 6000 at time t] = $6000/(10{,}000\ 1.1^t) = (0.6)(1.1^{-t})$.
Expected number of small losses by time 9 is:

$$\int_0^9 200(0.6)(1.1^{-t})\ dt = \{-120/\ln(1.1)\}\ 1.1^{-t} \Big]_{t=0}^{t=9} = \mathbf{725.1}.$$

5.40. C. If a flaw occurs at time r, then the probability it is still undetected at time 50 is $S(50 - r) = \{80/(80 + 50 - r)\}^5 = \{80/(130 - r)\}^5$. Mean number of undetected flaws is:

$$\int_0^{50} (0.03)\ S(50 - r)\ dr = (0.03)80^5 \int_0^{50} (130 - r)^{-5}\ dr = (0.03)80^5(1/4)(1/80^4 - 1/130^4) = 0.514.$$

The number of undetected flaws is Poisson with mean 0.514.
The probability of zero undetected flaws is: $e^{-0.514}$ = **59.8%**.
Comment: Similar to Exercise 5.61 in Introduction to Probability Models by Ross.

5.41. E. Acceptable offers arrive at a Poisson rate of 2S(x) per week.
Prob[no acceptable offer in 10 weeks] = $\exp[-20S(x)] = 0.05$.
$\Rightarrow S(x) = \ln(0.05)/(-20) = 0.1498$. $\Rightarrow (10000/x)^3 = 0.1498$. $\Rightarrow x =$ **18,830**.

5.42. C. Acceptable offers arrive at a Poisson rate of 2S(x) per week.
The expected time to sell the painting is: 1/{2S(x)},
with expected cost of 200/{2S(x)} = 100/S(x).

$$E[X \mid X > x] = \int_x^\infty t\, f(t)\, dt \ / S(x) = \int_x^\infty t\, 3\, (10000^3)/t^3\, dt \ / (10000/x)^3 = (1.5x^{-2})\, x^3 = 3x/2.$$

Expected net revenue is: $E[X \mid X > x] - 100/S(x) = 3x/2 - x^3/10{,}000{,}000{,}000$.
Set the derivative with respect to x equal to zero: $3/2 - 3x^2/10^{10} = 0. \Rightarrow x =$ **70,711**.
More generally, expected net revenue is: $E[X \mid X > x] - 100/S(x) =$

$$\int_x^\infty t\, f(t)\, dt \ / S(x) - 100/S(x) = \{\int_x^\infty t\, f(t)\, dt \ - 100\} / S(x).$$

Set the derivative with respect to x equal to zero:

$$0 = (-f(x)\, x\, S(x) + f(x)\{\int_x^\infty t\, f(t)\, dt\, t \ - 100\}) / S(x)^2.$$

$$\Rightarrow x\, S(x) = \int_x^\infty t\, f(t)\, dt \ - 100. \Rightarrow \int_x^\infty t\, f(t)\, dt - \int_x^\infty x\, f(t)\, dt = 100. \Rightarrow E[(X - x)_+] = 100.$$

For the Single Parameter Pareto Distribution with $\alpha = 3$ and $\theta = 10{,}000$:

$$E[(X - x)_+] = \int_x^\infty S(x)\, dx = \int_x^\infty (10000/x)^3\, dx = 5 \times 10^{11} / x^2.$$

Therefore, we want: $5 \times 10^{11} / x^2 = 100. \Rightarrow x =$ **70,711**.
<u>Comment</u>: Difficult. Similar to Example 5.15 in Ross.
The expected time to sell the painting is: 1/{2S(70,711)} = 177 weeks.
If the expenses per week had been larger relative to the selling price, then x would have been smaller, as would be the expected time to sell the painting.

5.43. D. Let t be the time of a claim occurring. Then the time period available for the claim to be reported is 24 - t, and the probability that the claim will be reported in the first 24 months is:
$1 - e^{-(24 - t)/20} = 1 - 0.3012e^{t/20}$.
The expected number of claims reported in the first 24 months is:

$$\int_0^{24} (10)(1 - 0.3012e^{t/20})\, dt = (10)(24) - (3.012)20e^{t/20} \Big]_{t=0}^{t=24} = 240 - 60.24(e^{24/20} - 1) = \mathbf{100.2}.$$

<u>Comment</u>: Similar to SOA M, 5/05, Q.25.

5.44. B. From the previous solution, the expected number of claims reported by time 24 months is 100.2.
The number of claims reported is Poisson, with mean 100.2 and therefore variance 100.2.
Prob[less than 90 claims] ≅ $\Phi[(89.5 - 100.2)/\sqrt{100.2}\,] = \Phi[-1.07]$ = **14.2%**.

5.45. D. If an express train shows up first, a 60% chance, then they take the same train and arrive at work at the same time. If a local train shows up first, a 40% chance, then Fiona takes this local and Ezra waits for the next express. Fiona arrives at work in 35 minutes. If the next express shows up within 35 - 20 = 15 minutes, then Ezra arrives before or the same time Fiona does. Otherwise Ezra shows up after Fiona.
Express trains are Poisson with λ = (60%)(15) = 9 per hour. Number of express trains that show up within 15 minutes is Poisson with mean: (9)(15/60) = 2.25. Prob[Fiona arrives first] = Prob[local shows up first] Prob[No express shows up within 15 minutes of this local] = $0.40e^{-2.25}$ = **4.2%**.
Comment: Similar to SOA3, 11/03, Q.11.

5.46. C. Ezra is expected to wait 60/9 = 6.67 minutes. His trip will take 20 minutes.
Expected value of E is: 6.67 + 20 = 26.67.
Fiona is expected to wait 60/15 = 4 minutes. Her trip will take on average:
(60%)(20) + (40%)(35) = 26 minutes. Expected value of F is: 4 + 26 = 30.
The expected value of F - E is: 30 - 26.67 = **3.33 minutes**.
Comment: Similar to 3, 11/02, Q.20.

5.47. A. E is the sum of 20 plus a waiting time that is Exponential with mean 6.67.
Var[E] = Var[Exponential with θ = 6.67] = 6.67^2. Standard deviation of E is: **6.67 minutes**.

5.48. D. F is the sum of two independent variables, the time of the trip plus a waiting time.
The waiting time is Exponential with mean 4 and variance of: 4^2 = 16.
The time of Fiona's trip has a second moment: $(60\%)(20^2) + (40\%)(35^2) = 730$.
The time of Fiona's trip has a variance: $730 - 26^2 = 54$.
Var[F] = 16 + 54 = 70. Standard deviation of F is: $\sqrt{70}$ = **8.37 minutes**.

5.49. E. Let t be the time of a claim occurring. Then the time period in which the claim can be reported is: 1 - t. If t ≤ 0.75 then the claim has been reported by the end of the year.
If t > 0.75, then the probability that the claim is reported by the end of the year is: (1 - t)/0.25.
The expected number of claims reported by the end of year is:

$$(50)(0.75) + \int_{0.75}^{1} (50)(4)(1 - t)\, dt = 37.5 + (100)(0.25)^2 = 43.75.$$

The number of claims reported by the end of year is Poisson, with mean 43.75 and therefore variance 43.75.
Prob[less than 40 claims] ≅ $\Phi[(39.5 - 43.75)/\sqrt{43.75}\,] = \Phi[-0.64]$ = **26.1%**.

5.50. B. During the year 2000, losses are Pareto with $\alpha = 2.5$ and
$\theta = (1.05^2)(400 \text{ million}) = 441$ million. S(250 million) = $\{441/(441 + 250)\}^{2.5} = 0.325$.
Alternately, deflating back to 1998, S(250 million) = $\{400/(400 + 250/1.05^2)\}^{2.5} = 0.325$.
In 2000, $\lambda = (0.325)(0.82) = 0.267$.
Similarly, in the year 2001, $\lambda = \{400/(400 + 250/1.05^3)\}^{2.5}(0.82) = 0.279$.
Over the first 7 years: $\lambda = 0.267 + 0.279 + 0.291 + 0.303 + 0.315 + 0.327 + 0.339 = 2.121$.
$e^{-2.121} = 12.0\% > 10\%$. Over the first **8** years:
$\lambda = 0.267 + 0.279 + 0.291 + 0.303 + 0.315 + 0.327 + 0.339 + 0.352 = 2.473$.
$e^{-2.473} = 8.4\% \le 10\%$.

5.51. C. F = 7:55 + 25 minutes = 8:20. Express trains are Poisson at a rate of: (10)(2/3) = 20/3.
The average wait for an express is: 3/20 hours = 9 minutes.
We start a new Poisson process at 7:55.
(The Poisson Process is memoryless and Ethel is waiting for the next express to leave after 7:55.)
Expected value of E is: 7:55 + 9 minutes + 13 minutes = 8:17.
Expected value of F - E is: 8:20 - 8:17 = **3** minutes.
Comment: When Ethel arrives at 7:48 her expected time of arrival is:
7:48 + 9 minutes + 13 minutes = 8:10. However, when Fred arrives at 7:51, Ethel is still waiting, since no express train has come. At 7:51, Ethel's expected arrival time is:
7:51 + 9 minutes + 13 minutes = 8:13. At 7:55, Ethel is still waiting since no express train has come. At 7:55, Ethel's expected arrival time is: 7:55 + 9 minutes + 13 minutes = 8:17.

5.52. E. Fred gets to work at 8:20.
Ethel arrives first if an express leaves the station before 8:20 - 13 = 8:07.
We start a new Poisson process at 7:55.
The wait for an express is Exponential with mean: 3/20 hours = 9 minutes.
Prob[wait < 12 minutes] = $1 - e^{-12/9}$ = **73.6%**.

5.53. C. Wind losses are Poisson with claims intensity 1/10 = 0.10.
Earthquake losses are Poisson with claims intensity 1/25 = 0.04.
Theft losses are Poisson with claims intensity 1/5 = 0.20.
Since the three types of losses are independent, their sum is another Poisson Process, with claims intensity equal to the sum of the individual intensities: 0.10 + 0.04 + 0.20 = 0.34.
The average time between loss occurrences is 1/0.34 = **2.94 years**.

5.54. B. The total claims intensity is 0.34. The time until the fifth claim is a Gamma Distribution with $\alpha = 5$ and $\theta = 1/0.34 = 2.94$. Thus $F(11.76) = \Gamma[5 ; 11.76/2.94] = \Gamma[5 ; 4]$.
Comment: $\Gamma[5 ; 4] = 0.371163$.

5.55. B. Overall we have a Poisson Process, with claims intensity 0.34. Therefore the interevent times are Exponentially distributed with mean 1/0.34. Therefore the survival function at 4 is: S(4) = exp(-(0.34)(4)) = **25.7%**.
Comment: The answer would have been the same if we asked for example about the time between the seventeenth and eighteenth losses, rather than the seventh and eighth losses.

5.56. C. The waiting time until the first wind loss is Exponential with mean 10. Thus the chance of at least one wind loss by time t, is $1-e^{-t/10} = 1-e^{-0.1t}$. Similarly, the chance of at least one earthquake loss by time t, is $1-e^{-t/25} = 1-e^{-0.04t}$. The chance of at least one theft loss by time t, is: $1-e^{-t/5} = 1 - e^{-0.2t}$. Since the processes are independent, the chance that at least one loss of each type has occurred by time t is the product of the three individual probabilities:
$(1-e^{-0.1t})(1-e^{-0.04t})(1-e^{-0.2t})$.
At t = 8 we get, $(1-e^{-0.8})(1-e^{-0.32})(1-e^{-1.6})$ = (0.5507)(0.2739)(0.7981) = **12.04%.**

5.57. C. From the previous solution, the chance that at least one loss of each type has occurred by time t is: $(1 - e^{-0.1t})(1 - e^{-0.04t})(1 - e^{-0.2t}) = (1 - e^{-0.1t})(1 - e^{-0.2t} - e^{-.04t} + e^{-0.24t}) =$
$1 - e^{-0.2t} - e^{-0.04t} - e^{-0.1t} + e^{-0.24t} + e^{-0.3t} + e^{-0.14t} - e^{-0.34t}$.
Now the chance that it takes more than time t for at least one loss of each type to occur is:
$S(t) = 1 - F(t) = e^{-0.2t} + e^{-0.04t} + e^{-0.1t} - e^{-0.24t} - e^{-0.3t} - e^{-0.14t} + e^{-0.34t}$. In general, one can calculate the expected value of x by taking an integral of the survival function S(x). Thus the expected time until at least one loss of each type has occurred is:

$$\int_0^\infty S(t)\,dt = \int_0^\infty e^{-0.2t} + e^{-0.04t} + e^{-0.1t} - e^{-0.24t} - e^{-0.3t} - e^{-0.14t} + e^{-0.34t}\,dt =$$

1/0.2 + 1/0.04 + 1/0.1 - 1/0.24 - 1/0.3 - 1/0.14 + 1/0.34 =
5 + 25 + 10 - 4.167 - 3.333 - 7.143 + 2.941 = **28.298**.
Comment: See Example 5.17 in Introduction to Probability Models by Ross.

5.58. B. The size of wind losses follows a LogNormal Distribution with parameters $\mu = 2$ and $\sigma = 3$. Thus the chance that a wind loss is of size greater than 1000 is $1 - \Phi((\ln(1000) - 2)/3) = 1 - \Phi(1.64) = 1 - 0.9495 = 0.0505$. The chance of a loss being a wind loss is 0.1/0.34 = 0.294. Thus the chance of a loss being a wind loss of size greater than 1000 is:
(0.294)(0.0505) = **1.5%**.
Comment: The fact that we were asked about the 4th claim has no effect.

5.59. D. The size of earthquake losses follows a Pareto Distribution as per Loss Models, with parameters $\alpha = 2$ and $\theta = 1500$. Thus the chance that a earthquake loss is of size greater than 1000 is: $(1500/(1500+1000))^2 = 0.36$. Earthquakes follow a Poisson Process with intensity 0.04. Thus Earthquakes of size greater than 1000 follow a Poisson Process with intensity (0.04)(0.36) = 0.0144. Therefore, the expected time between earthquakes of size greater than 1000 is 1/0.0144 = **69.44 years**.
Alternately, the expected time between earthquakes is 25 years. Therefore, the expected time between earthquakes of size greater than 1000 is 25/0.36 = 69.44 years.

5.60. C. The size of theft losses follows a Weibull Distribution as per Loss Models, with parameters $\tau = 2$ and $\theta = 600$. Thus the chance that a theft loss is of size greater than 1000 is $\exp[-(1000/600)^2] = 0.0622$. Thefts follow a Poisson Process with intensity 0.20. Thus thefts of size greater than 1000 follow a Poisson Process with intensity: (0.20)(0.0622) = 0.0124.
Therefore, the expected wait until the first theft loss of size greater than 1000 is:
1/0.0124 = **80.4 years**.

5.61. E. From the previous solutions, losses of size greater than 1000 are given by a Poisson Process, with intensity: 0.0051 + 0.0144 + 0.0124 = 0.0319. Therefore, the expected wait until the first loss of size greater than 1000 is observed is: 1/0.0319 = **31.3 years**.

5.62. E. The time until the fourth theft loss is the sum of four independent Exponentials each with $\theta = 5$, which is a Gamma Distribution with $\alpha = 4$ and $\theta = 5$. The time until the third wind loss is a Gamma Distribution with $\alpha = 3$ and $\theta = 10$. The time until the second earthquake loss is a Gamma Distribution with $\alpha = 2$ and $\theta = 25$.
Thus the chance of at least four theft losses by time 20, is $\Gamma[4 ; 20/5] = \Gamma[4 ; 4]$.
Thus the chance of at least 3 wind losses by time 20, is $\Gamma[3 ; 20/10] = \Gamma[3 ; 2]$.
Thus the chance of at least 2 earthquake losses by time 20, is $\Gamma[2 ; 20/25] = \Gamma[2 ; 0.8]$.
Since the three Poisson processes are independent, the chance there has been at least four theft losses, at least three wind losses and at least two earthquake losses by time 20 is the product: $\mathbf{\Gamma[4 ; 4]\ \Gamma[3 ; 2]\ \Gamma[2 ; 0.8]}$.
Comment: $\Gamma[4 ; 4]\ \Gamma[3 ; 2]\ \Gamma[2 ; 0.8]$ = (0.56653)(0.32332)(0.19121) = 3.50%.

5.63. E. There are three independent Poisson processes:
Scientists with ray guns with $\lambda = 0.2$ per day, or 1.4 per week.
Scientists who escape with $\lambda = 0.2$ per day, or 1.4 per week.
Scientists who are eaten with $\lambda = 0.4$ per day, or 2.8 per week.
In order to satisfy the condition, we must have zero from the first process, and at least 3 from the third process. It does not matter how many we have from the second process.
Prob[0 ray guns] Prob[at least 3 eaten] = $(e^{-1.4})(1 - e^{-2.8} - 2.8\, e^{-2.8} - 2.8^2\, e^{-2.8}/2)$ =
(0.2466)(0.5305) = **13.1%**.

5.64. E. Men customers are Poisson with mean: (40%)(20) = 8.
Women customers are Poisson with mean: (30%)(20) = 6.
Children customers are Poisson with mean: (30%)(20) = 6.
Men customers who buy Squishees are Poisson with mean: (20%)(8) = 1.6.
Women customers who buy Squishees are Poisson with mean: (10%)(6) = 0.6.
Children customers who buy Squishees are Poisson with mean: (60%)(6) = 3.6.
$\Rightarrow$ Customers who buy Squishees are Poisson with mean: 1.6 + 0.6 + 3.6 = 5.8 per hour.
$\Rightarrow$ The expected time until the 50th Squishee sale is: 50/5.8 = **8.6 hours**.
Comment:
First thin each Poisson Process, and then add the resulting Poisson Processes together.
The portion of customers who buy a Squishee is: (0.4)(0.2) + (0.3)(0.1) + (0.3)(0.6) = 29%.
(29%)(20) = 5.8.

5.65. E. The 3 Poisson processes are independent. Thus knowing how many nickels and dimes he lost over a given period of time, tells us nothing about the expected number of quarters.
Over 40 minutes the number of quarters found is Poisson with mean: (40)(20%)(0.2) = 1.6
Prob[at least 3 quarters] = $1 - e^{-1.6} - 1.6\, e^{-1.6} - e^{-1.6}\, 1.6^2 / 2$ = **21.66%**.

5.66. A. The expected time until Klem loses his first coin is: 1/0.2 = **5 minutes**.

5.67. D. Quarters are Poisson with rate: (20%)(0.2) = 0.04.
The expected time until Klem loses his first quarter is: 1/0.04 = **25 minutes**.

5.68. B. Non-Quarters are Poisson with rate: (80%)(0.2) = 0.16.
The expected time until Klem loses his first coin that is not a quarter is: 1/0.16 = **6.25 minutes**.

5.69. D. The expected time until Klem loses his tenth coin is: 10/0.2 = **50 minutes**.

5.70. C. Nickels are Poisson with rate: (50%)(0.2) = 0.1.
The expected time until Klem loses his tenth nickel is: 10/0.1 = **100 minutes**.

5.71. C. Nickels are Poisson with rate: (50%)(0.2) = 0.1.
Dimes are Poisson with rate: (30%)(0.2) = 0.06.
Quarters are Poisson with rate: (20%)(0.2) = 0.04.
Prob[lost at least one coin of each type] =
Prob[at least one nickel] Prob[at least one dime] Prob[at least one quarter] =
$(1 - e^{-0.1t})\ (1 - e^{-0.06t})\ (1 - e^{-0.04t})$ =
$1 - e^{-0.1t} - e^{-0.06t} - e^{-0.04t} + e^{-0.16t} + e^{-0.14t} + e^{-0.10t} - e^{-0.2t}$.
The expected time until he loses a coin of each type is the integral of the survival function:

$$\int_0^\infty e^{-0.1t} + e^{-0.06t} + e^{-0.04t} - e^{-0.16t} - e^{-0.14t} - e^{-0.10t} + e^{-0.2t}\ dt =$$

1/0.1 + 1/0.06 + 1/0.04 - 1/0.16 - 1/0.14 - 1/0.1 + 1/0.2 = **33.27 minutes**.

5.72. A. Nickels are Poisson with rate: (50%)(0.2) = 0.1.
Dimes are Poisson with rate: (30%)(0.2) = 0.06.
1 - Prob[lost at least two nickels and lost at least two dimes] =
1 - Prob[at least two nickels] Prob[at least two dimes]
$1 - (1 - e^{-0.1t} - 0.1t\, e^{-0.1t})\,(1 - e^{-0.06t} - 0.06t\, e^{-0.06t}) =$
$e^{-0.1t} + e^{-0.06t} - e^{-0.16t} + 0.1t\, e^{-0.1t} + 0.06t\, e^{-0.06t} - 0.16t\, e^{-0.16t} - 0.006t^2\, e^{-0.16t}$.
The expected time until he loses at least 2 nickels and at least 2 dimes is the integral of the survival function: $\int_0^\infty e^{-0.1t} + e^{-0.06t} - e^{-0.16t}\, dt$ +

$$\int_0^\infty 0.1t\, e^{-0.1t} + 0.06t\, e^{-0.06t} - 0.016t\, e^{-0.16t}\, dt + \int_0^\infty - 0.006t^2\, e^{-0.16t}\, dt$$

$= 1/0.1 + 1/0.06 - 1/0.16 + 0.1/0.1^2 + 0.06/0.06^2 - (0.16)(1/0.16^2) - (0.006)(2/0.16^3)$
= **37.90 minutes**.
Comment: One could do the integrals by parts.

$\int_0^\infty t\, \lambda\, e^{-\lambda t}\, dt$ is the mean of an Exponential Distribution with hazard rate λ, which is $1/\lambda$.

$\int_0^\infty t^2\, \lambda\, e^{-\lambda t}\, dt$ is the second moment of an Exponential Distribution with hazard rate λ, which is $2/\lambda^2$.

$\int_0^\infty t^3\, \lambda\, e^{-\lambda t}\, dt$ is the third moment of an Exponential Distribution with hazard rate λ, which is $6/\lambda^3$.

Thus $\int_0^\infty t\, e^{-\lambda t}\, dt = 1/\lambda^2$, $\int_0^\infty t^2\, e^{-\lambda t}\, dt = 2/\lambda^3$, and $\int_0^\infty t^3\, e^{-\lambda t}\, dt$ is $6/\lambda^4$.

Alternately, one could use the result for Gamma type integrals, discussed in a subsequent section:

$\int_0^\infty t^{\alpha-1}\, e^{-t/\theta}\, dt = \Gamma(\alpha)\, \theta^\alpha$, or for integer n: $\int_0^\infty t^n\, e^{-ct}\, dt = n! / c^{n+1}$.

5.73. B. Nickels are Poisson with rate: (50%)(0.2) = 0.1.
Dimes are Poisson with rate: (30%)(0.2) = 0.06.
Quarters are Poisson with rate: (20%)(0.2) = 0.04.
1 - Prob[lost at least two coins of each type] =
1 - Prob[at least two nickels] Prob[at least two dimes] Prob[at least two quarters] =
$1 - (1 - e^{-0.1t} - 0.1t\ e^{-0.1t})\ (1 - e^{-0.06t} - 0.06t\ e^{-0.06t})\ (1 - e^{-0.04t} - 0.04t\ e^{-0.04t}) =$
$e^{-0.1t} + e^{-0.06t} + e^{-0.04t} - e^{-0.16t} - e^{-0.14t} - e^{-0.10t} + e^{-0.2t}$
$+ 0.1t\ e^{-0.1t} + 0.06t\ e^{-0.06t} + 0.04t\ e^{-0.04t} - 0.16t\ e^{-0.16t} - 0.14t\ e^{-0.14t} - 0.01t\ e^{-0.10t} + 0.2t\ e^{-0.2t}$
$- 0.006t^2\ e^{-0.16t} - 0.004t^2\ e^{-0.14t} - 0.0024t^2\ e^{-0.10t} + 0.0124t\ e^{-0.2t} + 0.00024t^3\ e^{-0.2t}$.
The expected time until he loses a coin of each type is the integral of the survival function:

$$\int_0^\infty e^{-0.1t} + e^{-0.06t} + e^{-0.04t} - e^{-0.16t} - e^{-0.14t} - e^{-0.10t} + e^{-0.2t}\ dt\ +$$

$$\int_0^\infty 0.1t\ e^{-0.1t} + 0.06t\ e^{-0.06t} + 0.04t\ e^{-0.04t}\ dt\ +$$

$$\int_0^\infty -0.16t\ e^{-0.16t} - 0.14t\ e^{-0.14t} - 0.1t\ e^{-0.10t} + 0.2t\ e^{-0.2t}\ dt\ +$$

$$\int_0^\infty - 0.006t^2\ e^{-0.16t} - 0.004t^2\ e^{-0.14t} - 0.0024t^2\ e^{-0.10t} + 0.0124t^2\ e^{-0.2t} + 0.00024t^3\ e^{-0.2t}\ dt$$

$= 1/0.1 + 1/0.06 + 1/0.04 - 1/0.16 - 1/0.14 - 1/0.1 + 1/0.2$
$+ 0.1/0.1^2 + 0.06/0.06^2 + 0.04/0.04^2$
$- (0.16)(1/0.16^2) - (0.14)(1/0.14^2) - (0.1)(1/0.1^2) + (0.2)(1/0.2^2)$
$- (0.006)(2/0.16^3) - (0.004)(2/0.14^3) - (0.0024)(2/0.1^3) + (0.0124)(2/0.2^3) + (0.00024)(6/0.2^4)$
= **59.90 minutes**.

5.74. E. Take the current time as 0. If a satellite was launched at time t, then the probability that it is still in the air is 1 - F(-t) = S(-t).
The mean number of satellites launched more than 20 years ago that are still in the air is:

$$\int_{-\infty}^{-20} 10\ S(-t)\ dt = 10 \int_{20}^{\infty} S(t)\ dt = 10 \int_{20}^{\infty} e^{-t/5}\ dt = (10)\ (5\ e^{-20/5}) = 0.916.$$

The number of such satellites is Poisson.
Probability of no such satellites is: $e^{-0.916}$ = **40.0%**.
Comment: Similar to Exercise. 5.67 in Introduction to Probability Models by Ross.
Not intended to be realistic.

5.75. C. Shorter treatments are a Poisson process with rate: (70%)(3)/60 = 0.035 per minute.
Longer treatments are an independent Poisson process with rate:
(30%)(3)/60 = 0.015 per minute.
If at least one longer treatment arrives from time 0 to 20, then Doris will not be free at 10:20.
If a shorter treatment arrives from time 10 to 20, then Doris will not be free at 10:20.
If two or more shorter treatments arrive from time 0 to 20, then Doris will not be free at 10:20.
Thus we want:

- No longer treatments from time 0 to 20.
- No shorter treatments from time 10 to 20.
- At most one shorter treatments from time 0 to 10.

Prob[No longer treatments from time 0 to 20] = $e^{-(20)(0.015)} = e^{-0.3}$ = 0.7408.
Prob[No shorter treatments from time 10 to 20] = $e^{-(10)(0.035)} = e^{-0.35}$ = 0.7047.
Prob[0 or 1 shorter treatments from time 0 to 10] = $e^{-0.35} + 0.35e^{-0.35}$ = 0.9513.
Since these probabilities are all independent,
Prob[Doris is free at 10:20] = (0.7408)(0.7047)(0.9513) = **49.7%**.

5.76. B. Shorter treatments are a Poisson process with rate: (70%)(3)/60 = 0.035 per minute.
Longer treatments are an independent Poisson process with rate:
(30%)(3)/60 = 0.015 per minute.
If at least one longer treatment arrives from time 10 to 30, then Doris will not be free at 10:30.
If at least two longer treatments arrive from time 0 to 30, then Doris will not be free at 10:30.
If a shorter treatment arrives from time 20 to 30, then Doris will not be free at 10:20.
If two or more shorter treatments arrive from time 10 to 30, then Doris will not be free at 10:20.
If three or more shorter treatments arrive from time 0 to 30, then Doris will not be free at 10:20.
If no longer treatments arrive, then we want:

- No shorter treatments from time 20 to 30.
- Either one shorter treatment from 10 to 20 and at most one shorter treatment from 0 to 10, or no shorter treatments from 10 to 20 and at most two shorter treatment from 0 to 10.

If one longer treatments arrives from 0 to 10, then we want:

- No longer treatments from 10 to 30.
- No shorter treatments from 0 to 30.

Prob[No longer treatments from time 0 to 30] = $e^{-(30)(0.015)} = e^{-0.45}$ = 0.6376.
Prob[No longer treatments from time 10 to 30] = $e^{-(20)(0.015)} = e^{-0.3}$ = 0.7408.
Prob[One longer treatments from time 0 to 10] = 0.15 $e^{-0.15}$ = 0.1291.
Prob[No shorter treatments from time 0 to 30] = $e^{-(30)(0.035)} = e^{-1.05}$ = 0.3499.
Prob[No shorter treatments from time 20 to 30] = $e^{-(10)(0.035)} = e^{-0.35}$ = 0.7047.
Prob[No shorter treatments from time 10 to 20] = $e^{-(10)(0.035)} = e^{-0.35}$ = 0.7047.
Prob[One shorter treatments from time 10 to 20] = 0.35 $e^{-0.35}$ = 0.2466.
Prob[0 or 1 shorter treatments from time 0 to 10] = $e^{-0.35} + 0.35e^{-0.35}$ = 0.9513.
Prob[0, 1, or 2 shorter treatments from time 0 to 10] = $e^{-0.35} + 0.35e^{-0.35} + (0.35^2/2)e^{-0.35}$
= 0.9945.
Prob[Doris is free at 10:30] =
Prob[no long 0 to 30] Prob[no short 20 to 30] Prob[1 short 10 to 20] Prob[0 or 1 short 0 to 10] +
Prob[no long 0 to 30] Prob[no short 20 to 30] Prob[no short 10 to 20] Prob[0, 1, 2 short 0 to 10] +
Prob[1 long 0 to 10] Prob[no long 10 to 30] Prob[no short 0 to 30] =
(0.6376)(0.7047){(0.2466)(0.9513) + (0.7047)(0.9945)} + (0.1291)(0.7408)(0.3499) = **45.4%**.
Comment: What happens over disjoint time intervals is independent.

5.77. D. Uptown express trains are Poisson with λ = 80%/4 = 0.2.
Downtown express trains are Poisson with λ = 60%/6 = 0.1.
Therefore, express trains are Poisson with λ = 0.2 + 0.1 = 0.3.
The average wait until the third express train is: 3/0.3 = **10 minutes**.

5.78. C. Measuring everything in months, λ = 0.75 per month.
Call the disabilities that follow an Exponential Distribution with mean 3 months less serious, and those that follow an Exponential Distribution with mean 9 months, more serious.
Then by thinning the less serious and more serious injuries are independent Poisson Processes, with $\lambda = (0.7)(0.75) = 0.525$ and $\lambda = (0.3)(0.75) = 0.225$.
Given a less serious injury at time t, the probability it is not closed by t = 15 is $e^{-(15-t)/3}$, t < 15.
Thus we can thin the less serious injuries into closed and open; since the thinning factor depends on t we get independent nonhomogeneous Poisson Processes.
Similarly, the open and closed more serious injuries are independent nonhomogeneous Poisson Processes.
Adding the closed less serious and the closed more serious independent nonhomogeneous Poisson Processes we get another nonhomogeneous Poisson Processes.
The probability that a worker injured at time t is still disabled at time 15 months (April 1, 2014) is:
$0.7\ e^{-(15-t)/3} + 0.3\ e^{-(15-t)/9}$.
Therefore, as of April 1, 2014, the expected number of workers still disabled whose injuries occurred during 2013 is:

$$\int_0^{12} \{0.7\ e^{-(15-t)/3} + 0.3\ e^{-(15-t)/9}\}\ 0.75\ dt =$$

$(0.75)\ \{(0.7)(3)(e^{-3/3} - e^{-15/3}) + (0.3)(9)(e^{-3/9} - e^{-15/9})\} = 1.637$.
Thus the number of workers whose injuries occurred during 2013 and who are still disabled as of April 1, 2014 is Poisson with mean 1.637.
The probability of exactly three such workers is: $e^{-1.637}\ 1.637^3 / 6$ = **14.2%**.
Comment: Beyond what you are likely to be asked on your exam.

5.79. A. The nickels and non-nickels are independent Poisson processes.
The number of nickels found does not affect the expected number of non-nickels.
Over an hour we expect: (30% + 20%)(0.2)(60) = 6 non-nickels.
Prob[7 non-nickels] = $6^7\ e^{-6} / 7!$ = **13.8%**.

5.80. D. The quarters and non-quarters are independent Poisson processes.
Over an hour we expect: (20%)(0.2)(60) = 2.4 quarters,
and (80%)(0.2)(60) = 9.6 non-quarters.
Prob[3 quarters] = $2.4^3\ e^{-2.4} / 3!$ = 0.2090.
Prob[9 non-quarters] = $9.6^9\ e^{-9.6} / 9!$ = 0.1293.
Prob[3 quarters and 9 non-quarters] = (0.2090)(0.1293) = **2.70%**.

5.81. C. The nickel, dime, and quarter Poisson processes are independent, and the expected numbers over 10 minutes are: 1, 0.6, and 0.4.
Prob[total of 25 cents at time 10] = Prob[1 quarter] Prob[0 dimes] Prob[0 nickels]
+ Prob[0 quarters] Prob[0 dimes] Prob[5 nickels] + Prob[0 quarters] Prob[1 dime] Prob[3 nickels]
+ Prob[0 quarters] Prob[2 dimes] Prob[1 nickel]
$= (0.4\ e^{-0.4})\ (e^{-0.6})\ (e^{-1}) + (e^{-0.4})\ (e^{-0.6})\ (1^5\ e^{-1}/120) + (e^{-0.4})\ (0.6e^{-0.6})\ (1^3\ e^{-1} / 6)$
$+ (e^{-0.4})\ (0.6^2 e^{-0.6}/2)\ (1\ e^{-1})$ = **9.32%**.

5.82. C. We need to calculate the probabilities of both events.
One way to get 25 cents at time 10 is:
Prob[1 quarter] Prob[0 dimes] Prob[0 nickels] = $(0.4\ e^{-0.4})\ (e^{-1})\ (e^{-0.6})$.
If we have zero dimes over the first 10 minutes, then we need 2 dimes over the next 20 minutes, which has probability: $1.2^2 e^{-1.2}/2$.
Another way to get 25 cents at time 10 is:
Prob[0 quarters] Prob[0 dimes] Prob[5 nickels] = $(e^{-0.4})\ (e^{-0.6})\ (1^5\ e^{-1}/120)$.
If we have zero dimes over the first 10 minutes, then we need 2 dimes over the next 20 minutes, which has probability: $1.2^2 e^{-1.2}/2$.
Another way to get 25 cents at time 10 is:
Prob[0 quarters] Prob[1 dime] Prob[3 nickels] = $(e^{-0.4})\ (0.6e^{-0.6})\ (1^3\ e^{-1} / 6)$.
If we have one dime over the first 10 minutes, then we need 1 dime over the next 20 minutes, which has probability: $1.2e^{-1.2}$.
Another way to get 25 cents at time 10 is:
Prob[0 quarters] Prob[2 dimes] Prob[1 nickel] = $(e^{-0.4})\ (0.6^2 e^{-0.6}/2)\ (1\ e^{-1})$.
If we have two dimes over the first 10 minutes, then we need 0 dimes over the next 20 minutes, which has probability: $e^{-1.2}$.
Therefore, the probability of both events is:
$= (0.4\ e^{-0.4})\ (e^{-1})\ (e^{-0.6})(1.2^2 e^{-1.2}/2) + (e^{-0.4})\ (e^{-0.6})\ (1^5\ e^{-1}/120)(1.2^2 e^{-1.2}/2) +$
$(e^{-0.4})\ (0.6e^{-0.6})\ (1^3\ e^{-1} / 6)(1.2e^{-1.2}) + (e^{-0.4})\ (0.6^2 e^{-0.6}/2)\ (1\ e^{-1})(e^{-1.2}) = 2.42\%$
From the previous solution, Prob[25 cents at time 10] = 9.32%
Therefore, the condition probability of 2 dimes by time 30 is: 2.42%/9.32% = **26.0%**.

5.83. B. The three types of coins are independent Poisson Processes, and for each type of coin, the number of coins lost over disjoint time intervals are independent, and thus so is the amount of money lost over disjoint time intervals. (The process is memoryless.)
Thus what we want is the probability that Klem lost coins totaling 10 during the period from time 10 to time 20.
The nickel, dime, and quarter Poisson processes are independent, and the expected numbers over 10 minutes are: 1, 0.6, 0.4.
Prob[total of 10 cents in 10 minutes] =
Prob[0 quarters] Prob[0 dimes] Prob[2 nickels] + Prob[0 quarters] Prob[1 dime] Prob[0 nickels]
$= (e^{-0.4})\ (e^{-0.6})\ (1^2\ e^{-1}/2) + (e^{-0.4})\ (0.6e^{-0.6})\ (e^{-1}) =$ **14.89%**.

5.84. D. The number of nickels is Poisson with mean: (50%)(60)(0.2) = 6.0.
The number of quarters is Poisson with mean: (20%)(60)(0.2) = 2.4.
The number of nickels plus quarters is Poisson with mean: 6.0 + 2.4 = 8.4.
Thus, the variance the number of coins not worth 10 is **8.4**.

5.85. D. (60)(0.2) = 12 coins per hour. Thinning, we get three independent Poisson Processes: nickels with $\lambda = 6$, dimes with $\lambda = 3.6$, and quarters with $\lambda = 2.4$.
Prob[5 nickels] = $e^{-6}\ 6^5 / 5! = 0.1606$. Prob[4 dimes] = $e^{-3.6}\ 3.6^4 / 4! = 0.1912$.
Prob[2 quarters] = $e^{-2.4}\ 2.4^2 / 2! = 0.2613$.
Thus, the desired probability is: (0.1606)(0.1912)(0.2613) = **0.802%**.

5.86. E. A claim that occurred at time t ≤ 5, has a probability of being reported by time 5 of:
$F(5 - t) = 1 - \{10/((5-t) + 10)\}^3$. Expected number of reported claims:

$$\lambda \int_0^5 (1 - \{10/(15 - t)\}^3)\, dt = \lambda 5 - 0.5\, \lambda(10^3)/(15\text{-}t)^2 \Big]_{t=0}^{t=5} = \lambda(5 - 2.778) = 2.222\lambda.$$

Expected number of unreported claims:

$$\lambda \int_0^5 \{10/(15 - t)\}^3\, dt = 2.778\lambda.$$

Set $402 = \lambda 2.222$. ⇒ $\lambda = 181$.
Expected number of unreported claims: (2.778)(181) = **503**.

5.87. D. Type I supernovas follow a Poisson Process with $\lambda = (10\%)(1/50) = 1/500$.
Thus the time until the next event is Exponential with mean 500 years.
The 90th percentile is where the distribution function is 90%.
$0.90 = 1 - e^{-t/500}$. ⇒ t = **1151** years.

5.88. B. Type I supernovas follow a Poisson Process with $\lambda = (10\%)(1/50) = 1/500$.
Thus the time until the next event is Exponential with mean 500 years.
This has a variance of: $500^2 = 250{,}000$.
The interevent times are independent, identically distributed Exponentials.
Thus the variance of the time until the third event is: (3)(250,000) = 750,000.
Standard deviation is: $\sqrt{750{,}000}$ = **866** years.

Alternately, the time until the third event is Gamma with $\alpha = 3$ and $\theta = 500$.
It has a variance of: $\alpha\theta^2$, and a standard deviation of: $\theta\sqrt{\alpha} = 500\sqrt{3}$ = **866** years.

5.89. C. We can add the two independent Poisson Processes and get a new Poisson Process with mean: 2 + 1 = 3.
Prob[more than 30 minutes for an interruption] = Prob[0 interruptions by time 1/2 hour] =
exp[-(3)(1/2)] = **22.3%**.
<u>Comment</u>: Similar to ST, 5/14, Q.2.

5.90. For claims that occurred at time t, by time 10 the percent reported is F(10-t).
The portion unreported is: 1 - F(10-t) = S(10-t) = $\exp[-\{(10-t)/2\}^{0.6}]$.
Since the claims intensity is a constant 40, the expected unreported claims are:

$$40 \int_0^{10} \exp[-\{(10-t)/2\}^{0.6}]\, dt = (40)(2/0.6) \int_0^{5^{0.6}} e^{-z} z^{1/0.6-1}\, dz = (400/3)\, \Gamma(1/.6)\, \Gamma[1/.6 ; 5^{0.6}] =$$

(133.333) Γ[1.66667] Γ[1.66667 ; 2.62653] = (133.333)(0.902745)(0.812245) = 97.77.
Comment: Where I've made the change of variables $z = ((10-t)/2)^{0.6}$, and used a computer to calculate the Gamma Functions. Well beyond what you are likely to be asked on your exam.
We expect 97.77 unreported claims at time 10. By time 10, we expect: (10)(40) = 400 claims in total. So we expect: 99.97 / 400 = 24.44% of the claims to be unreported.

5.91. D. Probability that a claim is greater than 5000 is: (70%)(4%) + (30%)(13%) = 6.7%.
The number of large claims is Poisson with mean: (9)(6.7%) = 0.603.
Probability of more than 1 large claim is: $1 - e^{-0.603} - 0.603\, e^{-0.603}$ = **12.3%**.

5.92. C. F(70,000) = $1 - (40/70)^4$ = 0.8934.
Thus acceptable job offer arrive at a Poisson rate of: (0.8)(1 - 0.8934) = 0.0853.
The number of acceptable job offers in 10 weeks is Poisson with mean: (10)(0.0853) = 0.853.
Prob[no acceptable job offers in 10 weeks] = $e^{-0.853}$ = 0.426.
Then the number of acceptable job offers over the next 5 weeks is Poisson with mean: (5)(0.0853) = 0.4265.
Prob[at least one acceptable job offer over next 5 weeks] = $1 - e^{-0.4265}$ = 0.347.
Thus the desired probability is: (0.426)(0.347) = **0.148**.
Comment: Similar to Exam S, 5/17,Q.2.

5.93. E. The number of claims over 3 days is Poisson.
The expected number of claims is: (3)(1.5) + (3)(1/2) = 6.
Prob[at least 4 claims] = $1 - e^{-6} - 6e^{-6} - 6^2e^{-6}/2 - 6^3e^{-6}/3!$ = **0.8488**.
Comment: Similar to Exam S, 11/17,Q.2.

5.94. D. For the Weibull, $S(x) = \exp[-(x/\theta)^\tau] = \exp[-(x/156.25 \text{ million})^{0.5}]$. If frequency is given by a Poisson and severity is independent of frequency, then the number of claims above a certain amount is also a Poisson. Let the mean frequency for losses (including those of all sizes) be μ.
Then the windstorms greater than 5 million are Poisson with mean μS(5 million); the mean is observed to be: 12/30 = 0.40. Thus μ = 0.4 / S(5 million).
The windstorms greater than 1 billion are Poisson with mean:
μS(1 billion) = (0.4)S(1 billion) / S(5 million) = (0.4) $(e^{-2.530})$ / $(e^{-0.1789})$ = 0.0381.
For the Poisson distribution with parameter λ, the time until the next claim is distributed as per an exponential with parameter λ and mean 1/λ.
Thus the average time until the next windstorms greater than 1 billion is: 1/0.0381 = **26.2**.

5.95. D. For the Pareto, the chance of a claim being greater than x is: 1 - F(x) = $\{\theta/(\theta+x)\}^\alpha$ = $\{150 \text{ million}/(150 \text{ million}+x)\}^4$. If frequency is given by a Poisson and severity is independent of frequency, then the number of claims above a certain amount is also a Poisson. Let the mean frequency for ground up claims be μ. Then the earthquakes greater than 30 million are Poisson with mean μ\{1 - F(30 million)\}; this mean is observed to be 10/50.
Thus μ = (10/50) / \{1- F(30 million)\} = (10/50)/0.4823 = 0.4147.
The earthquakes greater than 100 million are Poisson with mean μ\{1 - F(100 million)\}.
Thus the number of claims greater than \$100 million is Poisson with mean:
(0.4147)\{1- F(100 million)\} = (0.4147) (0.1296) = 0.0537.
For the Poisson distribution with parameter λ, the interevent time between claims is distributed as per an exponential with mean $1/\lambda$.
Thus the average interevent time between such storms is 1/0.0537 = **18.6**.

5.96. C. If frequency is given by a Poisson and severity is independent of frequency, then the number of claims above a certain amount (in constant dollars) is also a Poisson. For the Pareto, the chance of a claim being greater than x is: S(x) = $\{\theta/(\theta+x)\}^\alpha = \{7000/(7000+x)\}^{1.05}$.
Thus the number of claims greater than \$5 million is Poisson with mean:
1.3 S(5 million) / S(500,000) = 1.3 $(507{,}000 / 5{,}007{,}000)^{1.05}$ = 0.1174.
For the Poisson distribution with parameter λ, the interevent time between claims is distributed as per an exponential with mean $1/\lambda$.
Thus the mean interevent time between such storms is 1/0.1174 = **8.52**.

5.97. B. The Poisson parameter for large hurricanes is 1/2 = 0.5. The Poisson parameter for large earthquakes is 1/5 = 0.2. The Poisson parameter for large fires is 1/10 = 0.1.
The large loss occurrences are given by a Poisson with parameter: 0.5 + 0.2 + 0.1 = 0.8.
Thus the mean interevent time between large loss occurrences is 1/0.8 = **1.25**.

5.98. C. The finding of the three different types of coins are <u>independent</u> Poisson processes. Over the course of 60 minutes, Tom expects to find (0.6)(0.5)(60) = 18 coins worth 1 each and (0.2)(0.5)(60) = 6 coins worth 10 each. Tom finds 10 coins worth 5 each. The expected worth of the coins he finds is: (18)(1) + (10)(5) + (6)(10) = **128**.

5.99. C. The nickels and dimes are independent Poisson Process, each with intensity: (0.2)(0.5) = 0.1.
The probability density function of finding dimes is that of an Exponential with hazard rate 0.1: $0.1\ e^{-0.1t}$.
The probability that Tom has not found at least 2 nickels by time t is the sum of probabilities of zero and one nickel: $e^{-0.1t} + 0.1t\ e^{-0.1t}$.
Thus the probability that Tom does <u>not</u> find his second nickel prior to finding his first dime is:

$$\int_0^\infty (e^{-0.1t} + 0.1t\ e^{-0.1t})\ 0.1\ e^{-0.1t}\ dt = 0.1 \int_0^\infty e^{-0.2t}\ dt + 0.05 \int_0^\infty t\ 0.2\ e^{-0.2t}\ dt =$$

(0.1)(5) + (0.05)(5) = 0.75, where the second integral is that to compute the mean of an Exponential Distribution with hazard rate 0.2.
Thus the desired probability is: 1 - 0.75 = **25%**.
Alternately, the time of the finding of the second nickel is a Gamma Distribution with $\alpha = 2$ and $\theta = 1/0.1 = 10$. It has density: $t\ e^{-t/10} / 10^2 = 0.01te^{-0.1t}$.
The probability of no dimes by time t is: $e^{-0.1t}$.
Thus the probability that Tom finds his second nickel prior to finding his first dime is:

$$\int_0^\infty e^{-0.1t}\ 0.01\ t\ e^{-0.1t}\ dt = 0.05 \int_0^\infty t\ 0.2\ e^{-0.2t}\ dt = (0.05)(5) = \mathbf{0.25},$$ where the integral is that to compute the mean of an Exponential Distribution with hazard rate 0.2.

5.100. C. Large claims are a Poisson Process with mean (2%)(100) = 2 per month.
Over m months, the number of large claims is Poisson distributed with mean 2m.
We require that Prob(# large claims < 3) ≤ 10%.
Thus we require that: $e^{-2m} + 2me^{-2m} + (2m)^2e^{-2m}/2 \le 0.1$.
Plugging in m = 1, 2, etc., this inequality is first satisfied when m = **3**.

# months	Exp[-2m](1 + 2m + 2m^2)
1	0.677
2	0.238
3	0.062
4	0.014
5	0.003

5.101. B. For this Lognormal Distribution, S(28,000) = 1 - Φ[ln(28000) - 10.12)/0.12] = 1 - Φ(1) = 1 - 0.8413 = 0.1587. Acceptable offers arrive via a Poisson Process at rate 2 S(28000) = (2)(0.01587) = 0.3174 per month. Thus the number of acceptable offers over the first 3 months is Poisson distributed with mean (3)(0.3174) = 0.9522.
The probability of no acceptable offers over the first 3 months is: $e^{-0.9522}$ = **0.386**.
Alternately, the probability of no acceptable offers in a month is: $e^{-0.3174}$.
Probability of no acceptable offers in 3 months is: $(e^{-0.3174})^3 = e^{-0.9522}$ = **0.386**.

5.102. D. Coins of worth 10 are found via a Poisson Process with mean:
(20%)(0.5) = 0.1 per minute or 1 per 10 minutes.
Prob[at least 2 in first 10 minutes and at least 3 in first 20 minutes] =
Prob[2 in first 10 minutes] Prob[at least 1 in second 10 minutes] +
Prob[at least 3 in the first 10 minutes] = $(e^{-1}/2)(1 - e^{-1}) + (1 - e^{-1} - e^{-1} - e^{-1}/2)$ = **0.197**.
Alternately, one can calculate the probability of Tom failing.
Tom fails during the first 10 minutes, if he finds fewer than 2 dimes (coins worth 10) in the first 10 minutes: f(0) + f(1) = $e^{-1} + e^{-1}$.
Tom fails during the second 10 minutes if he finds 2 dimes during the first 10 minutes and no dimes during the second 10 minutes: f(2)f(0) = $(e^{-1}/2)(e^{-1})$.
Total probability of failing is: $e^{-1} + e^{-1} + (e^{-1}/2)(e^{-1})$ = 0.8034.
Probability of Tom succeeding: 1 - 0.8034 = **0.197**.
<u>Comment</u>: Always work with disjoint intervals.
For a Poisson Process, what happens on disjoint intervals is independent.
(0 ,10) overlaps with (0, 20). (0, 10) is disjoint from (10, 20).
Do not worry about the endpoints, since the Poisson Process is a continuous time process.

5.103. C. The average wait for the first train to arrive is: 1/20 hours = 3 minutes.
Your average train ride is: (25%)(16) + (75%)(28) = 25 minutes.
Your expected time of arrival is: 3 + 25 = 28 minutes.
Express trains are Poisson with mean: (25%)(20) = 5 per hour.
The average wait for an express train is: 1/5 hours = 12 minutes.
Your co-worker's expected time of arrival is: 12 + 16 = 28 minutes.
Your expected arrival times are the same.

5.104. (a) Your trip is the sum of two independent variables, the time of the ride plus a waiting time. The waiting time until the first train is Exponential with mean 3 and variance of: $3^2 = 9$.
Your ride is either 16 minutes or 28 minutes, with probabilities 25% and 75%.
The time of your ride has a mean of: (25%)(16) + (75%)(28) = 25.
The time of your ride has a second moment of: $(25\%)(16^2) + (75\%)(28^2) = 652$.
The time of your ride has a variance: $652 - 25^2 = 27$.
Variance of your total time is: 9 + 27 = 36. Standard deviation is: $\sqrt{36}$ = **6 minutes**.
(b) Coworker's trip is the sum of two independent variables, the time of the ride plus a waiting time.
The waiting time until the first express is Exponential with mean 12 and variance of: $12^2 = 144$.
His ride is always 28 minutes; it has a variance of zero.
Variance of his total time is: 144 + 0 = 144. Standard deviation is: $\sqrt{144}$ = **12 minutes**.
(c) For your coworker to take the fifth train to arrive, there must have been four locals followed by an express. This has a probability of: $(75\%^4)$ (25%) = **7.91%**.
(d) The probability of a local showing up at time t, and that this is the first train, in other words no expresses have shown up is proportional to:
Prob[No express by t] Prob[local at t] = $(e^{-5t})\ (15e^{-15t}) = 15\ e^{-20t}$.
In order to get the conditional density of this event, we need to divide by its integral from zero to infinity, which is 15/20 = 3/4. Thus the conditional density is: $20\ e^{-20t}$.
Thus the distribution of this event is an Exponential with mean 1/20 hours = 60 minutes /20 = 3 minutes, the same as the distribution of waiting times for the first train, regardless of type.
By similar reasoning, conditional on the first train to show up being a express, the conditional distribution of its time of arrival is also an Exponential with mean 3 minutes.
(e) For your coworker to take the fourth train to arrive, there must have been three locals followed by an express. From the previous solution, the average waiting time for the arrival time of the first local, conditional on the first train being a local, is Exponential with mean 3 minutes.
Since the Poisson processes have no memory, the interevent time between the first local and second local, conditional on the first and second trains being local, follows an independent, identically distributed Exponential to the waiting time to the first local.
Similarly, the next interevent time is also an independent, identically distributed Exponential.
Since the Poisson processes have no memory, the time from the 3rd local to the express is the same as the conditional distribution of the time of arrival of the first train conditional on it being an express; thus it is also an Exponential with mean 3 minutes.
So the conditional time your coworker has to wait is the sum of 4 independently distributed Exponentials, each with mean 3.
Thus his conditional waiting time has the same distribution as waiting for the 4th event from a Poisson with rate 1/3 minutes; for this Poisson, the mean number of events by 30 minutes is: 30/3 = 10.
Prob[coworker waited more than 30 minutes] = Prob[fewer than 4 events from a Poisson $\lambda = 10$]
$= e^{-10} + 10e^{-10} + 10^2e^{-10}/2 + 10^3e^{-10}/6$ = **1.034%**.

5.105. D. Trucks arrive at a Poisson rate of: (30%)(20) = 6 per hour.
$f(0) = e^{-6}$. $f(1) = 6e^{-6}$. $f(2) = 6^2e^{-6}/2$. $1 - \{f(0) + f(1) + f(2)\} = 1 - 25e^{-6}$ = **0.938**.

5.106. A. If an express train shows up first, a 25% chance, then both of you take the same train and arrive at work at the same time.
If a local train shows up first, a 75% chance, then you take this local and your coworker waits for the next express. You arrive at work in 28 minutes. If the next express shows up within 28 - 16 = 12 minutes, then your coworker arrives before or the same time you do. Otherwise he shows up after you.
Prob[you arrive first] =
Prob[local shows up first] Prob[No express shows up within 12 minutes of this local] =
$(0.75)(e^{-(12)(1/12)}) = 0.75e^{-1}$ = **0.276**.
Comment: Express trains are Poisson with λ = (25%)(20) = 5 per hour = 1/12 per minute.
Number of express trains that show up within 12 minutes is Poisson with mean: (12)(1/12) = 1.

5.107. D. In two hours expect 120/3 = 40 coins.
Number of dimes in 2 hours is Poisson with mean: (0.2)(40) = 8.
Prob[at least 4 dimes in the first 2 hours] = $1 - \{e^{-8} + 8e^{-8} + (8^2/2)\ e^{-8} + (8^3/6)\ e^{-8}\}$ = **0.958**.

5.108. B. Claims of $100,000 or more are Poisson with mean: (5)(1 - 0.9) = 0.5 per year.
The number of large claims during 3 years is Poisson with mean: (3)(0.5) = 1.5.
$f(0) = e^{-1.5}$ = **0.2231**.

5.109. A. Let p = the proportion of severe claims.
average severity = $7,000,000/2000 = 3500 = (50%)(1000) + (50% - p)(5000) + p10000.
⇒ 500 = 5000p. ⇒ p = **10%**.

5.110. E. Since the interevent times are (independent and identically distributed) Exponential variables, the number of hurricanes is a Poisson Process.
The number of hurricanes over the season is Poisson with mean: 24/6 = 4.
The number of hurricanes over the season that hit Broward County is Poisson with mean:
(30%)(4) = 1.2. Prob[more than one hitting Broward] = $1 - e^{-1.2} - 1.2e^{-1.2}$ = **33.7%**.

5.111. D. Poisoned glasses of wine are Poisson with mean: (0.01)(2) = 0.02 per day.
The probability of no poisoned glasses over 30 days is: $e^{-(30)(0.02)} = e^{-0.6}$ = **0.549**.
Comment: Survival corresponds to zero poisoned glasses of wine.
The king can drink any number of non-poisoned glasses of wine.
The poisoned and non-poisoned glasses are independent Poisson Processes.

5.112. A. Given a local arrives first, you take the local and your coworker waits for the first express train. You arrive at work in 28 minutes. You arrive at work first provided your coworker has to wait more than 28 - 16 = 12 minutes.
Express trains are Poisson with mean: (25%)(20) = 5 per hour.
During twelve minutes = 1/5 of an hour, the number of express trains is Poisson with mean 1.
Prob[no express in 12 minutes] = e^{-1} = **0.368**.

5.113. E. Let t be the time of a deer being hit. Then the time period in which the deer can be discovered is: 10 - t, and the probability that the deer will be discovered in the first 10 days is:
$1 - e^{-(10 - t)/7} = 1 - 0.2397e^{t/7}$.
The expected number of deer discovered in the first ten days is:

$$\int_0^{10} (20)\,(1 - 0.2397\,e^{t/7})\,dt = (20)(10) - (4.794)7e^{t/7}\Big]_{t=0}^{t=10} = 200 - 33.558(e^{10/7} - 1) = \mathbf{93.5}.$$

5.114. A. From the previous solution, the expected number of deer discovered in the first ten days is 93.5.
The number of deer found is Poisson, with mean 93.5 and therefore variance 93.5.
Prob[more than 110 deer found] $\cong 1 - \Phi[(110.5 - 93.5)/\sqrt{93.5}\,] = 1 - \Phi[1.76]$ = **3.92%.**

5.115. A. Since the time between reported claims is exponentially distributed and the time between claims in independent, we have a Poisson process with $\lambda = 1/0.50 = 2$ per year.
We can thin this original Poisson Process; the claims on which there are payments are Poisson with mean (0.7)(2) = 1.4.
The probability of no such claims in 2 years is: $e^{-(2)(1.4)}$ = **6.1%.**
<u>Comment</u>: Alternately, this is a compound process with a Poisson with $\lambda = 2$ as primary (whether there is a claim) and a Bernoulli with q = 0.7 as secondary (whether there is a payment on a claim.) Over two years we have a Poisson primary with $\lambda = 4$.
The probability generating function for a Poisson is: $\exp[\lambda(z - 1)] = \exp[4(z - 1)]$.
Use the first step of the Panjer Algorithm, where s(0) is the density of the secondary at 0:
Prob[0 payments] = c(0) = P(s(0)) = P(0.3) = $\exp[4(0.3 - 1]\ e^{-2.8}$ = **6.1%.**

5.116. C. By thinning, the number of goals is Poisson with mean (30)(0.1) = 3.
This Poisson has variance 3, and standard deviation $\sqrt{3}$ = **1.732**.

5.117. D. Independent, identically distributed exponential interevent times implies a Poisson Process. Therefore, if we had an infinite number of batteries, over three years the number of failures is Poisson with mean 3. We are interested in the probability of fewer than 2 failures, which is:
$e^{-3} + 3e^{-3} = 4e^{-3}$ = **0.199**.
Alternately, the sum of two independent, identically distributed Exponentials is a Gamma Distribution, with $\alpha = 2$ and the same θ. Thus the time the probe continues to transmit has a Gamma Distribution with $\alpha = 2$ and $\theta = 1$. This Gamma has density $f(t) = te^{-t}$.

$$S(t) = \int_t^\infty f(t)\,dt = \int_t^\infty t\,e^{-t}\,dt = e^{-t} + te^{-t}.\quad S(3) = 4e^{-3} = \mathbf{0.199}.$$

5.118. D. Payouts of size one are Poisson with λ = (1/2)(5) = 2.5 per hour.
Payouts of size one are Poisson with λ = (1/4)(5) = 1.25 per hour.
Payouts of size one are Poisson with λ = (1/8)(5) = 0.625 per hour.
Prob[0 of size 1 over 1/3 of an hour] = $e^{-2.5/3}$.
Prob[0 of size 2 over 1/3 of an hour] = $e^{-1.25/3}$.
Prob[0 of size 3 over 1/3 of an hour] = $e^{-0.625/3}$.
The three Poisson Processes are independent, so we can multiply the above probabilities:
$e^{-2.5/3}e^{-1.25/3}e^{-0.625/3} = e^{-1.458}$ = **0.233**.
Alternately, payouts of sizes one, two, or three are Poisson with
λ = (1/2 + 1/4 + 1/8)(5) = 4.375 per hour.
Prob[0 of sizes 1, 2, or 3, over 1/3 of an hour] = $e^{-4.375/3}$ = **0.233**.

5.119. C. Catalogs are Poisson with mean over two days of: (2)(30%)(10) = 6.
Letters are Poisson with mean over two days of: (2)(20%)(10) = 4.
The Poisson processes are all independent. Therefore, knowing he got 20 applications tells us nothing about the number of catalogs or letters.
Prob[at least 3 letters] = 1 - e^{-4} - e^{-4} 4 - e^{-4} $4^2/2$ = 0.7619.
Prob[5 catalogs] = e^{-6} $6^5/120$ = 0.1606.
Prob[at least 3 letters and exactly 5 catalogs] = (0.7619)(.1606) = **12.2%**.

5.120. C. The number of endorsements and cancellations is Poisson with λ = (250)(60%) = 150.
Applying the normal approximation with mean and variance equal to 150:
Prob[more than 156] $\cong$ 1 - $\Phi[(156.5 - 150)/\sqrt{150}]$ = 1 - Φ[0.53] = **29.8%**.
Comment: Using the continuity correction, 156 is out and 157 is in:

156 156.5 157

|→

5.121. C. (See Comment). Let X = number of endorsements resulting in a premium increase.
Let Y = number of endorsements that do not result in a premium increase.
X is Poisson with mean: (45%)(200) = 90. Y is Poisson with mean: (55%)(200) = 110.
X and Y are independent.
X - Y has a mean of 90 - 110 = -20, and a variance of 90 + 110 = 200.
We want the probability that X > Y.

Prob[X - Y > 0] $\cong 1 - \Phi\left[\frac{0 - (-20)}{\sqrt{200}}\right]$ = 1 - Φ[1.414] = **7.86%**.

Alternately, Prob[X - Y ≥ 1] $\cong 1 - \Phi\left[\frac{1 - (-20)}{\sqrt{200}}\right]$ = 1 - Φ[1.485] = **6.88%**.

Comment: We are told not to use the continuity correction; I do not know why.
The first manner of not using the continuity correction was the one the Exam Committee intended.
The second manner is equally valid mathematically, and gives a different answer, choice B.
Using the continuity correction: Prob[X - Y > 0] $\cong 1 - \Phi\left[\frac{1/2 - (-20)}{\sqrt{200}}\right]$ = 1 - Φ[1.450] = 7.36%.

5.122. D. $S(50) = e^{-50/15} = 0.03567$. Large losses are Poisson with mean: $10\ S(50) = 0.3567$.
Over 30 days, large losses are Poisson with mean: $(30)(0.3567) = 10.70$.
Prob[exactly 9 large losses in a 30-day period] = $e^{-10.7}\ 10.7^9 / 9!$ = **11.4%**.

5.123. B. For the Exponential Distribution, $S(30,000) = e^{-30,000/10,000} = e^{-3}$.
Therefore, large losses are Poisson with mean: $10e^{-3} = 0.498$.
The waiting times between large losses are independent Exponentials with $\theta = 1/0.498 = 2.01$.
Each Exponential has variance: $2.01^2 = 4.04$.
Thus the time until the second large loss has variance: $(2)(4.04) = 8.08$. $\sqrt{8.08}$ = **2.84**.
Alternately, the waiting time until the second large claim is Gamma with $\alpha = 2$ and $\theta = 2.01$.
This Gamma has variance: $(2)(\ 2.01^2) = 8.08$. $\sqrt{8.08}$ = **2.84**.

5.124. B. For the Exponential severity: S(1 million) = Exp[-(1 million) / 160,000] = 0.001930.
Thus catastrophes are Poisson with $\lambda = (0.001930)(200) = 0.3861$.
Thus the time between catastrophes is Exponential with mean: $1/0.3861 = 2.590$.
To find the median: $0.5 = 1 - \text{Exp}[-x/2.590]$. $\Rightarrow$ x = **1.795**.
Comment: The median is where the distribution function is 50%; it is the 50th percentile.

5.125. C. 40% of claims are small, of size less than 7000.
Thus small claims follow a Poisson process with: $\lambda = (0.4)(8) = 3.2$.
The number of small claims over a length of time of 0.6 is Poisson with mean: $(0.6)(3.2) = 1.92$.
Prob[at least 2 small claims by time 0.6] =
1 - Prob[no small claims by time 0.6] - Prob[1 small claim by time 0.6] =
$1 - e^{-1.92} - 1.92\ e^{-1.92}$ = **57.2%**.
Comment: I have no idea why they said the severity was discrete; this does not affect the solution.

5.126. C. The number of red cars is Poisson with rate per hour: (75%)(20) = 15.
The expected number of red cars in the next five hours is: (5)(15) = 75.
The total expected number of red cars that will have arrived after eight hours have passed is:
28 + 75 = **103**.
Comment: The original exam question left out the word “after” in “arrived after eight hours”..
What happens from time 3 to 8 is independent of what happened from time 0 to 3.
The number of red cars is independent of the number of blue cars.

5.127. B. We can add the two independent Poisson Processes and get a new Poisson Process with mean: 5 + 10 = 15.
Prob[more than 10 minutes for a ride] = Prob[0 rides by time 1/6 hour] = exp[-(15)(1/6)] = **8.21%**.

5.128. C. Adding the three Poisson Processes, gives a Poisson Process with
λ = 1/10 + 4/30 + 2/60 = 4/15.
$\Rightarrow$ W is the time between vehicles, and is Exponential with mean 15/4. $\Rightarrow$
Var[W] = $(15/4)^2$ = **14.06**.
Alternately, the time until the arrival of the first vehicle of a given type is Exponential.
With t in minutes, Prob[no vehicle by time t] =
Prob[No taxi by time t] Prob[No bus by time t] Prob[No streetcar by time t] =
$e^{-t/10}\ e^{-4t/30}\ e^{-2t/60} = e^{-0.2667t}$.
This is the survival function of an Exponential Distribution with mean 1/0.2667.
Thus W is Exponential with mean 3.75 minutes.
Thus the variance of W is: 3.75^2 = **14.06**.
<u>Comment</u>:
You need to assume that the arrivals of the different types of vehicles are independent.
Then W is the minimum of three independent Exponential variables; this an example of an order statistic, which is discussed in “Mahler’s Guide to Statistics.”
We need to be careful to work in consistent units of time, in this case minutes.
Streetcars arrive at a rate of 2 per hour, which is 2 per 60 minutes. Thus streetcar arrivals are Exponential with hazard rate 2/60 and thus mean 30 minutes.

5.129. A. The rate at which buses show up is: 1/15 + 1/10 = 1/6.
Thus the average wait for a bus is 6 minutes.
(You will take the first one that arrives.)
The probability the first bus to arrive is Line 1 is: 10 / (10 + 25) = 40%,
while the probability the first bus to arrive is Line 2 is: 15 / (10 + 25) = 60%.
Thus your average ride is: (40%)(8) + (60%)(20) = 15.2 minutes.
Thus, the expected length of time that it will take you to reach your destination is:
6 + 15.2 = 21.2.
To the nearest minute this is **21**.
<u>Comment</u>: The last sentence in the exam question was missing the word “it”.
“to the nearest minute” is an unnecessary addition to the wording of an otherwise good question.

5.130. C. Dimes are Poisson with rate: (0.1)(0.15) = 0.015 per minute.
Number of dimes in 30 minutes is Poisson with mean: (30)(0.015) = 0.45.
Prob[at least 1 dime first 30 minutes and at least 2 dimes in first 60 minutes] =
Prob[1 dime in first 30 minutes] Prob[at least 1 dime in next 30 minutes] +
Prob[at least 2 dimes in first 30 minutes] =
$(0.45\ e^{-0.45})\ (1 - e^{-0.45}) + 1 - e^{-0.45} - 0.45\ e^{-0.45}$ = **0.1794**.
<u>Comment</u>: What happens over the first 30 minutes is independent of what happens over the second 30 minutes.

5.131. D. The number of cash transactions over 2 hours is Poisson with mean:
(2)(10%)(30) = 6.
Prob[at least 3] = 1 - f(0) - f(1) - f(2) = $1 - e^{-6}\ (1 + 6 + 6^2/2)$ = **93.80%**.

5.132. E. The number of calls from each region are independent Poissons with means:
2, 3, 1, and 4.
Prob[at least one call from each region] =
Prob[> 0 from Region 1] Prob[> 0 from Reg. 2] Prob[> 0 from Reg. 3] Prob[> 0 from Reg. 4] =
$(1 - e^{-2})\ (1 - e^{-3})\ (1 - e^{-1})\ (1 - e^{-4})$ = **50.98%**.

5.133. B. Thinning, we get three independent Poisson Processes:
PD with $\lambda = 3$, BI with $\lambda = 2$, and Both with $\lambda = 5$.
Prob[4 PD] = $e^{-3}\ 3^4 / 4!$ = 0.1680. Prob[2 BI] = $e^{-2}\ 2^2 / 2!$ = 0.2707.
Prob[6 Both] = $e^{-5}\ 5^6 / 6!$ = 0.1462.
Thus, the desired probability is: (0.1680)(0.2707)(0.1462) = **0.665%**.

5.134. C. The number of job offer over 4 months is Poisson with mean: (4)(3) = 12.
For the LogNormal, $F[50{,}000] = \Phi[\frac{\ln[50{,}000] - 10.5}{0.2}] = \Phi[1.60] = 0.9452$.
Thus the number of acceptable job offers during 4 months is Poisson with mean:
(1 - 0.9452)(12) = 0.6576.
⇒ The chance of no acceptable job offers during 4 months is: $e^{-0.6576}$ = **0.518**.
Comment: Using a computer with no rounding, the answer is 0.517333.
As shown in the tables attached to your exam, for the LogNormal Distribution:
$F(x) = \Phi\left[\frac{\ln(x) - \mu}{\sigma}\right]$.

5.135. C. The number of catastrophes over 6 months is Poisson.
The expected number of catastrophes over 6 months (one half year) is:
(1/2)(5) + (1/2)(1/5) + (1/2)(1) = 3.1.
Prob[at least 2 catastrophes] = $1 - e^{-3.1} - 3.1e^{-3.1}$ = **0.8153**.

5.136. E. Your average wait is: 1/20 hours = 3 minutes.
Your average ride is: (0.7)(30) + (0.3)(18) = 26.4 minutes.
X_1 = 3 + 26.4 = 29.4 minutes.
Express trains are Poisson with mean: (0.3)(20) = 6 per hour.
Your coworker's average wait is: 1/6 hours = 10 minutes.
X_2 = 10 + 18 = 28 minutes.
$X_1 - X_2$ = 29.4 - 28 = **1.4** minutes.
Comment: Similar to 3, 11/02, Q.20.

Section 6, Mixing Poisson Processes[64]

Assume there are two types of risks, each with claims given by a Poisson Process:

Type	Annual Rate	Probability
A	0.3	60%
B	0.5	40%

Exercise: For an insured of unknown type, what is the probability that this insured has one claim next year?
[Solution: For type A, f(1) = 0.3 $e^{-0.3}$ = 0.222. For type B, f(1) = 0.5 $e^{-0.5}$ = 0.303.
For the mixture, f(1) = (60%)(0.222) + (40%)(0.303) = 0.254.]

One can determine the other densities of this mixture in a similar manner:[65]

Number of Claims	Probability for Type A	Probability for Type B		Probability for Mixture
0	0.74082	0.60653		0.68710
1	0.22225	0.30327		0.25465
2	0.03334	0.07582		0.05033
3	0.00333	0.01264		0.00705
4	0.00025	0.00158		0.00078
5	0.00002	0.00016		0.00007
6	0.00000	0.00001		0.00001
SUM	1.00000	1.00000		1.00000

The density function of the mixed distribution, is the mixture of the density function for specific values of the parameter that is mixed.

Moments of Mixed Distributions:

In a similar manner the mean frequency for an insured of unknown type is:
(60%)(0.3) + (40%)(0.5) = 0.38.

The mean of a mixed distribution is the mixture of the means.

In general, **the nth moment of a mixed distribution is the mixture of the nth moments** for specific values of the parameter λ: $E[X^n] = E_\lambda[E[X^n \mid \lambda]]$.[66]

[64] See Section 1.1.3 of "Poisson Processes and Mixture Distributions" by Daniel.
[65] There is a very small probability of more than 6 claims.
[66] This result holds in general, not just for mixing Poissons.

Exercise: For this mixed distribution, what is the second moment?
[Solution: For type A, the second moment is: variance + $\text{mean}^2 = 0.3 + 0.3^2 = 0.39$.
For type B, the second moment is: variance + $\text{mean}^2 = 0.5 + 0.5^2 = 0.75$.
Therefore the second moment of the mixture is: (60%)(0.39) + (40%)(0.75) = 0.534.]

One could instead determine the moments of this mixture by working with the previously calculated densities:

Number of Claims	Probability for Type A	Probability for Type B	Probability for Mixture		Probability times Number of Claims	Probability times Square Number of Claims
0	0.74082	0.60653	0.68710		0.00000	0.00000
1	0.22225	0.30327	0.25465		0.25465	0.25465
2	0.03334	0.07582	0.05033		0.10066	0.20131
3	0.00333	0.01264	0.00705		0.02116	0.06349
4	0.00025	0.00158	0.00078		0.00313	0.01251
5	0.00002	0.00016	0.00007		0.00036	0.00180
6	0.00000	0.00001	0.00001		0.00003	0.00021
SUM	1.00000	1.00000	1.00000		0.38000	0.53398

Exercise: For this mixed distribution, what is the variance?
[Solution: $0.534 - 0.38^2 = 0.3896$.
Comment: For a mixture of Poissons, the variance is greater than the mean.]

Exercise: Each insured is Poisson with mean λ.
The lambda values vary across the portfolio via $u(\lambda)$.
What is the mean for a mixture of Poissons?
[Solution: For a given value of lambda, the mean of a Poisson Distribution is λ. We need to weight these first moments together via the density of lambda $u(\lambda)$: $\int \lambda\, u(\lambda)\, d\lambda$ = mean of u.]

In general, the mean of a mixture of Poissons is the mean of the mixing distribution.[67]

Exercise: Each insured is Poisson with mean λ. λ is uniformly distributed from 0.05 to 0.15.
What is the mean of the mixed distribution?
[Solution: The mean of the mixture is the mean of the uniform mixing distribution, 0.10.]

[67] This result will hold whenever the parameter being mixed is the mean, as it was in the case of the Poisson.

Mixing Versus Thinning:

Mixing involves different types of insureds or risks.

For example: Good and Bad, each of which follows a Poisson Process, but with $\lambda = 0.03$ for Good insureds and $\lambda = 0.07$ for Bad insureds.

In order to simulate this mixture:[68]
First simulate the type of insured.
Then simulate how many claims that insured had and when they occurred.

Thinning involves different types of claims or events.

For example, claims follow a Poisson Process.
Then the large and small claims are two independent Poisson Processes.

Another example, Lucky Tom finds coins at a Poisson rate.
The dimes, nickels, and pennies are three independent Poisson processes.

In order to simulate this example of thinning:
First simulate when the coin is found.
Then simulate the type of coin.

[68] Simulation is not on this exam.

Problems:

Use the following information for the next 6 questions:
An insurance company insures 2 indistinguishable populations.
The claims frequency of each insured follows a Poisson process.

Population (class)	Annual claims intensity	Probability of being in class
Good	0.05	2/3
Bad	0.10	1/3

6.1 (1 point) What is the expected number of claims next year for an insured picked at random?
A. 6% B. 6.7% C. 7.5% D. 8% E. 9%

6.2 (2 points) What is the variance of the distribution of the number of claims next year from an insured picked at random?
A. 0.061 B. 0.064 C. 0.067 D. 0.070 E. 0.073

6.3 (1 point) What is the probability of one claim next year from an insured picked at random?
A. 3% B. 4% C. 5% D. 6% E. 7%

6.4 (2 points) What is the probability of a total of two claims over the next five years from an insured picked at random?
A. 3% B. 4% C. 5% D. 6% E. 7%

6.5 (2 points) For an insured that had 1 claim in year 1, calculate this insured's probability of having 2 claims in year 2.
A. 0.025% B. 0.05% C. 0.1% D. 0.2% E. 0.3%

6.6 (2 points) For an insured that had 1 claim in year 1, calculate the probability that the time until the next claim from this insured is greater than 10 years.
A. 40% B. 43% C. 46% D. 49% E. 52%

Use the following information for the next 2 questions:
(i) Claims for individual insureds follow a Poisson Process.
(ii) Half of the insureds have expected annual claim frequency of 40%.
(iii) The other half of the insureds have expected annual claim frequency of 60%.

6.7 (2 points) What is the probability that the time until the next claim from a randomly selected insured will be more than 3 years?
(A) 19% (B) 20% (C) 21% (D) 22% (E) 23%

6.8 (2 points) A randomly selected insured has made 2 claims in each of the first two policy years. What is the probability that the time until the next claim from this insured will be more than 3 years?
(A) 19% (B) 20% (C) 21% (D) 22% (E) 23%

Use the following information for the next four questions:
(i) Claims for individual insureds follow a Poisson Process.
(ii) λ is uniformly distributed over the interval from 2% to 10%.

6.9 (1 point) What is the mean claim frequency.
A. 3% B. 4% C. 5% D. 6% E. 7%

6.10 (2 points) For an insured picked at random, calculate the probability that there are no claims over the next 3 years.
A. 82% B. 84% C. 86% D. 88% E. 90%

6.11 (3 points) For an insured picked at random, calculate the probability that there is one claim over the next 3 years.
A. 15% B. 17% C. 19% D. 21% E. 23%

6.12 (1 point) An insured is picked at random. What is the mean time until the first claim?
A. 17 B. 18 C. 19 D. 20 E. 21

6.13 (3 points) The number of minutes between uptown subway trains at the Canal Street Station is uniformly distributed from 2 to 8. Passengers for this uptown train arrive at this station at a Poisson rate of 10 per minute. Let X denote the number of passengers waiting when the next uptown train arrives. Determine the variance of X.
(A) 250 (B) 275 (C) 300 (D) 325 (E) 350

6.14 (2 points) A tyrannosaur eats only scientists.
It eats scientists at a Poisson rate that varies by day:
On a warm day the tyrannosaur eats an average of 1.5 scientists.
On a cold day the tyrannosaur eats an average of 0.5 scientists.
On a future day that is equally likely to be warm or cold, what is the probability that the tyrannosaur eats exactly two scientists?
(A) 12% (B) 14% (C) 16% (D) 18% (E) 20%

Use the following information for the next three questions:
An insurance company sells two types of policies with the following characteristics:

Type of Policy	Proportion of Total Policies	Annual Claim Frequency
I	25%	Poisson with λ = 0.25
II	75%	Poisson with λ = 0.50

6.15 (1 point)
What is the probability that an insured picked at random will have no claims next year?
A. 50% B. 55% C. 60% D. 65% E. 70%

6.16 (1 point)
What is the probability that an insured picked at random will have one claim next year?
A. less than 30%
B. at least 30% but less than 35%
C. at least 35% but less than 40%
D. at least 40% but less than 45%
E. at least 45%

6.17 (1 point)
What is the probability that an insured picked at random will have two claims next year?
A. 4% B. 6% C. 8% D. 10% E. 12%

6.18 (3 points)
The number of times Donald loses his temper on a given day is Poisson with mean λ.
However, λ varies randomly from day to day and is uniformly distributed from 1 to 3.
Using the normal approximation with continuity correction, calculate the probability of Donald losing his temper at least 5 times tomorrow.
A. 1% B. 2% C. 3% D. 4% E. 5%

Use the following information for the next 5 questions:

Each insured's claim frequency follows a Poisson process.
There are three types of insureds as follows:

Type	A Priori Probability	Mean Annual Claim Frequency (Poisson Parameter)
A	60%	1
B	30%	2
C	10%	3

6.19 (1 point) What is the chance of a single individual having 4 claims in a year?
A. less than 0.03
B. at least 0.03 but less than 0.04
C. at least 0.04 but less than 0.05
D. at least 0.05 but less than 0.06
E. at least 0.06

6.20 (1 point) What is the mean of this mixed distribution over one year?
A. 1.1 B. 1.2 C. 1.3 D. 1.4 E. 1.5

6.21 (2 points) What is the variance of this mixed distribution over one year?
A. less than 2.0
B. at least 2.0 but less than 2.1
C. at least 2.1 but less than 2.2
D. at least 2.2 but less than 2.3
E. at least 2.3

6.22 (1 point) An insured is picked at random. What is the mean time until the first claim?
A. 0.7 B. 0.8 C. 0.9 D. 1.0 E. 1.1

6.23 (2 points) An insured is picked at random.
What is the probability that the time until the first claim is more than 0.4?
A. 48% B. 51% C. 54% D. 57% E. 60%

Use the following information for the next 7 questions:
On his daily walk, Clumsy Klem loses coins at a Poisson rate.
At random, on half the days, Klem loses coins at a rate of 0.2 per minute.
On the other half of the days, Klem loses coins at a rate of 0.6 per minute.
The rate on any day is independent of the rate on any other day.

6.24 (2 points) Calculate the probability that Clumsy Klem loses exactly one coin during the sixth minute of today's walk.
(A) 0.21 (B) 0.23 (C) 0.25 (D) 0.27 (E) 0.29

6.25 (2 points) Calculate the probability that Clumsy Klem loses exactly one coin during the first two minutes of today's walk.
A. Less than 32%
B. At least 32%, but less than 34%
C. At least 34%, but less than 36%
D. At least 36%, but less than 38%
E. At least 38%

6.26 (2 points) Let A = the number of coins that Clumsy Klem loses during the first minute of today's walk. Let B = the number of coins that Clumsy Klem loses during the first minute of tomorrow's walk. Calculate Prob[A + B = 1].
(A) 0.30 (B) 0.32 (C) 0.34 (D) 0.36 (E) 0.38

6.27 (2 points) Calculate the probability that Clumsy Klem loses exactly one coin during the third minute of today's walk and exactly one coin during the fifth minute of tomorrow's walk.
(A) 0.05 (B) 0.06 (C) 0.07 (D) 0.08 (E) 0.09

6.28 (2 points) Calculate the probability that Clumsy Klem loses exactly one coin during the third minute of today's walk and exactly one coin during the fifth minute of today's walk.
(A) 0.05 (B) 0.06 (C) 0.07 (D) 0.08 (E) 0.09

6.29 (2 points) Calculate the probability that Clumsy Klem loses exactly one coin during the first four minutes of today's walk and exactly one coin during the first four minutes of tomorrow's walk.
A. Less than 8.5%
B. At least 8.5%, but less than 9.0%
C. At least 9.0%, but less than 9.5%
D. At least 9.5%, but less than 10.0%
E. At least 10.0%

6.30 (3 points) Calculate the probability that Clumsy Klem loses exactly one coin during the first two minutes of today's walk, and exactly 2 coins during the following 3 minutes of today's walk.
(A) 0.05 (B) 0.06 (C) 0.07 (D) 0.08 (E) 0.09

Solutions to Problems:

6.1. B. (0.05)(2/3) + (0.10)(1/3) = **0.0667**.

6.2. C. For a Good insured, the second moment is: variance + mean^2 = 0.05 + 0.05^2 = 0.0525.
For a Bad insured, the second moment is: variance + mean^2 = 0.1 + 0.1^2 = 0.11.
The second moment of the mixture is: (0.0525)(2/3) + (0.11)(1/3) = 0.0717.
The variance of the mixture is: 0.0717 - 0.0667^2 = **0.0672**.
Comment: The variance is greater than the mean.

6.3. D. For a Good insured, the probability of one claim next year is: 0.05 $e^{-0.05}$ = 0.04756.
For a Bad insured, the probability of one claim next year is: 0.1 $e^{-0.1}$ = 0.09048.
(0.04756)(2/3) + (0.09048)(1/3) = **0.0619**.

6.4. B. For a Good insured, the probability of 2 claims over 5 years is: 0.25^2 $e^{-0.25}$ / 2 = 0.0243.
For a Bad insured, the probability of 2 claims over 5 years is: 0.5^2 $e^{-0.5}$ / 2 = 0.0758.
(0.0243)(2/3) + (0.0758)(1/3) = **0.0415**.

6.5. E. Prob[1 claim | good] = 0.05 $e^{-0.05}$ = 0.04756.
Prob[1 claim | bad] = 0.1 $e^{-0.1}$ = 0.09048.
By Bayes Theorem, the probability that an insured that had one claim in year one is good is:

$$\frac{\text{Prob[Good] Prob[1 claim | Good]}}{\text{Prob[Good] Prob[1 claim | Good] + Prob[Bad] Prob[1 claim | Bad]}} =$$

$$\frac{(2/3)(0.04756)}{(2/3)(0.04756) + (1/3)(0.09048)} = 51.25\%.$$

By Bayes Theorem, the probability that an insured that had one claim in year one is bad is:

$$\frac{(1/3)(0.09048)}{(2/3)(0.04756) + (1/3)(0.09048)} = 48.75\% = 1 - 51.25\%.$$

Prob[2 claims | good] = 0.05$^2 e^{-0.05}$/2 = 0.001189. Prob[2 claims | bad] = 0.1$^2 e^{-0.1}$/2 = 0.004524.
Therefore, this insured's probability of having 2 claims in year 2 is:
(51.25%)(0.001189) + (48.75%)(0.004524) = **0.281%**.
Comment: Bayes Theorem is discussed briefly in "Mahler's Guide to Statistical Learning."

6.6. D. S(10 | good) = exp[-(10)(0.05)] = 0.6065. S(10 | bad) = exp[-(10)(0.1)] = 0.3679.
Using the results of Bayes Theorem from the previous solution:
S(10) = (51.25%)(0.6065) + (48.75%)(0.3679) = **49.0%**.

6.7. E. (0.5)exp[-(3)(0.4)] + (0.5)exp[-(3)(0.6)] = **23.3%.**

6.8. B. (0.228)exp[-(3)(0.4)] + (0.772)exp[-(3)(0.6)] = **19.6%.**

6.9. D. The mean of the mixed distribution is: E[λ] = (2% + 10%)/2 = **6%**.

6.10. B. Over three years, the number of claims is Poisson with mean 3λ.
The chance of no claim from this Poisson distribution is: $e^{-3\lambda}$.
We average over the possible values of λ:

$$\text{Prob(0 claims)} = (1/0.08) \int_{0.02}^{0.10} e^{-3\lambda}\, d\lambda = (1/0.08)\, (-e^{-3\lambda}/3) \Big]_{\lambda=0.02}^{\lambda=0.10} = \mathbf{83.7\%}.$$

6.11. A. Over three years, the number of claims is Poisson with mean 3λ.
The chance of one claim from this Poisson distribution is: $3\lambda e^{-3\lambda}$.
We average over the possible values of λ:

$$\text{Prob(1 claim)} = (1/0.08) \int_{0.02}^{0.10} 3\lambda\, e^{-3\lambda}\, d\lambda = (1/0.08)\, (-e^{-3\lambda}/3 - \lambda e^{-3\lambda}) \Big]_{\lambda=0.02}^{\lambda=0.10} = \mathbf{14.7\%}.$$

6.12. D. Given λ, the mean time until the first claim is $1/\lambda$.

$$\frac{1}{0.10 - 0.02} \int_{0.02}^{0.10} \frac{1}{\lambda}\, d\lambda = \{\ln(0.10) - \ln(0.02)\} / 0.08 = 12.5 \ln(5) = \mathbf{20.1}.$$

6.13. E. Given t, the time between trains, X is Poisson with mean 10t,
and 2nd moment: $10t + (10t)^2$.

$$E[X] = \int_2^8 10t\, (1/6)\, dt$$

$$E[X^2] = \int_2^8 (10t + 100t^2)\, (1/6)\, dt = 50 + (100/6)(512 - 8)/3 = 2850. \quad \text{Var}[X] = 2850 - 50^2 = \mathbf{350}.$$

Alternately, Var[X] = Var[E[X | t]] + E[Var[X | t]] = Var[10t] + E[10t] = 100Var[t] + 10E[t]
= $(100)(8 - 2)^2/12 + (10)(5)$ = **350**.
Comment: Similar to Exercise 5.50 in Introduction to Probability Models by Ross.

6.14. C. Prob[2 scientists | warm] = $1.5^2\, e^{-1.5} / 2$ = 25.10%.
Prob[2 scientists | cold] = $0.5^2\, e^{-0.5} / 2$ = 7.58%. (50%)(25.10%) + (50%)(7.58%) = **16.3%**.
Comment: The mean of the mixed distribution is: (1.5 + 0.5)/2 = 1.
However, $1^2 e^{-1}/2 = 18.4\% \neq 16.3\%$.
One should average the Poisson densities at 2 for the different types of days.

6.15. D. $(25\%)(e^{-0.25}) + (75\%)(e^{-0.5})$ = **65.0%**.

6.16. A. $(25\%)(0.25\, e^{-0.25}) + (75\%)(0.5\, e^{-0.5})$ = **27.6%**.

6.17. B. (25%)(0.25^2 $e^{-0.25}$/2) + (75%)(0.5^2 $e^{-0.5}$/2) = **6.3%**.

6.18. E. Each Poisson has mean λ and second moment $\lambda + \lambda^2$.
Thus the mixture has mean E[λ] = (1 + 3)/2 = 2.
The uniform distribution from 1 to 3 has second moment:

$$\int_1^3 x^2 /2 \, dx = (3^3 - 1^3)/6 = 13/3.$$

Thus the mixture has second moment of: E[$\lambda + \lambda^2$] = 2 + 13/3 = 19/3.
Thus the mixture has a variance of: 19/3 - 2^2 = 7/3.
Prob[mixed distribution ≥ 5] ≅ 1 - $\Phi[(4.5 - 2)/\sqrt{7/3}]$ = 1 - Φ[1.64] = **5%**.
Alternately, the variance of the mixture is:
E[Var[N | λ] + Var[E[N | λ] = E[λ] + Var[λ] = 2 + 2^2/12 = 7/3. Proceed as before.
Comment: Using the continuity correction, 4 is out and 5 is in:

4 4.5 5
|→

The uniform has a variance of: 13/3 - 2^2 = 1/3 = $(3 - 1)^2$/12 = width squared divided by 12.

6.19. D. Chance of observing 4 claims is $\lambda^4 e^{-\lambda}$/ 24.
Weight the chances of observing 4 claims by the a priori probabilities of λ.

Type	A Priori Probability	Poisson Parameter	Chance of 4 Claims
A	0.6	1	0.0153
B	0.3	2	0.0902
C	0.1	3	0.1680
Average			**0.053**

6.20. E. (60%)(1) + (30%)(2) + (10%)(3) = **1.5**.

6.21. A. For a Type A insured, the second moment is: variance + mean^2 = 1 + 1^2 = 2.
For a Type B insured, the second moment is: variance + mean^2 = 2 + 2^2 = 6.
For a Type C insured, the second moment is: variance + mean^2 = 3 + 3^2 = 12.
The second moment of the mixture is: (60%)(2) + (30%)(6) + (10%)(12) = 4.2.
The variance of the mixture is: 4.2 - 1.5^2 = **1.95**.
Comment: The variance is greater than the mean.

6.22. B. (60%)(1/1) + (30%)(1/2) + (10%)(1/3) = **0.783**.

6.23. D. (60%)($e^{-0.4}$) + (30%)($e^{-0.8}$) + (10%)($e^{-1.2}$) = **56.7%**.

6.24. C. For λ = 0.2, f(1) = $0.2e^{-0.2}$ = 0.1638. For λ = 0.6, f(1) = $0.6e^{-0.6}$ = 0.3293.
Prob[1 coin] = (0.5)(0.1638) + (0.5)(0.3293) = **24.65%**.

6.25. A. Over two minutes, the mean is either 0.4: f(1) = $0.4e^{-0.4}$ = 0.2681,
or the mean is 1.2: f(1) = $1.2e^{-1.2}$ = 0.3614.
Prob[1 coin] = (0.5)(0.2681) + (0.5)(0.3614) = **31.48%**.

6.26. C. Prob[0 coins during a minute] = $(0.5)e^{-0.2} + (0.5)e^{-0.6}$ = 0.6838.
Prob[1 coin during a minute] = $(0.5)0.2e^{-0.2} + (0.5)0.6e^{-0.6}$ = 0.2465.
Prob[A + B = 1] = Prob[A= 0]Prob[B] + Prob[A = 1]Prob[B = 0] = (2)(0.6838)(0.2465) = **33.71%**.
Comment: Since the minutes are on different days, their lambdas are picked independently.

6.27. B. Prob[1 coin during a minute] = $(0.5)0.2e^{-0.2} + (0.5)0.6e^{-0.6}$ = 0.2465.
Since the minutes are on different days, their lambdas are picked independently.
Prob[1 coin during 1 minute today and 1 coin during 1 minute tomorrow] =
Prob[1 coin during a minute] Prob[1 coin during a minute] = 0.2465^2 = **6.08%**.

6.28. C.
Prob[1 coin during third minute and 1 coin during fifth minute | λ = 0.2] =
$(0.2e^{-0.2})(0.2e^{-0.2})$ = 0.0268.
Prob[1 coin during third minute and 1 coin during fifth minute | λ = 0.6] =
$(0.6e^{-0.6})(0.6e^{-0.6})$ = 0.1084.
(0.5)(0.0268) + (0.5)(0.1084) = **6.76%**.
Comment: Since the minutes are on the same day, they have the same λ, whichever it is.

6.29. A. Prob[1 coin during 4 minutes] = $(0.5)0.8e^{-0.8} + (0.5)2.4e^{-2.4}$ = 0.2866.
Since the time intervals are on different days, their lambdas are picked independently.
Prob[1 coin during 4 minutes today and 1 coin during 4 minutes tomorrow] =
Prob[1 coin during 4 minutes] Prob[1 coin during 4 minutes] = 0.2866^2 = **8.33%**.

6.30. B. Prob[1 coin during two minutes and 2 coins during following 3 minutes | λ = 0.2] =
$(0.4e^{-0.4})\ (0.6^2e^{-0.6}/2)$ = 0.0265.
Prob[1 coin during two minutes and 2 coins during following 3 minutes | λ = 0.6] =
$(1.2e^{-1.2})(1.8^2e^{-1.8}/2)$ = 0.0968.
(0.5)(0.0265) + (0.5)(0.0968) = **6.17%**.

Section 7, Negative Binomial Distribution[69]

An important frequency distribution is the Negative Binomial, which has the Geometric as a special case.

Negative Binomial Distribution

Support: x = 0, 1, 2, 3 ... Parameters: $\beta > 0$, $r \geq 0$. **r = 1 is a Geometric Distribution**

The Negative Binomial is the discrete analog of the continuous Gamma Distribution with $\alpha = r$.

P. d. f. : $$\mathbf{f(x)} = \frac{r(r+1)...(r+x-1)}{x!} \frac{\beta^x}{(1+\beta)^{x+r}} = \binom{x+r-1}{x} \frac{\beta^x}{(1+\beta)^{x+r}}.$$

$$\mathbf{f(0)} = \frac{1}{(1+\beta)^r}.$$

$$f(1) = \frac{r\,\beta}{(1+\beta)^{r+1}}.$$

$$f(2) = \frac{r\,(r+1)\,\beta^2\,/2}{(1+\beta)^{r+2}}.$$

$$f(3) = \frac{r\,(r+1)\,(r+2)\,\beta^3\,/6}{(1+\beta)^{r+3}}.$$

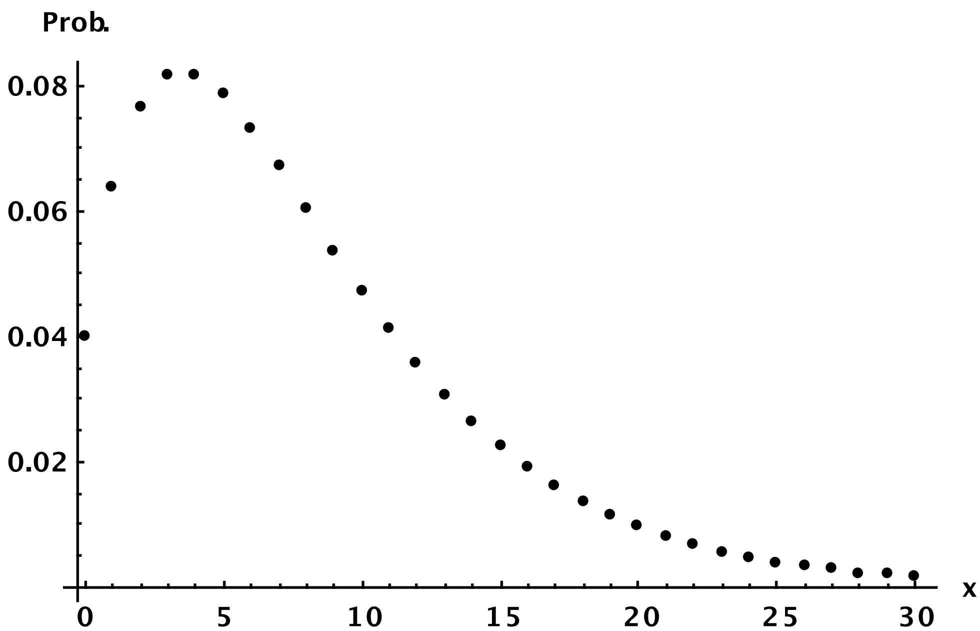
Mean = rβ **Variance = rβ(1+β)** **Variance / Mean = 1 + β > 1.**

[69] See Definition and Properties 1.21 in Daniel.

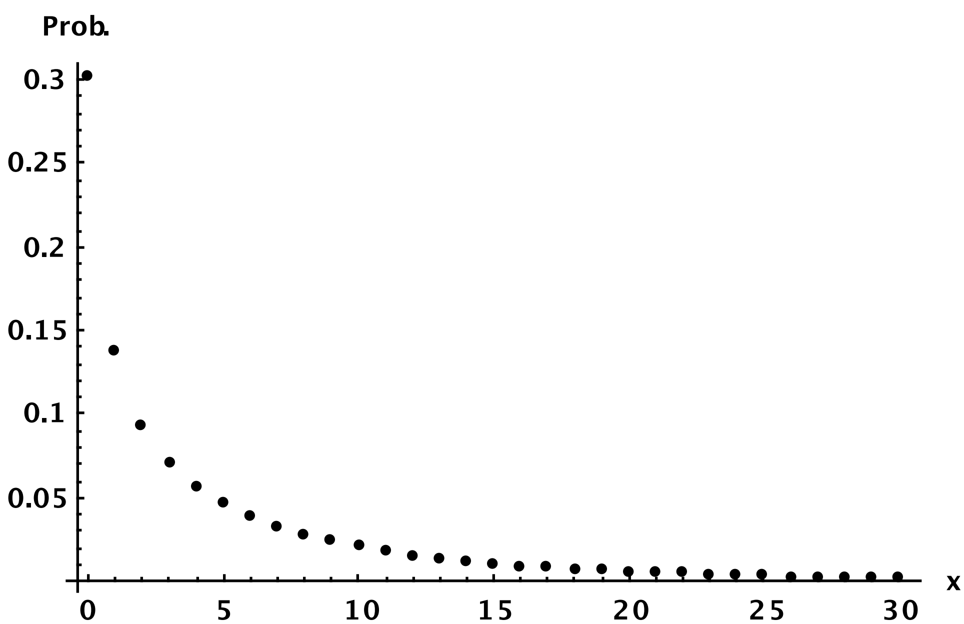
Examples of Negative Binomial Distributions:

A Negative Binomial Distribution with r = 2 and β = 4,

$f(x) = \frac{(2)(3)...(x+1)}{x!} \frac{4^x}{5^{x+2}} = (x+1)(0.04)(0.8^x)$:



A Negative Binomial Distribution with r = 0.5 and β = 10:



Here is a Negative Binomial Distribution with parameters β = 2/3 and r = 8:

Number of Claims	f(x)	F(x)	Number of Claims times f(x)	Square of Number of Claims times f(x)
0	0.0167962	0.0167962	0.00000	0.00000
1	0.0537477	0.0705439	0.05375	0.05375
2	0.0967459	0.1672898	0.19349	0.38698
3	0.1289945	0.2962843	0.38698	1.16095
4	0.1418940	0.4381782	0.56758	2.27030
5	0.1362182	0.5743964	0.68109	3.40546
6	0.1180558	0.6924522	0.70833	4.25001
7	0.0944446	0.7868968	0.66111	4.62779
8	0.0708335	0.8577303	0.56667	4.53334
9	0.0503705	0.9081007	0.45333	4.08001
10	0.0342519	0.9423527	0.34252	3.42519
11	0.0224194	0.9647721	0.24661	2.71275
12	0.0141990	0.9789711	0.17039	2.04465
13	0.0087378	0.9877089	0.11359	1.47669
14	0.0052427	0.9929516	0.07340	1.02757
15	0.0030757	0.9960273	0.04614	0.69204
16	0.0017685	0.9977959	0.02830	0.45275
17	0.0009987	0.9987946	0.01698	0.28863
18	0.0005548	0.9993494	0.00999	0.17977
19	0.0003037	0.9996531	0.00577	0.10964
20	0.0001640	0.9998171	0.00328	0.06560
21	0.0000875	0.9999046	0.00184	0.03857
22	0.0000461	0.9999507	0.00101	0.02232
23	0.0000241	0.9999747	0.00055	0.01273
24	0.0000124	0.9999872	0.00030	0.00716
25	0.0000064	0.9999935	0.00016	0.00398
26	0.0000032	0.9999968	0.00008	0.00218
27	0.0000016	0.9999984	0.00004	0.00119
28	0.0000008	0.9999992	0.00002	0.00064
29	0.0000004	0.9999996	0.00001	0.00034
30	0.0000002	0.9999998	0.00001	0.00018
Sum	0.9999998		5.33333	37.33314

For example, $f(5) = \frac{(2/3)^5}{(1+2/3)^{8+5}} \frac{12!}{5!\,7!} = \frac{(0.000171993)\,(479{,}001{,}600)}{(120)\,(5040)} = 0.136.$

The mean is: $r\beta$ = (8) (2/3) = 5.333. The variance is: 8(2/3)(1 + 2/3) = 8.89.

The variance can also be computed as: (mean)$(1+\beta)$ = 5.333(5/3) = 8.89.

The variance is indeed = $E[X^2] - E[X]^2$ = 37.333 - 5.3333^2 = 8.89.

Mean and Variance of the Negative Binomial Distribution:

The **mean of the Negative Binomial is rβ**
and the variance of the Negative Binomial is rβ(1+β).
Since $\beta > 0$, $1 + \beta > 1$, **for the Negative Binomial Distribution the variance is greater than the mean**.

Non-Integer Values of r:

Note that even if r is not integer, the binomial coefficient in the front of the Negative Binomial Density can be calculated as: $\binom{x+r-1}{x} = \frac{(x+r-1)!}{x!\,(r-1)!} = \frac{(x+r-1)\,(x+r-2)\,...\,(r)}{x!}$.

For example with r = 6.2 if one wanted to compute f(4), then the binomial coefficient in front is:

$$\binom{4+6.2-1}{4} = \binom{9.2}{4} = \frac{9.2!}{5.2!\,4!} = \frac{(9.2)\,(8.2)\,(7.2)\,(6.2)}{4!} = 140.32.$$

Note that the numerator has 4 factors; in general it will have x factors. These four factors are: 9.2! / (9.2-4)! = 9.2!/5.2!, or if you prefer: $\Gamma(10.2) / \Gamma(6.2)$ = (9.2)(8.2)(7.2)(6.2).

As shown in Loss Models, in general one can rewrite the density of the Negative Binomial as:

$$\mathbf{f(x) = \frac{r(r+1)...(r+x-1)}{x!} \frac{\beta^x}{(1+\beta)^{x+r}}}$$, where there are x factors in the product in the numerator.

Exercise: For a Negative Binomial with parameters r = 6.2 and β = 7/3, compute f(4).
[Solution: f(4) = {(9.2)(8.2)(7.2)(6.2)/4!} $(7/3)^4$ / $(1+ 7/3)^{6.2+4}$ = 0.0193.]

Negative Binomial as a Mixture of Poissons:

As discussed subsequently, when Poissons are mixed via a Gamma Distribution, the mixed distribution is always a Negative Binomial Distribution, with r = α = shape parameter of the Gamma and β = θ = scale parameter of the Gamma. The mixture of Poissons via a Gamma distribution produces a Negative Binomial Distribution and increases the variance above the mean.

Geometric Distribution

Support: x = 0, 1, 2, 3... Parameters: $\beta > 0$.

The Geometric Distribution is a special case of a Negative Binomial Distribution with r = 1.

The Geometric Distribution is the discrete analog of the continuous Exponential Distribution.

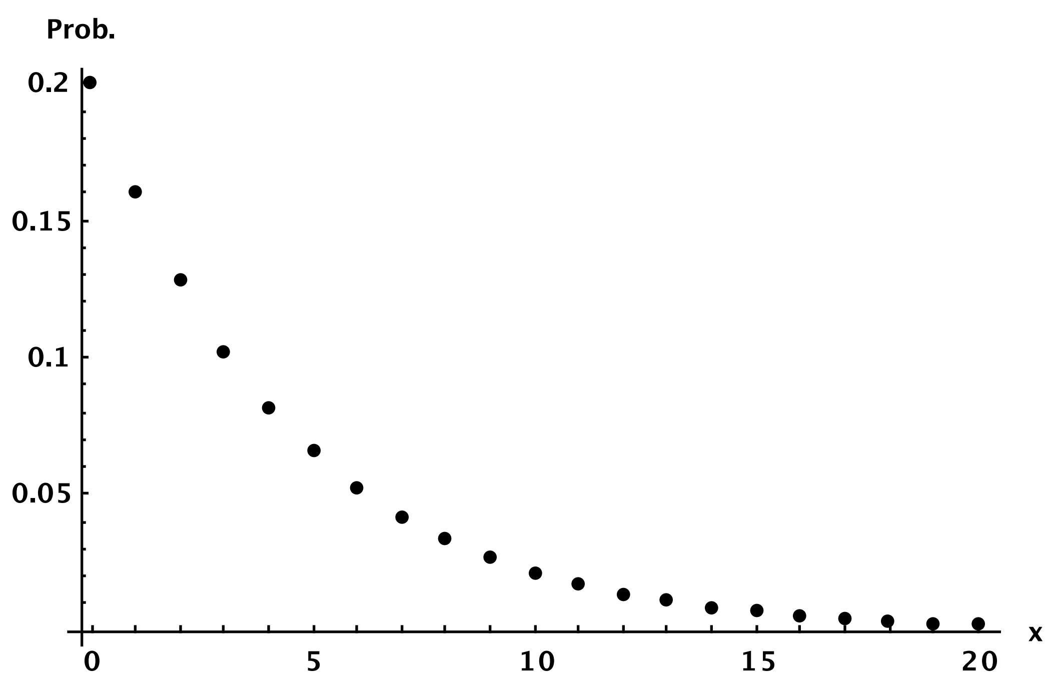
D. f. : $F(x) = 1 - \left(\frac{\beta}{1+\beta}\right)^{x+1}$

P. d. f. : $\mathbf{f(x)} = \frac{\beta^x}{(1+\beta)^{x+1}}$

Mean = β
Variance = β(1+β) **Variance / Mean = 1 + β > 1**.

The Negative Binomial Distribution with parameters β and r, with r integer, can be thought of as the sum of r independent Geometric distributions with parameter β.

The Negative Binomial Distribution for r = 1 is a Geometric Distribution.
A Geometric Distribution for $\beta = 4$:



<u>Adding Negative Binomial Distributions</u>:

Since the Negative Binomial is a sum of Geometric Distributions, if one sums independent Negative Binomials with the same β, then one gets another Negative Binomial, with the same β parameter and the sum of their r parameters.[70]

Exercise: X is a Negative Binomial with β = 1.4 and r = 0.8. Y is a Negative Binomial with β = 1.4 and r = 2.2. Z is a Negative Binomial with β = 1.4 and r = 1.7.
X, Y, and Z are independent of each other. What form does X + Y + Z have?
[Solution: X + Y + Z is a Negative Binomial with β = 1.4 and r = 0.8 + 2.2 + 1.7 = 4.7.]

If X is Negative Binomial with parameters β and r_1, and Y is Negative Binomial with parameters β and r_2, X and Y independent, then X + Y is Negative Binomial with parameters β and $r_1 + r_2$.

Specifically, the sum of n independent identically distributed Negative Binomial variables, with the same parameters β and r, is a Negative Binomial with parameters β and nr.

Exercise: X is a Negative Binomial with β = 1.4 and r = 0.8.
What is the form of the sum of 25 independent random draws from X?
[Solution: A random draw from a Negative Binomial with β = 1.4 and r = (25)(0.8) = 20.]

Thus if one had 25 exposures, each of which had an independent Negative Binomial frequency process with β = 1.4 and r = 0.8, then the portfolio of 25 exposures has a Negative Binomial frequency process with β = 1.4 and r = 20.

[70] This holds whether or not r is integral. This is analogous to adding independent Gammas with the same θ parameter. One obtains a Gamma, with the same θ parameter, but with the new α parameter equal to the sum of the individual α parameters.

Problems:

In the following four questions, the number of claims has a Negative Binomial distribution with parameters β = 0.5 and r = 4.

7.1 (1 point) What is the mean?
A. 0.5 B. 1 C. 2 D. 3 E. 4

7.2 (1 point) What is the variance?
A. 0.5 B. 1 C. 2 D. 3 E. 4

7.3 (2 points) What is the chance of having 3 claims?
A. less than 13%
B. at least 13% but less than 14%
C. at least 14% but less than 15%
D. at least 15% but less than 16%
E. at least 16%

7.4 (2 points) What is the chance of having 2 claims or less?
A. 60% B. 62% C. 64% D. 66% E. 68%

7.5 (2 points) For a Negative Binomial distribution with β = 0.8 and r = 1.4, what is the density at 4?
A. less than 3%
B. at least 3% but less than 4%
C. at least 4% but less than 5%
D. at least 5% but less than 6%
E. at least 6%

In the following four questions, the number of claims has a Geometric distribution with $\beta = 0.7$.

7.6 (1 point) What is the mean?
(A) 0.4 (B) 0.5 (C) 0.6 (D) 0.7 (E) 0.8

7.7 (1 point) What is the variance?
A. less than 1.0
B. at least 1.0 but less than 1.1
C. at least 1.1 but less than 1.2
D. at least 1.2 but less than 1.3
E. at least 1.3

7.8 (2 points) What is the probability of having 3 claims?
A. less than 3%
B. at least 3% but less than 4%
C. at least 4% but less than 5%
D. at least 5% but less than 6%
E. at least 6%

7.9 (2 points) What is the probability of having 3 claims or more?
A. 7% B. 8% C. 9% D. 10% E. 11%

Use the following information for the next two questions:
Six friends each have their own phone.
The number of calls each friend gets per night from telemarketers is Geometric with $\beta = 0.4$.
The number of calls each friend gets is independent of the others.

7.10 (2 points) Tonight, what is the probability that two of the friends get one or more calls from telemarketers, while the other four do not?
A. 30% B. 32% C. 34% D. 36% E. 38%

7.11 (2 points) Tonight, what is the probability that the friends get a total of two calls from telemarketers?
A. 11% B. 14% C. 17% D. 20% E. 23%

Solutions to Problems:

7.1. C. mean = $r\beta$ = (4)(0.5) = **2**.

7.2. D. variance = $r\beta(1+\beta)$ = (4)(0.5)(1.5) = **3**.

7.3. C. $f(3) = \{r(r+1)(r+2)/3!\}\beta^3/(1+\beta)^{3+r} = \{(4)(5)(6)/6\}\ 0.5^3/1.5^7$ = **14.6%**.

7.4. E. $f(0) = 1/1.5^4 = 0.1975$. $f(1) = (4)(0.5)/1.5^5 = 0.2634$. $f(2) = \{(4)(5)/2\}\ 0.5^2/1.5^6 = 0.2195$.
Prob[2 claims or less] = f(0) + f(1) + f(2) = 0.1975 + 0.2634 + 0.2195 = **68.0%**.

7.5. B. $f(4) = \{r(r+1)(r+2)(r+3)\} / 4!\}\ \beta^4 / (1+\beta)^{4+r} = \{(1.4)(2.4)(3.4)(4.4)/ 24\}\ 0.8^4/1.8^{5.4}$ = **3.6%**.

7.6. D. mean = β = **0.7**.

7.7. C. variance = $\beta(1+\beta)$ = (0.7)(1.7) = **1.19**.

7.8. C. $f(x) = \beta^x / (1+\beta)^{x+1}$. $f(3) = (0.7)^3 / (1.7)^{3+1}$ = **4.11%**.

7.9. A. 1 - {f(0) + f(1) + f(2)} = 1 - (0.5882 + 0.2422 + 0.0997) = **6.99%**.
Alternately, $S(x) = \{\beta/(1+\beta)\}^{x+1}$. $S(2) = (0.7/1.7)^3$ = **6.99%**.

7.10. B. For the Geometric, $f(0) = 1/(1+\beta) = 1/1.4$. 1 - f(0) = 0.4/1.4.
Prob[4 with 0 and 2 not with 0] = $\{6! / (4!\ 2!)\}\ (1/1.4)^4\ (0.4/1.4)^2$ **= 31.9%**.

7.11. E. The total number of calls is Negative Binomial with r = 6 and β = 0.4.
$f(2) = (r(r+1)/2!)\beta^2/(1+\beta)^{2+r} = ((6)(7)/2)0.4^2/1.4^8$ = **22.8%**.

Section 8, Gamma Distribution[71]

The mathematics of the Gamma Distribution comes up in many different applications. The Gamma Distribution can be used as a size of loss distribution. The distribution of the n^{th} event time from a homogeneous Poisson Process is Gamma. If each insured is Poisson with mean λ and λ is distributed via a Gamma, then one has the Gamma-Poisson frequency process to be discussed in a subsequent section.

Complete Gamma Function:[72]

The quantity $x^{\alpha-1}e^{-x}$ is finite for $x \geq 0$ and $\alpha \geq 1$. Since it declines quickly to zero as x approaches infinity, its integral from zero to ∞ exists. This is the much studied and tabulated (complete) Gamma Function.

$$\mathbf{\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}\, dt} = \theta^{-\alpha} \int_0^\infty t^{\alpha-1} e^{-t/\theta}\, dt \text{ , for } \alpha \geq 0,\ \theta \geq 0.$$

For alpha integer, $\Gamma(\alpha) = (\alpha - 1)!$ $\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$

$\Gamma(1) = 1$. $\Gamma(2) = 1$. $\Gamma(3) = 2$. $\Gamma(4) = 6$. $\Gamma(5) = 24$. $\Gamma(6) = 120$. $\Gamma(7) = 720$. $\Gamma(8) = 5040$.

Incomplete Gamma Function:[73]

The Incomplete Gamma Function is defined as:

$$\Gamma(\alpha\,;x) = \int_0^x t^{\alpha-1} e^{-t}\, dt \ / \ \Gamma(\alpha).$$

$\Gamma(\alpha\,;0) = 0$. $\Gamma(\alpha\,;\infty) = \Gamma(\alpha) / \Gamma(\alpha) = 1$. As discussed below, the Incomplete Gamma Function with the introduction of a scale parameter θ is the Gamma Distribution.

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[71] See Definition and Properties 1.20 in Daniel.
[72] See Appendix A of Loss Models. Also see the Handbook of Mathematical Functions, by M. Abramowitz, et. al.
[73] See Appendix A of Loss Models. Also see the Handbook of Mathematical Functions, by M. Abramowitz, et. al.

Gamma Distribution:

$F(x) = \Gamma(\alpha; x/\theta)$ **f(x) =** $\frac{(x/\theta)^{\alpha} e^{-x/\theta}}{x\,\Gamma(\alpha)} = \frac{x^{\alpha-1} e^{-x/\theta}}{\theta^{\alpha}\,\Gamma(a)}$, $x > 0$.

Mean = αθ **Variance = $\alpha\theta^2$** Second moment = $\alpha(\alpha+1)\theta^2$

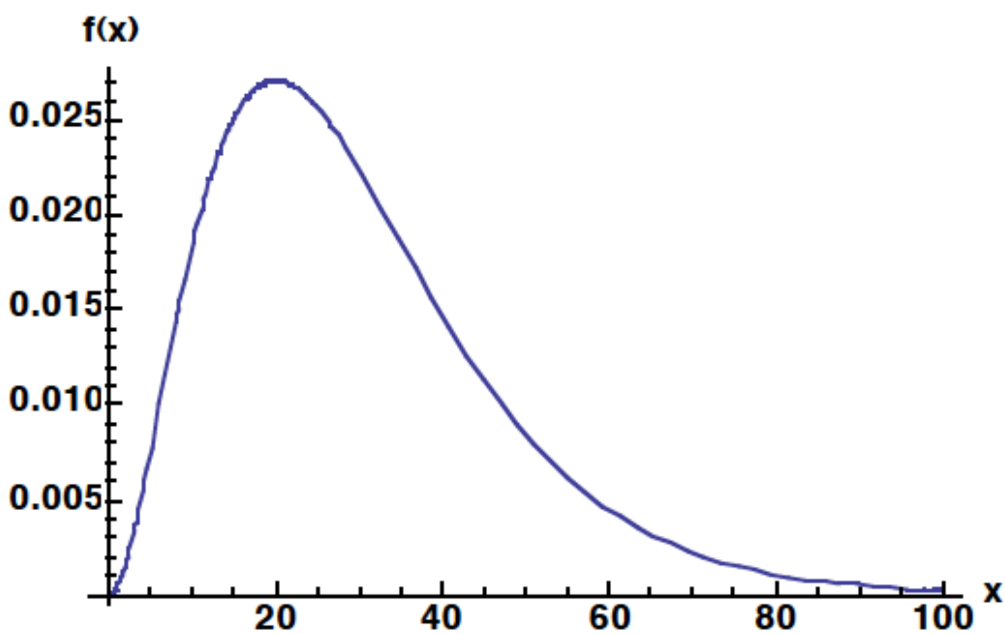
$E[X^n] = \theta^n \prod_{i=0}^{n-1} (\alpha + i) = \theta^n\,(\alpha)...(\alpha + n-1)$.

The sum of n independent identically distributed variables which are Gamma with parameters α and θ is a Gamma distribution with parameters nα and θ.
For α = a positive integer, the Gamma distribution is the sum of α independent variables each of which follows an Exponential distribution. **For α = 1 you get the Exponential Distribution.**

If one multiplies by a constant c a Gamma variable with parameters α and θ, then one obtains another Gamma variable but with parameters α and cθ.[74]

For very large α, the Gamma distribution approaches a symmetric Normal Distribution.

Here's a graph of the density function of a Gamma Distribution with α = 3 and θ = 10:



[74] For example, under 5% uniform inflation, a Gamma severity distribution would be multiplied by 1.05.
If the severity in the earlier year had parameters α and θ, then the severity in the later year has α and 1.05θ.

Erlang Distribution:

The special case of the Gamma where alpha is integer is called the Erlang Distribution.
For dealing with event times of Poisson Processes, we have alpha integer.[75]

Letting $\theta = 1/\lambda$ and $\alpha = n$, we have for the Erlang Distribution:

$f(x) = \lambda^n x^{n-1} e^{-\lambda x} / (n-1)!$, $x > 0$.

Mean = n/λ Variance = n/λ^2

Distribution of the Erlang at Time t = Prob[n^{th} event time ≤ t]

$$= \text{Prob[number of events by time } t \geq n] = 1 - \sum_{k=0}^{n-1} e^{-\lambda t} (\lambda t)^k / k!\ .$$

A Chi-Square Distribution with ν even is an Erlang Distribution with $\lambda = 1/2$ and $n = \nu/2$.

Integrals:

Integrals involving e^{-x} and powers of x can be written in terms of the Gamma function:

$$\int_0^\infty t^{\alpha-1} e^{-t/\theta} \, dt = \Gamma(\alpha)\, \theta^\alpha, \text{ or for integer n: } \int_0^\infty t^n e^{-ct} \, dt = n! / c^{n+1}.$$

Exercise: What is the integral from 0 to ∞ of: $t^3 e^{-t/10}$?
[Solution: With $\alpha = 4$ and $\theta = 10$, this integral is: $\Gamma(4)10^4 = (6)(10000) = 60{,}000$.]

This formula for "gamma-type" integrals is very useful for working with anything involving the Gamma distribution, for example the Gamma-Poisson process. It follows from the definition of the Gamma function and a change of variables.

The Gamma density in the Appendix of Loss Models is: $\theta^{-\alpha} x^{\alpha-1} e^{-x/\theta} / \Gamma(\alpha)$.
Since this probability density function must integrate to unity, the above formula for gamma-type integrals follows. This is a useful way to remember this formula on the exam.

Another useful way to remember this formula is from the moments of the Exponential Distribution: $E[X^k] = \theta^k k!$.

$$\Rightarrow \int_0^\infty t^k e^{-t/\theta} / \theta \, dt = \theta^k k! \Rightarrow \int_0^\infty t^k e^{-t/\theta} \, dt = \theta^{k+1} k!.$$

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75 For the mixing of Poissons via a Gamma, to be discussed subsequently, on the exam, alpha is more likely to be an integer, but need not be so.

Problems:

Use the following information for the next five questions:
X has a Gamma Distribution with parameters $\alpha = 4$ and $\theta = 10$.

8.1 (1 point) What is the density at 25?

Hint: $f(x) = \dfrac{x^{\alpha-1}\ e^{-x/\theta}}{\theta^{\alpha}\ \Gamma(\alpha)}$. $\Gamma[n] = (n-1)!$.

8.2 (1 point) What is the mean?

8.3 (1 point) What is the variance?

8.4 (1 point) What is the distribution of 2X?

8.5 (1 point) What is the distribution of X + X?

8.6 (2 points) X follows a Gamma Distribution, with $\alpha = 8$ and $\theta = 5$.
What is the probability that X is less than 25? (Use the Normal Approximation.)
A. less than 15%
B. at least 15% but less than 16%
C. at least 16% but less than 17%
D. at least 17% but less than 18%
E. at least 18%

Use the following information for the next three questions:
X has a Gamma Distribution with parameters $\alpha = 3$ and $\theta = 5$.

8.7 (1 point) What is the mean?

8.8 (1 point) What is the variance?

8.9 (1 point) What is the density at 10?

Hint: $\dfrac{x^{\alpha-1}\ e^{-x/\theta}}{\theta^{\alpha}\ \Gamma(\alpha)}$. $\Gamma[n] = (n-1)!$.

8.10 (2 points) Velma Dinkley uses her flashlight to explore haunted houses.
Velma's flashlight needs two batteries.
Each battery has a constant failure rate of 0.001.
In addition to the two batteries in her flashlight, Velma has a pack of four batteries.
When one of the batteries fails, Velma replaces it with a fresh one.
Let T be the total time her flashlight will function without buying any new batteries.
What is the distribution of T?

Solutions to Problems:

8.1. $f(25) = 25^3 \exp[-25/10]/ \{(10^4\ \Gamma(4)\} = 1282.58/ \{(10{,}000)(3!)\}$ = **0.0214**.

8.2. $E[X] = \alpha\theta$ = **40**.

8.3. $Var[X] = \alpha\theta^2$ = **400**.

8.4. When you multiply a Gamma variable by a constant you get another Gamma with the theta parameter multiplied by that constant.
2X has a **Gamma Distribution with $\alpha = 4$ and $\theta = 20$**.
Comment: This behavior of a Gamma Distribution is important under uniform inflation.

8.5. When you add two independent Gamma variables with the same theta, you get another Gamma with the sum of the alpha parameters.
X + X has a **Gamma Distribution with $\alpha = 8$ and $\theta = 10$**.
Comment: Each Gamma can be thought of as the sum of 4 independent, identically distributed Exponential Distributions with $\theta = 10$. The sum of two such Gammas is the sum of 8 independent, identically distributed Exponential Distributions with $\theta = 10$, or a Gamma Distribution with $\alpha = 8$ and $\theta = 10$.

8.6. A. For the Gamma Distribution: Mean = $\alpha\theta = 40$, Variance = $\alpha\theta^2 = 200$.
Thus the Standard Deviation is: $\sqrt{200} = 14.14$.
$Prob[X < 25] \cong \Phi[(25 - 40)/14.14] = \Phi[-1.06]$ = **14.5%**.
Comment: When applying the Normal Approximation to a continuous distribution, there is no "continuity correction" such as is applied when approximating a discrete distribution.

8.7. $E[X] = \alpha\theta$ = **15**.

8.8. $Var[X] = \alpha\theta^2$ = **75**.

8.9. $f(x) = \theta^{-\alpha} x^{\alpha-1}\ e^{-x/\theta} / \Gamma(\alpha)$.
$f(10) = 10^{3-1} \exp[-10/5] / \{(5^3\ \Gamma(3)\} = 13.53/ \{(125)(2!)\}$ = **0.0541**.

8.10. The time until the first battery fails is the minimum of two independent identically distributed Exponentials with $\lambda = 0.001$, which is an Exponential with failure rate 0.002 and mean 500.
A total of 5 batteries can fail, until her flashlight will not function without buying new batteries.
Thus T is the sum of 5 independent, identically distributed Exponentials,
a **Gamma Distribution with $\alpha = 5$ and $\theta = 500$.**
Comment: Similar to Exercise 5.23 in Introduction to Probability Models by Ross.

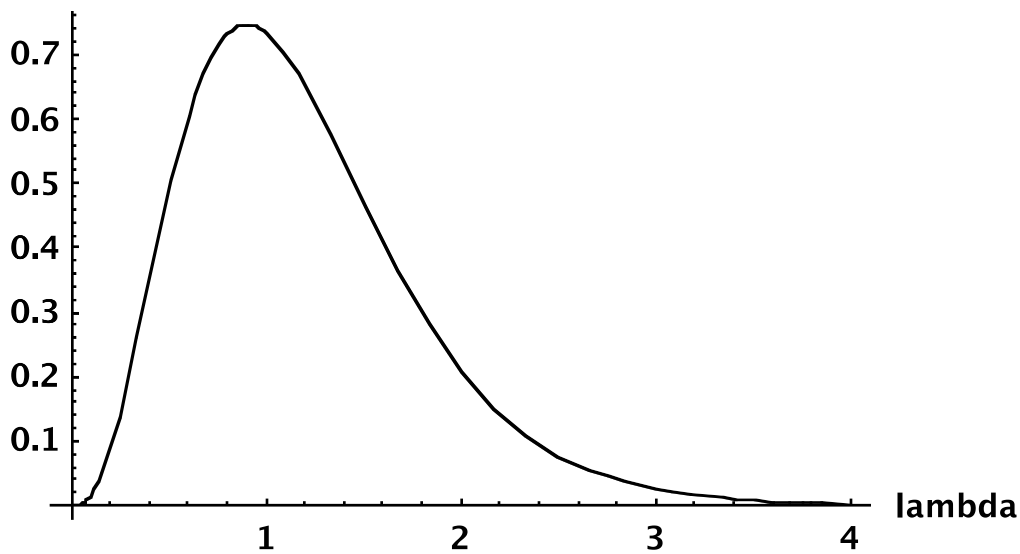
Section 9, Gamma-Poisson[76]

For a continuous mixture of homogeneous Poisson processes, the mixing distribution could be Gamma, including the special case of an Exponential ($\alpha = 1$).

The number of claims for a particular policyholder is a homogenous Poisson Process with rate λ.

The λ values over the portfolio of policyholders are Gamma distributed.

Given the value of λ, N is a homogeneous Poisson process with rate λ, but λ is itself random, with for example a Gamma Distribution with $\alpha = 4$ and $\theta = 0.3$:



The mean of this Gamma is: $\alpha\theta = (4)(0.3) = 1.2$.
The variance of this Gamma is: $\alpha\theta^2 = (4)(0.3^2) = 0.36$.

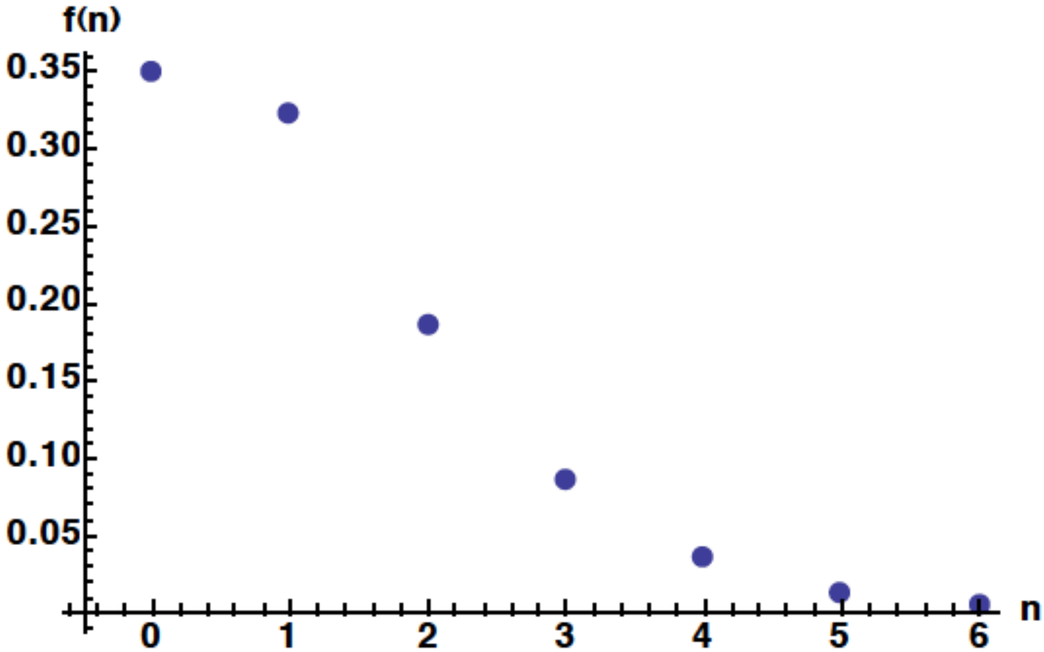
[76] See Page 12 of "Poisson Processes" by Daniel.
See also Example 5.29 in Introduction to Probability Models by Ross, not on the syllabus.

<u>Mixed Distribution Over One Year</u>:

Over a period of length one, for a homogenous Poisson process mixed by a Gamma Distribution, the mixed distribution is Negative Binomial with r = α and β = θ.

Beta rhymes with theta, while r goes with alpha.

For the example, the distribution of the number of claims next year from an insured picked at random is Negative Binomial, with $r = \alpha = 4$ and $\beta = \theta = 0.3$:



The mean of this Negative Binomial is: $r\beta = (4)(0.3) = 1.2$.
Variance of this Negative Binomial is: $r\beta(1+\beta) = (4)(0.3)(1.3) = 1.56$.

Exercise: Given the value of λ, N is a homogeneous Poisson process with rate λ, but λ is itself random, with a Gamma Distribution with $\alpha = 4$ and $\theta = 0.3$. What is Pr[N(1) = 2]?
[Solution: For one year, the mixed distribution is Negative Binomial with r = 4 and β = 0.3.
For an insured picked at random, the probability of 2 claims from this insured next year is:
$f(2) = \{r(r+1)/2!\}\ \beta^2/(1+\beta)^{2+r} = \{(4)(5)/2\}\ 0.3^2 / 1.3^6 = 18.6\%$.
<u>Comment</u>: This matches the value of f(x) shown in the above graph at x = 2.]

The Exponential Distribution is a special case of a Gamma Distribution, with α = 1.
Therefore, if λ is distributed via an Exponential with mean q, then N is distributed via a Negative Binomial with r = 1 and β = θ, in other words a Geometric Distribution with mean β = θ.

Mixed Distribution Over More than One Year:

Exercise: $u(\lambda) = 2e^{-2\lambda}$, $\lambda > 0$.
What is the probability of observing c claims by time Y from an individual picked at random?
[Solution: For a particular value of lambda, the probability of observing c claims by time Y is: $(\lambda Y)^c e^{-\lambda Y} / c!$. We take a weighted average over lambda, integrating versus $u(\lambda)$.

$$\int_0^\infty (\lambda Y)^c e^{-\lambda Y}/c! \ u(\lambda) \ d\lambda = (Y^c/c!) \int_0^\infty \lambda^c e^{-\lambda Y} 2e^{-2\lambda} \ d\lambda$$

$$= (2Y^c/c!) c!/(2 + Y)^{c+1} = (Y/2)^c / (1 + Y/2)^{c+1}.$$

Comment: This is a Geometric Distribution, with $\beta = Y/2$.]

In general, **if one has a (homogenous) Poisson Process mixed by a Gamma Distribution with parameters α and θ, then over a period of length Y, the mixed distribution is Negative Binomial with r = α and β = Yθ.**[77] [78]

Recall that the Exponential Distribution of a Gamma Distribution, with $\alpha = 1$.
Therefore, over a period of length Y, for a homogenous Poisson process mixed by an Exponential Distribution, the mixed distribution is Negative Binomial with $r = \alpha = 1$ and $\beta = Y\theta$, in other words a Geometric Distribution.

In the exercise, $\alpha = 1$, $\theta = 1/2$, $r = \alpha = 1$, and $\beta = Y\theta = Y/2$.

Exercise: $u(\lambda) = 2e^{-2\lambda}$, $\lambda > 0$. What is the probability of observing more than 2 claims by time 3 from an individual picked at random?
[Solution: The mixed distribution is a Geometric, with $\beta = 3/2$. $f(0) = 1/(1 + \beta) = 2/5$.
$f(1) = f(0)\beta/(1 + \beta) = f(0)3/5 = 6/25$. $f(2) = f(1)3/5 = 18/125$.
$1 - f(0) - f(1) - f(2) = 1 - 0.4 - 0.24 - 0.144 = 0.216$.]

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[77] See Example 1.24 in "Poisson Processes" by Daniel.
[78] For a Gamma-Poisson, the mixed distribution is Negative Binomial with $r = \alpha$ and $\beta = \theta$.
In this case, the number of claims is Poisson with mean $Y\lambda$.
Since λ is distributed via a Gamma with parameters α and θ, $Y\lambda$ is distributed via a Gamma with parameters α and $Y\theta$. Therefore, for this Gamma-Poisson, the mixed distribution is Negative Binomial with $r = \alpha$ and $\beta = Y\theta$.

Problems:

Use the following information to answer the next 2 questions:
The number of claims from a particular policyholder is a Poisson Process. The values of the claims intensities for the individual policyholders in a portfolio follow a Gamma distribution, with parameters $\alpha = 3$ and $\theta = 1/12$.

9.1 (2 points) What is the chance that an insured picked at random from the portfolio will have no claims over the next three years?
A. less than 35%
B. at least 35% but less than 40%
C. at least 40% but less than 45%
D. at least 50% but less than 55%
E. at least 55%

9.2 (2 points) What is the chance that an insured picked at random from the portfolio will have one claim over the next three years?
A. less than 35%
B. at least 35% but less than 40%
C. at least 40% but less than 45%
D. at least 50% but less than 55%
E. at least 55%

9.3 (3 points) When Blondie goes shopping, Dagwood stands outside on the sidewalk while he waits for her to finish. Pretty women walk by Dagwood at a Poisson rate of 2 per minute.
The time in minutes that Blondie spends shopping is Gamma distributed with $\alpha = 3$ and $\theta = 10$.
Using the Normal Approximation with continuity correction, what is the probability that at least 80 pretty women walk by Dagwood while he waits for Blondie?
A. 21% B. 23% C. 25% D. 27% E. 29%

9.4 (3 points) Given the value of λ, N is a homogeneous Poisson process with rate λ, but λ is itself random, with a Gamma Distribution with $\alpha = 2.5$ and $\theta = 1/2$. What is $\Pr[N(1) > 2]$?
A. less than 15%
B. at least 15% but less than 17%
C. at least 17% but less than 19%
D. at least 19% but less than 21%
E. at least 21%

Use the following information for the next 8 questions:
As he walks, Clumsy Klem loses coins at a Poisson rate.
The Poisson rate, expressed in coins per minute, is constant during any one day, but varies from day to day according to a gamma distribution with mean 0.2 and variance 0.016.
The denominations of coins are randomly distributed: 50% of the coins are worth 5;
30% of the coins are worth 10; and 20% of the coins are worth 25.

9.5 (2 points) Calculate the probability that Clumsy Klem loses exactly one coin during the tenth minute of today's walk.
(A) 0.09 (B) 0.11 (C) 0.13 (D) 0.15 (E) 0.17

9.6 (3 points) Calculate the probability that Clumsy Klem loses exactly two coins during the first 10 minutes of today's walk.
(A) 0.12 (B) 0.14 (C) 0.16 (D) 0.18 (E) 0.20

9.7 (4 points) Calculate the probability that the worth of the coins Clumsy Klem loses during his one-hour walk today is greater than 300.
A. 1% B. 3% C. 5% D. 7% E. 9%

9.8 (2 points) Calculate the probability that the sum of the worth of the coins Clumsy Klem loses during his one-hour walks each day for the next 5 days is greater than 900.
A. 1% B. 3% C. 5% D. 7% E. 9%

9.9 (2 points) During the first 10 minutes of today's walk, what is the chance that Clumsy Klem loses exactly one coin of worth 5, and possibly coins of other denominations?
A. 31% B. 33% C. 35% D. 37% E. 39%

9.10 (3 points) During the first 10 minutes of today's walk, what is the chance that Clumsy Klem loses exactly one coin of worth 5, and no coins of other denominations?
A. 11.6% B. 12.0% C. 12.4% D. 12.8% E. 13.2%

9.11 (3 points) Let A be the number of coins Clumsy Klem loses during the first minute of his walk today. Let B be the number of coins Clumsy Klem loses during the first minute of his walk tomorrow. What is the probability that A + B = 3?
A. 0.2% B. 0.4% C. 0.6% D. 0.8% E. 1.0%

9.12 (3 points) Let A be the number of coins Clumsy Klem loses during the first minute of his walk today. Let B be the number of coins Clumsy Klem loses during the first minute of his walk tomorrow. Let C be the number of coins Clumsy Klem loses during the first minute of his walk the day after tomorrow. What is the probability that A + B + C = 2?
A. 8% B. 10% C. 12% D. 14% E. 16%

Use the following information for the next four questions:
The claims from a particular insured follow a homogeneous Poisson Process.
The values of the rate λ for the individual insureds in a portfolio follow a Gamma distribution, with parameters $\alpha = 4$ and $\theta = 0.1$.

9.13 (1 point) For an insured picked at random from the portfolio, what is the expected number of claims over the next five years?

9.14 (2 points) What is the chance that an insured picked at random from the portfolio will have no claims over the next five years?
A. 20% B. 21% C. 22% D. 23% E. 24%

9.15 (2 points) What is the chance that an insured picked at random from the portfolio will have one claim over the next five years?
A. 22% B. 24% C. 26% D. 28% E. 30%

9.16 (2 points) For an insured picked at random, what is the average wait until the first claim?
Hint: For a Gamma Distribution, $E[X^k] = \theta^k \Gamma(\alpha + k) / \Gamma(\alpha)$, $k > -\alpha$.
A. 2.4 B. 2.7 C. 3.0 D. 3.3 E. 3.6

Use the following information for the next five questions:
$N \mid \Lambda$ is a Poisson random variable with mean Λ.
Λ is a Gamma random variable with parameters $\alpha = 2$ and $\theta = 5$.

9.17 (1 point) Determine E[N].

9.18 (1 point) Determine Var[N].

9.19 (2 points) Determine Pr[N = 0].

9.20 (2 points) Determine Pr[N = 1].

9.21 (2 points) Determine Pr[N = 2].

Use the following information for the next five questions:
N | Λ is a homogeneous Poisson Process with rate Λ.
Λ is a Gamma random variable with parameters $\alpha = 0.5$ and $\theta = 3$.

9.22 (1 point) Determine E[N(3)].

9.23 (1 point) Determine Var[N(3)].

9.24 (2 points) Determine Pr[N(3) = 0].

9.25 (2 points) Determine Pr[N(3) = 1].

9.26 (2 points) Determine Pr[N(3) = 2].

9.27 (3, 5/00, Q.4) (2.5 points) You are given:
(i) The claim count N has a Poisson distribution with mean Λ.
(ii) Λ has a gamma distribution with mean 1 and variance 2.
Calculate the probability that N = 1.
(A) 0.19 (B) 0.24 (C) 0.31 (D) 0.34 (E) 0.37

9.28 (3, 5/01, Q.3) (2.5 points) Glen is practicing his simulation skills.
He generates 1000 values of the random variable X as follows:
(i) He generates the observed value λ from the gamma distribution with $\alpha = 2$ and $\theta = 1$ (hence with mean 2 and variance 2).
(ii) He then generates x from the Poisson distribution with mean λ.
(iii) He repeats the process 999 more times: first generating a value λ, then generating x from the Poisson distribution with mean λ.
(iv) The repetitions are mutually independent.
Calculate the expected number of times that his simulated value of X is 3.
(A) 75 (B) 100 (C) 125 (D) 150 (E) 175

9.29 (3, 5/01, Q.15) (2.5 points) An actuary for an automobile insurance company determines that the distribution of the annual number of claims for an insured chosen at random is modeled by the negative binomial distribution with mean 0.2 and variance 0.4.
The number of claims for each individual insured has a Poisson distribution and the means of these Poisson distributions are gamma distributed over the population of insureds.
Calculate the variance of this gamma distribution.
(A) 0.20 (B) 0.25 (C) 0.30 (D) 0.35 (E) 0.40

9.30 (3, 11/01, Q.27) (2.5 points) On his walk to work, Lucky Tom finds coins on the ground at a Poisson rate. The Poisson rate, expressed in coins per minute, is constant during any one day, but varies from day to day according to a gamma distribution with mean 2 and variance 4. Calculate the probability that Lucky Tom finds exactly one coin during the sixth minute of today's walk.

(A) 0.22 (B) 0.24 (C) 0.26 (D) 0.28 (E) 0.30

9.31 (2 points) In 3, 11/01, Q.27, calculate the probability that Lucky Tom finds exactly one coin during the first two minutes of today's walk.

(A) 0.12 (B) 0.14 (C) 0.16 (D) 0.18 (E) 0.20

9.32 (3 points) In 3, 11/01, Q.27, let A = the number of coins that Lucky Tom finds during the first minute of today's walk. Let B = the number of coins that Lucky Tom finds during the first minute of tomorrow's walk. Calculate Prob[A + B = 1].

(A) 0.09 (B) 0.11 (C) 0.13 (D) 0.15 (E) 0.17

9.33 (3 points) In 3, 11/01, Q.27, calculate the probability that Lucky Tom finds exactly one coin during the third minute of today's walk and exactly one coin during the fifth minute of today's walk.

(A) 0.48 (B) 0.52 (C) 0.56 (D) 0.60 (E) 0.64

9.34 (3 points) In 3, 11/01, Q.27, calculate the probability that Lucky Tom finds exactly one coin during the first minute of today's walk, exactly two coins during the second minute of today's walk, and exactly three coins during the third minute of today's walk.

A. Less than 0.2%
B. At least 0.2%, but less than 0.3%
C. At least 0.3%, but less than 0.4%
D. At least 0.4%, but less than 0.5%
E. At least 0.5%

9.35 (2 points) In 3, 11/01, Q.27, calculate the probability that Lucky Tom finds exactly one coin during the first minute of today's walk and exactly one coin during the fifth minute of tomorrow's walk.

(A) 0.05 (B) 0.06 (C) 0.07 (D) 0.08 (E) 0.09

9.36 (2 points) In 3, 11/01, Q.27, calculate the probability that Lucky Tom finds exactly one coin during the first three minutes of today's walk and exactly one coin during the first three minutes of tomorrow's walk.

(A) 0.005 (B) 0.010 (C) 0.015 (D) 0.020 (E) 0.025

9.37 (3 points) In 3, 11/01, Q.27, calculate the probability that Lucky Tom finds exactly two more coins during the first minute of today's walk than the first minute of tomorrow's walk.

(A) 0.07 (B) 0.08 (C) 0.09 (D) 0.10 (E) 0.11

9.38 (3, 11/02, Q.5) (2.5 points) Actuaries have modeled auto windshield claim frequencies. They have concluded that the number of windshield claims filed per year per driver follows the Poisson distribution with parameter λ, where λ follows the gamma distribution with mean 3 and variance 3.

Calculate the probability that a driver selected at random will file no more than 1 windshield claim next year.

(A) 0.15 (B) 0.19 (C) 0.20 (D) 0.24 (E) 0.31

Solutions to Problems:

9.1. D. Over three years, the number of claims from a policy holder is Poisson with mean 3λ. 3λ follows a Gamma distribution with $\alpha = 3$ and $\theta = 3/12 = 1/4$.
The mixed distribution for the Gamma-Poisson is a Negative Binomial, with parameters $r = \alpha = 3$ and $\beta = \theta = 1/4$. $f(0) = 1/(1 + \beta)^r = 1/(5/4)^3 =$ **0.512**.

9.2. A. Negative Binomial, with parameters $r = \alpha = 3$ and $\beta = \theta = 1/4$.
Therefore $f(1) = r\beta/(1 + \beta)^{r+1} = (3)(1/4)/(5/4)^4 =$ **0.3072**.

9.3. E. Given Blondie spends t minutes shopping, the number of pretty women is Poisson with mean 2t. Since t follows a Gamma with $\alpha = 3$ and $\theta = 10$, 2t is Gamma with $\alpha = 3$ and $\theta = 20$.
Therefore, the mixed distribution is Negative Binomial with $r = 3$ and $\beta = 20$.
The number of pretty women has mean: (3)(20) = 60, and variance: (3)(20)(21) = 1260.
Prob[at least 80] $\cong 1 - \Phi[(79.5 - 60)/\sqrt{1260}] = 1 - \Phi[0.55] =$ **29.1%**.

9.4. B. Over one year, the mixed distribution is Negative Binomial with $r = 2.5$ and $\beta = 0.5$.
$f(0) = 1/(1+\beta)^r = 1/1.5^{2.5} = 36.29\%$. $f(1) = r\,\beta/(1+\beta)^{1+r} = (2.5)(0.5)/1.5^{3.5} = 30.24\%$.
$f(2) = \{r(r+1)/2!\}\beta^2 / (1+\beta)^{2+r} = \{(2.5)(3.5)/2\}\, 0.5^2 / 1.5^{4.5} = 17.64\%$.
$\Pr[N(1) > 2] = 1 - f(0) - f(1) - f(2) =$ **15.83%**.

9.5. D. For the Gamma, mean $= \alpha\theta = 0.2$, and variance $= \alpha\theta^2 = 0.016$.
Thus $\theta = 0.016/0.2 = 0.08$ and $\alpha = 2.5$.
This is a Gamma-Poisson, with mixed distribution a Negative Binomial, with $r = \alpha = 2.5$ and $\beta = \theta = 0.08$. $f(1) = r\beta/(1+\beta)^{1+r} = (2.5)(0.08)/(1+0.08)^{3.5} =$ **0.153**.
Comment: Similar to 3, 11/01, Q.27.

9.6. E. Over 10 minutes, the rate of loss is Poisson, with 10 times that for one minute.
λ has a Gamma distribution with $\alpha = 2.5$ and $\theta = 0.08$. $\Rightarrow$
10λ has a Gamma distribution with $\alpha = 2.5$ and $\theta = (10)(0.08) = 0.8$.
The mixed distribution is a Negative Binomial, with $r = \alpha = 2.5$ and $\beta = \theta = 0.8$.
$f(2) = \{r(r+1)/2\}\, \beta^2/(1+\beta)^{2+r} = \{(2.5)(3.5)/2\}\, 0.8^2 / (1+0.8)^{4.5} =$ **0.199**.

9.7. B. Mean value of a coin is: (50%)(5) + (30%)(10) + (20%)(25) = 10.5.
2nd moment of the value of a coin is: (50%)(5^2) + (30%)(10^2) + (20%)(25^2) = 167.5.
Over 60 minutes, the rate of loss is Poisson, with 60 times that for one minute.
λ has a Gamma distribution with α = 2.5 and θ = 0.08. $\Rightarrow$
60λ has a Gamma distribution with α = 2.5 and θ = (60)(0.08) = 4.8.
The mixed distribution is a Negative Binomial, with r = α = 2.5 and β = θ = 4.8.
Therefore, the mean number of coins: rβ = (2.5)(4.8) = 12,
and the variance of number of coins: rβ(1+β) = (2.5)(4.8)(5.8) = 69.6.
The mean worth is: (10.5)(12) = 126.
Variance of worth is: (12)(167.5 - 10.5^2) + (10.5^2)(69.6) = 8360.4.
Prob[worth > 300] ≅ 1 - $\Phi[(300.5 - 126)/\sqrt{8360.4}]$ = 1 - Φ(1.91) = **2.81%**.
Klem loses money in units of 5 cents or more.
Therefore, if he loses more than 300, he loses 305 or more.
Thus it might be better to approximate the probability as:
1 - $\Phi[(304.5 - 126)/\sqrt{8360.4}]$ = 1 - Φ(1.95) = **2.56%**.
Along this same line of thinking, one could instead approximate the probability by taking the probability from 302.5 to infinity: 1 - $\Phi((302.5 - 126)/\sqrt{8360.4})$ = 1 - Φ(1.93) = **2.68%**.

9.8. E. From the previous solution, for a day chosen at random, the worth has mean 126 and variance 8360.4. The worth over five days is the sum of 5 independent variables; the sum of 5 days has mean: (5)(126) = 630 and variance: (5)(8360.4) = 41,802.
Prob[worth > 900] ≅ 1 - $\Phi[(900.5 - 630)/\sqrt{41{,}802}]$ = 1 - Φ(1.32) = **9.34%**.
Klem loses money in units of 5 cents or more.
Therefore, if he loses more than 900, he loses 905 or more.
It might be better to approximate the probability as:
Prob[worth > 900] = Prob[worth ≥ 905] ≅ 1 - $\Phi((904.5 - 630)/\sqrt{41{,}802})$ = 1 - Φ(1.34) = **9.01%**.
One might have instead approximated as: 1 - $\Phi((902.5 - 630)/\sqrt{41{,}802})$ = 1 - Φ(1.33) = **9.18%**.

9.9. A. 50% of the coins are worth 5, so if the overall process is Poisson with mean λ, then losing coins of worth 5 is Poisson with mean 0.5λ.
Over 10 minutes it is Poisson with mean 5λ.
λ has a Gamma distribution with α = 2.5 and θ = 0.08. $\Rightarrow$
5λ has a Gamma distribution with α = 2.5 and θ = (5)(0.08) = 0.4.
The mixed distribution is a Negative Binomial, with r = α = 2.5 and β = θ = 0.4.
f(1) = $r\beta/(1 + \beta)^{r+1}$ = (2.5)(0.4)/($1.4^{3.5}$) = **30.8%**.

9.10. D. Losing coins of worth 5, 10, and 25 are three independent Poisson Processes.
Over 10 minutes losing coins of worth 5 is Poisson with mean 5λ.
Over 10 minutes losing coins of worth 10 is Poisson with mean 3λ.
Over 10 minutes losing coins of worth 25 is Poisson with mean 2λ.
Prob[1 coin @ 5]Prob[0 coins @ 10]Prob[0 coins @ 25] = $5\lambda e^{-5\lambda} e^{-3\lambda} e^{-2\lambda} = 5\lambda e^{-10\lambda}$.
λ has a Gamma distribution with $\alpha = 2.5$ and $\theta = 0.08$. $\Rightarrow f(\lambda) = 12.5^{2.5} \lambda^{1.5} e^{-12.5\lambda} / \Gamma(2.5)$.

$$\int_0^\infty 5\lambda e^{-10\lambda}\ f(\lambda)\ d\lambda = \int_0^\infty 5\lambda e^{-10\lambda}\ 12.5^{2.5}\ \lambda^{1.5}\ e^{-12.5\lambda} / \Gamma(2.5)\ d\lambda$$

$$= \{(5)12.5^{2.5}/\Gamma(2.5)\} \int_0^\infty \lambda^{2.5}\ e^{-22.5\lambda}\ d\lambda = \{(5)12.5^{2.5}/\Gamma(2.5)\}\{\Gamma(3.5)/22.5^{3.5}\}$$

$= (5)(2.5)12.5^{2.5}/22.5^{3.5} =$ **12.8%**.

<u>Comment</u>: While given lambda, each Poisson Process is independent, the mixed Negative Binomials are not independent, since each day we use the same lambda (appropriately thinned) for each denomination of coin. From the previous solution, the probability of one coin worth 5 is 30.80%. The distribution of coins worth ten is Negative Binomial with r = 2.5 and $\beta = (3)(0.08) = 0.24$. Therefore, the chance of seeing no coins worth 10 is: $1/1.24^{2.5} = 58.40\%$.
The distribution of coins worth 25 is Negative Binomial with r = 2.5 and $\beta = (2)(0.08) = 0.16$.
Therefore, the chance of seeing no coins worth 25 is: $1/1.16^{2.5} = 69.0\%$.
However, (30.80%)(58.40%)(69.00%) = 12.4% ≠ 12.8%, the correct solution.
One can not multiply the three probabilities together, because the three events are not independent. The three probabilities each depend on the same lambda value for the given day.

9.11. E. A is Poisson with mean λ_A, where λ_A is a random draw from a Gamma Distribution with $\alpha = 2.5$ and $\theta = 0.08$. B is Poisson with mean λ_B, where λ_B is a random draw from a Gamma Distribution with $\alpha = 2.5$ and $\theta = 0.08$. Since A and B are from walks on different days, λ_A and λ_B are independent random draws from the same Gamma.
Thus $\lambda_A + \lambda_B$ is from a Gamma Distribution with $\alpha = 2.5 + 2.5 = 5$ and $\theta = 0.08$.
Thus A + B is from a Negative Binomial Distribution with r = 5 and $\beta = 0.08$.
The density at 3 of this Negative Binomial Distribution is: $\{(5)(6)(7)/3!\}0.08^3/1.08^8$ = **0.97%**.
Alternately, A and B are independent Negative Binomials each with r = 2.5 and $\beta = 0.08$.
Thus A + B is a Negative Binomial Distribution with r = 5 and $\beta = 0.08$. Proceed as before.
Alternately, for A and B the densities for each are:
$f(0) = 1/(1+\beta)^r = 1/1.08^{2.5} = 0.825$, $f(1) = r\beta/(1+\beta)^{1+r} = (2.5)0.08/1.08^{3.5} = 0.153$,
$f(2) = \{r(r+1)/2\}\ \beta^2/(1+\beta)^{2+r} = \{(2.5)(3.5)/2\}0.08^2/1.08^{4.5} = 0.0198$,
$f(3) = \{r(r+1)(r+2)/3!\}\ \beta^3/(1+\beta)^{3+r} = \{(2.5)(3.5)(4.5)/6\}0.08^3/1.08^{5.5} = 0.00220$.
Prob[A + B = 3] =
Prob[A=0]Prob[B=3] + Prob[A=1]Prob[B=2] + Prob[A=2]Prob[B=1] + Prob[A=3]Prob[B=0]
= (0.825)(0.00220) + (0.153)(0.0198) + (0.0198)(0.153) + (0.00220)(0.825) = **0.97%**.
Comment: For two independent Gamma Distributions with the same θ:
$\text{Gamma}(\alpha_1, \theta) + \text{Gamma}(\alpha_2, \theta) = \text{Gamma}(\alpha_1 + \alpha_2, \theta)$.

9.12. B. $\lambda_A + \lambda_B + \lambda_C$ is from a Gamma Distribution with $\alpha = (3)(2.5) = 7.5$ and $\theta = 0.08$.
Thus A + B + C is from a Negative Binomial Distribution with r = 7.5 and $\beta = 0.08$.
The density at 2 of this Negative Binomial Distribution is: $\{(7.5)(8.5)/2!\}\ 0.08^2 / 1.08^{9.5}$ = **9.8%**.

9.13. Over five years we have a Poisson Distribution with mean 5λ.
$E[5\lambda] = 5\ E[\lambda]$ = (5)(mean of the Gamma) = (5)(4)(0.1) = **2**.
Alternately, 5λ follows a Gamma distribution with $\alpha = 4$ and $\theta = (5)(0.1) = 1/2$.
The mixed distribution is a Negative Binomial, with parameters $r = \alpha = 4$ and $\beta = \theta = 1/2$.
The mean of this Negative Binomial distribution is: $r\beta = (4)(1/2)$ = **2**.

9.14. A. Over five years we have a Poisson Distribution with mean 5λ.
5λ follows a Gamma distribution with $\alpha = 4$ and $\theta = (5)(0.1) = 1/2$.
The mixed distribution is a Negative Binomial, with parameters $r = \alpha = 4$ and $\beta = \theta = 1/2$.
$f(0) = 1/(1+\beta)^r = 1/1.5^4$ = **19.75%**.

9.15. C. From the previous solution, $f(1) = r\beta/(1+\beta)^{r+1} = (4)(1/2)/1.5^5$ = **26.34%**.

9.16. D. For a Gamma Distribution, $E[X^{-1}] = \theta^{-1}\ \Gamma(\alpha - 1) / \Gamma(\alpha) = 1/\{\theta(\alpha-1)\}$, $\alpha > 1$.
For an individual the average wait until the first claim is $1/\lambda$.
However, lambda is Gamma Distributed, thus $E[1/\lambda] = 1/\{\theta(\alpha-1)\} = 1/\{(0.1)(4-1)\}$ = **3.33**.
Comment: *$1/\lambda$ is Inverse Gamma with $\alpha = 4$ and $\theta = 1/0.1 = 10$.*
Thus $1/\lambda$ has a mean of 10/(4-1) = 3.33, and mode 10/(4+1) = 2.

9.17 to 9.21. The mixed distribution is Negative Binomial with $r = \alpha = 2$ and $\beta = \theta = 5$.
$E[N] = r\beta =$ **10**. $Var[N] = r\beta(1+\beta) = (2)(5)(6) =$ **60**.
$Pr[N = 0] = 1/(1+\beta)^r = 1/6^2 =$ **2.78%**.
$Pr[N = 1] = r\beta/(1+\beta)^{1+r} = (2)(5)/6^3 =$ **4.63%**.
$Pr[N = 2] = \{r(r+1)/2\}\beta^2/(1+\beta)^{2+r} = \{(2)(3)/2\}(5^2)/6^4 =$ **5.79%**.

9.22 to 9.26. The mixed distribution over three years is Negative Binomial with $r = \alpha = 0.5$ and $\beta = 3\theta = 9$. $E[N(3)] = r\beta =$ **4.5**. $Var[N(3)] = r\beta(1+\beta) = (0.5)(9)(10) =$ **45**.
$Pr[N(3) = 0] = 1/(1+\beta)^r = 1/10^{0.5} =$ **31.62%**.
$Pr[N(3) = 1] = r\beta/(1+\beta)^{1+r} = (0.5)(9)/10^{1.5} =$ **14.23%**.
$Pr[N(3) = 2] = \{r(r+1)/2\}\beta^2/(1+\beta)^{2+r} = \{(0.5)(1.5)/2\}(9^2)/10^{2.5} =$ **9.61%**.

9.27. A. mean of Gamma = $\alpha\theta = 1$ and variance of Gamma = $\alpha\theta^2 = 2$.
Therefore, $\theta = 2$ and $\alpha = 1/2$.
The mixed distribution is a Negative Binomial with $r = \alpha = 1/2$ and $\beta = \theta = 2$.
$f(1) = r\beta/(1+\beta)^{1+r} = (1/2)(2)/(3^{3/2}) =$ **0.192**.

9.28. C. This is a Gamma-Poisson with $\alpha = 2$ and $\theta = 1$.
The mixed distribution is Negative Binomial with $r = \alpha = 2$, and $\beta = \theta = 1$.
For a Negative Binomial, $f(3) = \{(r)(r+1)(r+2)/3!\}\beta^3/(1+\beta)^{r+3} = \{(2)(3)(4)/6\}(1^3)/(2^5) = 1/8$.
Thus we expect: (1000)(1/8) = **125** out of 1000 simulated values to be 3.

9.29. A. The mean of the Negative Binomial is $r\beta = 0.2$, while the variance is $r\beta(1+\beta) = 0.4$.
Therefore, $1 + \beta = 2. \Rightarrow \beta = 1$ and $r = 0.2$. For a Gamma-Poisson, $\alpha = r = 0.2$ and $\theta = \beta = 1$.
Therefore, the variance of the Gamma Distribution is: $\alpha\theta^2 = (0.2)(1^2) =$ **0.2**.
Alternately, for the Gamma-Poisson, the variance of the mixed Negative Binomial is equal to:
mean of the Gamma + variance of the Gamma. Variance of the Gamma =
Variance of the Negative Binomial - Mean of the Gamma =
Variance of the Negative Binomial - Overall Mean =
Variance of the Negative Binomial - Mean of the Negative Binomial = 0.4 - 0.2 = **0.2**.

9.30. A. For the Gamma, mean = $\alpha\theta = 2$, and variance = $\alpha\theta^2 = 4$. Thus $\theta = 2$ and $\alpha = 1$.
This is a Gamma-Poisson, with mixed distribution a Negative Binomial, with $r = \alpha = 1$ and $\beta = \theta = 2$. This is a Geometric with $f(1) = \beta/(1+\beta)^2 = 2/(1+2)^2 = 2/9 =$ **0.222**.
Alternately, λ is distributed via an Exponential with mean 2, $f(\lambda) = e^{-\lambda/2}/2$.

Prob[1 claim] = $\int \text{Prob}[1 \text{ claim} \mid \lambda]\, f(\lambda)\, d\lambda = \int \lambda e^{-\lambda}\, e^{-\lambda/2}/2\, d\lambda =$

$$(1/2) \int_0^\infty \lambda\, e^{-3\lambda/2}\, d\lambda = (1/2)\,(2/3)^2\, \Gamma(2) = (1/2)(4/9)(1!) = 2/9 = \mathbf{0.222}.$$

Alternately, for the Gamma-Poisson, the variance of the mixed Negative Binomial =
total variance = $E[\text{Var}[N \mid \lambda]] + \text{Var}[E[N \mid \lambda]] = E[\lambda] + \text{Var}[\lambda] =$
mean of the Gamma + variance of the Gamma = 2 + 4 = 6.
The mean of the mixed Negative Binomial = overall mean = $E[\lambda]$ = mean of the Gamma = 2.
Therefore, $r\beta = 2$ and $r\beta(1+\beta) = 6$. $\Rightarrow r = 1$ and $\beta = 2$.
$f(1) = \beta/(1+\beta)^2 = 2/(1+2)^2 = 2/9 =$ **0.222**.
Comment: The fact that it is the sixth rather than some other minute is irrelevant.

9.31. C. Over two minutes (on the same day) we have a Poisson with mean 2λ.
$\lambda \sim \text{Gamma}(\alpha, \theta) = \text{Gamma}(1, 2)$.
$2\lambda \sim \text{Gamma}(\alpha, 2\theta) = \text{Gamma}(1, 4)$.
Mixed Distribution is Negative Binomial, with $r = \alpha = 1$ and $\beta = \theta = 4$.
$f(1) = \beta/(1 + \beta)^2 = 4/(1 + 4)^2 =$ **16%**.
Comment: If one multiplies a Gamma variable by a constant, one gets another Gamma with the same alpha and with the new theta equal to that constant times the original theta.

9.32. D. A ~ Negative Binomial with $r = 1$ and $\beta = 2$.
B ~ Negative Binomial with $r = 1$ and $\beta = 2$.
A + B ~ Negative Binomial with $r = 2$ and $\beta = 2$.
$f(1) = r\,\beta/(1 + \beta)^{1+r} = (2)(2)/(1 + 2)^3 =$ **14.8%**.
Alternately, the number of coins found in the minutes are independent Poissons with means λ_1 and λ_2. Total number found is Poisson with mean $\lambda_1 + \lambda_2$.
$\lambda_1 + \lambda_2 \sim \text{Gamma}(2\alpha, \theta) = \text{Gamma}(2, 2)$.
Mixed Negative Binomial has $r = 2$ and $\beta = 2$. Proceed as before.
Comment: The sum of two independent Gamma variables with the same theta, is another Gamma with the same theta and with the new alpha equal to the sum of the alphas.

9.33. E. Prob[1 coin during minute 3 | λ] = $\lambda e^{-\lambda}$. Prob[1 coin during minute 5 | λ] = $\lambda e^{-\lambda}$.
The Gamma has θ = 2 and α = 1, an Exponential. $\pi(\lambda) = e^{-\lambda/2}/2$.
Prob[1 coin during minute 3 and 1 coin during minute 5] =

$$\int \text{Prob[1 coin during minute 3} \mid \lambda] \text{ Prob[1 coin during minute 5} \mid \lambda] \ \pi(\lambda) \ d\lambda =$$

$$\int_0^\infty (\lambda e^{-\lambda})(\lambda e^{-\lambda})(e^{-\lambda/2}/2) \ d\lambda = \int_0^\infty \lambda^2 e^{-2.5\lambda}/2 \ d\lambda = \Gamma(3)\ (1/2.5)^3 / 2 = (1/2)(2/2.5^3) = \mathbf{6.4\%}.$$

<u>Comment</u>: It is true that Prob[1 coin during minute 3] = Prob[1 coin during minute 5] = 2/9. (2/9)(2/9) = 4.94%. However, since the two probabilities both depend on the same lambda, they are <u>not</u> independent.

9.34. D. Prob[1 coin during minute 1 | λ] = $\lambda e^{-\lambda}$. Prob[2 coins during minute 2 | λ] = $\lambda^2 e^{-\lambda}/2$.
Prob[3 coins during minute 3 | λ] = $\lambda^3 e^{-\lambda}/6$.
The Gamma has θ = 2 and α = 1, an Exponential. $\pi(\lambda) = e^{-\lambda/2}/2$.
Prob[1 coin during minute 1, 2 coins during minute 2, and 3 coins during minute 3] =

$$\int \text{Prob[1 coin minute 1} \mid \lambda] \text{ Prob[2 coins minute 2} \mid \lambda] \text{ Prob[3 coins minute 3} \mid \lambda] \ \pi(\lambda) \ d\lambda =$$

$$\int_0^\infty (\lambda e^{-\lambda})(\lambda^2 e^{-\lambda}/2)(\lambda^3 e^{-\lambda}/6)(e^{-\lambda/2}/2) \ d\lambda = \int_0^\infty \lambda^6 e^{-3.5\lambda}/24 \ d\lambda = \Gamma(7)\ (1/3.5)^7 / 24 = (720/24)/3.5^7$$

= **0.466%**.
<u>Comment</u>: Prob[1 coin during minute 1] = 2/9. Prob[2 coins during minute 2] = 4/27.
Prob[3 coins during minute 3] = 8/81. (2/9)(4/27)(8/81) = 0.325%. However, since the three probabilities depend on the same lambda, they are <u>not</u> independent.

9.35. A.
From a previous solution, for one minute, the mixed distribution is Geometric with β = 2.
$f(1) = \beta/(1+\beta)^2 = 2/(1+2)^2 = 2/9 = 0.2222$.
Since the minutes are on different days, their lambdas are picked independently.
Prob[1 coin during 1 minute today and 1 coin during 1 minute tomorrow] =
Prob[1 coin during a minute] Prob[1 coin during a minute] = 0.2222^2 = **4.94%**.

9.36. C. Over three minutes (on the same day) we have a Poisson with mean 3λ.
λ ~ Gamma(α, θ) = Gamma (1, 2).
3λ ~ Gamma(α, 3θ) = Gamma (1, 6).
Mixed Distribution is Negative Binomial, with r = α = 1 and β = θ = 6.
$f(1) = \beta/(1 + \beta)^2 = 6/(1 + 6)^2 = 0.1224$.
Since the time intervals are on different days, their lambdas are picked independently.
Prob[1 coin during 3 minutes today and 1 coin during 3 minutes tomorrow] =
Prob[1 coin during 3 minutes] Prob[1 coin during 3 minutes] = 0.1224^2 = **1.50%**.

9.37. C. From a previous solution, the number of coins found in each of these minutes are independent Geometric Distributions with $\beta = 2$.
The probability of exactly two more coins coin during the first minute of today's walk than the first minute of tomorrow's walk is:
P[2 today] P[0 tomorrow] + P[3 today] P[1 tomorrow] + P[4 today] P[2 tomorrow] + ... =

$$\frac{\beta^2}{(1+\beta)^3}\ \frac{1}{1+\beta} + \frac{\beta^3}{(1+\beta)^4}\ \frac{\beta}{(1+\beta)^2} + \frac{\beta^4}{(1+\beta)^5}\ \frac{\beta^2}{(1+\beta)^3} + ... = \frac{\beta^2}{(1+\beta)^4} + \frac{\beta^4}{(1+\beta)^6} + \frac{\beta^6}{(1+\beta)^8} + ...$$

This is a geometric series, with sum: $\dfrac{\dfrac{\beta^2}{(1+\beta)^4}}{1 - \dfrac{\beta^2}{(1+\beta)^2}} = \dfrac{\beta^2}{(1+\beta)^2\ (1 + 2\beta)} = \dfrac{4}{(9)(5)}$ = 4/45 = **8.89%**.

<u>Comment</u>: The answer would be the same comparing any minutes on two <u>different</u> days.
If instead we compared two minutes on the <u>same</u> day, then each minute is Poisson with the <u>same</u> lambda. Therefore, given lambda, the desired probability is:
$(\lambda^2 e^{-\lambda} / 2!)\ (e^{-\lambda}) + (\lambda^3 e^{-\lambda} / 3!)(\lambda\ e^{-\lambda}) + (\lambda^4 e^{-\lambda} / 4!)(\lambda^2 e^{-\lambda} / 2!) + \lambda^5 e^{-\lambda} / 5!)(\lambda^3 e^{-\lambda} / 3!) + ...$
$= e^{-2\lambda} \{\lambda^2 / 2\lambda + \lambda^4 / 3! + \lambda^6 / (4!\ 2!) + \lambda^8 / (5!\ 3!) + ...\}$.
Now, the distribution of lambda is Exponential with mean 2: $\pi[\lambda] = e^{-\lambda/2} /2$.
In order to determine the expected probability, we need to integrate the probability given lambda time the prior distribution of lambda over its support from 0 to infinity.

$$\int_0^\infty (e^{-2\lambda}\ \lambda^2 / 2!)\ (e^{-\lambda/2}\ /2)\ d\lambda = (1/2) \int_0^\infty e^{-2.5\lambda}\ \lambda^2\ d\lambda\ /2! = (1/5) \int_0^\infty x^2\ 2.5\ e^{-2.5x}\ dx\ / 2! =$$

(1/5) (second moment of an Exponential with mean 1/2.5) / 2! = (1/5) $(2!\ 0.4^2)$ / 2!.
Similarly, $E[e^{-2\lambda} \lambda^4 / 3!]$ = (1/5)(fourth moment of an Exponential with mean 1/2.5) / 3! =
(1/5) $(4!\ 0.4^4)$ / 3!.
$E[e^{-2\lambda} \lambda^6 / (4!\ 2!)]$ = (1/5)(sixth moment of an Exponential with mean 1/2.5) / (4! 2!)! =
(1/5) $(6!\ 0.4^6)$ / (4! 2!).
$E[e^{-2\lambda} \lambda^8 / (5!\ 3!)]$ = (1/5)(eighth moment of an Exponential with mean 1/2.5) / (5! 3!) =
(1/5) $(8!\ 0.4^8)$ / (5! 3!).

Thus the desired probability is: $(1/5)\ \{0.4^2 + 0.4^4 \binom{4}{1} + 0.4^6 \binom{6}{2} + 0.4^8 \binom{8}{3} + ...\}$.

Using a computer this probability is: 8.333% = 1/12.

9.38. E. Gamma has mean = $\alpha\theta = 3$ and variance = $\alpha\theta^2 = 3 \Rightarrow \theta = 1$ and $\alpha = 3$.
The Negative Binomial mixed distribution has $r = \alpha = 3$ and $\beta = \theta = 1$.
$f(0) = 1/(1+\beta)^3 = 1/8$. $f(1) = r\beta/(1+\beta)^4 = 3/16$. $F(1) = 1/8 + 3/16 = 5/16 =$ **0.3125**.

Section 10, Compound Poisson Processes[79]

In a **Compound Poisson Process**, a Poisson Process determines when events occur. Given an event, there is a random draw from some distribution in order to determine an amount attached to that event. Then one is interested in the total of the amounts from all the events by time t, S(t).

For actuaries, aggregate losses are the most common applications of Compound Poisson Processes. For example, assume claims are given by a Poisson Process with $\lambda = 0.4$. Assume frequency and severity are independent. For simplicity, assume there are only three sizes of claims: 100 with chance 25%, 200 with chance 60%, and 300 with chance 15%.

Then one could simulate this process by first simulating when each claim occurs.
For example, the first 5 claims might be at times: 0.23, 0.87, 0.92, 1.08, 1.70.
Then for each claim we would simulate a random claim size.
Assume that the first 5 claims have sizes: 200, 200, 300, 200, 100.[80]

Compound Poisson Process:
- **A Poisson Process, with intensity λ, determines when events occur**.
- **Given an event, there is a random draw from some distribution in order to determine an amount attached to that event**.

S(t) is the total of the amounts from all the events by time t.
If the events are claims and the amounts are sizes of claims,
then **S(t) is the aggregate loss by time t**.

Then in this case, we have for example: S(0) = 0, S(0.2) = 0, S(0.5) = 200, S(1) = 700, S(1.70) = 1000. S(t) has a jump discontinuity each time there is a claim.[81] In between those times S(t) is constant. We note that S(1.25) - S(0.9) = 900 - 400 = 500, is the aggregate losses in the time interval (0.9, 1.25].

In general, S(t) - S(s) is the aggregate losses that occur after time s and at or prior to time t.

Note that in this case, since there are a finite number of possible claim amounts, one can compute S(t) by keeping track of how many of each size there have been by time t. For example, by time 1.5 there have been 0 claims of size 100, 3 claims of size 200, and 1 claim of size 300. Thus S(1.5) = (0)(100) + (3)(200) + (1)(300) = 900.

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[79] See Section 1.2 of "Poisson Processes" by Daniel,
and Section 5.4.2 of Introduction to Probability Models by Ross.
[80] This is just one possible outcome of simulating five values from the given discrete severity distribution.
[81] In general, there will be a jump discontinuity in S(t) every time there is a claim, except when there a claim of size zero. (In some lines of insurance, claims closed without payment are common.) Some Compound Poisson Processes can have zero or negative amounts attached to some events.

In the process that generates aggregate losses, we could have just as easily have had a continuous size of loss distribution, such as a Pareto or a Uniform.
Aggregate losses with a Poisson frequency Distribution, when frequency and severity are independent, is just one example of what is called a Compound Poisson Process.

Definition:[82]

For a Compound Poisson Process, S(t):

- The number of events by time t, N(t), is a (homogenous) Poisson Process.
- Each event has an amount X_j.
- The amounts are independent, identically distributed variables.
- N(t) and X_j are independent.
- S(0) = 0.
- If N(t) = 0, then S(t) = 0.
- $S(t) = \sum_{j=1}^{N(t)} X_j$.

Mean:

As before, assume claims are given by a Poisson Process with $\lambda = 0.4$.
Assume frequency and severity are independent. For simplicity, assume there are only three sizes of claims: 100 with chance 25%, 200 with chance 60%, and 300 with chance 15%.

Then the number of claims by time 5 is Poisson Distributed with mean (5)(0.4) = 2.
The mean severity is: (100)(0.25) + (200)(0.60) + (300)(0.15) = 190.
Thus the expected dollars by time 5 are: (2)(190) = 380.

Mean of S(t) = (Mean of the Poisson Distribution)(Mean of the Amount Distribution)
= λt (Mean of the Amount Distribution)

In other words, **E[S(t)] = E[N(t)] E[X]**.[83]

In the example, E[S(5)] = (0.4)(5)(190) = 380.

[82] See Definition 1.8 in "Poisson Processes" by Daniel.
[83] See Fact 1.11 in "Poisson Processes" by Daniel.

Variance:

Variance of S(t) = λt (Second Moment of the Amount Distribution).
In other words, **$Var[S(t)] = E[N(t)] E[X^2]$**.[84]

In the previous example, the second moment of the severity is:
$(100^2)(0.25) + (200^2)(0.60) + (300^2)(0.15) = 40{,}000$.
Therefore, Var[S(5)] = (0.4)(5)(40,000) = 80,000.

Exercise: One has a Compound Poisson Process, with λ = 3.5. The amount of each claim follows a Pareto distribution, $F(x) = 1 - \{\theta/(\theta+x)\}^{\alpha}$, x > 0, with parameters α = 4 and θ = 60.
The mean of this Pareto Distribution is: θ/(α-1) = 60/3 = 20.
The second moment of this Pareto Distribution is: $2\theta^2 / \{(\alpha-1)(\alpha-2)\} = 7200/6 = 1200$.
What are the mean and variance of the aggregate losses by time 6, S(6)?
[Solution: The mean of S(6) is: (3.5)(6)(20) = 420.
The variance of S(6) is: (3.5)(6)(1200) = 25,200.]

Second moment of S(t) is: $E[S(t)^2] = Var[S(t)] + E[S(t)]^2 = E[N(t)] E[X^2] + \{E[N(t)] E[X]\}^2$.

Derivation of the Variance of a Compound Poisson:

This is a special case of the formula for the variance of aggregate losses:
$\sigma_{agg}^2 = \mu_F \sigma_S^2 + \mu_S^2 \sigma_F^2$.
Since we have a Poisson Distribution, the formula simplifies somewhat.

Variance of S(t) = (Mean of Poisson Distribution) (Variance of Amount Distribution)
+ (Mean of Amount Distribution)^2 (Variance of Poisson Distribution)
= λt (Variance of the Amount Distribution + Square of the Mean of the Amount Distribution)

= λt (Second Moment of the Amount Distribution).

Normal Approximation:

As t approaches infinity, S(t) approaches a Normal Distribution.[85]
Thus one can approximate a Compound Poisson Distribution by a Normal Distribution.

As previously, for a Compound Poisson Process with λ = 0.4, there only three sizes of claims:
100 with chance 25%, 200 with chance 60%, and 300 with chance 15%.
We had before that: E[S(5)] = 380, and Var[S(5)] = (0.4)(5)(40,000) = 80,000.

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[84] See Fact 1.11 in "Poisson Processes" by Daniel.
[85] The approximation is useful when the mean number of events is "large". *The more skewed the amount distribution, the higher the frequency has to be for the Normal Approximation to produce worthwhile results.*

Thus, Prob[S(5) > 400] $\cong$ 1 - Φ[(400 - 380) / $\sqrt{80,000}$] = 1 - Φ[0.07] = 1 - 0.5279 = 47.21%.

Exercise: One has a Compound Poisson Process, with λ = 3.5. The amount of each claim follows a Pareto distribution as per Loss Models, with parameters α = 4 and θ = 60.
Estimate the chance that S(6) is greater than 500. Use the Normal Approximation.
[Solution: From the solution to a previous exercise, the mean is 420 and the variance is 25,200.
Therefore, the chance that S(6) > 500 is approximately:
1 - Φ((500 - 420) / 158.75] = 1 - Φ[0.50] = 31%.
Comment: Do not use the continuity correction; the aggregate losses are continuously distributed.]

Exercise: The number of automobile accidents is given by a Poisson Process, with λ = 0.13.
The number of persons injured in each automobile accident follows a Binomial distribution as per Loss Models, with parameters q = 0.4 and m = 5.
What are the mean and variance of the total number of people injured by time 10, S(10)?
[Solution: The mean and variance of the Binomial are: (5)(0.4) = 2, and (5)(0.4)(0.6) = 1.2.
The second moment of the Binomial Distribution is: 1.2 + 2^2 = 5.2.
Thus the mean of S(10) is: (0.13)(10)(2) = 2.6.
The variance of S(10) is: (0.13)(10)(5.2) = 6.76.]

Exercise: In the previous exercise, estimate the chance that S(10) is less than 4.
Use the Normal Approximation.
[Solution: Prob[S(10) < 4] $\cong$ Φ[(3.5 - 2.6) / $\sqrt{6.76}$] = Φ[0.35] = 64%.
Comment: Use the continuity correction; the total number of people injured is discrete.]

Independent Poisson Processes:

For example, assume you are sitting by the side of a country road watching cars pass in one direction. Assume cars are either red or blue. Red cars pass with a Poisson Process with λ = 0.2, while blue cars pass with λ = 0.3. The number of red and blue cars are independent.
Red and blue cars have a combined claims intensity of 0.5.
Thus 0.2/0.5 = 40% of the cars that pass are red. 0.3/0.5 = 60% of the cars that pass are blue.

If one has **two independent Poisson Processes with claims intensities λ_1 and λ_2, then the chance that an event is from the first process is: $\lambda_1 / (\lambda_1 + \lambda_2)$**.
Similarly, the chance an event is from the second process is: $\lambda_2 / (\lambda_1 + \lambda_2)$.
More generally, if one has n independent Poisson Processes with rates $\lambda_1, \lambda_2, \ldots, \lambda_n$, then the chance that an event is from the first process is: $\lambda_1 / (\lambda_1 + \lambda_2 + \ldots + \lambda_n)$.
The chance that an event comes from process j is proportional to its intensity λ_j.

Adding Two Independent Compound Poisson Processes:[86]

Continuing this example, assume that the number of people in red cars has a distribution:
1 @ 70%, 2 @ 30%.
The number of people in blue cars has a distribution: 1@ 40%, 2 @ 40%, 3 @ 20%.

Exercise: Determine the mean and variance of the number of people in red cars that pass by time t.
[Solution: The mean of the amount distribution is: (0.7)(1) + (0.3)(2) = 1.3.
The second moment of the amount distribution is: $(0.7)(1^2) + (0.3)(2^2) = 1.9$.
$E[S_{red}(t)] = (0.2)t(1.3) = 0.26t$. $Var[S_{red}(t)] = (0.2)t(1.9) = 0.38t$.]

Exercise: Determine the mean and variance of the number of people in blue cars that pass by time t.
[Solution: The mean of the amount distribution is: (0.4)(1) + (0.4)(2) + (0.2)(3) = 1.8.
The second moment of the amount distribution is: $(0.4)(1^2) + (0.4)(2^2) + (0.2)(3^2) = 3.8$.
$E[S_{blue}(t)] = (0.3)t(1.8) = 0.54t$. $Var[S_{blue}(t)] = (0.3)t(3.8) = 1.14t$.]

Let S(t) be the total number of people that pass by time t. S(t) is the sum of two Compound Poisson Processes, one from the red cars and one from the blue cars. $S(t) = S_{red}(t) + S_{blue}(t)$.

S(t) is a Compound Poisson Process.
S(t) has rate $\lambda = \lambda_{red} + \lambda_{blue} = 0.2 + 0.3 = 0.5$.
S(t) has an amount distribution which is a mixture, with 40% weight to the red amount distribution and 60% weight to the blue amount distribution.

More specifically, the amount distribution of S(t) is:
1 @ (0.4)(70%) + (0.6)(40%) = 52%, 2 @ (0.4)(30%) + (0.6)(40%) = 36%,
and 3 @ (0.6)(20%) = 12%.

Exercise: Determine the mean and variance of S(t), the total number of people that pass by time t.
[Solution: The mean of the amount distribution is: (0.52)(1) + (0.36)(2) + (0.12)(3) = 1.6.
The second moment of the amount distribution is: $(0.52)(1^2) + (0.36)(2^2) + (0.12)(3^2) = 3.04$.
$E[S(t)] = (0.5)t(1.6) = 0.8t$. $Var[S(t)] = (0.5)t(3.04) = 1.52t$.
Alternately, $E[S(t)] = E[S_{red}(t)] + E[S_{blue}(t)] = 0.26t + 0.54t = 0.8t$.
$Var[S(t)] = Var[S_{red}(t)] + Var[S_{blue}(t)] = 0.38t + 1.14t = 1.52t$.]

[86] See Section 1.4.2 of "Poisson Processes" by Daniel.

<u>Covariances and Correlations</u>:

The **Covariance** of X and Y is defined as: **Cov[X, Y] ∫ E[XY] - E[X] E[Y]**.
Cov[X, X] = Var[X]. If X and Y are independent, Cov[X, Y] = 0.
Cov[X, Y] = Cov[Y, X]. Cov[X, Y + Z] = Cov[X, Y] + Cov[X, Z]. Cov[X, bY] = b Cov[X, Y].

Var[X + Y] = Var[X] + Var[Y] + 2Cov[X, Y].

If X and Y are independent then: Cov[X, Y] = 0 & Var[X + Y] = Var[X] + Var[Y].

The **Correlation** of two random variables is: **Corr[X, Y]** ∫ $\frac{\text{Cov}[X, Y]}{\sqrt{\text{Var}[X]\ \text{Var}[Y]}}$.

Correlation is always in the interval [-1, +1].
Two variables that are proportional with a <u>positive</u> proportionality constant have a correlation of +1.

<u>Correlations of Poisson Processes</u>:

One can use the independence of increments of Poisson processes in order to calculate correlations.

In general, S(t) is independent of S(t+s) - S(t), since the number of losses from time 0 to t is independent of the number of losses from time t to t+s.[87]
Therefore, Cov[S(t), S(t+s) - S(t)] = 0.
Thus Cov[S(t), S(t+s)] = Cov[S(t), S(t)] + Cov[S(t), S(t+s) - S(t)] = Var[S(t)].

Exercise: For a compound Poisson Process S(t), what is the correlation of S(9) - S(2) and S(20) - S(5)?
[Solution: S(5) - S(2) is independent of S(9) - S(5). ⇒ Cov[S(5) - S(2), S(9) - S(5)] = 0.
S(9) - S(5) is independent of S(20) - S(9). ⇒ Cov[S(9) - S(5), S(20) - S(9)] = 0.
S(5) - S(2) is independent of S(20) - S(9). ⇒ Cov[S(5) - S(2), S(20) - S(9)] = 0.
Cov[S(9) - S(2), S(20) - S(5)] = Cov[S(5) - S(2) + S(9) - S(5), S(9) - S(5) + S(20) - S(9)]
= Cov[S(5) - S(2), S(9) - S(5)] + Cov[S(5) - S(2), S(20) - S(9)]
+ Cov[S(9) - S(5), S(9) - S(5)] + Cov[S(9) - S(5), S(20) - S(9)]
= Cov[S(9) - S(5), S(9) - S(5)] = Var[S(9) - S(5)] = 4 λ E[X^2].
Var[S(9) - S(2)] = 7 λ E[X^2]. Var[S(20) - S(5)] = 15 λ E[X^2].
Therefore, Corr[S(9) - S(2), S(20) - S(5)] = $4\lambda E[X^2]$ / $\sqrt{7\lambda E[X^2] 15\ \lambda E[X^2]}$ = 4 / $\sqrt{(7)(15)}$ = 0.39.
<u>Comment</u>: In general, the correlation is:
(length of overlap of the two intervals) / $\sqrt{\text{product of the length of the two intervals}}$.]

[87] Also each severity is independent of frequency and each size of loss is independent of any other.

Problems:

Using the following information for the next 5 questions:
- An insurer pays out claims via a Poisson Process with claims intensity 15 per week.
- The amount of money per claim is exponentially distributed with mean $1000.
- One year = 52 weeks.

10.1 (1 point) What is the expected amount paid per year?
A. Less than $775,000
B. At least $775,000 but less than $800,000
C. At least $800,000 but less than $825,000
D. At least $825,000 but less than $850,000
E. At least $850,000

10.2 (1 point) What is the standard deviation of the amount paid per year?
A. Less than $38,000
B. At least $38,000 but less than $39,000
C. At least $39,000 but less than $40,000
D. At least $40,000 but less than $41,000
E. At least $41,000

10.3 (2 points) What is the covariance of the amount paid over the first quarter year (13 weeks) and the amount paid over the whole year (52 weeks)?
A. Less than 350 million
B. At least 350 million but less than 375 million
C. At least 375 million but less than 400 million
D. At least 400 million but less than 425 million
E. At least 425 million

10.4 (1 point) If the insurer has paid 200 claims for total of $210,000 over the first 13 weeks, what is the amount expected to be paid for the entire year (52 weeks)?
A. Less than $770,000
B. At least $770,000 but less than $780,000
C. At least $780,000 but less than $790,000
D. At least $790,000 but less than $800,000
E. At least $800,000

10.5 (3 points) Let time be measured in weeks. What is the covariance of the amount paid from time 5 to 30 with the amount paid from time 20 to 50?
A. 225 million B. 250 million C. 275 million D. 300 million E. 325 million

10.6 (2 points) You are given the following:
- Number of customers follows a Poisson Process with intensity 0.0125 per minute.
- The amount that a single customer spends has a uniform distribution on [0, 5000].
- Number of customers and the amount each customer spends are independent.

Determine the variance of the amount spent over 20 minutes.
A. Less than 2.1 million
B. At least 2.1 million but less than 2.2 million
C. At least 2.2 million but less than 2.3 million
D. At least 2.3 million but less than 2.4 million
E. At least 2.4 million

Use the following information for the next 3 questions:
- Number of claims follows a Poisson Process with intensity 5.
- Claim severity is independent of the number of claims and has the following probability density function: $f(x) = 3.5\, x^{-4.5}$, $x > 1$.

10.7 (2 points) Determine the variance of the losses paid over one year.
A. Less than 11.0
B. At least 11.0 but less than 11.3
C. At least 11.3 but less than 11.6
D. At least 11.6 but less than 11.9
E. At least 11.9

10.8 (2 points) Using the Normal Approximation, determine the approximate probability that the aggregate losses in a year will exceed 10.
A. Less than 14%
B. At least 14%, but less than 16%
C. At least 16%, but less than 18%
D. At least 18%, but less than 20%
E. At least 20%

10.9 *(2 points) Determine the variance of the aggregate losses paid over one year if the amount paid per loss is limited to a maximum of 5.*
A. Less than 11.0
B. At least 11.0 but less than 11.3
C. At least 11.3 but less than 11.6
D. At least 11.6 but less than 11.9
E. At least 11.9

Use the following information for the next five questions:
- For the State of West Dakota, the claims follow a Poisson Process
  with annual claims intensity 8200.
- The severity distribution is LogNormal, with parameters $\mu = 4$ and $\sigma = 0.8$.
- Frequency and severity are independent.

10.10 (1 point) Determine the expected aggregate losses per year.
A. Less than 600 thousand
B. At least 600 thousand but less than 605 thousand
C. At least 605 thousand but less than 610 thousand
D. At least 610 thousand but less than 615 thousand
E. At least 615 thousand

10.11 (2 points) Determine the variance of the aggregate losses in a year.
A. 82 million B. 84 million C. 86 million D. 88 million E. 90 million

10.12 (2 points) Determine the chance that the observed aggregate losses in a year will be more than 632 thousand. (Use the Normal Approximation.)
A. 5% B. 7% C. 9% D. 11% E. 13%

10.13 (2 points) Determine the chance that the observed aggregate losses in a year will be less than 601 thousand. (Use the Normal Approximation.)
A. 3% B. 5% C. 7% D. 9% E. 11%

10.14 (1 point) Determine the chance that the observed aggregate losses in a year will be within ±2.5% of its expected value. (Use the Normal Approximation.)
A. 80% B. 85% C. 90% D. 95% E. 98%

10.15 (1 point) You are given the following information:
- An insurance company pays claims according to a Poisson process at a rate of 10 per day.
- Claims are subdivided into three categories: Minor, Major, and Severe,
  with claim amounts provided below:

Category	Mean Claim Amount
Minor	500
Major	5000
Severe	100,000

- It is known that the proportion of claims in the Severe category is 0.1.
- The total expected claim payment amount in one day is 118,000.

Calculate the proportion of Major claims.
A. 0.22 B. 0.24 C. 0.26 D. 0.28 E. 0.30

Use the following information for the next three questions:
As he walks, Clumsy Klem loses coins at a Poisson rate of 0.2 coins/minute.
The denominations are randomly distributed:
(i) 50% of the coins are worth 5;
(ii) 30% of the coins are worth 10; and
(iii) 20% of the coins are worth 25.

10.16 (2 points) Calculate the variance of the value of the coins Klem loses during a one-hour walk.
A. Less than 2000
B. At least 2000, but less than 2100
C. At least 2100, but less than 2200
D. At least 2200, but less than 2300
E. At least 2300

10.17 (2 points) If Klem loses 2 coins worth 10, calculate the conditional variance of the value of the coins Klem loses during a one-hour walk.
A. Less than 1700
B. At least 1700, but less than 1800
C. At least 1800, but less than 1900
D. At least 1900, but less than 2000
E. At least 2000

10.18 (2 points) During a 15 minute walk, what is the probability that Klem loses coins worth in total 10?
A. 6% B. 8% C. 10% D. 12% E. 14%

10.19 (2 points) There are four types of claims. Each type of claim follows an independent Poisson Process. The table below describes the characteristics of the four types of claims.

Type of Claim	Mean Annual Frequency	Severity	
		Mean	Variance
W	0.04	150	2,500
X	0.03	350	75,000
Y	0.02	250	6,000
Z	0.01	600	125,000

Calculate the variance of the annual aggregate losses.
A. Less than 15,000
B. At least 15,000 but less than 16,000
C. At least 16,000 but less than 17,000
D. At least 17,000 but less than 18,000
E. At least 18,000

10.20 (3 points) You are given the following:
- A store is open 14 hours a day.
- Customers arrive at a Poisson rate of 100 per hour.
- There are two types of customers: “browsers” and “shoppers with a purpose”.
- At random, 70% are browsers and 30% are shoppers with a purpose.
- 20% of browsers buy something, with the value of their purchases being distributed with mean 50 and variance 5000.
- 60% of shoppers with a purpose buy something, with the value of their purchases being distributed with mean 100 and variance 1000.

Calculate the variance of the total daily sales of this store.

A. Less than 4.1 million
B. At least 4.1 million but less than 4.2 million
C. At least 4.2 million but less than 4.3 million
D. At least 4.3 million but less than 4.4 million
E. At least 4.4 million

10.21 (3 points) You are given the following:
- The number of families who immigrate to Vancougar from Honwan is given by a Poisson Process with intensity per <u>day</u> of 0.8.
- The sizes of these families are distributed as follows:

1	2	3	4	5	6	7	8	9	10
10%	15%	22%	20%	12%	8%	6%	4%	2%	1%

- The number of families and the sizes of the families are independent.

What is the variance of the <u>annual</u> number of immigrants to Vancougar from Honwan?

A. 5000 B. 5200 C. 5400 D. 5600 E. 5800

10.22 (1 point) You are given the following:
- Claims follows a Poisson Process with annual claims intensity λ.
- The amount of a single claim has an exponential distribution given by:

$$f(x) = e^{-x/\theta} / \theta, \ x > 0, \ \theta > 0.$$

- Number of claims and claim severity distributions are independent.

Determine the variance of the losses paid in one year.

A. $\lambda\theta$ B. $\lambda\theta^2$ C. $2\lambda\theta$ D. $2\lambda\theta^2$
E. None of A, B, C, or D.

10.23 (4 points) You are given the following:
- A store is open 10 hours a day.
- Customers arrive at a Poisson rate of 20 per hour.
- 40% of customers are male and 60% of customers are female, at random.
- Males do not ask for assistance.
- 30% of male customers buy nothing.
- 70% of male customers buy something, with the value of their purchases being distributed via a Gamma Distribution with $\alpha = 3$ and $\theta = 10$. $E[X] = \alpha\theta$. $E[X^2] = \alpha(\alpha+1)\theta^2$.
- Females ask for assistance from the salesperson Sybil.
- Sybil has a split personality: good and bad, equally likely.
- Each personality is in control of Sybil for short random periods of time.
- 20% of females assisted by "good" Sybil buy nothing.
- 80% of females assisted by "good" Sybil buy something, with the value of their purchases being distributed via a Gamma Distribution with $\alpha = 2$ and $\theta = 20$.
- 50% of females assisted by "bad" Sybil buy nothing.
- 50% of females assisted by "bad" Sybil buy something, with the value of their purchases being distributed via a Gamma Distribution with $\alpha = 2$ and $\theta = 10$.

Calculate the variance of the total daily sales of this store.

(A) 160,000 (B) 170,000 (C) 180,000 (D) 190,000 (E) 200,000

10.24 (2 points) Use the following information:
- Medical losses follow a homogeneous Poisson Process with $\lambda = 2$.
- The size of medical losses are uniform from 0 to 2000.
- Dental losses follow a homogeneous Poisson Process with $\lambda = 1$.
- The size of dental losses is uniform from 0 to 500.

What is the probability that a loss is less than 300?

A. 25% B. 30% C. 35% D. 40% E. 45%

10.25 (3 points) Generous Gail finds coins on her way to work at a Poisson rate of 0.2 coins per minute. The coins are: nickels worth 5, dimes worth 10, and quarters worth 25.
The denominations are randomly distributed:
(i) 40% of the coins are nickels;
(ii) 30% of the coins are dimes;
(iii) 30% of the coins are quarters.
Gail donates any quarters she finds to charity. Let G be the total value of the coins Gail finds on her 40 minute walk to work tomorrow, which she does <u>not</u> donate to charity.
Using the Normal Approximation, what is the probability that G is greater than 50?
A. Less than 20%
B. At least 20% but less than 30%
C. At least 30% but less than 40%
D. At least 40% but less than 50%
E. At least 50%

Use the following information for the next three questions:
A Compound Poisson Process has claims intensity 10
and a LogNormal Severity, with $\mu = 4$ and $\sigma = 1.5$.

10.26 (3 points) What is the correlation between the aggregate amount between time 0 to 30 and the aggregate amount between time 3 and 12?
A. 50% B. 55% C. 60% D. 65% E. 70%

10.27 (3 points) What is the correlation between the aggregate amount between time 0 to 10 and the aggregate amount between time 3 and 12?
A. 45% B.55% C. 65% D. 75% E. 85%

10.28 (3 points) What is the covariance of the aggregate amount between time 0 to 20 and the number of claims between time 0 and 20?
A. Less than 25,000
B. At least 25,000 but less than 30,000
C. At least 30,000 but less than 35,000
D. At least 35,000 but less than 40,000
E. At least 40,000

10.29 (2 points)
Fortunate Phyllis finds coins on her way to work at a Poisson rate of 0.1 coins per minute.
The coins are equally likely to be: nickels worth 5, dimes worth 10, and quarters worth 25.
If she finds a nickel, then the next coin she finds will be a quarter.
If she finds a dime, then the next coin she finds will be a nickel.
If she finds a quarter, then the next coin she finds will be a dime.
Determine the probability that during her half hour walk to work, the total of the value of the coins that Phyllis finds exceeds 25.
(A) 64% (B) 67% (C) 70% (D) 73% (E) 76%

10.30 (2 points) The hare challenges the tortoise to a race. As they race, the hare quickly gets a very large lead, so he decides to take a nap by the side of the road with 4000 feet left to go.
By the time the hare finally wakes up, the tortoise has passed him and is only 2 minutes from the finish line!
The hare moves via jumps.
The number of his jumps is via a Poisson Process, with $\lambda = 300$ per minute.
The size of his jumps are Exponential with mean 6 feet.
What is the probability that the hare wins the race?
(A) 0.5% (B) 1% (C) 2% (D) 3% (E) 4%

10.31 (3 points) Customers arrive at a bank at a Poisson rate of 20 per hour.
70% of them deposit money and 30% withdraw money.
Individual deposits are distributed with mean 3 and standard deviation 4.
Individual withdrawals are distributed with mean 5 and standard deviation 10.
The number arriving and the amounts of deposits and withdrawals are independent.
Using the Normal approximation, calculate the probability that the amount of money the bank has at the end of eight hours will be at least 150 more than at the beginning.
(A) 0.18 (B) 0.22 (C) 0.24 (D) 0.28 (E) 0.32

10.32 (3 points) Claims for the Rocky Insurance Company have a Compound Poisson Process, with λ = 500 per year, and an Exponential Severity with mean 100.
Claims for the Balboa Insurance Company have a Compound Poisson Process, with λ = 300 per year, and an Exponential Severity with mean 200.
These two insurance companies merge.
What is the probability that over the next three years the merged company will have aggregate losses greater than 350,000? Use the Normal Approximation.
A. Less than 1.0%
B. At least 1.0% but less than 1.5%
C. At least 1.5% but less than 2.0%
D. At least 2.0% but less than 2.5%
E. At least 2.5%

10.33 (3 points) Taxicabs arrive at the Metropolis City Convention Center via a Poisson Process with rate 6.5 per minute. The number of passengers dropped off by each taxicab is Binomial with q = 0.5 and m = 4. The number of passengers dropped off by each taxicab is independent of the number of taxicabs that arrive and is independent of the number of passengers dropped off by any other taxicab. Using the normal approximation, determine the probability that at least 1500 passengers will be dropped off in two hours.
A. 83% B. 87% C. 91% D. 95% E. 99%

10.34 (2 points) Rosetta Stone has automobile accidents at a Poisson rate of 0.04 per year. The damage to her automobile due to any single accident is uniformly distributed over the interval from 0 to 6000. What is the coefficient of variation, standard deviation divided by mean, for her aggregate damage over 10 years?
A. 1.0 B. 1.2 C. 1.4 D. 1.6 E. 1.8

10.35 (2 points) For a Compound Poisson Distribution, S(t):
- The rate is 7.
- Mean of the severity distribution = 5.
- Variance of the severity distribution = 20.

Using the Normal Approximation, what is the probability that S(3) < 100?
A. 28% B. 32% C. 36% D. 40% E. 44%

10.36 (2 points) Dairy Airlines flies sightseeing tours.
The number of passengers carried by Dairy Airlines follows a compound Poisson process:
- The rate at which flights leave is 1.5 per hour.
- The expected total number of passengers on all flights in one hour is 18.
- The variance of the total number of passengers on all flights in one hour is 300.

Calculate the variance of the number of passengers per flight.
A. 48 B. 50 C. 52 D. 54 E. 56

Use the following information for the next two questions:
- A workers compensation insurance policy for a large employer covers both temporary and permanent disability claims due to work related injuries to employees.
- The number of temporary and permanent disability claims are independent of each other.
- Temporary disability claims follow a compound Poisson process.
- The average time between temporary disability claims is four weeks.
- The cost of individual temporary disability claims follows a Weibull distribution with $\tau = 0.5$ and $\theta = 5000$.
- Permanent disability claims follow a compound Poisson process.
- The average time between permanent disability claims is eighty weeks.
- The cost of a permanent disability claim follows a Pareto distribution with $\alpha = 2$ and $\theta = 200,000$.

10.37 (3 points) Calculate the probability that the cost of a claim exceeds 20,000.
A. Less than 15%
B. At least 15%, but less than 16%
C. At least 16%, but less than 17%
D. At least 17%, but less than 18%
E. At least 18%

10.38 (2 points)
A reinsurance contract provides coverage for any individual claim amount in excess of 50,000.
Calculate the variance of the annual number of claims covered by the reinsurance contract.
Assume that a year has 52 weeks.
A. Less than 0.70
B. At least 0.70, but less than 0.80
C. At least 0.80, but less than 0.90
D. At least 0.90, but less than 1.00
E. At least 1.00

<u>10.39</u> (3 points) You are given the following information:
- The number of accidents for a group of insureds follows a Poisson Process.
- Accidents occur at a rate of 25 claims per day on Friday and Saturday.
- Accidents occur at a rate of 15 claims per day on the other 5 days of the week.
- The size of each claim follows a lognormal distribution with the following parameters:
  $\mu = 9$ and $\sigma = 0.6$.
- Claim frequency and severity are independent.

Calculate the standard deviation of the total loss amount for a seven day period.

A. 120,000 B. 125,000 C. 130,000 D. 135.000 E. 140,000

<u>10.40</u> (3 points) Let $S(t) = X_1 + X_2 + ... + X_{N(t)}$ be a compound Poisson process where:
- X_i is Inverse Gamma with $\alpha = 4$ and $\theta = 30$.
- N(t) is a Poisson process with rate 0.7.

Calculate $E[S(5)^2]$.

A. Less than 1700
B. At least 1700, but less than 1800
C. At least 1800, but less than 1900
D. At least 1900, but less than 2000
E. At least 2000

10.41 (2 points) You are given the following information:
- Accidents follow a compound Poisson process
- Accidents occur at the rate of $\lambda = 30$ per day
- Accident severity follows an exponential distribution with $\theta = 800$
- The insurance payment for each accident is subjected to a deductible
- V_1 is the variance of daily aggregate payments with a deductible of 100 per accident
- V_2 is the variance of daily aggregate payments with a deductible of 250 per accident

Calculate the ratio V_2 / V_1.

A. Less than 0.5
B. At least 0.5, but less than 0.6
C. At least 0.6, but less than 0.7
D. At least 0.7, but less than 0.8
E. At least 0.8

10.42 (2 points) You are given the following information:
- S is a compound Poisson process with mean number of claims equal to 10.
- Individual claim amounts have mean 500 and variance 300,000.

Determine the standard deviation of S.

10.43 (5A, 11/98, Q.23) (1 point) The distribution of aggregate claims, S, is compound Poisson with $\lambda = 3$. Individual claim amounts are distributed as follows:

x	p(x)
1	0.40
2	0.20
3	0.40

Which of the following is the closest to the normal approximation of Pr[S > 9]?
A. 8% B. 11% C. 14% D. 17% E. 20%

10.44 (5A, 5/99, Q.24) (1 point) You are given the following information concerning the claim severity, X, and the annual aggregate amount of claims, S:
E[X] = 50,000. Var[X] = 500,000,000. Var[S] = 30,000,000.
Assume that the claim sizes (X_1, X_2, ...) are identically distributed random variables and the number of claims sizes are mutually independent.
Assume that the number of claims (N) follows a Poisson distribution.
What is the likelihood that there will be at least one claim next year?
A. Less than 5%
B. At least 5%, but less than 50%
C. At least 50%, but less than 95%
D. At least 95%
E. Cannot be determined from the above information.

10.45 (5A, 5/99, Q.38) (2.5 points) For a particular line of business, the aggregate claim amount S follows a compound Poisson distribution. The aggregate number of claims N, has a mean of 350. The dollar amount of each individual claim, x_i, i = 1,...,N is uniformly distributed over the interval from 0 to 1000 . Assume that N and the X_i are mutually independent random variables. Using the Normal Approximation, calculate the probability that S > 180,000.

10.46 (3, 5/00, Q.10) (2.5 points) Taxicabs leave a hotel with a group of passengers at a Poisson rate $\lambda = 10$ per hour. The number of people in each group taking a cab is independent and has the following probabilities:

Number of People	Probability
1	0.60
2	0.30
3	0.10

Using the normal approximation, calculate the probability that at least 1050 people leave the hotel in a cab during a 72-hour period.
(A) 0.60 (B) 0.65 (C) 0.70 (D) 0.75 (E) 0.80

10.47 (3, 5/01, Q.4) (2.5 points) Lucky Tom finds coins on his way to work at a Poisson rate of 0.5 coins per minute. The denominations are randomly distributed:
(i) 60% of the coins are worth 1;
(ii) 20% of the coins are worth 5;
(iii) 20% of the coins are worth 10.
Calculate the variance of the value of the coins Tom finds during his one-hour walk to work.
(A) 379 (B) 487 (C) 566 (D) 670 (E) 768

10.48 (3, 5/01, Q.36) (2.5 points)
The number of accidents follows a Poisson distribution with mean 12.
Each accident generates 1, 2, or 3 claimants with probabilities 1/2, 1/3, 1/6, respectively.
Calculate the variance in the total number of claimants.
(A) 20 (B) 25 (C) 30 (D) 35 (E) 40

10.49 (3, 11/01, Q.19) (2.5 points)
A Poisson claims process has two types of claims, Type I and Type II.
(i) The expected number of claims is 3000.
(ii) The probability that a claim is Type I is 1/3.
(iii) Type I claim amounts are exactly 10 each.
(iv) The variance of aggregate claims is 2,100,000.
Calculate the variance of aggregate claims with Type I claims excluded.
(A) 1,700,000 (B) 1,800,000 (C) 1,900,000 (D) 2,000,000 (E) 2,100,000

10.50 (3, 11/01, Q.30) (2.5 points) The claims department of an insurance company receives envelopes with claims for insurance coverage at a Poisson rate of λ = 50 envelopes per week. For any period of time, the number of envelopes and the numbers of claims in the envelopes are independent. The numbers of claims in the envelopes have the following distribution:

Number of Claims	Probability
1	0.20
2	0.25
3	0.40
4	0.15

Using the normal approximation, calculate the 90^{th} percentile of the number of claims received in 13 weeks.
(A) 1690 (B) 1710 (C) 1730 (D) 1750 (E) 1770

10.51 (3, 11/02, Q.15) (2.5 points) Bob is an overworked underwriter.
Applications arrive at his desk at a Poisson rate of 60 per day.
Each application has a 1/3 chance of being a “bad” risk and a 2/3 chance of being a “good” risk.
Since Bob is overworked, each time he gets an application he flips a fair coin.
If it comes up heads, he accepts the application without looking at it.
If the coin comes up tails, he accepts the application if and only if it is a “good” risk.
The expected profit on a “good” risk is 300 with variance 10,000.
The expected profit on a “bad” risk is -100 with variance 90,000.
Calculate the variance of the profit on the applications he accepts today.
(A) 4,000,000 (B) 4,500,000 (C) 5,000,000 (D) 5,500,000 (E) 6,000,000

10.52 (CAS3, 11/03, Q.30) (2.5 points) Speedy Delivery Company makes deliveries 6 days a week. Accidents involving Speedy vehicles occur according to a Poisson process with a rate of 3 per day and are independent.
In each accident, damage to the contents of Speedy's vehicles is distributed as follows:

Amount of damage	Probability
$0	1/4
$2,000	1/2
$8,000	1/4

Using the normal approximation, calculate the probability that Speedy's weekly aggregate damages will not exceed $63,000.
A. 0.24 B. 0.31 C. 0.54 D. 0.69 E. 0.76

10.53 (SOA3, 11/03, Q.20) *(2.5 points) The RIP Life Insurance Company specializes in selling a fully discrete whole life insurance of 10,000 to 65 year olds by telephone.*
For each policy:
(i) The annual contract premium is 500.
(ii) Mortality follows the Illustrative Life Table.
(iii) i = 0.06
The number of telephone inquiries RIP receives follows a Poisson process with mean 50 per day. 20% of the inquiries result in the sale of a policy. The number of inquiries and the future lifetimes of all the insureds who purchase policies on a particular day are independent.
Using the normal approximation, calculate the probability that S, the total prospective loss at issue for all the policies sold on a particular day, will be less than zero.
(A) 0.33 (B) 0.50 (C) 0.67 (D) 0.84 (E) 0.99

10.54 (CAS3, 5/04, Q.26) (2.5 points) On Time Shuttle Service has one plane that travels from Appleton to Zebrashire and back each day.
Flights are delayed at a Poisson rate of two per month.
Each passenger on a delayed flight is compensated $100.
The numbers of passengers on each flight are independent and distributed with mean 30 and standard deviation 50.
(You may assume that all months are 30 days long and that years are 360 days long.)
Calculate the standard deviation of the annual compensation for delayed flights.
A. Less than $25,000
B. At least $25,000, but less than $50,000
C. At least $50,000, but less than $75,000
D. At least $75,000, but less than $100,000
E. At least $100,000

10.55 (SOA M, 5/05, Q.6) (2.5 points)
Insurance losses are a compound Poisson process where:
(i) The approvals of insurance applications arise in accordance with a Poisson process at a rate of 1000 per day.
(ii) Each approved application has a 20% chance of being from a smoker and an 80% chance of being from a non-smoker.
(iii) The insurances are priced so that the expected loss on each approval is -100.
(iv) The variance of the loss amount is 5000 for a smoker and is 8000 for a non-smoker.
Calculate the variance for the total losses on one day's approvals.
(A) 13,000,000 (B) 14,100,000 (C) 15,200,000 (D) 16,300,000 (E) 17,400,000

10.56 (CAS3, 11/05, Q.27) (2.5 points) The number of accidents reported to a local insurance adjusting office is a Poisson process with parameter $\lambda = 3$ claims per hour.
The number of claimants associated with each reported accident follows a negative binomial distribution with parameters $r = 3$ and $\beta = 0.75$.
If the adjusting office opens at 8:00 a.m., calculate the variance in the distribution of the number of claimants before noon.
A. 9 B. 16 C. 47 D. 108 E. 189

10.57 (SOA M, 11/05, Q.7) (2.5 points)
Customers arrive at a bank according to a Poisson process at the rate of 100 per hour.
20% of them make only a deposit, 30% make only a withdrawal and the remaining 50% are there only to complain. Deposit amounts are distributed with mean 8000 and standard deviation 1000. Withdrawal amounts have mean 5000 and standard deviation 2000.
The number of customers and their activities are mutually independent.
Using the normal approximation, calculate the probability that for an 8-hour day the total withdrawals of the bank will exceed the total deposits.
(A) 0.27 (B) 0.30 (C) 0.33 (D) 0.36 (E) 0.39

10.58 (SOA M, 11/05, Q.40) (2.5 points) Lucky Tom deposits the coins he finds on the way to work according to a Poisson process with a mean of 22 deposits per month.
5% of the time, Tom deposits coins worth a total of 10.
15% of the time, Tom deposits coins worth a total of 5.
80% of the time, Tom deposits coins worth a total of 1.
The amounts deposited are independent, and are independent of the number of deposits.
Calculate the variance in the total of the monthly deposits.
(A) 180 (B) 210 (C) 240 (D) 270 (E) 300

10.59 (CAS3, 5/07, Q.2) (2.5 points)
Customers enter a supermarket at a Poisson rate of 3 customers per minute.
Each person spends \$50 on average but the amount spent by anyone individual follows a Normal distribution with coefficient of variation, (σ/μ), of 25%.
The amount spent by a customer is revenue to the supermarket.
Calculate the standard deviation of the revenue generated by customers who enter the supermarket in a single hour.
A. Less than \$500
B. At least \$500, but less than \$550
C. At least \$550, but less than \$600
D. At least \$600, but less than \$650
E. At least \$650

10.60 (SOA MLC, 5/07, Q.6) (2.5 points)
People arrive at a food bank at a Poisson rate of 10 per day.
80% of them donate nonperishable units of food and 20% withdraw units of food.
Individual food donations are distributed with mean 15 and variance 75 and individual food withdrawals are distributed with mean 40 and variance 533.
The number arriving and the amounts of donations and withdrawals are independent.
Using the normal approximation, calculate the probability that the amount of food units at the end of seven days will be at least 600 more than at the beginning of the week.
(A) 0.07 (B) 0.09 (C) 0.11 (D) 0.13 (E) 0.15

10.61 (CAS3, 11/07, Q.3) (2.5 points)
The number of hurricanes for State XYZ follows a Poisson process with a mean of 0.02 per month. The amount of damage caused by a hurricane follows a Pareto distribution with mean 5000 and variance 25,000,000.
You have set the risk load to be 10% of the standard deviation of losses.
If the hurricane season is 4 months long, how much is the risk load?
A. Less than 50
B. At least 50, but less than 150
C. At least 150, but less than 250
D. At least 250, but less than 350
E. At least 350

10.62 (CAS3L, 5/08, Q.12) (2.5 points) You are given the following information:
- Hurricane occurrences in Texas follow a Poisson process with λ = 2 per year.
- An insurance company sells 1,000,000 homeowners policies in Texas.
- The insurance company's losses per hurricane follow an exponential distribution with a mean of $10 million.
- The "risk load" is calculated as 10% of the sum of the expected losses per year and the standard deviation per year.

Calculate the risk load per policy.
A. Less than 5 dollars
B. At least 5 dollars, but less than 10 dollars
C. At least 10 dollars, but less than 15 dollars
D. At least 15 dollars, but less than 20 dollars
E. At least 20 dollars

10.63 (CAS3L, 11/08, Q.3) (2.5 points)
You are given the following information about a compound Poisson process that describes the number of items purchased by shoppers in a store:
- The rate at which shoppers arrive at the store is 10 per hour.
- The expected number of items purchased by all shoppers entering the store in one hour is 10.
- The variance of the number of items purchased by all shoppers entering the store in one hour is 100.

Calculate the variance of the number of items purchased by an individual shopper.
A. Less than 6.5
B. At least 6.5, but less than 7.5
C. At least 7,5, but less than 8.5
D. At least 8.5, but less than 9.5
E. At least 9.5

10.64 (CAS3L, 5/09, Q.10) (2.5 points)
The premium production of an insurance agent can be modeled by the following process:
- The number of policies the agent sells follows a Poisson distribution with λ = 13 policies per day.
- The amount of premium (in $) on any given policy follows a gamma distribution with α = 0.5 and θ = 1000.

Calculate the standard deviation of the total amount of premium written by the agent over a five-day period.
A. Less than $6,000
B. At least $6,000, but less than $7,000
C. At least $7,000, but less than $8,000
D. At least $8,000, but less than $9,000
E. At least $9,000

10.65 (CAS3L, 5/10, Q.14) (2.5 points) You are given the following:
● House fires in a city occur at a homogeneous Poisson rate of 22 per month.
● The cost (in thousands of dollars) to repair each house where a fire has occurred has a lognormal distribution with the following parameters:
$\mu = 2$
$\sigma = 0.75$
● The cost to repair fire-damaged houses is observed for the city over a two month period.
Using the normal approximation, calculate the probability that the total repair costs in this city (in thousands of dollars) are greater than 508 over this period.
A. Less than 18%
B. At least 18%, but less than 19%
C. At least 19%, but less than 20%
D. At least 20%, but less than 21%
E. At least 21%

10.66 (CAS3L, 11/10, Q.12) (2.5 points) You are given the following information:
● An insurance policy covers a dwelling for damage only due to hail storms and floods.
● Aggregate hail storm damage follows a compound Poisson process.
● The average time between hail storms is three months.
● The damage caused by an individual hail storm follows an exponential distribution with $\theta = 2{,}000$.
● Aggregate flood damage follows a compound Poisson process.
● The average time between floods is 12 months.
● The damage caused by an individual flood follows an exponential distribution with $\theta = 10{,}000$.
Calculate the probability that the damage caused by an insured event exceeds 7,500.
A. Less than 5.0%
B. At least 5.0%, but less than 7.5%
C. At least 7.5%, but less than 10.0%
D. At least 10.0%, but less than 12.5%
E. At least 12.5%

10.67 (2 points) In the previous question, CAS3L,11/10, Q.12, what is the variance of the aggregate annual losses from hail storms and floods combined?
A. Less than 220 million
B. At least 220 million, but less than 230 million
C. At least 230 million, but less than 240 million
D. At least 240 million, but less than 250 million
E. At least 250 million

10.68 (CAS3L, 5/11, Q.11) (2.5 points) For a collection of insured vehicles, windshield cracks are repaired at a Poisson rate of 150 per month.
Windshield crack repairs fall into two categories:
- 90% of the cracks are minor and cost \$100 to repair
- 10% of the cracks are major and cost \$1,100 to repair

Using the normal approximation, calculate the probability that total windshield crack repair cost in one month is more than \$40,000.
A. Less than 0.9%
B. At least 0.9%, but less than 1.0%
C. At least 1.0%, but less than 1.1%
D. At least 1.1%, but less than 1.2%
E. At least 1.2%

10.69 (CAS3L, 11/11, Q.10) (2.5 points) You are given the following information:
- An insurance policy covers claims arising from two independent perils, Fire and Wind.
- Claim frequency for each peril follows a Poisson process.
- For Fire, the average number of claims reported each year is 700.
  Fire claim severities are independent and follow an exponential distribution with θ = 35,000.
- For Wind, the average number of claims reported each year is 2.
  Wind claim severities are independent and follow an exponential distribution with θ = 200,000.

A reinsurance contract provides coverage for any individual claim amount in excess of 250,000.
Calculate the variance of the annual number of claims covered by the reinsurance contract.
A. Less than 0.25
B. At least 0.25, but less than 0.50
C. At least 0.50, but less than 0.75
D. At least 0.75, but less than 1.00
E. At least 1.00

10.70 (CAS3L, 5/12, Q.11) (2.5 points) In a manufacturing company, work-related accidents occur at a constant Poisson rate of 10 per month.
This company purchases a Workers Compensation policy from an insurance company.
The insurance company assumes that the payment for each claim follows an Exponential distribution and the average payment is 500 per claim.
Using the Normal approximation, calculate the probability that the insurance company will pay more than 70,000 within a year.
A. Less than 0.0960
B. At least 0.0960, but less than 0.0970
C. At least 0.0970, but less than 0.0980
D. At least 0.0980, but less than 0.0990
E. At least 0.0990

10.71 (CAS3L, 5/13, Q.11) (2.5 points) You are given the following information:
● Assume all claims are either auto or homeowners claims.
● Auto claims occur under a Poisson process and $\lambda = 0.3$.
● Homeowners claims occur under a Poisson process and $\lambda = 0.4$.
● The number of auto and homeowners claims are independent.
● The distributions of the number of people involved in a given claim are as follows:

Number of People Auto Claim	Probability
1	0.6
2	0.4

Number of People Homeowners Claim	Probability
1	0.5
2	0.3
3	0.2

Determine the variance of the total number of people that are involved in claims by time 1.
A. Less than 0.5
B. At least 0.5, but less than 1.0
C. At least 1.0, but less than 1.5
D. At least 1.5, but less than 2.0
E. At least 2.0

10.72 (CAS3L, 11/13, Q.11) (2.5 points) You are given the following information:
● The number of claims received by an insurance company follows a Poisson Process.
● Claims arrive at a rate of 20 claims per day on Mondays.
● Claims arrive at a rate of 10 claims per day on the other 6 days of the week.
● The size of each claim follows a Gamma distribution with $\alpha = 5$ and $\theta = 200$.
● Claim frequency and severity are independent.
Calculate the standard deviation of the total loss amount for a seven day period.
A. Less than 10,000
B. At least 10,000, but less than 11,000
C. At least 11,000, but less than 12,000
D. At least 12,000, but less than 13,000
E. At least 13,000

10.73 (CAS ST, 11/15, Q.3) (2.5 points) The number of customers who use an ATM for withdrawals follows the Poisson process with the rate equal to 100 per day.
The amounts withdrawn are distributed as follows:

Amount Withdrawn	Probability
\$20	0.25
\$40	0.50
\$50	0.10
\$100	0.15

Calculate the standard deviation of the total daily withdrawals.
A. Less than \$500
B. At least \$500, but less than \$510
C. At least \$510, but less than \$520
D. At least \$520, but less than \$530
E. At least \$530

10.74 (CAS S, 11/15, Q.4) (2.2 points)
You are given the following information on a workers' compensation policy:
● Claims occur according to a Poisson process with rate λ = 10 per week.
● Claims are classified independently into the following claim types:

Claim Type	Probability
1	0.2
2	0.3
3	0.5

● For each claim a random amount X_i, i = 1, 2, 3, is added to a fund.
● The fund starts with an amount of zero at time zero.
● The random variables X_i follow the Exponential distribution with the following means:

Claim Type	Mean
1	500
2	300
3	200

Calculate the 95th percentile of the amount of the fund after 13 weeks, using the Normal approximation.
A. Less than 43,000
B. At least 43,000, but less than 45,000
C. At least 45,000, but less than 47,000
D. At least 47,000, but less than 49,000
E. At least 49,000

10.75 (CAS S, 5/16, Q.5) (2.2 points)
You are given the following information on a health insurance policy:
● Claims occur according to a Poisson process with mean λ = 10 per week.
● Claim amounts follow the Pareto distribution with probability density function

$$f(x) = \frac{(4)\,(1000^4)}{(x + 1000)^5},\ 0 < x$$

Calculate the probability that the total claim amount in 52 weeks exceeds 200,000 using the Normal approximation.
A. Less than 0.02
B. At least 0.02, but less than 0.03
C. At least 0.03, but less than 0.04
D. At least 0.04, but less than 0.05
E. At least 0.05

10.76 (CAS S, 11/16, Q.1) (2.2 points) You are given the following information:
● An insurance company pays claims according to a Poisson process at a rate of 5 per day.
● Claims are sub-divided into three categories: Minor, Major, and Severe, with claim amounts provided below:

Category	Claim Amount
Minor	1
Major	4
Severe	10

● It is known that the proportion of claims in the Severe category is 0.15.
● The total expected claim payment amount in one day is 17.
Calculate the proportion of Major claims.
A. Less than 0.25
B. At least 0.25, but less than 0.40
C. At least 0.40, but less than 0.55
D. At least 0.55, but less than 0.70
E. At least 0.70

10.77 (CAS S, 11/16, Q.4) (2.2 points) Steve catches fish at a Poisson rate of 3 per hour.
The price Steve gets at the market for each fish is randomly distributed as follows:

Price	Probability
$10	20%
$20	60%
$30	20%

Using the normal approximation without a continuity correction, calculate the probability that Steve will receive at least $300 for fish caught in a four hour-period.
A. Less than 0.19
B. At least 0.19, but less than 0.20
C. At least 0.20, but less than 0.21
D. At least 0.21, but less than 0.22
E. At least 0.22

10.78 (CAS S, 11/17, Q.4) (2.2 points) You are given the following information:
● Taxicabs leave a hotel with a group of passengers at a Poisson rate of 10 per hour.
● The number of people in each group taking a cab is independent and has the following probability:

Number of people	Probability
1	0.60
2	0.25
3	0.10
4	0.05

Calculate the probability that at least 800 people leave the hotel in a cab during a 48-hour period using the normal approximation.
A. Less than 0.200
B. At least 0.200, but less than 0.210
C. At least 0.210, but less than 0.220
D. At least 0.220, but less than 0.230
E. At least 0.230

10.79 (MAS-1, 5/18, Q.4) (2.2 points)
You are given the following information about flights on ABC Airlines:
● Flights are delayed at a Poisson rate of four per month.
● The number of passengers on each flight is independently distributed with mean of 100 and standard deviation of 50.
● Each passenger on a delayed flight is compensated with a payment of 200.
Calculate the standard deviation of the total annual compensation for delayed flights.
A. Less than 100,000
B. At least 100,000, but less than 125,000
C. At least 125,000, but less than 150,000
D. At least 150,000, but less than 175,000
E. At least 175,000

10.80 (MAS-1, 11/18, Q.4) (2.2 points)
You are given the following information about a Poisson process:
● Claims occur at a rate of 4 per month
● Each claim is independent and takes on values with probabilities below:

Probability	Claim Amount
0.25	5
0.50	50
0.25	95

● The monthly claim experience is independent
● X is the aggregate claim amount for twelve months
Calculate Var[X].
A. Less than 40,000
B. At least 40,000, but less than 80,000
C. At least 80,000, but less than 120,000
D. At least 120,000, but less than 160,000
E. At least 160,000

10.81 (MAS-1, 5/19, Q.3) (2.2 points) You are given the following information:
- Accidents follow a compound Poisson process
- Accidents occur at the rate of $\lambda = 40$ per day
- Accident severity follows an exponential distribution with $\theta = 1{,}000$
- The insurance payment for each accident is subjected to a deductible
- V_1 is the variance of daily aggregate payments with a deductible of 100 per accident
- V_2 is the variance of daily aggregate payments with a deductible of 500 per accident

Calculate the ratio V_2 / V_1.

A. Less than 0.5
B. At least 0.5, but less than 0.6
C. At least 0.6, but less than 0.7
D. At least 0.7, but less than 0.8
E. At least 0.8

10.82 (MAS-1, 11/19, Q.3) (2.2 points)
You are given the following information about an online retailer:
- Orders are placed on the website according to a homogeneous Poisson process with mean 50 per hour
- The number of items purchased in each order is independent and has the following distribution:

Number of Items	Probability
1	0.50
2	0.40
3	0.08
4	0.02

Calculate the variance of the total number of items purchased in a four-hour period.
A. Less than 100
B. At least 100, but less than 300
C. At least 300, but less than 500
D. At least 500, but less than 700
E. At least 700

Solutions to Problems:

10.1. B. The mean of X(52) = (52)(15)($1000) = **$780,000**.

10.2. C. The second moment of the Exponential Distribution is: $(2)(1000^2)$ = 2,000,000.
Thus the variance of X(52) = (52)(15)(2 million) = 1.56×10^9.
Thus the standard deviation is: $\sqrt{1.56 \times 10^9}$ = **39,497**.

10.3. C. X(13) is independent from X(52)-X(13), therefore Cov[X(13), X(52)-X(13)] = 0.
Thus Cov[X(13), X(52)] = Cov[X(13), X(13) + {X(52) - X(13)}] =
Cov[X(13), X(13)] + Cov[X(13), X(52)-X(13)] = Var[X(13)] + 0 =
(13)(15)(2 million) = **390 million**.
Comment: The correlation between X(13) and X(52) is: $390/\sqrt{390/1560} = \sqrt{13/52} = 1/2$.
Cov[X, X] = Var[X]. Cov[X, Y + Z] = Cov[X, Y] + Cov[X, Z].

10.4. D. X(13) is independent from X(52)-X(13). Therefore the expected value of X(52)-X(13) does not depend on the observation. The expected value of X(52)-X(13) is: (39)(15)($1000) = $585,000. Thus for the entire year we expect $210,000 + $585,000 = **$795,000**.
Comment: *Resembles the Bornhuetter-Ferguson Loss Development Method, covered on the CAS Exam on Basic Reserving.*

10.5. D. X(30) - X(5) overlaps with X(50) - X(20). However, X(20) - X(5), X(30) - X(20), and X(50) - X(30), are mutually disjoint, and thus independent.
Therefore, Cov[X(20) - X(5), X(30) - X(20)] = 0, Cov[X(20) - X(5), X(50) - X(30)] = 0, and Cov[X(30) - X(20), X(50) - X(30)] = 0.
Thus Cov[X(30) - X(5) , X(50) - X(20)] =
Cov[{X(20) - X(5)} + {X(30) - X(20)} , {X(30) - X(20)} + {X(50) - X(30)}] =
Cov[X(20) - X(5), X(30) - X(20)] + Cov[X(20) - X(5), X(50) - X(30)]
+ Cov[X(30) - X(20), X(30) - X(20)] + Cov[X(30) - X(20), X(50) - X(30)]
= 0 + 0 + Var[X(30) - X(20)] + 0 = (10)(15)(2 million) = **300 million**.
Comment: We make use of the independent increments property of Poisson Processes.
The covariance is equal to the variance for the period of overlap.

10.6. A. This is a Compound Poisson Process.
The variance of X(20) = (20)(.0125) (2nd moment of the amounts spent)

$$= (0.25 / 5000) \int_0^{5000} x^2 \, dx = (0.25 / 5000)\,(5000)^3 / 3 = \mathbf{2{,}083{,}333}.$$

10.7. D. The variance of X(1) is: 5 (2nd moment of the severity) =

$$= 5 \int_1^\infty x^2 \, (3.5\, x^{-4.5}) \, dx = -(5)(3.5/1.5)\, x^{-1.5} \Big]_{x=1}^{x=\infty} = (5)(2.333) = \mathbf{11.67}.$$

Comment: The severity distribution is a Single Parameter Pareto Distribution, as per Appendix A.4.1.3 of Loss Models.

10.8. D. The mean severity is:

$$\int_1^\infty x(3.5\, x^{-4.5}) \, dx = -(3.5/2.5)\, x^{-2.5} \Big]_{x=1}^{x=\infty} = 1.4.$$

Thus the mean of X(1) is: (5)(1.4) = 7. From the solution to the prior question, the variance of X(1) is: 11.67. Thus the standard deviation of X(1) is $\sqrt{11.67}$ = 3.416. To apply the Normal Approximation we subtract the mean and divide by the standard deviation.

Prob[X(1) > 10] ≅ 1 - Φ[(10 - 7)/ 3.416] = 1 - Φ[0.88] = 1 - 0.8108 = **18.92%**.

10.9. B. The severity distribution is Single Parameter Pareto. F(x) = 1 - $x^{-3.5}$, prior to the effects of capping the payment at 5.

The second moment of the severity after the capping is:

$$\int_1^\infty x^2 \, (3.5\, x^{-4.5}) \, dx + (5^2\, S(5) = -(3.5/1.5)\, x^{-1.5} \Big]_{x=1}^{x=\infty} + (25)(5^{-3.5}) =$$

2.125 + 0.089 = 2.214.

Therefore variance of the losses paid over one year = (5)(2.214) = **11.07**.

Comment: Using the formula in Appendix A.4.1.3 of Loss Models, for the Single Parameter Pareto the limited second moment, $E[(X \wedge x)^2] = \alpha\theta^2/(\alpha-2) - 2\theta^\alpha x^{2-\alpha}/(\alpha-2)$.

For α = 3.5, θ =1, and x = 5, this is: $(3.5)(1^2)/(3.5-2) - (2)(1^2)(5^{2-3.5})/(3.5-2)$ = 2.214.

10.10. E. The mean severity = $\exp(\mu + 0.5\sigma^2)$ = exp(4.32) = 75.19.

Thus the mean aggregate losses is (8200)(75.19) = **616,547**.

10.11. D. The second moment of the severity = $\exp(2\mu + 2\sigma^2)$ = exp(9.28) = 10,721.

Thus since the frequency is Poisson and independent of the severity:

$\sigma_{PP}^2 = \mu_F$(2nd moment of the severity) = (8200)(10721) = **87.91 million**.

10.12. A. Since the variance is 87.91 million, the standard deviation of the aggregate losses is 9376. Prob[aggregate losses > 632,000] ≅ 1 - Φ[(632,000 - 616,547) / 9376] =

1 - Φ[1.65] = 1 - 0.9505 = **4.95%**.

10.13. B. Since the variance is 81.59 million, the standard deviation of the aggregate losses is 9376. Thus the probability of the observed aggregate losses being less than 601 thousand is approximately: Φ[(601 - 616.547) / 9.376] = 1 - Φ(1.66) = 1 - 0.9515 = **4.85%**.

10.14. C. Using the solutions to the prior two questions: 1 - 4.95% - 4.85% ≅ **90%**.
Comment: *If one were asked for the Full Credibility criterion for Pure Premiums (Aggregate Losses) corresponding to a 90% chance of being within ±2.5% of the aggregate losses, as explained in Classical Credibility, the answer would be*
$(y_p/r)^2 (1+CV^2) = (1.645/0.025)^2 \exp(\sigma^2) = 4330(1.8965) = 8212$ *claims.*
For the LogNormal Distribution: $1+CV^2 = \exp(\sigma^2) = \exp(0.8^2) = 1.8965$.
That is just another way of saying there is about a 90% chance of being within ±2.5% of the expected pure premium for about 8200 expected claims.

10.15. E. Let the proportion of Major claims be p.
118,000 = (10) {(500)(0.9 - p) + 5000p + (100,000)(0.1)}. ⇒ p = **0.3**.
Comment: Similar to CAS S, 11/16, Q.1.

10.16. B. Over one hour, the mean frequency is: (0.2)(60) = 12.
The second moment of the severity is: $(0.5)(5^2) + (0.3)(10^2) + (0.2)(25^2) = 167.5$.
Variance = (mean freq.)(2nd moment of severity) = (12)(167.5) = **2010**.
Alternately, the losing of nickels is a Poisson process with mean: (12)(50%) = 6 and therefore variance 6. The value of nickels lost has variance: $(5^2)(6) = 150$.
Similarly, the value of dimes (each worth 10) lost has variance: $(10^2)(3.6) = 360$.
The value of quarters (each worth 25) lost has variance: $(25^2)(2.4) = 1500$.
The three processes are independent, so the variance of the total value of coins lost is:
150 + 360 + 1500 = **2010**.
Comment: Similar to 3, 5/01, Q.4.

10.17. A. The nickels, dimes, and quarters are three independent Poisson processes.
The losing of nickels is a Poisson process with mean: (12)(50%) = 6 and therefore variance 6.
The value of nickels (each worth 5) lost has variance: $(5^2)(6) = 150$.
Similarly, the value of quarters (each worth 25) lost has variance: $(25^2)(2.4) = 1500$.
We know how many dimes were lost, thus the value of the dimes does not contribute to the conditional variance of the value of the coins lost.
So the conditional variance of the total value of coins lost is: 150 + 1500 = **1650**.

10.18. C. Over 15 minutes, nickels are Poisson with $\lambda = (50\%)(15)(0.2) = 1.5$, dimes are Poisson with $\lambda = (30\%)(15)(0.2) = 0.9$, and quarters are Poisson with $\lambda = (20\%)(15)(0.2) = 0.6$.
Prob[2@5]Prob[0@10]Prob[0@25] + Prob[0@5]Prob[1@10]Prob[0@25] =
$(e^{-1.5}1.5^2/2)e^{-0.9}e^{-0.6} + e^{-1.5}(0.9e^{-0.9})e^{-0.6} = 2.025\ e^{-3} =$ **0.101**.

10.19. A. For a Poisson Process, the variance for each type of claim is given by the mean frequency times the second moment of the severity. For example, for Claim Type Z, the variance of the aggregate losses for one year is: (0.01)(360,000 + 125,000) = 4850.
Then the process variances for each type of claim add to get the total variance, 13,145.

Type of Claim	Mean Frequency	Mean Severity	Square of Mean Severity	Variance of Severity	Variance of Aggregate Losses
W	0.04	150	22,500	2,500	1,000
X	0.03	350	122,500	75,000	5,925
Y	0.02	250	62,500	6,000	1,370
Z	0.01	600	360,000	125,000	4,850
SUM					**13,145**

Comment: This is like adding up four independent die rolls; the variances add. For example this could be a nonrealistic model of homeowners insurance with the four types of claims being: Fire, Liability, Theft and Windstorm.

10.20. C. There are an average of 1400 customers per day. The original Poisson Process is divided into two independent Poisson Processes, browsers with λ = (70%)(1400) = 980, and shoppers with a purpose with λ = (30%)(1400) = 420.
The browsers are divided into two independent Poisson Processes, one for nonpurchasers with λ = (80%)(980) = 784, and the other related to purchasers with λ = (20%)(980) = 196.
The shoppers with a purpose are divided into two independent Poisson Processes, one for nonpurchasers with λ = (40%)(420) = 168, and the other related to purchasers, with λ = (60%)(420) = 252.
Each Compound Poisson Process has variance of λ(second moment of severity).
Each nonpurchaser has total sales of zero, and second moment of zero.
There are four independent compound Poisson Processes, whose variances add:

Poisson Process	λ	mean severity	variance of severity	2nd moment of severity	Variance of Compound
Browser NonPurchasers	784			0	0
Browser Purchasers	196	50	5000	7500	1,470,000
Shoppers NonPurchasers	168			0	0
Shoppers Purchasers	252	100	1000	11000	2,772,000
Total	1400				**4,242,000**

Alternately, severity is a mixed distribution, with (80%)(70%) + (40%)(30%) = 68% weight to zero (no purchase), (20%)(70%) = 14% weight to a distribution with mean 50, variance 5000, and second moment 7500, and (60%)(30%) = 18% weight to a distribution with mean 100, variance 1000, and second moment 11000.
The second moment of severity is a weighted average of the individual second moments:
(68%)(0) + (14%)(7500) + (18%)(11000) = 3030.
Variance of the Compound Poisson Process is: (1400)(3030) = **4,242,000**.
Comment: Similar to 3, 11/02, Q. 15.

10.21. E. Second moment of the amounts: (0.1)(1) + (0.15)(4) + (0.22)(9) + (0.2)(16) + (0.12)(25) + (0.08)(36) + (0.06)(49) + (0.04)(64) + (0.02)(81) + (0.01)(100) = 19.88.
Variance = (365)(λ)(2nd moment of the amounts) = (365)(0.8)(19.88) = **5805**.

10.22. D. Variance = $\mu_F \sigma_S^2 + \mu_S^2 \sigma_F^2 = \lambda(\theta^2) + (\theta)^2\lambda = 2\lambda\theta^2$.
Alternately, variance of X(1) = (1)(λ) (2nd moment of Exponential) = $\lambda 2\theta^2$.

10.23. E. There are an average of 200 customers per day. Gender divides the original Poisson Process into two independent Poisson Processes, male with λ = (40%)(200) = 80, and female with λ = (60%)(200) = 120.
The male Poisson Process is in turn divided into two independent Poisson Processes, one for nonpurchasers with λ = (30%)(80) = 24, and the other related to purchasers, with λ = (70%)(80) = 56.
The females are divided into two independent Poisson Processes, one related to good Sybil with λ = (1/2)(120) = 60, and the other related to bad Sybil with λ = (1/2)(120) = 60.
The good Sybil process is divided into two independent Poison processes, nonpurchasers with λ = (20%)(60) = 12, and purchasers with λ = (80%)(60) = 48.
The bad Sybil process is divided into two independent Poison processes, nonpurchasers with λ = (50%)(60) = 30, and purchasers with λ = (50%)(60) = 30.
Each Compound Poisson Process has variance of λ(second moment of severity).
Each Gamma Distribution has a second moment of: $\alpha(\alpha+1)\theta^2$.
Each nonpurchaser has total sales of zero, and second moment of zero.
There are six independent compound Poisson Processes, whose variances add:

Poisson Process	λ	α	θ	second moment of severity	variance
Male NonPurchasers	24			0	0
Male Purchasers	56	3	10	1200	67,200
Female/Good NonPurchasers	12			0	0
Female/Good Purchasers	48	2	20	2400	115,200
Female/Bad NonPurchasers	30			0	0
Female/Bad Purchasers	30	2	10	600	18,000
Total	200				**200,400**

Alternately, severity is a mixed distribution, with (30%)(40%) + (20%)(50%)(60%) + (50%)(50%)(60%) = 33% weight to zero (no purchase), (70%)(40%) = 28% weight to Gamma(3, 10), (80%)(50%)(60%) = 24% weight to Gamma(2, 20), and (50%)(50%)(60%) = 15% weight to Gamma(2, 10). The second moment of severity is a weighted average of the individual second moments: (33%)(0) + (28%)(1200) + (24%)(2400) + (15%)(600) = 1002.
Variance of the Compound Poisson Process is: (200)(1002) = **200,400**.
Comment: Similar to 3, 11/02, Q. 15.

10.24. B. For Medical, the probability a loss is less than 300 is: 300/2000 = 3/20.
For Dental, the probability a loss is less than 300 is: 300/500 = 3/5.
Of all losses, the proportion that are from Medical is: $\frac{2}{2 + 1}$ = 2/3.
Prob[X < 300] = Prob[Med.] Prob[X < 300 | Med.] + Prob[Dent.] Prob[X < 300 | Dent.] =
(2/3)(3/20) + (1/3)(3/5) = **30%**.
Alternately, small medical losses are Poisson with intensity: (2)(3/20) = 0.3.
Small dental losses are Poisson with intensity: (1)(3/5) = 0.6.
Thus small losses are Poisson with intensity: 0.3 + 0.6 = 0.9.
All losses are Poisson with intensity: 2 + 1 = 3
Thus the probability that a loss is less than 300 is: 0.9 / 3 = **30%**.

10.25. B. She expects to find: (0.2)(40) = 8 coins.
The nickels are Poisson with mean: (40%)(8) = 3.2.
Their worth has mean: (5)(3.2) = 16, and variance: (5^2)(3.2) = 80.
The dimes are Poisson with mean: (30%)(8) = 2.4.
Their worth has mean: (10)(2.4) = 24, and variance: (10^2)(2.4) = 240.
The sum of the worth of nickels and dimes has mean: 16 + 24 = 40,
and variance: 80 + 240 = 320.
Prob[G > 50] ≅ 1 - $\Phi[(50.5 - 40)/\sqrt{320}]$ = 1 - Φ[0.587] = **28%**.
Alternately, since the smallest unit of money available is 5,
Prob[G > 50] ≅ 1 - $\Phi[(52.5 - 40)/\sqrt{320}]$ = 1 - Φ[0.699] = **24%**.
Alternately, the nickels plus dimes is a Poisson Process with λ = (0.2)(0.7) = 0.14.
The mean severity is: {(5)(0.4) + (10)(0.3)} / (0.4 + 0.3) = 7.143.
The second moment of severity is: {(5^2)(0.4) + (10^2)(0.3)} / (0.4 + 0.3) = 57.143.
Mean of G is: (40)(0.14)(7.143) = 40.
Variance of G is: (40)(0.14)(57.143) = 320. Proceed as before.

10.26. B. The second moment of the LogNormal Distribution is: $\exp[2\mu+2\sigma^2]$ = exp[8 + 4.5] = 268,337. Let X(t) be the aggregate by time t.
The variance of X(30) is: (30)(10)(268,337) = 80,501,100.
The variance of X(12) - X(3) is: (12 - 3)(10)(268,337) = 24,150,330.
X(3) is independent of X(12) - X(3). X(30) - X(12) is independent of X(12) - X(3).
Cov[X(12) - X(3), X(30)] = Cov[X(12) - X(3), X(3) + {X(12) - X(3)} + {X(30) - X(12)}] =
Cov[X(12) - X(3), X(3)] + Cov[X(12) - X(3), X(12) - X(3)] +
Cov[X(12) - X(3), X(30) - X(12)] = 0 + Var[X(12) - X(3)] + 0 = 24,150,330.
Corr[X(12) - X(3), X(30)] = 24,150,330 / $\sqrt{(24,150,330)(80,501,100)}$ = **54.8%**.

<u>Comment</u>: Corr[X(12) - X(3), X(30)] = $\sqrt{9/30}$ = 54.8%.

10.27. D. The variance of X(10) is: (10)(10)(268,337) = 26,833,700.
The variance of X(10) - X(3) is: (7)(10)(268,337) = 18,783,590.
The variance of X(12) - X(3) is: (12 - 3)(10)(268,337) = 24,150,330.
X(3) is independent of X(10) - X(3). X(10) - X(3) is independent of X(12) - X(10).
X(3) is independent of X(12) - X(10).
Cov[X(12) - X(3), X(10)] = Cov[{X(10) - X(3)} + {X(12) - X(10)}, X(3) + {X(10) - X(3)}] =
Cov[X(10) - X(3), X(3)] + Cov[X(10) - X(3), X(10) - X(3)] +
Cov[X(12) - X(10), X(3)] + Cov[X(12) - X(10), X(10) - X(3)] = 0 + Var[X(10) - X(3)] + 0 + 0 =
18,783,590. Corr[X(12) - X(3), X(10)] = 18,783,590 / $\sqrt{(24{,}150{,}330)(26{,}833{,}700)}$ = **73.8%**.

<u>Comment</u>: Corr[X(12) - X(3), X(10)] = 7/ $\sqrt{(9)(10)}$ = 73.8%, where 7 is the length of the overlap of the two time intervals of lengths 9 and 10. Cov[X, X] = Var[X].
Cov[X, Y + Z] = Cov[X, Y] + Cov[X, Z]. Cov[X, Y] = 0 for X and Y independent.

10.28. C. E[N(20)] = 20λ. E[X(20)] = 20λE[X].
E[N(20) X(20) | N(20) = n] = n E[X(20) | N(20) = n] = n(n E[X]) = E[X] n^2.
E[N(20) X(20)] = E[E[X] n^2] = E[X] E[n^2] = E[X](second moment of Poisson with mean 20λ) =
E[X]{20λ + 400λ^2} = 20λE[X] + 400λ^2E[X].
Cov[N(20), X(20)] = E[N(20) X(20)] - E[N(20)]E[X(20)] = 20λE[X] + 400λ^2E[X] - (20λ)(20λE[X])
= 20λE[X]. The mean of the LogNormal Distribution is: exp[$\mu+\sigma^2/2$] = exp[4 + $1.5^2/2$] = 168.17.
Cov[N(20), X(20)] =20λE[X] = (20)(10)(168.17) = **33,635**.
<u>Comment</u>: Similar to Q. 5.46 in <u>Introduction to Probability Models</u> by Ross.
In general, Cov[N(t), X(t)] = (mean severity)λt.
Corr[N(t), X(t)] = (mean severity)λt / $\sqrt{\lambda t\ \lambda t(\text{second moment of severity})}$ =
(mean severity) / $\sqrt{\text{second moment of severity}}$.

10.29. D. The number of coins she finds is Poisson with mean: (0.1)(30) = 3.
The total worth is 25 or less if:
She finds 0 coins. Probability: e^{-3} = 4.98%.
She finds 1 coin. Probability: $3e^{-3}$ = 14.94%.
She find 2 coins, the first a dime and the next a nickel. Probability: $(1/3)(3^2/2)e^{-3}$ = 7.47%.
Prob[worth > 25] = 1 - (4.98% + 14.94% + 7.47%) = **72.6%**.
<u>Comment</u>: If she finds 2 coins, then the total worth is equally likely to be 30, 15, or 35, and depends solely on the type of coin she finds first. If she finds 3 coins, then the total worth is 40.

10.30. **D**. Mean of the Compound Poisson Process is: (300t)(6) = 1800t.
Variance of the Compound Poisson Process is: $(300t)(2)6^2$ = 21,600t.
Prob[hare wins] = Prob[Compound Poisson Process over 2 minutes > 4000] ≅
1 - Φ[(4000 - 3600) / $\sqrt{43{,}200}$] = 1 - Φ(1.92) = **2.74%**.

10.31. D. The number of deposits over eight hours is Poisson with mean: (70%)(20)(8) = 112.
Mean of the distribution of the amount deposited is: (112)(3) = 336.
Variance of the distribution of the amount of food donated is: $(112)(4^2 + 3^2) = 2800$.
The number of withdrawals over seven days is Poisson with mean: (30%)(20)(8) = 48.
Mean of the distribution of the amount of food withdrawn is: (48)(5) = 240.
Variance of the distribution of the amount of food withdrawn is: $(48)(10^2 + 5^2) = 6000$.
Mean net amount of money is: 336 - 240 = 96.
Variance of the net amount of money is: 2800 + 6000 = 8800.
Probability that net amount is at least 150 is: $1 - \Phi[(150 - 96) / \sqrt{8800}] = 1 - \Phi[0.58] =$ **28.1%**.
Comment: Similar to SOA MLC, 5/07, Q.6.

10.32. D.
Over three years the aggregate losses for Rocky have mean: (500)(3)(100) = 150,000.
Over three years the aggregate losses for Rocky have variance: $(500)(3)\{(2)100^2\}$ = 30 million.
Over three years the aggregate losses for Balboa have mean: (300)(3)(200) = 180,000.
Over three years the aggregate losses for Balboa have variance: $(300)(3)\{(2)200^2\}$ = 72 million.
Over three years the aggregate losses for the merged insurer have mean:
150,000 + 180,000 = 330,000.
Over three years the aggregate losses for the merged insurer have variance:
30 million + 72 million = 102 million. The Standard Deviation is: $\sqrt{102 \text{ million}}$ = 10,100.

Prob[3 year Aggregate > 350,000] $\cong 1 - \Phi[(350{,}000 - 330{,}000)/10{,}100] = 1 - \Phi[1.98] =$ **2.39%.**
Alternately, the merged company has a Compound Poisson Process with rate λ = 500 + 300 = 800. The amount distribution for the merged company is a mixture of Exponentials with means 100 and 200, and weights 500/800 = 5/8 and 300/800 = 3/8.
This mixture has mean: (5/8)(100) + (3/8)(200) = 137.5.
This mixture has second moment: $(5/8)\{(2)(100^2) + (3/8)\{(2)(200^2)\} = 42{,}500$.
Therefore, the merged company has E[S(3)] = (800)(3)(137.5) = 330,000,
and Var[S(3)] = (800)(3)(42,500) = 102 million. Proceed as before.
Comment: See Example 1.28 in "Poisson Processes" by Daniel.
An Exponential with mean θ, has a second moment of $2\theta^2$.

10.33. A. The passengers per cab has mean: (0.5)(4) = 2, and variance: (0.5)(1 - 0.5)(4) = 1.
The mean aggregate number of passengers in 120 minutes is: (6.5)(120)(2) = 1560.
The variance of the aggregate number of passengers in 120 minutes is:
$(6.5)(120)(1 + 2^2) = 3900$.
Thus the probability that the aggregate number of passengers is ≥ 1500 is approximately:
$1 - \Phi[(1499.5 - 1560)/\sqrt{3900}] = 1 - \Phi[-0.97] =$ **83.4%.**
Comment: Similar to 3, 5/00, Q.10.
I used the continuity correction, since the number of passengers is integer.

10.34. E. This uniform distribution has mean 3000 and variance $6000^2/12$ = 3 million.
E[S(10)] = (0.04)(10)(3000) = 1200. Var[S(10)] = $(0.04)(10)(3 \text{ million} + 3000^2)$ = 4.8 million.
Coefficient of variation is: $\sqrt{4.8 \text{ million}}$ / 1200 = **1.826.**

10.35. E. E[S(3)] = (7)(3)(5) = 105. Var[S(3)] = (7)(3)(20 + 5^2) = 945.
Prob[S(3) < 100] ≅ $\Phi[100 - 105)/\sqrt{945}\,] = \Phi[-0.16]$ = **43.6%**.

10.36. E. Let X be the number of passengers on an individual flight.
18 = Mean of aggregate = λ E[X] = 1.5 E[X]. ⇒ E[X] = 12.
300 = Variance of aggregate = λ E[X^2] = 1.5 E[X^2]. ⇒ E[X^2] = 200.
Var[X] = E[X^2] - E[X]^2 = 200 - 12^2 = **56**.
Comment: Similar to CAS 3L, 11/08, Q.3.

10.37. C. For temporary disability, S(20,000) = $\exp[-\sqrt{20{,}000/5000}\,]$ = 13.54%.

For permanent disability, S(20,000) = $\left(\frac{200{,}000}{200{,}000 + 20{,}000}\right)^2$ = 82.64%.

For temporary disability, λ = 1/4. For permanent disability, λ = 1/80.

Thus of the claims, the proportion that are temporary disability is: $\frac{1/4}{1/4 + 1/80}$ = 20/21.

Prob[X > 20000] = Prob[Temp.] Prob[X > 20000 | Temp.] + Prob[Perm.] Prob[X > 20000 | Perm.]
= (20/21)(13.54%) + (1/21)(82.64%) = **16.83%**.
Alternately, large temporary disability claims are Poisson with intensity:
(1/4)(13.54%) = 0.03385 per week.
Large permanent disability claims are Poisson with intensity: (1/80)(82.64%) =
0.01033 per week.
Thus large claims are Poisson with intensity: 0.03385 + 0.01033 = 0.04418 per week.
All claims are Poisson with intensity: 1/4 + 1/80 = 0.2625 per week
Thus the probability that a claim exceeds 20,000 is: 0.04418 / 0.2625 = **16.83%**.
Comment: Similar to CAS3L, 11/10, Q.12.

10.38. D. For temporary disability, F(x) = 1 - $\exp[-\sqrt{x/5000}\,]$.
S(50,000) = $\exp[-\sqrt{50{,}000/5000}\,]$ = 4.233%.

For permanent disability, F(x) = 1 - $\left(\frac{200{,}000}{200{,}000 + x}\right)^2$.

S(50,000) = $\left(\frac{200{,}000}{200{,}000 + 50{,}000}\right)^2$ = 64%.

Expected annual number of temporary disability claims: 52/4 = 13.
Expected annual number of large temporary disability claims: (4.233%)(13) = 0.550.
Expected annual number of permanent disability claims: 52/80 = 0.65.
Expected annual number of large permanent disability claims: (64%)(0.65) = 0.416.
Thus the number of large losses of either type is Poisson with mean: 0.550 + 0.416 = 0.966.
⇒ The variance of the annual number of claims covered by the reinsurance contract is **0.966**.
Comment: Similar to CAS3L, 11/11, Q.10.

10.39. C. 2nd moment of the LogNormal Distribution is: $\exp[(2)(9) + (2)(0.6^2)]$ = 134.9 million.
Thus the variance of the aggregate on Friday or Saturday is: (25)(134.9 million) = 3372.5 million.
The variance of the aggregate of each of the other 5 days is: (15)(134.9 million) = 2023.5 million.
Since the days are independent, the variance of the total loss amount for a seven day period is:
(2)(3372.5 million) + (5)(2023.5 million) = 16,862.5 million. $\sqrt{16{,}862.5 \text{ million}}$ = **130,000**.

Alternately, the number of accidents for the week is Poisson with mean: (2)(25) + (5)(15) = 125.
The variance of the total loss amount for a seven day period is:
(125)(134.9 million) = 16,862.5 million. $\sqrt{16{,}862.5 \text{ million}}$ = **130,000**.

Comment: Similar to CAS 3L 11/13, Q.11.

10.40. B. The number of claims by time 5 is Poisson with mean (5)(0.7) = 3.5.
The mean severity is: $\theta / (\alpha-1)$ = 30 / (4-1) = 10.

Thus the second moment of severity is: $\frac{\theta^2}{(\alpha-1)\ (\alpha-2)}$ = 30^2 / {(3)(2)} = 150.

The mean aggregate by time 5 is: (3.5) (10) = 35.
The variance of the aggregate by time 5 is: (3.5) (150) = 525.
Thus the second moment of the aggregate by time 5 is: 525 + 35^2 = **1750**.
Comment: Similar to ST, 5/14, Q.3.

10.41. E. Due to the memoryless property of the Exponential, the non-zero payments follow the same Exponential Distribution with θ = 800.
Thinning the Poisson, the number of non-zero payments is also Poisson with mean: S(d) 30.
Then the variance of the daily aggregate payments is:
S(d) 30 (second moment of the Exponential) = S(d) (30) (2) (800^2) = 38.4 million S(d).
Thus $V_2 / V_1 = S(d_2) / S(d_1)$ = exp[-250/800] / exp[-100/800] = exp[-0.1875] = **0.829**.
Comment: Similar to MAS-1, 5/19, Q.3.

10.42. $E[X^2] = Var[X] + E[X]^2$ = 300,000 + 500^2 = 550,000.
Var[S] = E[N] Var[X] = (10)(550,0000) = 5,500,000.
The standard deviation of S is: $\sqrt{5{,}500{,}000}$ = **2345**.

10.43. D. Mean aggregate loss = (3){(0.4)(1) + (0.2)(2) + (0.4)(3)} = 6.
For a compound Poisson, variance of aggregate losses =
(mean frequency)(second moment of severity) = (3){(0.4)(1^2) + (0.2)(2^2) + (0.4)(3^2)} = 14.4.
Since the severity is discrete, one should use the continuity correction.
Pr[S>9] $\cong 1 - \Phi((9.5 - 6)/\sqrt{14.4}) = 1 - \Phi(0.94)$ = 1 - 0.8264 = **17.4%**.

10.44. A. Since frequency is Poisson,
Var[S] = (mean frequency)(second moment of the severity).
30,000,000 = λ (500,000,000 + $50{,}000^2$). λ = 1/100.
Prob(N≥1) = 1 - Prob(N = 0) = 1 - $e^{-\lambda}$ = 1 - $e^{-0.01}$ = **0.995%**.

10.45. Mean Aggregate Loss = (350)(500) = 175,000. Since frequency is Poisson,
Variance of Aggregate Loss = (350)(2nd moment of severity) = $(350)(1000^2/3)$ =
116867 million.
Prob(S > 180,000) ≅ 1 - $\Phi[(180,000 - 175,000)/\sqrt{116.67 \text{ million}}]$ = 1 - Φ[0.46] = **32.3%**.
Comment: The second moment of the uniform distribution (a , b) is: $(b^3 - a^3) / (3(b-a))$.

10.46. D. The mean number of people per cab is: (0.6)(1) + (0.3)(2) + (0.1)(3) = 1.5.
The second moment of the number of people per cab is: $(0.6)(1^2) + (0.3)(2^2) + (0.1)(3^2) = 2.7$.
Over 72 hours the frequency is Poisson with mean number of cabs: (72)(10) = 720.
The mean aggregate number of people are: (720)(1.5) = 1080.
For a Poisson process, variance of aggregate number of people =
(mean number of cabs)(2nd moment of number of people per cab) = (720)(2.7) = 1944.
Thus the probability that the aggregate number of people is at least 1050 is approximately:
1 - $\Phi[(1049.5 - 1080)/\sqrt{1944}]$ = 1 - Φ(-0.69) = **0.7549**.
Comment: Since the number of people is an integer, use the continuity correction.

10.47. E. Over one hour, the mean frequency is (0.5)(60) = 30.
The second moment of the severity is: $(0.6)(1^2) + (0.2)(5^2) + (0.2)(10^2) = 25.6$.
For a compound Poisson, variance = (mean freq.)(2nd moment of severity) = (30)(25.6) = **768.**
Alternately, the finding of pennies is a Poisson process with mean: (30)(60%) = 18 and therefore variance 18. The value of pennies found has variance: $(1^2)(18) = 18$.
Similarly, the value of nickels found has variance: $(5^2)(6) = 150$.
The value of dimes found has variance: $(10^2)(6) = 600$.
The three processes are independent, so the variance of the total value of coins found is:
18 + 150 + 600 = **768**.

10.48. E. The second moment of the number of claimants per accident is:
$(1/2)(1^2) + (1/3)(2^2) + (1/6)(3^2) = 3.333$. The variance of a Compound Poisson Distribution is:
λ(2nd moment of the secondary distribution) = (12)(3.333) = **40**.
Alternately, thinning the original Poisson, those accidents with 1, 2, or 3 claimants are independent Poissons. Their means are: (1/2)(12) = 6, (1/3)(12) = 4, and (1/6)(12) = 2.
Number of accidents with 3 claimants is Poisson with mean 2. ⇒
The variance of the number of accidents with 3 claimants is 2.
Number of claimants for those accidents with 3 claimants = (3)(# of accidents with 3 claimants)
⇒ The variance of the # of claimants for those accidents with 3 claimants is: $(3^2)(2)$.
Due to independence, the variances of the three processes add: $(1^2)(6) + (2^2)(4) + (3^2)(2)$ = **40**.

10.49. D. We have thinned the original Poisson process;
Type I and Type II are two independent Poisson processes.
2,100,000 = Total Variance = Variance of Type I + Variance of Type II.
Expect 3000/3 = 1000 claims of Type I.
For Type I the severity is constant at 10 and the frequency is Poisson with mean 1000.
Therefore, Variance of Type I = $(10^2)(1000)$ = 100,000.
Therefore, Variance of Type II = 2,100,000 - 100,000 = **2 million**.
Alternately, for the whole process: $3000E[X^2]$ = 2,100,000. Therefore, $E[X^2]$ = 700.
700 = $E[X^2] = (1/3)(10^2)$ + (2/3)(2nd moment of severity of Type II).
Therefore, 2nd moment of severity of Type II = 1000.
Variance of Type II = (2/3)(3000)(1000) = **2 million**.
Comment: Expect 3000(2/3) = 2000 claims of Type II. 2 million = Variance of Type II = (2000)(2nd moment of severity of Type II). 2nd moment of severity of Type II = 1000.

10.50. B. Mean # claims / envelope = (1)(0.2) + (2)(0.25) + (3)(0.4) + (4)(0.15) = 2.5.
2nd moment # claims / envelope = $(1^2)(0.2) + (2^2)(0.25) + (3^2)(0.4) + (4^2)(0.15)$ = 7.2.
Over 13 weeks, the number of envelopes is Poisson with mean: (13)(50) = 650.
Mean of the compound distribution = (650)(2.5) = 1625.
Variance of the aggregate number of claims = Variance of a compound Poisson distribution = (mean primary Poisson distribution)(2nd moment of the secondary distribution) = (650)(7.2) = 4680. $\Phi(1.282) = 0.90$. Estimated 90^{th} percentile = $1625 + 1.282\sqrt{4680}$ = **1713.**

10.51. C. The coin flips divide the original Poisson Process into two independent Poisson Processes, each with λ = 60/2 = 30. The Poisson Process related to heads is in turn divided into two independent Poisson Processes, one related to good risks, with λ = (2/3)(30) = 20, and the other related to bad risks, with λ = (1/3)(30) = 10.
Similarly, the Poisson Process related to tails is divided into two independent Poisson Processes, one related to good risks, with λ = (2/3)(30) = 20, and the other related to rejected (bad) risks, with λ = (1/3)(30) = 10.
Each Compound Poisson Process has variance of λ(second moment of severity).
Good risks have a second moment of severity of: 10000 + 300^2 = 100,000.
Bad risks have a second moment of severity of: 90000 + 100^2 = 100,000.
Rejected risks always result in zero profit for the insurer, for a second moment of zero.
There are four independent compound Poisson Processes, whose variances add:

Poisson Process	λ	second moment of severity	variance
Head/Good	20	100,000	2,000,000
Head/Bad	10	100,000	1,000,000
Tail/Good	20	100,000	2,000,000
Tail/Reject	10	0	0
Total			**5,000,000**

Alternately, severity is a mixed distribution, with 50% weight to heads and 50% weight to tails. These are each in turn mixed distributions.
Heads is 2/3 weight to good and 1/3 weight to bad, with second moment:
(2/3)(10000 + 300^2) + (1/3)(90000 + 100^2) = 100,000.
Tails is 2/3 weight to good and 1/3 weight to zero, with second moment:
(2/3)(10000 + 300^2) + (1/3)(0^2) = 66,667.
The overall second moment of severity is: (50%)(100,000) + (50%)(66,667) = 83,333.
Variance of the Compound Poisson Process is: (60)(83,333) = **5 million**.
Comment: Difficult!

10.52. D. For a week, frequency is Poisson with mean: (6)(3) = 18.
The mean severity is: (1/4)(0) + (1/2)(2000) + (1/4)(8000) = 3000.
The 2nd moment of severity is: (1/4)(0^2) + (1/2)(2000^2) + (1/4)(8000^2) = 18,000,000.
Mean aggregate loss = (18)(3000) = 54,000.
Variance of aggregate loss = (18)(18,000,000) = 324,000,000.
Prob[Aggregate ≤ 63,000] ≅ $\Phi[(63{,}000 - 54{,}000)/\sqrt{324{,}000{,}000}\,] = \Phi[0.5]$ = **0.6915**.

10.53. C. A_{65} = 0.43980. $^2A_{65}$ = 0.23603. d = 0.06/1.06 = 0.0566038.
Frequency is Poisson with mean: (20%)(50) = 10 policies per day.
Given curtate future lifetime K, the prospective loss variable is:
L = present value of losses - present value of premiums =
$10000v^{K+1} - 500(1 - v^{K+1})/d = 18{,}833.33v^{K+1} - 8833.33$.
$E[L] = 18{,}833.33E[v^{K+1}] - 8833.33 = 18{,}833.33A_{65} - 8833.33 = -550.43$.
$Var[L] = 18{,}833.33^2Var[v^{K+1}] = 18{,}833.33^2Var[A_{65}] = 18{,}833.33^2(^2A_{65} - A_{65}^2) = 15{,}112{,}092$.
Mean aggregate = 10E[L] = (10)(-550.43) = -5504.3.
Variance of aggregate = $(10)E[L^2] = (10)(15{,}112{,}092 + 550.43^2) = 154{,}150{,}652$.
Prob[Aggregate < 0] $\cong \Phi[(0 - -5504.3)/\sqrt{154{,}150{,}652}] = \Phi[0.44]$ = **0.6700**.

Comment: The mean prospective loss per policy is: $10{,}000A_{65} - 500\ddot{a}_{65}$ =
(10,000)(0.43980) - (500)(9.8969) = -550.45. One can start with equation 6.3.1 in
Actuarial Mathematics; however, one can not use equation 6.3.3, which only applies when the premiums are chosen so that E[L] = 0, which is not the case here. Here we do have that
$E[L] = (1 + P/d)A_{65} - P/d$, and $Var[L] = (1 + P/d)^2Var[A_{65}] = (1 + P/d)^2(^2A_{65} - A_{65}^2)$.

10.54. B. The total number of delayed passengers is an aggregate distribution, with frequency the number of delayed flights, and the severity the number of passengers on a flight.
The number of flights delayed per year is Poisson with mean: (2)(12) = 24.
The second moment of the severity distribution is: $50^2 + 30^2 = 3400$.
The variance of the number of passengers delayed per year is: (24)(3400) = 81,600.
The standard deviation of the number of passengers delayed per year is: $\sqrt{81{,}600}$ = 285.66.
The standard deviation of the annual compensation is: (100)(285.66) = **28,566**.

10.55. E. The smoker process is Poisson with mean (20%)(1000) = 200.
The variance of the smoker losses is: $(200)(5000 + 100^2)$ = 3.0 million.
The nonsmoker process is Poisson with mean (80%)(1000) = 800.
The variance of the nonsmoker losses is: $(800)(8000 + 100^2)$ = 14.4 million.
The two processes are independent, so their variances add:
3.0 million + 14.4 million = **17.4 million**.
Comment: For each compound Poisson Process, variance = l(second moment of severity).

10.56. D. Second Moment of the Negative Binomial = variance + mean^2 = $r\beta(1 + \beta) + (r\beta)^2$ =
$(3)(0.75)(1.75) + \{(3)(0.75)\}^2 = 9$. For a Compound Poisson Process,
Variance = (λt)(second moment of severity distribution) = (3)(4)(9) = **108**.

10.57. A. This is a Compound Poisson Process, with amount distribution that is a 3-point mixture. The amount distribution has mean: (20%)(8000) + (30%)(-5000) + (50%)(0) = 100.
To get the second moment of the amount distribution, we weight together the individual second moments: (20%)($1000^2 + 8000^2$) + (30%)($2000^2 + 5000^2$) + (50%)(0) = 21.7 million.
Therefore, the mean of the Compound Poisson Process is: (8)(100)(100) = 80,000.
The variance of the Compound Poisson Process is: (8)(100)(21.7 million) = 17,360 million.
Prob[total withdrawals of the bank will exceed the total deposits] =
Prob[Compound Poisson Process is less than 0] ≅
$\Phi[(0 - 80000)/\sqrt{17{,}360 \text{ million}}] = \Phi[-0.61] = 1 - 0.7291 =$ **0.2709**.

Alternately, one can thin the original Poisson Process into three independent Poisson Processes: deposit, withdrawal, and complaint.
The deposit process has an intensity of: (20%)(100) = 20, mean of: (8)(20)(8000) = 1,280,000, and variance of: (8)(20)($1000^2 + 8000^2$) = 10,400 million.
The withdrawal process has an intensity of: (30%)(100) = 30, mean of: (8)(30)(-5000) = -1,200,000, and variance of: (8)(30)($2000^2 + 5000^2$) = 6960 million.
The complaint process has a mean and variance of 0.
Therefore, the original process has a mean of: 1,280,000 - 1,200,000 + 0 = 80,000, and a variance of: 10,400 million + 6960 million + 0 = 17,360 million. Proceed as before.

10.58. B. 2nd moment of the amount distribution is: (5%)(10^2) + (15%)(5^2) + (80%)(1^2) = 9.55.
Variance of the compound Poisson Process is: (22)(9.55) = **210.1**.

10.59. E. $\mu = 50$. $\sigma = (25\%)\mu = 12.5$.
2nd moment of severity = $\sigma^2 + \mu^2 = 12.5^2 + 50^2 = 2656.25$.
Variance of the Compound Poisson is: λ (Second moment of severity) = (3)(60)(2656.25) = 478,125. Standard deviation is: $\sqrt{478{,}125}$ = **$691**.

10.60. A. The number of donations over seven days is Poisson with mean: (80%)(10)(7) = 56.
Mean of the distribution of the amount of food donated is: (56)(15) = 840.
Variance of the distribution of the amount of food donated is: (56)($75 + 15^2$) = 16,800.
The number of withdrawals over seven days is Poisson with mean: (20%)(10)(7) = 14.
Mean of the distribution of the amount of food withdrawn is: (14)(40) = 560.
Variance of the distribution of the amount of food withdrawn is: (14)($533 + 40^2$) = 29,862.
Mean net amount of food is: 840 - 560 = 280.
Variance of the net amount of food is: 16,800 + 29,862 = 46,662.
Probability that net amount is at least 600 is:
$1 - \Phi[(599.5 - 280)/\sqrt{46{,}662}] = 1 - \Phi[1.48] =$ **6.94%**.

10.61. C. Variance for 4 months = λt(second moment of the amount distribution)
= (0.02)(4)($25{,}000{,}000 + 5000^2$) = 4 million. Standard deviation = 2000. (10%)(2000) = **200**.
<u>Comment</u>: *This Pareto distribution has a coefficient of variation of:* $\sqrt{25{,}000{,}000}$ */ 5000 = 1.*
This is impossible, since any Pareto has a coefficient of variation > 2.

10.62. A. Second Moment of Severity = $2\theta^2$ = (2)(10 million)^2.
Mean Aggregate Loss = λ(mean severity) = (2)(10 million) = $20 million.
Variance of Aggregate Loss = λ(second moment of severity) = (2)(2)(10 million)^2.
Standard Deviation of Aggregate Loss = $20 million.
Risk Load = (10%)(20 million + 20 million) = $4 million.
The risk load per policy = ($4 million)/(1 million) = **$4**.

10.63. D. Let X be the number of items purchased by an individual shopper.
10 = Mean of aggregate = λ E[X] = 10 E[X]. $\Rightarrow$ E[X] = 1.
100 = Variance of aggregate = λ E[X^2] = 10 E[X^2]. $\Rightarrow$ E[X^2] = 10.
Var[X] = E[X^2] - E[X]^2 = 10 - 1^2 = **9**.

10.64. B. The Gamma has a second moment of: $\alpha(\alpha+1)\theta^2$ = 750,000.
Variance of the aggregate amount is:
λt (second moment of the amount distribution) = (13)(5)(750,000) = 48,750,000.
$\sqrt{48{,}750{,}000}$ = **6982**.

10.65. B. The number of fires over two months is Poisson with mean: (2)(22) = 44.
The mean of the LogNormal is: exp[2 + 0.75^2/2] = 9.7889.
The second moment of the LogNormal is: exp[(2)(2) + (2)(0.75^2)] = 168.174.
The mean aggregate cost is: (44)(9.7889) = 430.71.
The variance of the aggregate cost is: (44)(168.174) = 7399.66.
Prob[Aggregate Cost > 508] = $1 - \Phi[\frac{508 - 430.71}{\sqrt{7399.66}}]$ = 1 - Φ[0.90] = 1 - 0.8159 = **18.41%**.
<u>Comment</u>: Aggregate losses are continuous, so there is no need to use the continuity correction.

10.66. D. For Hail S(7500) = $e^{-7500/2000}$ = 2.352%.
For Flood S(7500) = $e^{-7500/10,000}$ = 47.237%.
For Hail, λ = 1/3. For Flood, λ = 1/12.
Thus of the claims, the proportion that are from Hail is: $\frac{1/3}{1/3 + 1/12}$ = 80%.
Prob[X > 7500] = Prob[Hail] Prob[X > 7500 | Hail] + Prob[Flood] Prob[X > 7500 | Flood] =
(80%)(2.352%) + (20%)(47.237%) = **11.329%**.
Alternately, large hail losses are Poisson with intensity: (1/3)(2.352%) = 0.00784 per month.
Large flood losses are Poisson with intensity: (1/12)(47.237%) = 0.03936 per month.
Thus large losses are Poisson with intensity: 0.00784 + 0.03936 = 0.04720 per month.
All losses are Poisson with intensity: 1/3 + 1/12 = 0.41667 per month.
Thus the probability that the damage caused by an insured event exceeds 7,500 is:
0.04720 / 0.41667 = **11.33%**.
<u>Comment</u>: The survival function of the mixture is the mixture of the survival functions.

10.67. C. Hail storms are Poisson with $\lambda = 4$ per year.
The second moment of severity for hail storms is: $2\theta^2 = (2)(2000^2) = 8$ million.
The variance of losses from hail storms is: (4)(8 million) = 32 million.
Floods are Poisson with $\lambda = 1$ per year.
The second moment of severity for floods is: $2\theta^2 = (2)(10{,}000^2) = 200$ million.
The variance of losses from floods is: (1)(200 million) = 200 million.
The variance of the aggregate annual losses from hail storms and floods combined is:
32 million + 200 million = **232 million**.

10.68. D. The first moment of severity is: (0.9)(100) + (0.1)(1100) = 200.
The second moment of severity is: $(0.9)(100^2) + (0.1)(1100^2) = 130{,}000$.
Mean Aggregate = (150)(200) = 30,000.
Variance of Aggregate = (150)(130,000) = 19.5 million.
Prob[Aggregate > 40,000] $\cong 1 - \Phi[(40{,}000 - 30{,}000) / \sqrt{19.5 \text{ million}}] = 1 - \Phi[2.265]$ = **1.18%**.

10.69. E. For Fire, S(250,000) = exp[-250/35] = 0.0007905.
Thus the number of large fire losses is Poisson with mean: (0.0007905)(700) = 0.553.
For Wind, S(250,000) = exp[-250/200] = 0.2865.
Thus the number of large fire losses is Poisson with mean: (0.2865)(2) = 0.573.
Thus the number of large losses of either type is Poisson with mean: 0.553 + 0.573 = 1.126.
$\Rightarrow$ The variance of the annual number of claims covered by the reinsurance contract is **1.126**.
Comment: Getting the variance of the amount paid rather than the number of covered claims is harder and beyond what you would expect to be asked on this exam.
The non-zero payments on fire claims are also Exponential with $\theta = 35{,}000$,
and second moment: $2\theta^2 = 2450$ million.
Therefore, the variance of the payments on fire claims is: (0.553)(2450 million) = 1355 million.
The non-zero payments on wind claims are also Exponential with $\theta = 200{,}000$,
and second moment: $2\theta^2 = 80{,}000$ million.
Therefore, the variance of the payments on wind claims is:
(0.573)(80,000 million) = 45,840 million.
Since fire and wind are independent, the variance of the total amount paid by the reinsurer is:
1355 million + 45,840 million = 47,195 million. $\Rightarrow$ standard deviation is 217,244.
The mean amount paid paid by the reinsurer is: (0.553)(35,000) + (0.573)(200,000) = 133,955.

10.70. D. Over 12 months, the mean of the aggregate is: (120)(500) = 60,000.
The variance of the aggregate is: (120)(2nd moment of severity) = $(120)(2)(500^2)$.
Standard deviation is: $500\sqrt{240} = 7746$.
Prob[Aggregate > 70,000] $\cong 1 - \Phi[(70{,}000 - 60{,}000) / 7746] = 1 - \Phi[1.291]$ = **9.84%**.

10.71. E. The second moment for the number of people for Auto is: $(0.6)(1^2) + (0.4)(2^2) = 2.2$.
Variance of Auto Compound Poisson is: $(0.3)(2.2) = 0.66$.
The second moment for the number of people for H.O. is: $(0.5)(1^2) + (0.3)(2^2) + (0.2)(3^2) = 3.5$.
Variance of H.O. Compound Poisson is: $(0.4)(3.5) = 1.40$.
The variances of the two independent Poisson Processes add: 0.66 + 1.40 = **2.06**.

10.72. A. The second moment of the Gamma Distribution is:
$\alpha(\alpha+1)\theta^2 = (5)(6)(200^2) = 1{,}200{,}000$.
Thus the variance of the aggregate on Monday is: (20)(1.2 million) = 24 million.
The variance of the aggregate of each of the other 6 days is: (10)(1.2 million) = 12 million.
Since the days are independent, the variance of the total loss amount for a seven day period is:
24 million + (6)(12 million) = 96 million. $\sqrt{96 \text{ million}}$ = **9798**.
Alternately, the number of claims for the week is Poisson with mean: 20 + (6)(10) = 80.
The variance of the total loss amount for a seven day period is:
(80)(1.2 million) = 96 million. $\sqrt{96 \text{ million}}$ = **9798**.

10.73. C. The second moment of the amount distribution is:
$(0.25)(20^2) + (0.50)(40^2) + (0.10)(50^2) + (0.15)(100^2) = 2650$.
Variance of this Compound Poisson is: $\lambda \, E[X^2] = (100)(2650) = 265{,}000$.
Standard deviation of the total daily withdrawals is: $\sqrt{265{,}000}$ = **514.8**.

10.74. C. The severity is a mixture of three Exponentials.
Mean of the mixture is the mixture of the means:
(0.2)(500) + (0.3)(300) + (0.5)(200) = 290.
The second moment of the mixture is the mixture of the second moments:
$(0.2)(2)(500^2) + (0.3)(2)(300^2) + (0.5)(2)(200^2) = 194{,}000$.
Mean of the compound Poisson is: (13)(10)(290) = 37,700.
Variance of the compound Poisson is: (13)(10)(194,000) = 25,220,000.
95^{th} percentile of the amount of the fund after 13 weeks: $37{,}700 + 1.645\sqrt{25{,}220{,}000}$ = **45,961**.

10.75. B. Looking in the tables attached to the exam, the Pareto Distribution has $\alpha = 4$ and $\theta = 1000$. The Pareto has mean: $\theta/(\alpha-1) = 1000/(4-1) = 333.333$.
The Pareto has second moment: $\dfrac{2\,\theta^2}{(\alpha-1)(\alpha-2)} = \dfrac{(2)\,(1000^2)}{(4-1)\,(4-2)}$ = 1/3 million.
Thus the mean aggregate loss is: (52)(10)(333.333) = 173,333.
The variance of aggregate loss is: (52)(10)(1/3 million) = 173.333 million.
Probability that the total claim amount in 52 weeks exceeds 200,000 is approximately:
$1 - \Phi[\dfrac{200{,}000 - 173{,}333}{\sqrt{173.333 \text{ million}}}] = 1 - \Phi[2.026]$ = **2.14%**.

10.76. B. Let the proportion of Major claims be p.
$17 = (5)\,\{(1)(0.85 - p) + 4p + (10)(0.15)\}$. $\Rightarrow$ p = **0.35**.

10.77. C. Mean severity is: (20%)(10) + (60%)(20) + (20%)(30) = 20.
Second moment of severity is: (20%)(10^2) + (60%)(20^2) + (20%)(30^2) = 440.
For 4 hours, mean aggregate is: (3)(4)(20) = 240.
Variance of aggregate is: (3)(4)(440) = 5280.
Without using the continuity correction:
Prob[at least 300] = $1 - \Phi[\frac{300 - 240}{\sqrt{5280}}] = 1 - \Phi[0.83] = 1 - 0.7967 =$ **0.2033**.

10.78. C. Mean severity is: (0.6)(1) + (0.25)(2) + (0.1)(3) + (0.05)(4) = 1.6.
Second moment of severity is: (0.6)(1^2) + (0.25)(2^2) + (0.1)(3^2) + (0.05)(4^2) = 3.3.
The mean aggregate is: (48)(10)(1.6) = 768.
The variance of the aggregate is: (48)(10)(3.3) = 1584.
Thus using the continuity correction, the probability of at least 800 people is approximately:
$1 - \Phi[\frac{799.5 - 768}{\sqrt{1584}}] = 1 - \Phi[0.7915] =$ **21.4%**.

10.79. D. The number of delayed flights per year is Poisson with mean: (4)(12) = 48.
The second moment of the distribution of number of passengers per flight is:
$100^2 + 50^2$ = 12,500.
Thus the variance of the aggregate number of passengers is: (48)(12,500) = 600,000.
The standard deviation of the total annual compensation is: 200 = **154,919**.
Comment: Multiplying a variable by a constant, such as 200,
multiplies its standard deviation by 200.

10.80. E. The second moment of severity is: (0.25)(5^2) + (0.50)(50^2) + (0.25)(95^2) = 3512.5.
Over twelve months, the average frequency is: (4)(12) = 48.
The variance of the Compound Poisson is: (48)(3512.5) = **168,600**.
Comment: It is common to use S rather than X for aggregate losses.

10.81. C. Due to the memoryless property of the Exponential, the non-zero payments follow the same Exponential Distribution with θ = 1,000.
Thinning the Poisson, the number of non-zero payments is also Poisson with mean: S(d) 40.
Then the variance of the daily aggregate payments is:
S(d) 40 (second moment of the Exponential) = S(d) (40) (2) (1000^2) = 80 million S(d).
Thus $V_2 / V_1 = S(d_2) / S(d_1)$ = exp[-500/1000] / exp[-100/1000] = exp[-0.4] = **0.670**.

10.82. D.
The second moment of severity is: (0.5)(1^2) + (0.4)(2^2) + (0.08)(3^2) + (0.02)(4^2) = 3.14.
Variance of the compound Poisson: (4)(50)(3.14) = **628**.

Section 11, Comparing Poisson Processes

For example, assume you are sitting by the side of a country road watching cars pass in one direction. Assume cars pass with a Poisson Process with intensity $\lambda = 0.10$. Assume 20% of the cars that pass are red, 30% of the cars that pass are blue and 50% of the cars that pass are green. Then the passing of red cars is a Poisson process with intensity $\lambda = 0.02$. The Poisson process for red cars is independent of the Poisson Process for blue cars with intensity 0.03 and of the Poisson Process for green cars with intensity 0.05.

The red cars pass with claims intensity 0.02, while blue cars pass with claims intensity 0.03. Red and blue cars have a combined claims intensity of 0.05. Thus 0.02/0.05 = 40% of the red and blue cars that pass are red. Since we have independent and constant claims intensities, the color of the first red or blue car is red 40% of the time.

If one has **two independent Poisson Processes with claims intensities λ_1 and λ_2, then the chance that a claim from the first process appears before a claim from the second process is** $\frac{\lambda_1}{\lambda_1 + \lambda_2}$.[88] Similarly, the chance that a claim from the second process appears before a claim from the first process is: $\frac{\lambda_2}{\lambda_1 + \lambda_2}$.[89]

Exercise: What is the probability that a blue car passes before a red car passes?
[Solution: 0.03 / (0.02 + 0.03) = 60%.]

The same type of statement can be made with respect to the three different colors of cars. The probability that the first car seen is red is: (0.02) / (0.02 + 0.03 + 0.05) = 20%. The probabilities that the first car seen is blue or green are respectively: 0.03/0.1 = 30%, and 0.05/0.1 = 50%.

Exercise: What is the probability that the third car to pass is red?
[Solution: Since we have independent and stationary increments, it is the same as the chance that the first car is red, 20%.]

More generally, if one has n independent Poisson Processes with claims intensities λ_1, λ_2, ... , λ_n, then the chance that a claim from the first process appears before a claim from any of the other processes is: $\frac{\lambda_1}{\lambda_1 + \lambda_2 + ... + \lambda_n}$. The chance that the first claim comes from process i is proportional to its intensity λ_i.

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88 See Page 14 of "Poisson Processes" by Daniel, and Page 309 of Ross.
89 Note that these probabilities only depend on the relative sizes of λ_1 and λ_2.

<u>Number of Events from a Poisson Process prior to the 1st Event from another Poisson Process</u>:

Assume that cars are only red or blue. The red cars pass with claims intensity 0.02, while blue cars pass with claims intensity 0.03. Then 40% of the cars that pass are red and 60% of the cars are blue, independent of the color of any other car.

The chance of seeing four red cars in a row is: $0.4^4 = 0.0256$.
The chance of seeing four red cars and then a blue car is: $(0.4^4)(0.6) = 0.01536$.

Exercise: What is the probability of seeing exactly x red cars prior to seeing the first blue car?
[Solution: See x red cars in a row followed by a blue car. Probability is: $(0.6)(0.4^x)$.
<u>Comment</u>: This is a Geometric Distribution with $\beta / (1+\beta) = 0.4$. Thus $\beta = 0.4 / 0.6 = 2/3$.]

Mathematically, we have a series of Bernoulli trials, with a blue car as a success and a red car as a failure. In general, for a series of independent identical Bernoulli trials, the chance of the first success following x failures is given by a Geometric Distribution with mean:

$$\beta = \frac{\text{chance of a failure}}{\text{chance of a success}}.$$

If one has two independent Poisson Processes with intensities λ_1 and λ_2, then the chance that the next event will be from the first process is: $\frac{\lambda_1}{\lambda_1 + \lambda_2}$.

Thus the number of events from the first process prior to the first event from the second process is Geometric with $\beta = \frac{\text{chance of a failure}}{\text{chance of a success}} = \frac{\text{Prob[type 1]}}{\text{Prob[type 2]}} = \frac{\frac{\lambda_1}{\lambda_1 + \lambda_2}}{\frac{\lambda_2}{\lambda_1 + \lambda_2}} = \frac{\lambda_1}{\lambda_2}$.

Exercise: Uptown and downtown trains are independent Poisson processes with the same rate. You are waiting for a downtown train.
What is the probability that more than two uptown trains pass while you are waiting?
[Solution: The number of uptown trains prior to the first downtown train is Geometric with $\beta = 1$.
$1 - f(0) - f(1) - f(2) = 1 - 1/2 - 1/4 - 1/8 = 1/8$.]

The number of events from the first process prior to the <u>second</u> event from the second process is the sum of two independent, identically distributed Geometrics and thus a Negative Binomial with r = 2. Thus the number of events from the first process prior to the r^{th} event from the second process is Negative Binomial with $\beta = \lambda_1/\lambda_2$ and r.

n Events from One Poisson Process Before k Events from Another Poisson Process:

One can ask somewhat more complex questions. For example, what is the chance that we see 2 red cars prior to seeing any blue cars? Since we have independent and stationary increments, this is just the product of the chance of the first car being red and the chance that the second car is red: (0.4)(0.4) = 0.16.

Exercise: What is the chance of seeing 3 red cars before seeing a blue car?
[Solution: The chance of a red car before a blue car is 40%. Since we have independent and stationary increments, the desired probability is (0.4)(0.4)(0.4) = 0.064.]

Exercise: What is the chance of seeing 3 red cars before seeing 2 blue cars?
[Solution: The chance of a red car before a blue car is 40%. The chance of a blue car before a red car is 60%. There are the following four possibilities in which we observe 3 red cars before seeing two blue cars: r,r,r; b,r,r,r; r,b,r,r; r,r,b,r. The probability of the first outcome, three red cars, is $0.4^3 = 0.064$. The probability of each of the next three possibilities is: $(0.4^3)(0.6) = 0.0384$.
Thus the total probability that we observe 3 red cars before seeing two blue cars is:
$0.4^3 + 3\ (0.4^3)(0.6) = 0.1792$.]

There are the following four possibilities in which we observe 3 red cars before seeing 2 blue cars: r,r,r; b,r,r,r; r,b,r,r; r,r,b,r.[90]
This is also a list of all the situations in which at least 3 of the first 4 cars is red.
We see 3 red cars before seeing 2 blue cars if and only if at least 3 of the first 4 cars is red. The number of red cars among the first 4 is Binomial with q = 0.4 and m = 4. Thus, the probability of at least 3 red cars among the first 4 is: $(4)(0.4^3)(0.6) + 0.4^4 = 0.1792$.

In general, n red cars before k blue cars. ⇔ at least n of the first n + k - 1 cars is red.
The chance that we observe n red cars before seeing k blue cars is the same as the chance that at least n of the first n + k - 1 cars is red. This is the sum of Binomial terms from n to n + k - 1, with $q = \lambda_r / (\lambda_r + \lambda_b) = 0.02 / (0.02 + 0.03) = 0.4$ and m = n + k - 1.

Alternately, think of a red car as a "success" and a blue car as a "failure". Then the chance that we observe 3 red cars before seeing 2 blue cars, is the chance of fewer than 2 "failures" prior to our 3rd "success", in a series of independent Bernoulli trials. The chance of x failures prior to the 3rd success is given by a Negative Binomial Distribution with r = 3.[91] The chance of a red car or a success is 40%. Thus the other parameter of the Negative Binomial is:
β = chance of failure / chance of success = 0.6/0.4 = 1.5.

Thus the chance that we observe 3 red cars before seeing 2 blue cars is the sum of terms from this Negative Binomial from x = 0 to 1; it is F(1) for this Negative Binomial with r = 3 and $\beta = 1.5$.
For this Negative Binomial, $f(0) = 1/(1+\beta)^r = 1/2.5^3 = 0.064$,
$f(1) = r\beta/(1+\beta)^{r+1} = (3)(1.5)/2.5^4 = 0.1152$.
F(1) = 0.064 + 0.1152 = 0.1792, matching the previous result.

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[90] Through the first four cars these are 5 possibilities: r,r,r,r; r,r,r,b; b,r,r,r; r,b,r,r; r,r,b,r.
[91] See my section on the Negative Binomial Distribution

Thus we see how we can calculate the probability of seeing 3 red cars before 2 blue cars either as a sum of Binomial densities or a sum of Negative Binomial densities.

Exercise: What is the chance that we seeing 4 blue cars before seeing 7 red cars?
[Solution: The chance of a blue car before a red car is 60%.

n	0	1	2	3	4	5	6
f(n)	0.1296	0.2074	0.2074	0.1659	0.1161	0.0743	0.0446
F(n)	0.1296	0.3370	0.5443	0.7102	0.8263	0.9006	0.9452

We want F(7-1) = F(6) for a Negative Binomial with parameters r = 4 and
β = chance of failure / chance of success = (0.4/0.6) = 2/3. F(6) = 0.9452.
Alternately, it is the sum of Binomial terms from 4 to 4 + 7 - 1 = 10, with q = 0.6 and
m = 10. This is: $210(0.6^4)(0.4^6) + 252(0.6^5)(0.4^5) + 210(0.6^6)(0.4^4) + 120(0.6^7)(0.4^3) + 45(0.6^8)(0.4^2) + 10(0.6^9)(0.4^1) + 1(0.6^{10}) = 0.9452$.]

More generally, one can solve this type of problem by either of two techniques, one involving a sum of Binomial densities and the other a sum of Negative Binomial densities. **If one has two independent Poisson Processes with claims intensities λ_1 and λ_2, then the chance that n claims from the first process occur before k claims from the second process is either the sum of Binomial terms from n to n + k - 1, with $q = \lambda_1 / (\lambda_1 + \lambda_2)$ and m = n + k - 1, or F(k-1) for a Negative Binomial Distribution with parameters r = n and $\beta = \lambda_2 / \lambda_1$.**[92]

Exercise: Claims from illness are a Poisson Process with claims intensity 13.
Claims from accident are a Poisson Process with claims intensity 7.
The two processes are independent.
What is the probability of 5 claims from illness before 3 claims due to accident?
[Solution: Each claim has a 13/(13 + 7) = 0.65 chance of being from illness.
There are 5 claims from illness before 3 claims due to accident if and only if at least 5 of the first 7 claims is from illness. (If fewer than 5 of the first 7 are from illness, then at least 3 are from accident.) The number of the first 7 claims that are from illness is Binomial with q = 0.65 and m = 7.
The desired probability is the sum of the densities at 5, 6 and 7:
$21(0.65^5)(0.35^2) + 7(0.65^6)(0.35) + 0.65^7 = 0.2985 + 0.1848 + 0.0490 = 0.5323$.
Alternately, we want F(3-1) = F(2) for a Negative Binomial with parameters r = 5 and
β = chance of failure / chance of success = 0.35/0.65 = 7/13.
$f(0) = 1/(1+\beta)^r = 1/(20/13)^5 = 0.1160$,
$f(1) = r\beta/(1+\beta)^{r+1} = (5)(7/13)/(20/13)^6 = 0.2031$.
$f(2) = r(r+1)\beta^2/ 2(1+\beta)^{r+2} = (5)(6)(7/13)^2/ 2(20/13)^7 = 0.2132$.
F(2) = 0.1160 + 0.2031 + 0.2132 = 0.5323.]

[92] This is equivalent to the formula near the bottom of p. 230 of Introduction to Probability Models by Ross.

<u>Problems</u>:

Use the following information for the next four questions:
- You and your friend Nancy are waiting together for your buses.
- Nancy is waiting for the Bayside bus and you are waiting for the Whitestone bus.
- Bayside buses arrive via a Poisson Process at a rate of 6 per hour.
- Whitestone buses arrive via a Poisson Process at a rate of 4 per hour.
- The arrival of Bayside buses and Whitestone buses are independent.

11.1 (2 points) What is the average time until the first one of you catches her bus?
(A) 4 minutes (B) 6 minutes (C) 8 minutes (D) 10 minutes (E) 12 minutes

11.2 (2 points) What is the average time until the last one of you catches their bus?
(A) 15 minutes (B) 17 minutes (C) 19 minutes (D) 21 minutes (E) 23 minutes

11.3 (2 points) What is the average time you wait without Nancy?
(A) 4 minutes (B) 6 minutes (C) 8 minutes (D) 9 minutes (E) 10 minutes

11.4 (2 points) What is the average time Nancy waits without you?
(A) 4 minutes (B) 6 minutes (C) 8 minutes (D) 9 minutes (E) 10 minute

Use the following information for the next 3 questions:
Taxicabs leave a hotel with a group of passengers at a Poisson rate $\lambda = 10$ per hour. The number of people in each group taking a cab is independent and has the following probabilities:

Number of People	Probability
1	0.60
2	0.30
3	0.10

11.5 (1 point) What is the probability that the first taxicab to leave has 2 people?
A. 10% B. 15% C. 20% D. 25% E. 30%

11.6 (3 points) What is the probability that 3 taxicabs leave with 2 people each before 3 taxicabs with 1 person each?
A. 21% B. 23% C. 25% D. 27% E. 29%

<u>**11.7**</u> (3 points) What is the probability that 5 taxicabs with 1 person each leave before 2 taxicabs with multiple people?
A. 21% B. 23% C. 25% D. 27% E. 29%

Use the following information for the next 9 questions:

- A loss occurrence may be caused by wind, earthquake or theft.
- Wind, earthquake and theft losses occur independently of one another.
- Wind losses follow a Poisson Process.
  The expected amount of time between wind losses is 10 years.
- Earthquake losses follow a Poisson Process.
  The expected amount of time between earthquake losses is 25 years.
- Theft losses follow a Poisson Process.
  The expected amount of time between theft losses is 5 years.
- The size of wind losses follows a LogNormal Distribution, with parameters $\mu = 2$ and $\sigma = 3$.
- The size of earthquake losses follows a Pareto Distribution, with parameters $\alpha = 2$ and $\theta = 1500$.
- The size of theft losses follows a Weibull Distribution, with parameters $\tau = 2$ and $\theta = 600$.

11.8 (1 point) What is the chance of the first theft loss occurring prior to a wind loss?
A. 1/2 B. 2/3 C. 7/10 D. 3/4 E. 4/5

11.9 (1 point) What is the chance that the fifth loss is an earthquake loss?
A. Less than 0.12
B. At least 0.12, but less than 0.14
C. At least 0.14, but less than 0.16
D. At least 0.16, but less than 0.18
E. At least 0.18

11.10 (1 point) What is the chance of the both the fifth and tenth losses being from earthquakes?
A. Less than 1.0%
B. At least 1.0%, but less than 1.1%
C. At least 1.1%, but less than 1.2%
D. At least 1.2%, but less than 1.3%
E. At least 1.3%

11.11 (3 points) A loss of size greater than 1000 is observed.
What is the chance that the loss is an earthquake?
A. Less than 0.44
B. At least 0.44, but less than 0.46
C. At least 0.46, but less than 0.48
D. At least 0.48, but less than 0.50
E. At least 0.50

<u>11.12</u> (3 points) Calculate the probability that a loss occurrence exceeds 1000.
A. Less than 6%
B. At least 6%, but less than 7%
C. At least 7%, but less than 8%
D. At least 8%, but less than 9%
E. At least 9%

11.13 (1 point) What is the standard deviation of the annual number of loss occurrences that exceed 1000?
A. 0.16 B. 0.18 C. 0.20 D. 0.22 E. 0.24

<u>11.14</u> (1 point) What is the chance of the fourth loss being a theft and the seventh loss being from wind?
A. 17% B. 19% C. 21% D. 23% E. 25%

<u>11.15</u> (3 points) What is the chance of observing 4 wind losses before observing 6 theft losses?
A. 33% B. 35% C. 37% D. 39% E. 41%

11.16 (3 points) What is the chance that we observe 3 earthquake losses before seeing 8 theft losses?
A. Less than 0.16
B. At least 0.16, but less than 0.18
C. At least 0.18, but less than 0.20
D. At least 0.20, but less than 0.22
E. At least 0.22

11.17 (3 points) In the State of Grace, the male drivers each have their claim frequency given by a Poisson Process with $\lambda = 0.05$, while the female drivers each have their claim frequency given by a Poisson Process with $\lambda = 0.03$.
You insure in the State of Grace 2000 male drivers and 1000 female drivers.
Assume the claim frequency processes of the individual drivers are independent.
What is the chance of observing 2 claims from female drivers before 3 claims from male drivers?
A.15% B. 17% C. 19% D. 21% E. 23%

11.18 (2 points) Buses and streetcars are independent Poisson processes.
The rate for streetcars is twice that for busses.
You are waiting for a bus.
What is the probability that more than two streetcars pass while you are waiting?

11.19 (4 points) An insurance company has two insurance portfolios.
● Claims in Portfolio 1 occur in accordance with a Poisson process with mean 50 per year.
● Claims in Portfolio 2 occur in accordance with a Poisson process with mean 80 per year.
● The two processes are independent.
Calculate the probability that 4 claims occur in Portfolio 1 before 6 claims occur in Portfolio 2.
A. Less than 47%
B. At least 47%, but less than 50%
C. At least 50%, but less than 53%
D. At least 53%, but less than 56%
E. At least 59%

11.20 (1, 5/00, Q.10) (1.9 points)
An insurance company sells two types of auto insurance policies: Basic and Deluxe.
The time until the next Basic Policy claim is an exponential random variable with mean two days. The time until the next Deluxe Policy claim is an independent exponential random variable with mean three days.
What is the probability that the next claim will be a Deluxe Policy claim?
(A) 0.172 (B) 0.223 (C) 0.400 (D) 0.487 (E) 0.500

11.21 (3, 11/00, Q.6) (2.5 points) An insurance company has two insurance portfolios.
Claims in Portfolio P occur in accordance with a Poisson process with mean 3 per year.
Claims in Portfolio Q occur in accordance with a Poisson process with mean 5 per year.
The two processes are independent.
Calculate the probability that 3 claims occur in Portfolio P before 3 claims occur in Portfolio Q.
(A) 0.28 (B) 0.33 (C) 0.38 (D) 0.43 (E) 0.48

11.22 (CAS3, 5/05, Q.12) (2.5 points) The number of cars entering a tunnel is 10 times the number of trucks. The interarrival time of each vehicle follows an exponential distribution.
Determine the probability that 20 cars enter the tunnel before 4 trucks enter.
A. Less than 72%
B. At least 72%, but less than 77%
C. At least 77%, but less than 82%
D. At least 82%, but less than 87%
E. 87% or more

11.23 (CAS3, 11/05, Q.25) (2.5 points) ABC Insurance Company plans to sell insurance policies that provide coverage for damage due to hail storms.
The annual premium of $125 per policy is collected at the moment the policy is sold.
ABC expects policies to be sold according to a Poisson process, with a rate of 200 per year.
Hail storm events also follow a Poisson process, with a rate of 10 per year.
Calculate the probability that ABC collects at least $10,000 in premiums by the time the first storm occurs.
A. 0.01 B. 0.02 C. 0.03 D. 0.04 E. 0.05

11.24 (CAS3, 5/06, Q.34) (2.5 points)
The number of claims arriving each day in the Montana and Nevada claim offices follow independent Poisson processes with parameters $\lambda_M = 2$ and $\lambda_N = 3$, respectively.
Calculate the probability that the Montana office receives three claims before the Nevada office receives two claims.
A. Less than 0.15
B. At least 0.15 but less than 0.20
C. At least 0.20 but less than 0.25
D. At least 0.25 but less than 0.30
E. At least 0.30

11.25 (SOA MLC, 5/07, Q.8) (2.5 points)
Kevin and Kira excel at the newest video game at the local arcade, "Reversion".
The arcade has only one station for it. Kevin is playing. Kira is next in line. You are given:
(i) Kevin will play until his parents call him to come home.
(ii) Kira will leave when her parents call her. She will start playing as soon as Kevin leaves if he is called first.
(iii) Each child is subject to a constant force of being called: 0.7 per hour for Kevin; 0.6 per hour for Kira.
(iv) Calls are independent.
(v) If Kira gets to play, she will score points at a rate of 100,000 per hour.
Calculate the expected number of points Kira will score before she leaves.
(A) 77,000 (B) 80,000 (C) 84,000 (D) 87,000 (E) 90,000

11.26 (CAS S, 5/17, Q.3) (2.2 points)
You are given the following information about emergency room visits at a local hospital:
- The amount of time between emergency room visits for broken bones is exponentially distributed with a mean of two hours.
- The amount of time between emergency room visits for the flu is exponentially distributed with a mean of five hours.
- Emergency room visits for broken bones and emergency room visits for the flu are independent.

Calculate the probability that two patients will come into the hospital with broken bones before three patients come into the hospital with the flu.
A. Less than 0.875
B. At least 0.875, but less than 0.900
C. At least 0.900, but less than 0.925
D. At least 0.925, but less than 0.950
E. At least 0.950

11.27 (CAS S, 5/17, Q.7) *(2.2 points) You are given:*
- *A factory machine is processing parts sequentially in an assembly line.*
- *After a part is processed, the machine moves to the next part in line.*
- *There is a chance that while waiting to be processed, a part will fall to the floor and drop out of the processing queue. The probability a part drops on the floor in a given time is distributed exponentially with rate θ for each part. Once a part falls on the floor, it drops out of the processing line permanently.*
- *The time to complete processing, once the machine has begun work on a part, is exponentially distributed with mean 5 and is independent for each part.*
- *A part is now 5^{th} in line for processing and has a probability of being processed of 0.10.*
- *P_7 is the probability that the part that is seventh in line for processing will eventually be processed.*

Calculate P_7.

A. Less than 0.060
B. At least 0.060, but less than 0.065
C. At least 0.065, but less than 0.070
D. At least 0.070, but less than 0.075
E. At least 0.075

11.28 (CAS S, 11/17, Q.1) (2.2 points) You are given the following information:
- Coins are tossed into a fountain according to a Poisson process at a rate of one every two minutes.
- The coin denominations are independently distributed as follows:

Coin Denomination	Probability
Penny	0.50
Nickel	0.20
Dime	0.20
Quarter	0.10

Calculate the probability that two Quarters will be tossed into the fountain before four non-Quarter coins.
A. Less than 0.06
B. At least 0.06, but less than 0.07
C. At least 0.07 but less than 0.08
D. At least 0.08, but less than 0.09
E. At least 0.09

11.29 (MAS-1, 11/18, Q.3) (2.2 points)
You are given the following information regarding two portfolios of ABC Insurance Company:
- Claims in Portfolio 1 occur according to a Poisson process with a rate of four per year
- Claims in Portfolio 2 occur according to a Poisson process with a rate of two per year
- The two Poisson processes are independent

Calculate the probability that three claims occur in Portfolio 1 before three claims occur in Portfolio 2.
A. Less than 0.755
B. At least 0.755, but less than 0.765
C. At least 0.765, but less than 0.775
D. At least 0.775, but less than 0.785
E. At least 0.785

Solutions to Problems:

11.1. B. Prob[neither bus arrived by time t] = $e^{-6t}e^{-4t} = e^{-10t}$ = S(t).
Mean time until the first bus is the integral of its survival function:
Integral from 0 to ∞ of e^{-10t} is: 1/10 hour = **6 minutes**.

11.2. C. Prob[both bus arrived by time t] = $(1 - e^{-6t})(1 - e^{-4t}) = 1 - e^{-4t} - e^{-6t} + e^{-10t}$ = F(t).
Mean time until the bus train is the integral of its survival function:
Integral from 0 to ∞ of $e^{-4t} + e^{-6t} - e^{-10t}$ is: 1/4 + 1/6 - 1/10 hour = **19 minutes**.

11.3. D. Probability of an event from the first process before an event from the second process is: $\lambda_1/(\lambda_1+\lambda_2)$ = 6/(6 + 4) = 0.6 = probability that the Bayside bus arrives first.
When an Bayside bus arrives, the average wait until the next Whitestone bus is 1/4 hours = 15 minutes. (In fact, this is true whenever we start observing.)
⇒ Average time you wait without Nancy = (0.6)(15 minutes) = **9 minutes**.

11.4. A. Probability of an event from the second process before an event from the first process is: $\lambda_2/(\lambda_1+\lambda_2)$ = 4/(6 + 4) = 0.4 = probability that the Whitestone bus arrives first.
When an Whitestone bus arrives, the average wait until the next Bayside bus is 1/6 hours = 10 minutes. (In fact, this is true whenever we start observing.)
⇒ Average time you wait without Nancy = (0.4)(10 minutes) = **4 minutes**.
Comment: E[arrival of first bus] + E[you wait alone] + E[Nancy waits alone] = 6 + 9 + 4 = 19 = E[arrival of both buses].

11.5. E. **30%** of the taxicabs have 2 people.

11.6. A. Ignore taxicabs with 3 people.
1/3 of taxicabs with either one or two people have two people.
The number of taxicabs out of the first five such taxicabs that have 2 people is Binomial with parameters q = 1/3 and m = 5.
3 taxicabs leave with 2 people each before 3 taxicabs with 1 person each, if and only if 3 or more of the first 5 such taxicabs have 2 people.
This has probability: $10(1/3)^3(2/3)^2 + 5(1/3)^4(2/3) + (1/3)^5$ = **21.0%**.
Alternately, if one has two independent Poisson Processes with claims intensities λ_1 and λ_2, then the chance that n claims from the first process occur before m claims from the second process is F(m-1) for a Negative Binomial Distribution with parameters r = n and $\beta = \lambda_2 / \lambda_1$. In this case we want F(3-1) = F(2) for a Negative Binomial with parameters r = 3 and β = 0.6/0.3 = 2. F(2) = **21.0%**.

n	0	1	2
f(n)	0.0370	0.0741	0.0988
F(n)	0.0370	0.1111	0.2099

11.7. B. 0.6 of taxicabs have one person. The number of taxicabs out of the first six taxicabs that have 1 person is Binomial with parameters q = 0.6 and m = 6.
5 taxicabs leave with 1 person each before 2 taxicabs with multiple people, if and only if 5 or more of the first 6 taxicabs have 1 person. This has probability: $6(0.6)^5(0.4) + (0.6)^6$ = **23.3%**.
Alternately, if one has two independent Poisson Processes with claims intensities λ_1 and λ_2, then the chance that n claims from the first process occur before m claims from the second process is F(m-1) for a Negative Binomial Distribution with parameters
r = n and $\beta = \lambda_2 / \lambda_1$. In this case we want F(2-1) = F(1) for a Negative Binomial with parameters r = 5 and β = 0.4/0.6 = 2/3. F(1) = f(0) + f(1) = 0.0778 + 0.1555 = **23.3%**.

11.8. B. Wind losses have claims intensity 0.10. Theft losses have claims intensity 0.20.
⇒ There is a chance of: 0.2/(0.1 + 0.2) = **2/3** of the first theft loss occurring prior to a wind loss.

11.9. A. Earthquake losses have claims intensity 0.04. The total claims intensity is 0.34.
Thus 0.04/0.34 = **11.8%** of the losses are from earthquake.
<u>Comment</u>: Due to independent and stationary increments, the chance of any loss being an earthquake is the same.

11.10. E. The chance any loss being earthquake is 0.04/0.34. The types of any two losses are independent. Therefore, the chance of the both the fifth and tenth losses being from earthquakes is $(0.04/0.34)^2$ = **1.38%**.

11.11. B. The size of wind losses follows a LogNormal Distribution with $\mu = 2$ and $\sigma = 3$.
Thus the chance that a wind loss is of size greater than 1000 is:
$1 - \Phi[\{\ln(1000) - 2\}/3] = 1 - \Phi(1.64) = 1 - 0.9495 = 0.0505$.
Wind losses > 1000 follow a Poisson Process with intensity (0.051)(0.10) = 0.0051.
The size of earthquake losses follows a Pareto Distribution, with parameters $\alpha = 2$ and $\theta = 1500$.
Thus the chance that a earthquake loss is of size greater than 1000 is:
$\{1500/(1500+1000)\}^2 = 0.36$.
Earthquakes follow a Poisson Process with intensity 0.04. Thus Earthquakes of size greater than 1000 follow a Poisson Process with intensity (0.04)(0.36) = 0.0144.
The size of theft losses follows a Weibull Distribution, with parameters $\tau = 2$ and $\theta = 600$.
Thus the chance that a theft loss is of size greater than 1000 is $\exp[-(1000/600)^2] = 0.0622$.
Thefts follow a Poisson Process with intensity 0.20.
Thus thefts of size greater than 1000 follow a Poisson Process with intensity:
(0.20)(0.0622) = 0.0124.
Thus the chance that the loss of size greater than 1000 is an earthquake is:
0.0144 / (0.0051 + 0.0144+ 0.0124) = **0.451**.
<u>Comment</u>: The chance that it is a theft is: 0.0124 / (0.0051 + 0.0144 + 0.0124) = 0.389.
The chance that it is a wind loss is: 0.0051 / (0.0051 + 0.0144 + 0.0124) = 0.160.

11.12. E. From the previous solution, losses of size greater than 1000 have an intensity of:
0.0051 + 0.0144 + 0.0124 = 0.0319.
All losses have an intensity of: 0.10 + 0.04 + 0.20 = 0.34.
Thus the probability that a loss occurrence exceeds 1000 is: 0.0319/0.34 = **9.38%**.
Alternately, the probability that a loss is from wind is: 0.10/0.34 = 5/17.
The probability that a loss is from earthquake is: 0.04/0.34 = 2/17.
The probability that a loss is from theft is: 0.20/0.34 = 10/17.
The chance that a wind loss is of size greater than 1000 is:
$1 - \Phi[\{\ln(1000) - 2\}/3] = 1 - \Phi(1.64) = 1 - 0.9495 = 0.0505$.
The chance that a earthquake loss is of size greater than 1000 is: $\{1500/(1500+1000)\}^2 = 0.36$.
The chance that a theft loss is of size greater than 1000 is: $\exp[-(1000/600)^2] = 0.0622$.
Thus the probability that a loss occurrence exceeds 1000 is:
Prob[W] Prob[X > 1000 | W] + Prob[EQ] Prob[X > 1000 | EQ] + Prob[T] Prob[X > 1000 | T] =
(5/17)(0.0505) + (2/17)(0.36) + (10/17)(0.0622) = **9.38%**.
Comment: Similar to CAS3L, 11/10, Q. 12.

11.13. B. From a previous solution, losses of size greater than 1000 are Poisson with
$\lambda = 0.0051 + 0.0144 + 0.0124 = 0.0319$.
The number of large losses per year are Poisson with mean 0.0319.
⇒ Variance is 0.0319. ⇒ Standard Deviation is **0.179**.

11.14. A. The chance any loss being a theft is 0.20/0.34. The chance any loss being from wind is 0.10/0.34. The types of any two losses are independent. Therefore, the chance of the fourth loss being a theft and the seventh loss being from wind is: (0.20/0.34)(0.10/0.34) = **17.3%**.

11.15. B. The chance of a wind loss before a theft loss is: 0.1/(0.1+0.2) = 1/3.
We want F(6-1) = F(5) for a Negative Binomial with parameters r = 4 and
β = chance of failure / chance of success = (2/3)/(1/3) = 2.

n	0	1	2	3	4	5
f(n)	0.0123	0.0329	0.0549	0.0732	0.0854	0.0910
F(n)	0.0123	0.0453	0.1001	0.1733	0.2586	**0.3497**

Alternately, 4 wind losses occur before 6 theft losses occur, if and only if 4 or more of the first 9 claims are wind, ignoring losses other than theft or wind. This has probability equal to the sum of Binomial terms from 4 to 9, with q = 1/3 and m = 9. This is: $126(1/3)^4(2/3)^5 + 126(1/3)^5(2/3)^4 + 84(1/3)^6(2/3)^3 + 36(1/3)^7(2/3)^2 + 9(1/3)^8(2/3) + (1/3)^9$ = **0.3497**.

11.16. E. The chance of a earthquake loss before a theft loss is: 0.04/(0.04+0.20) = 1/6.
We want F(8-1) = F(7) for a Negative Binomial with parameters r = 3 and
β = chance of failure / chance of success = (5/6)/(1/6) = 5.

n	0	1	2	3	4	5	6	**7**
f(n)	0.0046	0.0116	0.0193	0.0268	0.0335	0.0391	0.0434	0.0465
F(n)	0.0046	0.0162	0.0355	0.0623	0.0958	0.1348	0.1783	**0.2248**

Alternately, 3 earthquake losses occur before 8 theft losses occur, if and only if 3 or more of the first 10 claims are earthquake, ignoring losses other than earthquake or wind. This has probability equal to the sum of Binomial terms from 3 to 10, with q = 1/6 and m = 10.
This is: $120(1/6)^3(5/6)^7 + 210(1/6)^4(5/6)^6 + 252(1/6)^5(5/6)^5 + 210(1/6)^6(5/6)^4 + 120(1/6)^7(5/6)^3 + 45(1/6)^8(5/6)^2 + 10(1/6)^9(5/6) + (1/6)^{10}$ = **0.2248**.

11.17. E. The 2000 male drivers have a Poisson Process with λ = (2000)(0.05) = 100.
The 1000 female drivers have a Poisson Process with λ = (1000)(0.03) = 30.
30/(100 + 30) = 3/13 of the claims are from female drivers.
The number of claims out of the first four that are from females is Binomial with parameters q = 3/13 and m = 4.
2 claims from female drivers occur before 3 claims from male drivers, if and only if 2 or more of the first 4 claims are from females.
This has probability: $6(3/13)^2(10/13)^2 + 4(3/13)^3(10/13) + (3/13)^4$ = **23.0%**.
Alternately, if one has two independent Poisson Processes with claims intensities λ_1 and λ_2, then the chance that n claims from the first process occur before m claims from the second process is F(m-1) for a Negative Binomial Distribution with parameters r = n and $\beta = \lambda_2 / \lambda_1$.
In this case we want F(3-1) = F(2) for a Negative Binomial with parameters:
r = 2 and β = 100/30 = 10/3. F(2) = 0.0533 + 0.0819 + 0.0945 = **23.0%**.

11.18. The number of streetcars prior to the first bus is Geometric with $\beta = \dfrac{\text{rate of streetcars}}{\text{rate of buses}} = 2$.

1 - f(0) - f(1) - f(2) = 1 - 1/3 - 2/9 - 4/27 = 8/27 = **29.6%**.

11.19. B. If one has two independent Poisson Processes with claims intensities λ_1 and λ_2, then the chance that a claim is from the first process is $\lambda_1 / (\lambda_1 + \lambda_2)$.
Therefore, the chance that a claim is from Portfolio 1 is 5/13, independent of the other claims.
The number of claims out of the first 9 that are from Portfolio 1 is Binomial with parameters q = 5/13 and m = 9.
4 claims occur in Portfolio 1 before 6 claims occur in Portfolio 2, if and only if 4 or more of the first 9 claims are from Portfolio P. This has probability:
$1 - \{(8/13)^9 + 9(8/13)^8(5/13) + 36(8/13)^7(5/13)^2 + 84(8/13)^6(5/13)^3\} =$ **0.479**.
Alternately, if one has two independent Poisson Processes with claims intensities λ_1 and λ_2, then the chance that n claims from the first process occur before m claims from the second process is F(m-1) for a Negative Binomial Distribution with parameters r = n and $\beta = \lambda_2 / \lambda_1$.
In this case we want F(6-1) = F(5) for a Negative Binomial with parameters r = 4 and β = 8/5.
F(5) = **0.479**.

n	0	1	2	3	4	5	6
f(n)	0.0219	0.0539	0.0829	0.1020	0.1098	0.1082	0.0887
F(n)	0.0219	0.0757	0.1586	0.2606	0.3705	0.4786	0.5673

Comment: Similar to 3, 11/00, Q.6.

11.20. C. These are two independent Poisson Processes, with λ = 1/2 and 1/3.
Probability that the next claim will be a Deluxe Policy claim is: (1/3)/{1/2 + 1/3} = 2/5 = **0.4**.

11.21. A. If one has two independent Poisson Processes with claims intensities λ_1 and λ_2, then the chance that a claim is from the first process is $\lambda_1 / (\lambda_1 + \lambda_2)$.
Therefore, the chance that a claim is from Portfolio P is 3/8, independent of the other claims.
The number of claims out of the first five that are from Portfolio P is Binomial with parameters q = 3/8 and m = 5.
3 claims occur in Portfolio P before 3 claims occur in Portfolio Q, if and only if 3 or more of the first 5 claims are from Portfolio P.
This has probability: $10(3/8)^3(5/8)^2 + 5(3/8)^4(5/8) + (3/8)^5$ = **0.275**.
Alternately, if one has two independent Poisson Processes with claims intensities λ_1 and λ_2, then the chance that n claims from the first process occur before m claims from the second process is: F(m-1) for a Negative Binomial Distribution with parameters r = n and $\beta = \lambda_2 / \lambda_1$.
In this case we want F(3-1) = F(2) for a Negative Binomial with parameters r = 3 and β = 5/3.
F(2) = **0.2752**.

n	0	1	2	3	4	5	6
f(n)	0.0527	0.0989	0.1236	0.1287	0.1207	0.1056	0.0880
F(n)	0.0527	0.1516	0.2752	0.4040	0.5247	0.6303	0.7183

Alternately, the number of claims by time t from Q is Poisson with mean 5t.
Prob(Number of claims by time t from Q < 3) = $e^{-5t} + 5te^{-5t} + 25t^2e^{-5t}/2$.
The time of the third claim from P is the sum of three independent Exponentials each with θ = 1/3, which is a Gamma Distribution with α = 3 and θ = 1/3. This Gamma has density of:
$3^3t^2e^{-3t}/2 = 13.5t^2e^{-3t}$. Prob[3 claims from P before 3 claims from Q] =

$$\int \text{Prob(3rd claim from P at time t) Prob(Number of claims by time t from Q < 3) } dt =$$

$$\int_0^\infty 13.5\, t^2\, e^{-3t}\, (e^{-5t} + 5te^{-5t} + 25t^2e^{-5t}/2)\, dt =$$

$$13.5 \int_0^\infty t^2e^{-8t}\, dt + 67.5 \int_0^\infty t^3e^{-8t}\, dt + 168.75 \int_0^\infty t^4e^{-8t}\, dt =$$

$(13.5)(2!/8^3) + (67.5)(3!/8^4) + (168.75)(4!/8^5)$ = 0.0527 + 0.0989 + 0.1236 = **0.2752**.

Comment: For Gamma type integrals: $\int_0^\infty t^{\alpha-1}\, e^{-t/\theta}\, dt = \Gamma(\alpha)\theta^\alpha = (\alpha-1)!\,\theta^\alpha$.

11.22. D. Prob[car] = 10/11. Prob[truck] = 1/11.
Prob[20 cars before 4 trucks] = Prob[at least 20 of first 23 vehicles are cars] =
$(10/11)^{20}(1/11)^3(21)(22)(23)/6 + (10/11)^{21}(1/11)^2(22)(23)/2 + (10/11)^{22}(1/11)(23) + (10/11)^{23}$
= **84.89%**.
Alternately, let a car be a success and a truck be a failure.
Then the number of failures before 20 successes is Negative Binomial with r = 20 and
β = (chance of failure) / (chance of success) = (1/11)/(10/11) = 1/10.
Prob[fewer than 4 failures] = f(0) + f(1) + f(2) + f(3) = $1/1.1^{20} + 20(0.1)/1.1^{21}$
$+ \{(20)(21)/2\}(0.1)^2/1.1^{22} + \{(20)(21)(22)/6\}(0.1)^3/1.1^{23}$ = **84.89%**.
Comment: Since the interarrival times are exponential, this is a Poisson Process.
You have to assume that the interarrival times are independent.
If exactly 20 of the first 23 vehicles are cars, then we are OK.
If exactly 21 of the first 23 vehicles are cars, then we are OK.
If exactly 22 of the first 23 vehicles are cars, then we are OK.
If exactly 23 of the first 23 vehicles are cars, then we are OK.
These are disjoint events, therefore we can add their probabilities.
If exactly 19 of the first 23 vehicles are cars, then we are not OK.

11.23. B. $10,000 in premiums. ⇔ 80 policies sold.
Thus the question is asking for the probability that we get at least 80 events from the policy sale process before one event from the storm process. The probability that each event will be a policy sale rather than a storm is: 200/(200 + 10) = 20/21.
Prob[at least 80 policy sale before one storm] = Prob[80 of first 80 events are policy sales] =
$(20/21)^{80}$ = **2.0%.**
Alternately, define a storm as a "success" and a policy sale as a "failure." Then we have a series of independent Bernoulli trials with a 1/21 chance of success. If we stop when we get our first success, then the number of failures is Geometric with β = (20/21)/(1/21) = 20. We want the probability of least 80 failures, which is $S(79) = \{\beta/(1 + \beta)\}^{80} = (20/21)^{80}$ = **2.0%.**
Comment: It is not obvious at first reading that the solution involves comparing two Poisson Processes. If one did not realize that, one can instead get a solution via integration.

11.24. B. Each claim has a probability of being from Montana of: 2/(2 + 3) = 0.4.
Prob[3 M before 2 N] = Prob[at least 3 of the first 4 are M] =
Prob[3 of the first 4 are M] + Prob[4 of the first 4 are M] = $(4)(0.4^3)(0.6) + 0.4^4$ = **0.1792**.
Alternately, define M as a success, and N as a failure.
The number failures (Ns) before the third success (M) is a Negative Binomial with r = 3 and
β = (chance of failure) / (chance of success) = 0.6/0.4 = 1.5.
Prob[less than 2 failures before the 3rd success] = f(0) + f(1) = $1/2.5^3 + (1.5)(3)/(2.5^4)$ = **0.1792**.

11.25. E. Since we are only interested in the first event, we can treat this mathematically as two independent Poisson Processes; for Kevin $\lambda = 0.7$ and for Kira $\lambda = 0.6$.
The probability that Kevin is called first is: 0.7/(0.7 + 0.6) = 7/13.
If Kevin leaves first, then Kira will play on average 1/0.6 hours.
(Starting a new Poisson Process with average waiting time of 1/0.6 hours.)
Thus the average time Kira plays is: (1/0.6)(7/13).
The average number of points she scores is: (100,000)(1/0.6)(7/13) = **89,744**.
Alternately, the density of when Kevin is called is $0.7e^{-0.7t}$.
The probability that Kira has not been called by time t, her survival function is: $e^{-0.6t}$.
Therefore, the probability that Kevin is called first is:

$$\int_0^\infty 0.7\, e^{-0.7t}\, e^{-0.6t}\, dt = 0.7/0.13 = 7/13.$$

Assuming Kevin is called first, then the time Kira plays is Exponential with mean 1/0.6 hours.
The average number of points she scores is: (100,000)(1/0.6)(7/13) = **89,744**.
Comment: Intended to be a Life Contingencies (Survival Models) question.

11.26. D. $\lambda_{bones} = 1/2$. $\lambda_{flu} = 1/5$. Prob[bones | visit] = $\frac{\lambda_{bones}}{\lambda_{bones} + \lambda_{flu}} = \frac{1/2}{1/2 + 1/5} = 5/7$.

Prob[at least 2 of first 4 visits are due to broken bones] =
Prob[exactly 2 due to broken bones] + Prob[exactly 3 due to broken bones]
+ Prob[exactly 4 due to broken bones] =

$$\binom{4}{2} (5/7)^2 (2/7)^2 + \binom{4}{3} (5/7)^3 (2/7) + \binom{4}{4} (5/7)^4 =$$

(6)(100/2401) + (4)(250/2401) + (1)(625/2401) = 2225/2401 = **0.9267**.
Alternately, define flu as a "failure" and broken bones as a "success".
Then the chance that we observe 2 broken bones before seeing 3 flus, is the chance of fewer than 3 "failures" prior to our 2nd "success", in a series of independent Bernoulli trials. The number of failures before 2 successes is given by a Negative Binomial with r = 2 and

$$\beta = \frac{\text{chance of failure}}{\text{chance of success}} = (2/7) / (5/7) = 0.4.$$

We want the probability of fewer than three "failures":

$$f(0) + f(1) + f(2) = \frac{1}{1.4^2} + 2\frac{0.4}{1.4^3} + \frac{(2)(3)}{2!}\frac{0.4^2}{1.4^4} = \mathbf{0.9267}.$$

Comment: Since the mean time between visits due to broken bones is shorter than that due to flu, more than half of the visits are due to broken bones.
For this type of problem one can either add up Binomial or Negative Binomial densities.
The chance of x failures prior to success number r is given by a Negative Binomial Distribution.

11.27. D. I will use λ for the hazard rate of the Exponential Distribution of falling to the floor.
(θ is usually the mean.) For the first part in line, there are two independent Exponential Distributions competing to operate on it first; will it be processed or will it fall to the floor.
$\Rightarrow$ Prob[first part is processed] = (1/5) / (λ + 1/5) = 0.2 / (λ + 0.2).
The second part has to wait for something to happen to the first part; something happens to the first part via an Exponential with rate: λ + 0.2.
Thus the probability that the 2nd part makes it to the head of the line prior to falling to the floor is: $\frac{\lambda + 0.2}{(\lambda + 0.2) + \lambda} = \frac{\lambda + 0.2}{2\lambda + 0.2}$.

By the memoryless property, if the 2nd part makes it to the head of the line, then its probability of being processed is equal to that calculated previously for the first part: 0.2 / (λ + 0.2).
Prob[2nd part is processed] =
Prob[2nd part makes it to head of line] Prob[2nd part is processed | makes it to head of line] =
$\frac{\lambda + 0.2}{2\lambda + 0.2} \frac{0.2}{\lambda + 0.2} = \frac{0.2}{2\lambda + 0.2}$.

In order for the third part to get into the 2nd spot in line, something has to happen to either the first or second part before the third part falls on the floor. The combined things happening to the first part (λ + 0.2) and second part λ has a rate of: 2λ + 0.2. The rate at which the third part falls on the floor is λ.

Thus the chance that the 3rd part makes it into the second spot in line is: $\frac{2\lambda + 0.2}{3\lambda + 0.2}$.

Based on the memoryless property, if the third part makes it into the second spot in line, then its chance of being processed is the previously calculated: $\frac{0.2}{2\lambda + 0.2}$.

Thus the chance of the third part being processed is:
Prob[3rd part makes it to 2nd spot in line] Prob[processed | in 2nd spot in line] =
$\frac{2\lambda + 0.2}{3\lambda + 0.2} \frac{0.2}{2\lambda + 0.2} = \frac{0.2}{3\lambda + 0.2}$.

The pattern is: Prob[part n is processed] = $\frac{0.2}{n\lambda + 0.2}$.

We are given: 0.1 = Prob[5th part is processed] = $\frac{0.2}{5\lambda + 0.2}$. $\Rightarrow \lambda = 0.36$

Prob[7th part is processed] = $\frac{0.2}{7\lambda + 0.2} = \frac{0.2}{(7)(0.36) + 0.2}$ = **0.0735**.

<u>Comment</u>: Difficult! Beyond what you should be asked on your exam.
This is an example of a continuous time Markov Chain, formerly on the syllabus.

11.28. D. The probability of a quarter is 0.1 and the probability if a non-quarter is 0.9.
We require that at least 2 of the first 6-1 = 5 coins be quarters.
The number of coins out of the first five that are quarters is Binomial with parameters q = 0.1 and m = 5. Thus the desired probability is:

$\binom{5}{2}(0.1^2)(0.9^3) + \binom{5}{3}(0.1^3)(0.9^2) + \binom{5}{4}(0.1^4)(0.9) + 0.1^5 =$ **0.08146**.

Alternately, the chance that n claims from the first process occur before m claims from the second process is F(m-1) for a Negative Binomial Distribution with parameters r = n and $\beta = \lambda_2 / \lambda_1$.
In this case, we want F(4-1) = F(3) for a Negative Binomial with parameters r = 2 and β = 0.9/0.1 = 9. $F(3) = 1/10^2 + (2)\ 9/10^3 + \{(2)(3)/2\}\ 9^2/10^4 + \{(2)(3)(4)/6\}\ 9^3/10^5 =$ **0.08146**.
<u>Comment</u>: The given rate of the Poisson process is irrelevant.

11.29. E. Probability that a claim is from Portfolio 1 is: $\frac{\lambda_1}{\lambda_1 + \lambda_2}$ = 4/(4+2) = 2/3.

Prob[3 claims from Portfolio 1 before 3 claims from Portfolio 2] =
Prob[at least 3 of the first 5 claims from Portfolio 1].
The number of claims from Portfolio 1 out of the first 5 is Binomial with q = 2/3 and m = 5.
The desired probability is: $f(3) + f(4) + f(5) = \binom{5}{3}(2/3)^3(1/3)^2 + \binom{5}{4}(2/3)^4(1/3) + (2/3)^5 =$

(10)(8/243) + (5)(16/243) + 32/243 = **79.01%**.
Alternately, we want F(3-1) for a Negative Binomial Distribution with parameters r = 3 and
$\beta = \frac{\lambda_2}{\lambda_1}$ = 2/4 = 1/2. $F(2) = f(0) + f(1) + f(2) = \frac{1}{(1+\beta)^r} + \frac{r\,\beta}{(1+\beta)^{r+1}} + \frac{r\,(r+1)\,\beta^2/2}{(1+\beta)^{r+2}} =$

$1/1.5^3 + (3)(1/2)/1.5^4 + \{(3)(4)\ (1/2)^2 / 2\} / 1.5^5 =$ **79.01%**.

Section 12, Known Number of Claims

Exercise: Assume we know a claim from a Poisson Process has occurred on the interval (0, 3). What is the distribution of the time of that claim?
[Solution: Since the claims intensity is a constant,
the claim time is uniformly distributed on (0, 3).]

If we know we have a total of n claims from a Poisson Process on (0, T), then the n claim times are independently uniformly distributed on (0, T), due to the constant independent claims intensity.[93]

For example, assume we have a Poisson Process on (0, 5) and we observe 4 claims.
Then the 4 times of the claims are independent and each is uniform on (0, 5).

Exercise: Claims follow a Poisson process. Harry, Hermione, and Ron each make a claim sometime during the period 0 to 100. If Harry's claim is at time 37 and Hermione's claim is at time 82, what is the chance that Ron's claim occurs between the other two?
[Solution: The three claim times are independent. Ron's claim time is uniform on (0, 100).
The probability that it is in (37, 82) is: (82 - 37)/100 = 45%.]

A similar result applies if we know the nth claim occurred at time S_n. Then the arrival times of the first n-1 claims are uniformly distributed on $(0, S_n)$.[94] For example, assume we have a Poisson Process and we observe the 6th claim at time 1.7564. Then the times of the first five claims are independent and each is uniform on (0, 1.7564).

Exercise: Claims are given by a Poisson Process.
4 claims occur between time 0 and time 10.
Find the expected time of occurrence of the third claim to occur.
[Solution: Prob[third claim occurred by time t] = Prob[at least three claims by time t] =
Prob[exactly 3 claims by time t] + Prob[4 claims by time t] =
$4(0.1t)^3(1 - 0.1t) + (0.1t)^4 = 0.004t^3 - 0.0003t^4$.

Expected time of third claim = $\int_0^{10} S(t)\ dt = \int_0^{10} (1 - 0.004t^3 + 0.0003t^4)\ dt = 10 - 10 + 6 = 6$.]

In the above situation, the expected time of the first claim is: (10)(1/5). The expected time of the second claim is: (10)(2/5). The expected time of the 4th claim is: (10)(4/5). The expected times of the four claims are 2, 4, 6, and 8; they divide the time period into 5 equal segments.

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[93] See Theorem 5.2 in Introduction to Probability Models by Sheldon M. Ross.
Symmetry establishes that each claim time is uniformly distributed on (0,T).
Due to independent increments the claim times are independent.

[94] See Proposition 5.4 in Introduction to Probability Models by Sheldon M. Ross.

In general, **if there have been N claims from time 0 to T, the expected time of the i^{th} claim is: $T \frac{i}{N+1}$.**

Given there have been N claims, the expected times of occurrence divide the total time period into N+1 equal segments. This is an example of a general result: for a random sample of size N from a uniform distribution, the expected values of the order statistics are spread evenly over the support.[95]

Exercise: Claims are given by a Poisson Process.
4 claims occur between time 0 and time 10.
Find the probability that the third claim to occur, occurred between time 5 and 8.
[Solution: The claim times are independent uniform distributions on 0 to 10.
Prob[third claim occurred by time 5] = Prob[at least three claims by time 5] =
Prob[exactly 3 claims by time 5] + Prob[4 claims by time 5] = $4(1/2)^3(1/2) + (1/2)^4 = 5/16$.
Prob[third claim occurred by time 8] = Prob[at least three claims by time 8] =
$4(0.8)^3(0.2) + (0.8)^4 = 0.8192$.
Prob[3rd claim occurs between 5 and 8] =
Prob[3rd claim occurred by t = 8] - Prob[3rd claim occurred by t = 5] = 0.8192 - 5/16 = 0.5067.]

Derivation of the Expected Time of the i^{th} Claim:

If a Poisson Process has N claims from time 0 to T, then the times of occurrence are independent uniform distributions from 0 to T.

Prob[ith claim after time t] = Prob[fewer than i claims by time t]

$$= \sum_{j=0}^{i-1} \text{Prob[j claims by time t]} = \sum_{j=0}^{i-1} \frac{N!}{j!\,(N-j)!} (t/T)^j (1 - t/T)^{N-j}.$$

Expected time of the i^{th} claim is the integral of its survival function, Prob[i^{th} claim after time t]:

$$\int_{t=0}^{T} \sum_{j=0}^{i-1} \frac{N!}{j!\,(N-j)!} (t/T)^j (1 - t/T)^{N-j}\, dt = \sum_{j=0}^{i-1} \frac{N!}{j!\,(N-j)!} \int_{s=0}^{1} s^j (1-s)^{N-j}\, T\, ds =$$

$$T \sum_{j=0}^{i-1} \frac{N!}{j!\,(N-j)!} \beta[j + 1, (N - j) + 1] = T \sum_{j=0}^{i-1} \frac{N!}{j!\,(N-j)!} \frac{j!\,(N-j)!}{(N+1)!} = T \sum_{j=0}^{i-1} \frac{1}{N+1} = T \frac{i}{N+1}.$$ [96]

[95] See my section on Order Statistics in "Mahler's Guide to Statistics."
[96] I used a change of variables, s = t/T.
Note that the integral from zero to one of $s^{a-1} (1 - s)^{b-1}$ ds is: $\beta(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a + b) = (a-1)!(b-1)!/(a+b-1)!$.
This is the complete Beta Function. The Beta Distribution, as shown in Appendix A of Loss Models, integrates to one. For $\theta = 1$, this implies this integral equation.

Problems:

12.1 (1 point) A Poisson Process has a claims intensity of 0.15.
One observes two claims during the time interval (0, 5).
What is the probability that the first claim occurs before time 1?
A. 36% B. 37% C. 38% D. 39% E. 40%

Use the following information for the next four questions:
Claims follow a Poisson Process with $\lambda = 81$.
Bob and Ray each make a claim during the year 2002.

12.2 (1 point) What is the probability that Ray made his claim before Bob?
(A) 1/16 (B) 1/6 (C) 1/4 (D) 1/3 (E) 1/2

12.3 (1 point) What is the probability that Ray made his claim during the first quarter of 2002?
(A) 1/16 (B) 1/6 (C) 1/4 (D) 1/3 (E) 1/2

12.4 (1 point) What is the probability that Bob and Ray each made their claims during the last quarter of 2002?
(A) 1/16 (B) 1/6 (C) 1/4 (D) 1/3 (E) 1/2

12.5 (1 point) If Ray made his claim on July 23, 2002, what is the probability that Bob made his claim during May or June 2002?
(A) 1/16 (B) 1/6 (C) 1/4 (D) 1/3 (E) 1/2

12.6 (2 points) Claims are received by the Symphonic Insurance Company via a Poisson Process with mean 0.017. During the time interval (0, 100) it receives three claims, one from Beethoven, one from Tchaikovsky, and one from Mahler. The claim from Beethoven was received at time 27 and the claim from Mahler was received at time 91. What is probability that the claim from Tchaikovsky was received between the other two claims?
(A) 64% (B) 66% (C) 68% (D) 70% (E) 72%

12.7 (2 points) Events follow a Poisson Process with rate of 7 per week.
If there are a total of 7 events in a week, determine the probability of having exactly one event on each day of the week.

12.8 (3 points)
There are N independent, identically distributed insureds, each with Poisson frequency.
Conditional on observing C total claims, what is the frequency distribution for an individual insured?

Use the following information for the next 3 questions:
One has a counting process N(t) with stationary and independent increments. N(0) = 0.
Prob[N(x) = 1] = x/4 + o(x). Prob[N(x) ≥ 2] = o(x). N(3) = 9.

12.9 (2 points) What is the mean time between the ninth event and time = 3?
(A) 0.30 (B) 0.31 (C) 0.32 (D) 0.33 (E) 0.34

12.10 (3 points) What is the variance of the time between the ninth event and time = 3?
(A) 0.01 (B) 0.03 (C) 0.05 (D) 0.07 (E) 0.09

12.11 (2 points) What is the variance of the time between the ninth event and tenth event?
(A) 0.06 (B) 0.08 (C) 0.10 (D) 0.12 (E) 0.14

12.12 (2 points) Claims follow a Poisson Process with λ = 20 per year.
The average size of each claim is 1000.
Each claim is paid immediately when it occurs.
The annual force of interest is δ = 0.05.
Determine the actuarial present value of those claims occurring over the next 10 years.
A. Less than 120,000
B. At least 120,000, but less than 130,000
C. At least 130,000, but less than 140,000
D. At least 140,000, but less than 150,000
E. At least 150,000

Use the following information for the next three questions:
- Claims are given by a Poisson Process.
- The fifth loss occurred at time 31.

12.13 (1 point) What is the chance that the first loss occurred by time 18?
(A) 95% (B) 96% (C) 97% (D) 98% (E) 99%

12.14 (2 points) What is the chance that the second loss occurred by time 18?
(A) 74% (B) 76% (C) 78% (D) 80% (E) 82%

12.15 (2 points) What is the chance that the third loss occurred by time 18?
(A) 40% (B) 42% (C) 44% (D) 46% (E) 48%

Use the following information for the next 3 questions:

- A loss occurrence may be caused by wind, earthquake or theft.
- Wind, earthquake and theft losses occur independently of one another.
- Wind losses follow a Poisson Process.
  The expected amount of time between wind losses is 10 years.
- Earthquake losses follow a Poisson Process.
  The expected amount of time between earthquake losses is 25 years.
- Theft losses follow a Poisson Process.
  The expected amount of time between theft losses is 5 years.
- The size of wind losses follows a LogNormal Distribution, with parameters $\mu = 2$ and $\sigma = 3$.
- The size of earthquake losses follows a Pareto Distribution, with parameters $\alpha = 2$ and $\theta = 1500$.
- The size of theft losses follows a Weibull Distribution, with parameters $\tau = 2$ and $\theta = 600$.

12.16 (3 points) You observe the ninth loss at time 31.62.
What is the chance that the third loss occurred before time 8?
A. Less than 0.20
B. At least 0.20, but less than 0.25
C. At least 0.25, but less than 0.30
D. At least 0.30, but less than 0.35
E. At least 0.35

12.17 (1 point) The fourth loss is a theft loss which occurs at time = 9.2.
The first three losses included exactly one wind loss.
What is the chance that the wind loss occurred before time 2.7?
A. 0.30 B. 0.35 C. 0.40 D. 0.45 E. 0.50

12.18 (1 point) You observe exactly three losses, one of each type, prior to time = 8 years.
What is the chance that the earthquake loss occurred before time 5?
A. Less than 0.55
B. At least 0.55, but less than 0.60
C. At least 0.60, but less than 0.65
D. At least 0.65, but less than 0.70
E. At least 0.70

12.19 (2 points) You are given that N(t) follows a Poisson process.
Calculate $\Pr[N(3) = 4 \mid N(8) = 10]$.
A. 18% B. 20% C. 21% D. 23% E. 25%

Use the following information for the next three questions:

- Trucks pass a given spot via a Poisson Process via a rate of 8 per hour.
- 5 trucks passed between 1 and 2 o'clock.

12.20 (3 points) What is the probability that the third truck passed between 1:30 and 1:45?
(A) 25% (B) 30% (C) 35% (D) 40% (E) 45%

12.21 (2 points) What is the expected value of the time at which the fourth truck passed?
(A) 1:32 (B) 1:36 (C) 1:40 (D) 1:44 (E) 1:48

12.22 (2 points) What is the expected number of minutes between when the fourth truck passes and the fifth truck passes?
(A) 9 (B) 10 (C) 11 (D) 12 (E) 13

12.23 (CAS S, 11/15, Q.2) (2.2 points) You are given the following:

- A Poisson process with $\lambda = 1.2$ has its 4^{th} event occur at time 10.
- This process began operating at time 0.

Calculate the expected value of the time at which the 3^{rd} event occurred.
A. Less than 2
B. At least 2, but less than 4
C. At least 4, but less than 6
D. At least 6, but less than 8
E. At least 8

12.24 (CAS ST, 5/16, Q.1) (2.5 points)
You are given that N(t) follows the Poisson process with rate $\lambda = 2$.
Calculate $\Pr[N(2) = 3 \mid N(5) = 7]$.
A. Less than 0.25
B. At least 0.25, but less than 0.35
C. At least 0.35, but less than 0.45
D. At least 0.45, but less than 0.55
E. At least 0.55

12.25 (MAS-1, 5/18, Q.2) (2.2 points)
Insurance claims are made according to a Poisson process with rate λ.
Calculate the probability that exactly 3 claims were made by time t = 1,
given that exactly 6 claims are made by time t = 2.
A. Less than 0.3
B. At least 0.3, but less than 0.4
C. At least 0.4, but less than 0.5
D. At least 0.5, but less than 0.6
E. At least 0.6

Solutions to Problems:

12.1. A. The times of the two claims are independently, uniformly distributed from (0, 5). Therefore, they each have a 4/5 = 0.8 chance of being after time 1.
Prob[1st claim before 1] = 1 - Prob[both claims after time 1] = 1 - 0.8^2 = **0.36**.

12.2. E. The times of their claims are uniform and independent, so Ray has a **1/2** chance of making his claim first.

12.3. C. The time of Ray's claim is uniformly distributed, so the probability of it being made during the first quarter is **1/4**.

12.4. A. The times of their claims are uniform and independent. They each have a 1/4 chance of making their claim during the last quarter, so probability is: (1/4)(1/4) = **1/16**.

12.5. B. The times of their claims are uniform and independent. The time of Ray's claim has no effect on that of Bob. Probability of a claim in May or June is: 2/12 = **1/6**.

12.6. A. The time of Tchaikovsky's claim is uniformly distributed over (0, 100), independent of the time of the other two. The probability it is in the interval (27, 91) is: (91 - 27)/100 = **64%**.

12.7. The number of events from time 0 to time 1 is independent of the number of events from time 1 to time 2, etc. The probability of one event from time 0 to time 1 is: $1\ e^{-1}$.
The probability of 7 events in a week is: $7^7\ e^{-7}/ 7!$.
Thus the Prob[1 event each day | total of seven events] =

$$\frac{\text{Prob[Monday = 1] Prob[Tuesday = 1]} \cdots \text{Prob[Sunday = 1]}}{\text{Prob[Total of 7 for the week]}}$$

$e^{-1}\ e^{-1}\ e^{-1}\ e^{-1}\ e^{-1}\ e^{-1}\ e^{-1} / (7^7\ e^{-7}/ 7!) = 7! / 7^7 =$ **0.612%**.
Alternately, conditional on there being seven events, they are independent, uniformly distributed. If we wish to arrange the events each into the seven different days, we can put an event in 7 different days, then another event in any of 6 different days, then another event in any of 5 different days, etc. Thus the number of ways to arrange the events to satisfy our criteria is 7!. The total number of arrangements is 7^7. Thus the desired probability is: $7! / 7^7$ = **0.612%**.

12.8. This is mathematically equivalent to a (homogeneous) Poisson Process from time 0 to N, with the number of claims on time interval from 0 to 1 representing the claims from the first insured, the number of claims on time interval from 1 to 2 representing the claims from the second insured, etc. Then conditional on a total of C claims, the claims are independently and uniformly distributed from 0 to N. Each claim has independent chance of 1/N of being in the time interval 0 to 1. Thus the number of claims in the time interval (0, 1) is Binomial with m = C and q = 1/N. This is the number of claims for the first insured; there is nothing special about the first insured, so this is the distribution for each insured. **The number of claims for each insured is Binomial with m = C and q = 1/N**.
Comment: The Binomials are not independent; if for example the first insured has more than the expected number of claims, then there are fewer of the C claims to be had by other insureds. The joint distribution is a Multinomial.
If N = 2 and C = 2, there are three possibilities:
0 claims for first insured, 2 claims for second insured: $(e^{-\lambda})(e^{-\lambda}\ \lambda^2 /2) = e^{-2\lambda}\ \lambda^2 / 2$.
1 claim for first insured, 1 claim for second insured: $(\lambda e^{-\lambda})(\lambda e^{-\lambda}) = e^{-2\lambda}\ \lambda^2$.
2 claims for first insured, 0 claims for second insured: $(e^{-\lambda}\ \lambda^2 /2)(e^{-\lambda}) = e^{-2\lambda}\ \lambda^2 / 2$.
These probabilities are proportional to: 1/4, 1/2, 1/4.
The conditional distribution is Binomial with m = 2 and q = 1/2.
If each insured were instead Geometric, then with N = 2 and C = 2 for the three possibilities:

0 claims for first insured, 2 claims for second insured: $\left(\frac{1}{1+\beta}\right)\left(\frac{\beta^2}{(1+\beta)^3}\right) = \frac{\beta^2}{(1+\beta)^4}$.

1 claim for first insured, 1 claim for second insured: $\left(\frac{\beta}{(1+\beta)^2}\right)\left(\frac{\beta}{(1+\beta)^2}\right) = \frac{\beta^2}{(1+\beta)^4}$.

2 claims for first insured, 0 claims for second insured: $\left(\frac{\beta}{(1+\beta)^2}\right)\left(\frac{1}{1+\beta}\right) = \frac{\beta^2}{(1+\beta)^4}$.

These probabilities are equal. The conditional distribution is uniform on 0, 1, and 2.

12.9. A. This is a definition of a Poisson Process, with $\lambda = 4$.
Given there are nine claims by time 3, the times of those claims are independent, uniform distributions from 0 to 3.
Let X = time of the 9th claim. Prob[X < t] = Prob[each claim at time < t] = $(t/3)^9$.

Mean = $\int_0^3 \{1 - (t/3)^9\}\, dt = 3 - 0.3 = 2.7$. Expected time between X and 3 is: 3 - 2.7 = **0.3**.

12.10. D. Density of X is: $9t^8/3^9$.

Second Moment of X: $\int_0^3 t^2\, 9t^8/3^9\, dt = 3^{11}(9/3^9)/11 = 7.3636$.

Var[X] = $7.3636 - 2.7^2 = 0.0736$. Var[3 - X] = Var[X] = **0.0736**.

12.11. E. Assuming we start observing at time 3, the Poisson process has no memory.
Therefore, the time until the next event, which will be the tenth event, is Exponential with mean $1/\lambda = 1/4$.
What happens before time 3 is independent of what happens after time 3.
Therefore, if X = time of the 9th claim, and Y = the time of the tenth claim, 3 - X and Y - 3 are independent. Var[Y - X] = Var[Y] + Var[X] = $1/4^2 + 0.0736$ = **0.136**.

12.12. E. The time of occurrence of each claim independently has a uniform distribution from 0 to 10. The expected present value of each claim is:

$1000 \int_0^{10} e^{-0.05t}\, (1/10)\, dt = (1000/10)(1 - e^{-0.5})/0.05 = 786.94$.

We expect on average: (10)(20) = 200 claims. (200)(786.94) = **157,388**.
Comment: Similar to Example 5.21 in Introduction to Probability Models by Ross.

12.13. C. There are 4 losses independent and uniformly distributed on (0, 31).
Chance of each loss being in (18, 31) is 13/31.
Prob(first loss occurred by time 18) = 1 - Prob(4 losses after time 18) =
$1 - (13/31)^4$ = **96.9%**.
Comment: There is one claim at time 31 and four claims prior to time 31.
Thus there are four claims between times 0 and 31.

12.14. D. There are 4 losses independent and uniformly distributed on (0, 31).
Chance of each loss being in (18, 31) is 13/31. Prob(second loss occurred by time 18) =
1 - Prob(3 losses after time 18) - Prob(4 losses after time 18) =
$1 - (4)(13/31)^3(18/31) - (13/31)^4$ = **79.8%**.

12.15. C. Prob(third loss occurred by time 18) =
1 - Prob(2 losses after 18) - Prob(3 losses after 18) - Prob(4 losses after 18) =
$1 - (6)(13/31)^2(18/31)^2 - (4)(13/31)^3(18/31) - (13/31)^4$ = **44.2%**.
Alternately, Prob(3rd loss occurred by time 18) = Prob(3 losses by 18) + Prob(4 losses by 18) =
$4(18/31)^3(13/31) + (18/31)^4$ = **44.2%**.

12.16. D. The first eight claims are uniformly and independent distributed on (0, 31.62).
What we want is the third order statistic for 8 independent identically distributed variables, each uniform on (0, 31.62).

$$\sum_{j=3}^{8} \binom{8}{j} (x/31.62)^j\ (1-x/31.62)^{8-j} = \sum_{j=3}^{8} \binom{8}{j} (8/31.62)^j\ (1-8/31.62)^{8-j} =$$

$56(0.253^3)(0.747^5) + 70(0.253^4)(0.747^4) + 56(0.253^5)(0.747^3) + 28(0.253^6)(0.747^2) + 8(0.253^7)(0.747) + 0.253^8 = 0.2109 + 0.0893 + 0.0242 + 0.00411 + 0.0004 + 0.00001$ = **0.329**.
Comment: One can do the sum of Binomial terms somewhat faster by instead summing the terms of this Binomial Distribution from 0 to 2 and subtracting from unity. 1 - 0.671 = 0.329.

n	f(n)	F(n)
0	0.097	0.097
1	0.263	0.360
2	0.311	0.671

Alternately, the desired probability is β(3, 8+1-3; 8/31.62) = β(3, 6; 0.253) = 0.3289.

12.17. A. The times of the first three claims are independently and uniformly distributed on the interval (0, 9.2). Thus the chance that the wind loss occurred before time 2.7 is: 2.7/9.2 = **0.293**.
Comment: The fact that the fourth loss is a theft loss has no effect on the answer.

12.18. C. The times of the three claims are independently and uniformly distributed on the interval (0, 8). Thus the chance that the earthquake loss occurred before time 5 is: **5/8 = 0.625**.
Comment: The answer is the same if they had asked instead about the wind or the theft loss.

12.19. E. Given we have 10 claims by time 8, their times are independently, identically distributed uniform distributions on (0, 8). Thus there is a 3/8 chance that each one is before time 3.
Thus the number of claims from time 0 to 3 is Binomial with m = 10 and q = 3/8.
Pr[N(3) = 4 I N(8) = 10] is the density at 4 of this Binomial Distribution:

$$\binom{10}{4} (3/8)^4\ (5/8)^6 = (210)\ (3/8)^4\ (5/8)^6 = \mathbf{24.75\%}.$$

Alternately, $\Pr[N(3) = 4 \mid N(8) = 10] = \dfrac{\Pr[N(3) = 4 \text{ and } N(8) = 10]}{\Pr[N(8) = 10]} =$

$$\frac{\Pr[N(3) = 4]\ \Pr[N(8) - N(3) = 6]}{\Pr[N(8) = 10]} = \frac{\{(3\lambda)^4\ e^{-3\lambda} / 4!\}\{(5\lambda)^6\ e^{-5\lambda} / 6!\}}{(8\lambda)^{10}\ e^{-8\lambda} / 10!} = \frac{(3^4)\ (5^6)}{8^{10}}\ \frac{10!}{4!\ 6!}$$

= **24.75%**.
Comment: Similar to CAS Exam ST, 5/16, Q. 1.

12.20. D. Since we know we have 5 trucks in the interval from 0 to 1 in hours, we have 5 independent, identically distributed uniform distributions on [0, 1].
Prob[any given truck by time t] = t.
Prob[exactly 3 trucks out of 5 by time t] = $10t^3(1-t)^2 = 10t^3 - 20t^4 + 10t^5$.
Prob[exactly 4 trucks out of 5 by time t] = $5t^4(1-t) = 5t^4 - 5t^5$.
Prob[exactly 5 trucks out of 5 by time t] = t^5.
Prob[at least 3 trucks by time t] = $10t^3 - 20t^4 + 10t^5 + 5t^4 - 5t^5 + t^5 = 10t^3 - 15t^4 + 6t^5$.
Prob[at least 3 trucks by time 1/2] = 0.5.
Prob[at least 3 trucks by time 3/4] = 0.896.
Prob[3rd truck at time 1/2 to 3/4] = 0.896 - 0.5 = **0.396**.
Comment: The rate of the Poisson Process is irrelevant, once we know we have 5 trucks during the hour.

12.21. C. Expected time until 4th truck out of 5 is: (1 hour) 4/(5+1) = 2/3 hour from start. ⇔ **1:40**.
Alternately, Prob[exactly 4 trucks out of 5 by time t] = $5t^4(1-t) = 5t^4 - 5t^5$.
Prob[exactly 5 trucks out of 5 by time t] = t^5.
Prob[at least 4 trucks by time t] = $5t^4 - 5t^5 + t^5 = 5t^4 - 4t^5$.
survival function for 4th truck = $S(t) = 1 - 5t^4 + 4t^5$.

E[time of fourth truck] = $\int_0^1 S(t)\ dt$ = 1 - 5/5 + 4/6 = 2/3 hour from start ⇔ **1:40**.

Comment: Similar to CAS S, 11/15, Q.2.

12.22. B. Survival function for 5th truck = $S(t) = 1 - t^5$.

E[time of fifth truck] = $\int_0^1 S(t)\ dt$ = 1 - 1/6 = 5/6.

Expected time between when the fourth truck passes and the fifth truck passes =
E[time of fifth truck] - E[time of fourth truck] **=** 5/6 - 2/3 = 1/6 hour = **10 minutes**.
Comment: The expected time of the arrivals are: 10, 20, 30, 40, and 50 minutes. On average, the five trucks divide the hour into 5 + 1 = 6 equal segments of 10 minutes each. If there are N trucks during an hour, then the expected time for the i^{th} truck is i/(N + 1) in hours.

12.23. D. Given the fourth event has occurred at time 10, the expected times of the first three events are spread evenly over the interval (0, 10).
Conditional expected time of 1st event is: 2.5.
Conditional expected time of 2nd event is: 5.
Conditional expected time of 3rd event is: **7.5**.

12.24. B. Given we have 7 claims by time 5, their times are independently, identically distributed uniform distributions on (0, 5). Thus there is a 2/5 = 40% chance that each one is before time 2.
Thus the number of claims from time 0 to 2 is Binomial with m = 7 and q = 40%.
Pr[N(2) = 3 I N(5) = 7] is the density at 3 of this Binomial Distribution:

$\binom{7}{3}$ (0.4^3) (0.6^4) = (35) (0.4^3) (0.6^4) = **29.0%**.

Alternately, Pr[N(2) = 3 I N(5) = 7] = $\frac{\text{Pr[N(2) = 3 and N(5) = 7]}}{\text{Pr[N(5) = 7]}} = \frac{\text{Pr[N(2) = 3] Pr[N(5) - N(2) = 4]}}{\text{Pr[N(5) = 7]}}$

$= \frac{\{(3\lambda)^4 e^{-3\lambda} / 4!\}\{(5\lambda)^6 e^{-5\lambda} / 6!\}}{(8\lambda)^{10} e^{-8\lambda} / 10!} = \frac{(4^3)(6^4)}{10^7} \frac{7!}{3!\,4!}$ = **29.0%**.

Comment: The answer does not depend on the value of λ.

12.25. B. Given there are 6 claims in total, each claim has an independent probability of 1/2 of occurring between t = 0 and t = 1.
Prob[exactly 3 claims between t = 0 and t = 1] =
density at 3 of Binomial Distribution with n = 6 and θ = 1/2:

$\binom{6}{3}$ $(1/2)^3$ $(1- 1/2)^3$ = (20) $(1/2)^3$ $(1/2)^3$ = **0.3125**.

Section 13, Estimating Software Reliability[97]

Ross has an application of Poisson Processes to a model of computer software.

Assume one runs a software package for a period of time t, and observes how many bugs there were and how many errors each of those bugs produced. Then all of these detected bugs are fixed. Using the theory of Poisson Processes, Ross develops an estimate of the future expected error rate, after all the bugs that have been detected have been fixed.[98]

Given:
- t = testing time
- $M_1(t)$ = the number of bugs causing one error during the testing period
- $M_2(t)$ = the number of bugs causing two errors during the testing period.

Then using properties of Poisson Processes, Ross shows that:
- Estimated future rate of errors is: $M_1(t) / t$
- Variance of this estimate is: $\{M_1(t) + 2M_2(t)\} / t^2$

Exercise: During a test period of 1000 hours, 110 bugs are discovered in the new Underwriter-in-a-Box software package.

Bugs	45	35	20	10
Errors per bug	1	2	3	4

Each of these 110 bugs is successfully fixed.
Give an interval estimate of the future rate of errors from this software package.
Use plus or minus 2 standard deviations.
[Solution: Estimated future rate of errors = $M_1(t)/t$ = 45/1000 = 0.045.
Variance of this estimate = $\{M_1(t) + 2M_2(t)\}/t^2 = (45 + (2)(35))/1000^2 = 0.000115$.
Standard deviation of this estimate = 0.011.
Error rate is: 0.045 ± 0.022 = (0.023, 0.067).
Comment: Note that we only use information on how many bugs produced one and two errors during the testing period. The point estimate only requires the number of bugs that produced exactly one error during the testing period.]

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[97] While on the Syllabus, you are unlikely to be asked about this.
See Section 5.3.6 of Introduction to Probability Models by Sheldon M. Ross.
[98] These future errors would be caused by bugs that happened not to cause any errors during the finite testing period. Presumably such bugs have low rates of producing errors, and/or we were unlucky enough to have them not produce any errors during the testing period.

Derivation of the Rate of Future Software Errors:

Note that the future rate of errors depends only on the number of detected bugs that produced one error during the test period. This result follows from the special properties of the Poisson process and the specific assumptions made by Ross. Ross assumes:

- Prior to testing the software, there are m bugs.
- The errors produced by each bug are independent Poisson Processes with intensity λ_i.
- There is a testing period of length t.
- During the testing period, we observe $M_j(t)$ bugs that produced j errors.
- After the testing period, all detected bugs are fixed.
- Fixing the detected bugs introduces no additional bugs.

Exercise: During the testing period what is the chance that bug #3 produces exactly two errors?
[Solution: The number of errors produced by bug #3 during period t is a Poisson Distribution with mean: $\lambda_3 t$. Therefore, the chance that bug #3 produces exactly two errors is the density of this Poisson Distribution at two: $\exp[-\lambda_3 t]\ (\lambda_3 t)^2 / 2$.]

The expected number bugs that produce exactly two errors during the test period is:

$$\sum_{i=1}^{m} \exp[-\lambda_i t]\ (\lambda_i t)^2 / 2\ .$$

The expected number bugs that produce exactly j errors during the test period is:

$$E[M_j(t)] = \sum_{i=1}^{m} \exp[-\lambda_i t]\ (\lambda_i t)^j / j! = t^j \sum_{i=1}^{m} \lambda_i^j \exp[-\lambda_i t] / j!\ .$$

Specifically: $E[M_1(t)] = t \sum_{i=1}^{m} \lambda_i \exp[-\lambda_i t]$. $\qquad E[M_2(t)] = t^2 \sum_{i=1}^{m} \lambda_i^2 \exp[-\lambda_i t] / 2$.

The chance that bug i will not produce an error during time t is the density of a Poisson Distribution with mean $\lambda_i t$ at zero: $\exp[-\lambda_i t]$. If bug i produces no errors during the testing period, then it will not be fixed. If bug i is not fixed, it will produce future errors at an average rate of λ_i. Since the Poisson Processes are independent, we can add up the expected future errors produced by each bug that has not been fixed.

Expected Future Rate of Errors = $\sum_{i=1}^{m}$ (Chance that bug i is not fixed) (Rate of errors from bug i) =

$$\sum_{i=1}^{m} \exp[-\lambda_i t]\ \lambda_i = E[M_1(t)] / t.$$

Therefore, Expected Future Rate of Errors = $E[M_1(t)] / t$.

Let $\Lambda(t)$ be the future rate of errors after a testing period of time t. We have shown that as stated by Ross, $M_1(t)/t$ is an unbiased estimator of $\Lambda(t)$, the future rate of errors. Therefore, the variance of that estimator is the same as the mean squared error (MSE) of that estimator.[99]
We will now derive an estimator for that MSE.

$E[\{\Lambda(t) - M_1(t) / t\}^2] = E[\Lambda(t)^2] + E[(M_1(t) / t)^2] - 2\ E[\Lambda(t)\ M_1(t) / t]$.

Looking at just bug i, there are two possibilities. If bug i produced exactly one error during the test period, then it contributes one to $M_1(t)$, otherwise it contributes zero. The chance that bug i produced exactly one error is: $\lambda_i t \exp[-\lambda_i t]$.

Thus the contribution to $M_1(t)$ of bug i is a Bernoulli, with $q = \lambda_i t \exp[-\lambda_i t]$.
Thus the second moment of the contribution of bug i to $M_1(t)$ is: $\lambda_i t \exp[-\lambda_i t]$.[100]

Since the bugs are independent processes, the second moment of $M_1(t)$ is just the sum of the second moments of the different contributions.

$$E\{(M_1(t)/t\}^2] = E[M_1(t)^2] / t^2 = \sum_{i=1}^{m} \lambda_i t \exp[-\lambda_i t]\ / t^2 = \sum_{i=1}^{m} \lambda_i \exp[-\lambda_i t] / t\ = E[M_1(t)]/t^2.$$

Looking at just bug i, there are two possibilities. If bug i produced one or more errors during the test period, then it is fixed and does not contribute to $\Lambda(t)$, the future rate of errors. If bug i produced no error during the test period, which has a probability of $\exp[-\lambda_i t]$, then bug i contributes λ_i to $\Lambda(t)$, the future rate of errors.

Thus the contribution to $\Lambda(t)$ of bug i is λ_i times a Bernoulli, with $q = \exp[-\lambda_i t]$.
Therefore the second moment of the contribution of bug i to $\Lambda(t)$ is: $\lambda_i^2 \exp[-\lambda_i t]$.[101]

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[99] See my section on Properties of Estimators in "Mahler's Guide to Statistics."
[100] The variance of a Bernoulli is q(1-q). Therefore, the second moment of Bernoulli is: $q(1-q) + q^2 = q$.
[101] The second moment of a Bernoulli is q. $E[(cX)^2] = c^2 E[X^2]$.

Since the bugs are independent processes, the second moment of $\Lambda(t)$ is just the sum of the second moments of the different contributions.

$$E[\Lambda(t)^2] = \sum_{i=1}^{m} \lambda_i^2 \exp[-\lambda_i t] = 2\ E[M_2(t)] / t^2.$$

If bug i produced one error during the test period and therefore contributes to $M_1(t)$, then that bug is fixed and does not contribute to $\Lambda(t)$, the future rate of errors. Thus if bug i contributes to $M_1(t)$, then it does not contribute to $\Lambda(t)$, and vice versa. Therefore, the contribution to $E[\Lambda(t)\ M_1(t)]$ from each bug is zero. Since the bugs are independent processes, $E[\Lambda(t)\ M_1(t)]$ is just the sum of the contributions of individual bugs, each of which contributions is zero. Therefore, $E[\Lambda(t)\ M_1(t)] = 0$.

Putting the three pieces together:
$E[\{\Lambda(t) - M_1(t) / t\}^2] = E[\Lambda(t)^2] + E[(M_1(t) / t)^2] - 2\ E[\Lambda(t)\ M_1(t) / t]$
$= 2\ E[M_2(t)] / t^2 + E[M_1(t)] / t^2 - (2)(0/t) = \{M_1(t) + 2M_2(t)\} / t^2.$

Thus as stated by Ross, the variance of the estimate of the future rate of errors is:
$\{M_1(t) + 2M_2(t)\} / t^2.$

<u>Problems</u>:

13.1 (2 points) During a test period of 50 hours, 122 bugs are discovered in the new "Loss Development for Dummies" computer software package.

Bugs	32	51	28	8	3
Errors per bug	1	2	3	4	5

Each of these 122 bugs is successfully fixed.
We assume that this does not introduce any new bugs into the software.
Give an interval estimate of the future rate of errors from this software package, using plus or minus 2 standard deviations.
A. (0.18, 1.10) B. (0.28, 1.00) C. (0.38, 0.90) D. (0.48, 0.80) E. (0.58, 0.70)

13.2 (2 points) You are testing a computer model of the insurance losses due to catastrophes.
During 800 hours of testing you discover 9 bugs in the software that cause one error each and 3 bugs in the software that produce 2 errors each.
All of the bugs that were found during testing are fixed.
We assume that this does not introduce any new bugs into the software.
(a) Estimate the future rate of errors.
(b) What is the variance of this estimate?

Solutions to Problems:

13.1. A. Estimated future rate of errors = $M_1(t)/t$ = 32/50 = 0.64 errors per hour.
Variance of this estimate = $\{M_1(t)+2M_2(t)\}/t^2$ = {32 + (2)(51)} / 50^2 = 0.0536.
Standard deviation of this estimate = 0.232.
Estimated future error rate is: 0.64 ± 0.46 = **(0.18, 1.10)**.

13.2. (a) Estimated future rate of errors = $M_1(t)/t$ = 9/800 = 0.01125 errors per hour.
(b) Variance of this estimate = $\{M_1(t)+2M_2(t)\}/t^2$ = {9 + (2)(3)} / 800^2 = 0.00002344.

Section 14, Nonhomogeneous Poisson Processes[102]

A **Nonhomogeneous Poisson Process** differs from the homogeneous Poisson Process, in that the claims intensity (rate) **λ(t) is a function of time**, rather than a constant .[103] Thus a (homogeneous) Poisson Process is a special case of a nonhomogeneous Poisson Process.

For example, let λ(t) = t on (0, 3). Then as time passes, the expected claim frequency increases.[104] Alternately, let λ(t) = 1 + sin(2πt), then the claims intensity fluctuates seasonally.

One definition of the claims intensity λ(t) is:

$$\lambda(t) = \lim_{\delta \to 0} \{\text{Expected Number of Claims in } (t - \delta/2,\ t + \delta/2)\} / \delta.$$

Therefore, **λ(t) is the derivative of the expected number of claims**.
Therefore, the expected number of claims is the integral of λ(t) .

Exercise: For an nonhomogeneous Poisson Process with $\lambda(t) = t^2/200$, what is the expected number of events by time 15?
[Solution: One integrates the claims intensity from 0 to 15:

$$\int_0^{15} \lambda(t)\ dt = \int_0^{15} t^2/200\ dt = t^3/600 \Big]_{t=0}^{t=15} = 5.625.$$]

In general, in order to get the expected number of claims in the time interval (a, b), one integrates the claims intensity λ(t) from a to b.[105]

In addition,
the number of claims from time a to b is given by a Poisson Distribution with a mean of: the integral the claims intensity λ(t) from a to b.[106]

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[102] See Section 5.4.1 in Introduction to Probability Models by Ross, and Page 4 of "Poisson Processes" by Daniel.
[103] Note, the claims intensity is not dependent of the number of claims already observed.
[104] Perhaps the insured lives are getting older.
[105] This is similar to equation 3.3.7 in Actuarial Mathematics. However, in Life Contingencies, the force of mortality is defined per exposure. Also, when a death occurs the number of exposures is reduced, unlike in the case of either a homogeneous or nonhomogeneous Poisson Process.
Thus for the Life Contingencies, the average number of deaths between time a and b is the integral from a to b of the product of the force of mortality and the number of lives.
[106] Over very small subintervals, the claims intensity is approximately constant, and therefore the number of claims is approximately Poisson. We can then add up small disjoint subintervals, and due to independent increments, approximate adding up independent Poissons. The sum of independent Poissons is a Poisson. Thus by letting the size of the subintervals go to zero, we get a Poisson for the whole interval.

Define the mean value function as:[107]

$$\mathbf{m(t)} = \int_0^t \lambda(s)\, ds .$$

Note that since the claims intensity is nonnegative, m(t) is a nondecreasing function of t.

$m(0) = 0$. $m(t) \geq 0, t > 0$. $m'(t) = \lambda(t) \geq 0, t > 0$.[108]

λ(t) is analogous to velocity, while m(t) is analogous to distance.

m(t) is the mean number of claims from time 0 to t.
The number of claims from time a to b is Poisson Distributed with mean: m(b) - m(a).

Exercise: Given a nonhomogeneous Poisson Process with claims intensity $\lambda(t) = 3t^2$, what is the chance of three claims between time 2 and 2.5?
[Solution: The mean number of claims from time 2 to 2.5 is:

$$\int_2^{2.5} \lambda(t)dt = \int_2^{2.5} 3t^2 dt = t^3 \Big]_{t=2}^{t=2.5} = 7.625.$$

Thus the number of claims from time 2 to 2.5 is Poisson Distributed with mean 7.625.
$f(3) = e^{-7.625}\, 7.625^3 / 3! = 3.61\%$.
Comment: $m(t) = t^3$. $m(2.5) = 15.625$. $m(2) = 8$. $m(2.5) - m(2) = 7.625$.]

Exercise: Given a nonhomogeneous Poisson Process with claims intensity $\lambda(t) = 3t^2$, what is the variance of the number of claims between time 1 and 3?
[Solution: The mean number of claims from time 1 to 3 is:

$$\int_1^3 \lambda(t)dt = \int_1^3 3t^2 dt = t^3 \Big]_{t=1}^{t=3} = 26.$$

The number of claims from time 1 to 3 is Poisson Distributed with mean 26, and variance 26.]

Exercise: Given a nonhomogeneous Poisson Process with claims intensity $\lambda(t) = 3t^2$, what is the probability of more than 40 claims between time 1 and 3?
Use the Normal Approximation.
[Solution: The number of claims from time 1 to 3 is Poisson Distributed with mean 26.
Therefore, $\text{Prob}(n > 40) \cong 1 - \Phi[(40.5 - 26)/\sqrt{26}] = 1 - \Phi(2.844) = 0.2\%$.]

[107] Page 4 of Daniel. m(t) is also called the operational time.
[108] m(t) is analogous to the distance traveled, while λ(t) is analogous to the velocity.

Definition:

For a Poisson Process with claims rate (intensity) $\lambda(t)$:[109]

1. It is a counting process that starts at time = 0 with zero events; N(0) = 0.

2. It has independent increments.[110]

3. The increment N[t + h] - N[t] is Poisson with mean: $m(t + h) - m(t) = \int_t^{t+h} \lambda(s)\, ds$.

4. If $\lambda(t)$ is constant, then we have a homogeneous Poisson Process.

Compound Distributions:[111]

Just as in the homogeneous case, one can have an amount attached to each event.
Let S(t) be the aggregate amount by time t. Then:

Mean of S(t) = (Mean of the Poisson Distribution) (Mean of the Amount Distribution)
= m(t) (Mean of the Amount Distribution).
Variance of S(t) = m(t) (Second Moment of the Amount Distribution).

Exercise: The amount of each claim is uniformly distributed from 100 to 300.
The number of claims follows a nonhomogeneous Poisson Process with $\lambda(t) = 4 + t$.
Calculate the mean and standard deviation of the total claim payment by time 5.

[Solution: $m(5) = \int_0^5 (4 + t)\, dt = 20 + 5^2/2 = 32.5$.

E[S(5)] = (32.5)(200) = 6500.
Variance of the uniform is: $\text{width}^2 / 12 = (300 - 100)^2 / 12 = 3333.33$.
$\Rightarrow$ Second moment of the uniform is: $3333.33 + 200^2 = 43{,}333.33$.
$\Rightarrow$ Var[S(5)] = (32.5)(43,333.33) = 1,408,333.
$\Rightarrow$ Standard deviation is: $\sqrt{1{,}408{,}333} = 1187$.]

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[109] See Definition 1.2 in “Poisson Processes” by Daniel.
[110] $N[t_1 + h_1] - N[t_1]$ and $N[t_2 + h_2] - N[t_2]$ are independent if $(t_1, t_1 + h_1]$ does not overlap with $(t_2, t_2 + h_2]$.
Touching at endpoints is okay.
[111] See for example: 3L, 11/12, Q.11; ST, 5/14, Q.3; ST, 11/14, Q.3.

Problems:

Use the following information for the next four questions:
There is a nonhomogeneous Poisson Process with claims intensity $\lambda(t) = 1 / (7 + t)$.

14.1 (2 points) What is the mean number of claims expected by time 5?
A. Less than 0.45
B. At least 0.45, but less than 0.50
C. At least 0.50, but less than 0.55
D. At least 0.55, but less than 0.60
E. At least 0.60

14.2 (1 point) What is the mean number of claims expected by time 2?
A. Less than 0.10
B. At least 0.10, but less than 0.15
C. At least 0.15, but less than 0.20
D. At least 0.20, but less than 0.25
E. At least 0.25

14.3 (1 point) What is the mean number of claims expected from time 2 to time 5?
A. Less than 0.15
B. At least 0.15, but less than 0.20
C. At least 0.20, but less than 0.25
D. At least 0.25, but less than 0.30
E. At least 0.30

14.4 (1 point) What is the chance of observing exactly 2 claims between time 2 and time 5?
A. 1/2% B. 1% C. 2% D. 3% E. 4%

14.5 (2 points) A tyrannosaur eats only scientists.
It eats scientists at a Poisson rate per hour that varies by the time of day:
From 12 midnight to 6 a.m. $\lambda(t) = 0$.
From 6 a.m. to 9 a.m. $\lambda(t) = 0.1$.
From 9 a.m. to 3 p.m. $\lambda(t) = 0.2$.
From 3 p.m. to 6 p.m. $\lambda(t) = 0.1$.
From 6 p.m. to 12 midnight $\lambda(t) = 0$.
What is the probability that the tyrannosaur eats exactly two scientists tomorrow?
(A) 27% (B) 29% (C) 31% (D) 33% (E) 35%

<u>**14.6**</u> (2 points) Given a nonhomogeneous Poisson Process with claims intensity $\lambda(t) = 8t^{2.4}$, what is the probability of more than 608 claims by time 5?
A. 1/4% B. 1/2% C. 1% D. 3/2% E. 2%

Use the following information for the next two questions:
Traffic accidents in a city follow a nonhomogeneous Poisson Process, with the rate per hour varying by time of day:

Midnight - 6AM	6AM-9AM	9AM-3PM	3PM-6PM	6PM-9PM	9PM-Midnight
1%	10%	6%	8%	4%	8%

14.7 (2 points) What is the probability there are at least 3 accidents tomorrow?
(A) 11% (B) 13% (C) 15% (D) 17% (E) 19%

14.8 (2 points) Tomorrow, what is the probability of two accidents from midnight (12AM) to noon (12PM) and one accident from noon to midnight?
(A) 2.5% (B) 3.0% (C) 3.5% (D) 4.0% (E) 4.5%

14.9 (4 points) Each day, traffic passing through the Washington Tunnel increases during the morning and afternoon rush hours, and decreases at other times as follows:
(i) From 12 a.m. to 8 a.m., the numbers of cars follows a Poisson distribution with an increasing hourly rate of $\lambda(t) = 12 + 3.5t$ for $0 \le t \le 8$.
(ii) From 8 a.m. to 12 p.m., the numbers of cars follows a Poisson distribution with a decreasing hourly rate of $\lambda(t) = 60 - 2.5t$ for $8 \le t \le 12$.
(iii) From 12 p.m. to 6 p.m., the numbers of cars follows a Poisson distribution with an increasing hourly rate of $\lambda(t) = -30 + 5t$ for $12 \le t \le 18$.
(iv) From 6 p.m. to 12 a.m., the numbers of cars follows a Poisson distribution with a decreasing hourly rate of $\lambda(t) = 204 - 8t$ for $18 \le t \le 24$.
What is the probability that more than 900 cars will pass through the tunnel tomorrow?
A. 0.1% B. 0.5% C. 1.0% D. 1.5% E. 2.0%

14.10 (3 points) Don receives phone calls at work via a Poisson Process at a rate of 2 per hour.
Before 3:30 there is a 10% chance any call will be from his wife.
After 3:30 there is a 30% chance any call will be from his wife.
Determine the probability that Don received exactly two calls from his wife between 2:00 and 4:30.
A. Less than 18%
B. At least 18%, but less than 20%
C. At least 20%, but less than 22%
D. At least 22%, but less than 24%
E. At least 24%

14.11 (3 points) The Beatrice Spelling Insurance Agency keeps a petty cash fund on hand.
The number of withdrawals from the petty cash fund is Poisson, with mean of 1 per day on Mondays and Fridays, 0.5 per day on Tuesdays, Wednesdays and Thursdays, and 1.5 per day on Saturdays. Beatrice closes her agency on Sundays.
The amounts of withdrawals from the petty cash fund are:
\$1 with probability 50%, \$2 with probability 30%, and \$5 with probability 20%.
How much should be in the fund at the beginning of the week, in order to have a 95% probability that there will be enough money to last the entire week? Use the Normal Approximation.
A. 20 B. 22 C. 24 D. 26 E. 28

14.12 (3 points) At Cheers, their neighborhood bar, Norm and Cliff are playing darts.
The rate per minute with which Norm scores points follows a nonhomogeneous Poisson Process:
$\lambda(t) = t^2 + 3t$.
The rate per minute with which Cliff scores points follows an independent nonhomogeneous Poisson Process: $\lambda(t) = t^2/2 + 5t$.
After they play for 8 minutes, what is the probability that Cliff is ahead?
A. Less than 20%
B. At least 20%, but less than 25%
C. At least 25%, but less than 30%
D. At least 30%, but less than 35%
E. At least 35%

14.13. (3 points) A nonhomogeneous Poisson process N has a rate function

$$\lambda(t) = \begin{cases} 20 - (4 - t)^{1.5}, & 0 < t \le 4 \\ 20 - (t - 4)^{1.5}, & 4 \le t \le 10 \end{cases}$$

Calculate Var[N(9) I N(2) = 37].
A. Less than 110
B. At least 110, but less than 120
C. At least 120, but less than 130
D. At least 130, but less than 140
E. At least 140

14.14 (2 points) A commercial insurance policy provides coverage for occurrences during 2013.
Claims reported under this policy follow a Poisson process with rate function:

$$\lambda(t) = \begin{cases} 10t, & 0 < t \le 1 \\ 10, & 1 < t \le 2 \\ 90/(1+t)^2, & t > 2 \end{cases}$$

, where t is the time (in years) after January 1, 2013.

Calculate the expected percent of claims reported during 2015.
A. 9% B. 11% C. 13% D. 15% E. 17%

14.15 (3 points) The number of claims $\{N(t), t \ge 0\}$ is a nonhomogeneous Poisson process with intensity function:
$\lambda(t) = 1,\ 0 \le t < 10$ $\lambda(t) = 2,\ 10 \le t < 20$ $\lambda(t) = 3,\ 20 \le t$.
Calculate Var[N(26) I N(15) = 13].
(A) 20 (B) 22 (C) 24 (D) 26 (E) 28

14.16 (3 points) You are given the following information for an insurance policy:
- Accidents occur under a Poisson process at a rate of
  2 per month during the first half of the year,
  4 per month during the next 3 months of the year,
  and 3 per month during the last 3 months of the year,
- The payment for each accident follows an Inverse Gamma distribution with
  $\alpha = 6$ and $\theta = 5000$.

Calculate the standard deviation of the total claim payment within a year.
A. Less than 6400
B. At least 6400, but less than 6500
C. At least 6500, but less than 6600
D. At least 6600, but less than 6700
E. At least 6700

14.17 (3 points) Failures in a large group of aging industrial robots is given by a nonhomogeneous Poisson process with $\lambda(t) = \frac{0.2\,t + 1}{t + 10}$.
What is the probability of exactly 2 failures by time 20?
A. 19% B. 20% C. 21% D. 22% E. 23%

14.18 (2 points) Let $S(t) = X_1 + X_2 + ... + X_{N(t)}$ be a compound Poisson process where:
- X_i is uniform on [4, 9].
- N(t) is a Poisson process with rate $\lambda(t) = \frac{2}{3 + t}$, $t > 0$

Calculate Var[S(7)].
A. Less than 80
B. At least 80, but less than 90
C. At least 90, but less than 100
D. At least 100, but less than 110
E. At least 110

<u>**14.19**</u> (3 points) The number of persons injured in car accidents in a city follows a compound Poisson process. You are given:
- The expected number of accidents per hour = $\begin{cases} 3 \text{ for 5am to 11pm} \\ 1 \text{ for 11pm to 5am} \end{cases}$
- The number of persons injured per accident is:
  Prob[0] = 30%, Prob[1] = 30%, Prob[2] = 20%, Prob[3] = 15%, Prob[4] = 5%.

Calculate the probability that more than 90 persons will be injured in a 24 hour period using a normal approximation.
A. 5% B. 10% C. 15% D. 20% E. 25%

14.20 (2 points) You are given:
- A Poisson process N has a rate function: $\lambda(t) = t^{1.5}$.
- You have already observed 17 events by time t = 4.

Calculate the conditional probability, Pr[N(6) = 37 I N(4) = 17].
A. Less than 5%
B. At least 5%, but less than 7%
C. At least 7%, but less than 9%
D. At least 9%, but less than 11%
E. At least 11%

14.21 (CAS3, 5/04, Q.15) (2.5 points) You are given:
- The number of broken pipe claims that occur in a short span of time is proportional to the length of time, but the constant of proportionality varies as the temperature varies.
- The number of claims that occur in a given time period is independent of the number occurring in any disjoint time period.
- Broken pipe claims occur one at a time.
- Each broken pipe claim generates 1, 2, or 3 "reports" with equal probability.
- Each "report" corresponds to a loss uniformly distributed between 1 and 50 dollars.

Which of the following random variables could satisfy the definition of a non-homogeneous Poisson random variable?
1. The number of broken pipe claims
2. The number of "reports"
3. The total dollars of loss

A. 1 only B. 2 only C. 3 only D. 1 and 2 only E. 1, 2, and 3

14.22 (CAS3, 5/04, Q.27) (2.5 points)
Each day, traffic passing through the Washington Tunnel increases during the morning and afternoon rush hours, and decreases at other times as follows:
(i) From 12 a.m. to 8 a.m., the numbers of cars follows a Poisson distribution with an increasing hourly rate of $\lambda(t) = 12 + 3.5t$ for $0 \le t \le 8$.
(ii) From 8 a.m. to 12 p.m., the numbers of cars follows a Poisson distribution with a decreasing hourly rate of $\lambda(t) = 60 - 2.5t$ for $8 \le t \le 12$.
(iii) From 12 p.m. to 6 p.m., the numbers of cars follows a Poisson distribution with an increasing hourly rate of $\lambda(t) = -30 + 5t$ for $12 \le t \le 18$.
(iv) From 6 p.m. to 12 a.m., the numbers of cars follows a Poisson distribution with a decreasing hourly rate of $\lambda(t) = 204 - 8t$ for $18 \le t \le 24$.
What is the probability that exactly 25 cars pass through the tunnel between 11:30 a.m. and 12:30 p.m.?
A. 0.0187 B. 0.0273 C. 0.0357 D. 0.0432 E. 0.0511

14.23 (SOA3, 11/04, Q.26) (2.5 points) Customers arrive at a store at a Poisson rate that increases linearly from 6 per hour at 1:00 p.m. to 9 per hour at 2:00 p.m.
Calculate the probability that exactly 2 customers arrive between 1:00 p.m. and 2:00 p.m.
(A) 0.016 (B) 0.018 (C) 0.020 (D) 0.022 (E) 0.024

14.24 (CAS3, 5/05, Q.14) (2.5 points) The number of accidents on a highway follows a Poisson process, with the rate of accidents varying during the day, as shown below.

Time Interval	Accidents per hour
12 midnight - 7 am	0.05
7 am - 10 am	0.10
10 am - 4 pm	0.08
4 pm - 7 pm	0.07
7 pm - 12 midnight	0.02

Calculate the probability that three or more accidents occur in one day.
A.0.12 B. 0.18 C. 0.24 D. 0.27 E. 0.32

14.25 (CAS3, 11/05, Q.26) (2.5 points) The number of reindeer injuries on December 24 follows a Poisson process with intensity function: $\lambda(t) = \sqrt{\frac{t}{12}}$, $0 \le t \le 24$, where t is measured in hours.

Calculate the probability that no reindeer will be injured during the last hour of the day.
A Less than 30%
B. At least 30%, but less than 40%
C. At least 40%, but less than 50%
D. At least 50%, but less than 60%
E. At least 60%

14.26 (CAS3, 5/06, Q.33) (2.5 points) While on vacation, an actuarial student sets out to photograph a Jackalope and a Snipe, two animals common to the local area. A tourist information booth informs the student that daily sightings of Jackalopes and Snipes follow independent Poisson processes with intensity parameters:

$\lambda_J(t) = t^{1/3}/5$ for Jackalopes

$\lambda_S(t) = t^{1/2}/10$ for Snipes

where: $0 \le t \le 24$ and, t is the number of hours past midnight

If the student takes photographs between 1 pm and 5 pm, calculate the probability that he will take at least 1 photograph of each animal.
A. Less than 0.45
B. At least 0.45, but less than 0.60
C. At least 0.60, but less than 0.75
D. At least 0.75, but less than 0.90
E. At least 0.90

14.27 (2 points) In the previous question, CAS3, 5/06, Q.33, what is the probability that the student takes exactly four photographs?
A. 19% B. 20% C. 21% D. 22% E. 23%

14.28 (CAS3, 11/06, Q.28) (2.5 points) Customers arrive to buy lemonade according to a Poisson distribution with $\lambda(t)$, where t is time in hours, as follows:

$$\lambda(t) = \begin{cases} 2 + 6t & 0 \le t \le 3 \\ 20 & 3 < t \le 4 \\ 36 - 4t & 4 < t \le 8 \end{cases}$$

At 9:00 a.m., t is 0.
Calculate the number of customers expected to arrive between 10:00 a.m. and 2:00 p.m.
A. Less than 63
B. At least 63, but less than 65
C. At least 65, but less than 67
D. At least 67, but less than 69
E. At least 69

14.29 (3 points) In the previous question, CAS3, 11/06, Q.28, what is the probability that fewer than 70 customers arrive between 11:00 am and 4:00 p.m.? Use the Normal Approximation.
A. 6% B. 8% C. 10% D. 12% E. 14%

14.30 (SOA MLC, 5/07, Q.25) (2.5 points)
Subway trains arrive at a certain station according to a nonhomogeneous Poisson process.
l(t), the intensity function (trains per minute), varies with t, the time in minutes after 7:00 AM:
(i) $\lambda(t) = 0.05$, $0 \le t < 10$
(ii) $\lambda(t) = t/200$, $10 \le t < 20$
(iii) $\lambda(t) = 0.10$, $20 \le t$
Calculate the probability that exactly four trains arrive between 7:00 AM and 7:25 AM.
(A) 0.05 (B) 0.07 (C) 0.09 (D) 0.11 (E) 0.13

14.31 (CAS3L, 5/08, Q.10) (2.5 points)
Car accidents follow a Poisson process, as described below:
- On Monday and Friday, the expected number of accidents per day is 3.
- On Tuesday, Wednesday, and Thursday, the expected number of accidents per day is 4.
- On Saturday and Sunday, the expected number of accidents per day is 1.

Calculate the probability that exactly 18 accidents occur in a week.
A. Less than 0.06
B. At least 0.06 but less than 0.07
C. At least 0.07 but less than 0.08
D. At least 0.08 but less than 0.09
E. At least 0.09

14.32 (CAS3L, 5/08, Q.11) (2.5 points)
A customer service can center operates from 9:00 AM to 5:00 PM.
The number of calls received by the call center follows a Poisson process whose rate function varies according to the time of day, as follows:

Time of Day	Call Rate (per hour)
9:00 AM to 12:00 PM	30
12:00 PM to 1:00 PM	10
1:00 PM to 3:00 PM	25
3:00 PM to 5:00 PM	30

Using a normal approximation, what is the probability that the number of calls received from 9:00 AM to 1:00 PM exceeds the number of calls received from 1:00 PM to 5:00 PM?
A. Less than 10%
B. At least 10%, but less than 20%
C. At least 20%, but less than 30%
D. At least 30%, but less than 40%
E. At least 40%

14.33 (CAS3L, 11/08, Q.1) (2.5 points) The number of accidents on a highway from 3:00 PM to 7:00 PM follows a nonhomogeneous Poisson process with rate function
$\lambda = 4 - (t - 2)^2$, where t is the number of hours since 3:00 PM.
How many more accidents are expected from 4:00 PM to 5:00 PM than from 3:00 PM to 4:00 PM?
A. Less than 0.75
B. At least 0.75, but less than 1.25
C. At least 1.25, but less than 1.75
D. At least 1.75, but less than 2.25
E. At least 2.25

14.34 (CAS3L, 5/10, Q.12) (2.5 points) Downloads of a song on a musician's Web site follow a heterogeneous Poisson process with the following Poisson rate function:

$$\lambda(t) = e^{-0.25t}.$$

Calculate the probability that there will be more than two downloads of this song between times t = 1 and t = 5.
A. Less than 29%
B. At least 29%, but less than 30%
C. At least 30%, but less than 31%
D. At least 31%, but less than 32%
E. At least 32%

14.35 (CAS3L, 11/10, Q.11) (2.5 points) You are given the following information:

- A Poisson process N has a rate function $\lambda(t) = 3t^2$.
- You have observed 50 events by time t = 2.1.

Calculate Var[N(3) I N(2.1) = 50].
A. Less than 10
B. At least 10, but less than 20
C. At least 20, but less than 30
D. At least 30, but less than 40
E. At least 40

14.36 (CAS3L, 5/12, Q.9) (2.5 points) Claims reported for a group of policies follow a non-homogeneous Poisson process with rate function:

$\lambda(t) = 100 / (1+t)^3$, where t is the time (in years) after January 1, 2011.

Calculate the expected number of claims reported after January 1, 2011 for this group of policies.
A. Less than 45
B. At least 45, but less than 55
C. At least 55, but less than 65
D. At least 65, but less than 75
E. At least 75

14.37 (1 point) In the previous question, CAS3L, 5/12, Q.9, determine the expected percent of claims reported more than two years from January 1, 2011.

14.38 (CAS3L, 11/12, Q.11) (2.5 points)
You are given the following information for a Workers Compensation policy:

- Accidents occur under a Poisson process at a rate of 10 per month during the first half of the year, and 15 per month during the second half of the year.
- The payment for each accident follows a Pareto distribution with $\alpha = 5$ and $\theta = 2{,}000$.

Calculate the standard deviation of the total claim payment within a year.
A. Less than 9,985
B. At least 9,985, but less than 9,995
C. At least 9,995, but less than 10,005
D. At least 10,005, but less than 10,015
E. At least 10,015

14.39 (CAS3L, 5/13, Q.9) (2.5 points) You are given the following:

- An actuary takes a vacation where he will not have access to email for eight days.
- While he is away, emails arrive in the actuary's inbox following a non-homogeneous Poisson process where $\lambda(t) = 8t - t^2$ for $0 \le t \le \infty$. (t is in days)

Calculate the variance of the number of emails received by the actuary during this trip.
A. Less than 60
B. At least 60, but less than 70
C. At least 70, but less than 80
D. At least 80, but less than 90
E. At least 90

14.40 (CAS3L, 11/13, Q.9) (2.5 points)
You are given that claim counts follow a non-homogeneous Poisson Process with
$\lambda(t) = 30t^2 + t^3$.
Calculate the probability of at least two claims between time 0.2 and 0.3.
A. Less than 1%
B. At least 1%, but less than 2%
C. At least 2%, but less than 3%
D. At least 3%, but less than 4%
E. At least 4%

14.41 (CAS ST, 5/14, Q.3) (2.5 points)
Let $S(t) = X_1 + X_2 + ... + X_{N(t)}$ be a compound Poisson process where:

- X_i is uniform on [0, 3.14].
- N(t) is a Poisson process with rate $\lambda(t) = e^{-t}$, $t > 0$.

Calculate $E[S(4)^2]$.
A. Less than 2.5
B. At least 2.5, but less than 3.5
C. At least 3.5, but less than 4.5
D. At least 4.5, but less than 5.5
E. At least 5.5

14.42 (CAS ST, 11/14, Q.3) (2.5 points) The number of tow trucks needed for car accidents in a city follows a compound Poisson process. You are given:

- The expected number of accidents per hour = $\begin{cases} 5 \text{ for 7am to 7pm} \\ 2 \text{ for 7pm to 7am} \end{cases}$
- The number of tow trucks needed per accident follows the Poisson distribution with a mean of 1.5.
- The expected number of accidents per hour follows the Poisson distribution.

Calculate the probability that more than 150 tow trucks will be needed in a 24 hour period using a normal approximation.

A. Less than 1%
B. At least 1%, but less than 5%
C. At least 5%, but less than 10%
D. At least 10%, but less than 20%
E. At least 20%

14.43 (CAS ST, 5/15, Q.3) (2.5 points) You are given:

- Claim frequency follows the Poisson process with a rate $\lambda(t) = 3t$, $t > 0$.
- Frequency and severity of claims are independent.
- Claim severity follows the distribution given in the following table:

Amount	Probability
5	0.6
10	0.3
75	0.1

- The aggregate claim amount by $t = 5$ was 505.

Calculate the variance of the aggregate claim amount at $t = 25$.

A. Less than 500,000
B. At least 500,000, but less than 520,000
C. At least 520,000, but less than 540,000
D. At least 540,000, but less than 560,000
E. At least 560,000

14.44 (CAS ST, 11/15, Q.1) (2.5 points) For two Poisson processes, N_1 and N_2, you are given:

• N_1 has intensity function $\lambda_1(t) = \begin{cases} 2t & \text{for } 0 < t \le 1 \\ t^3 & \text{for } t > 1 \end{cases}$

• N_2 is a homogenous Poisson process.

• $\text{Var}[N_1(3)] = 4\ \text{Var}[N_2(3)]$

Calculate the intensity of N_2 at t = 3.

A. Less than 1
B. At least 1, but less than 3
C. At least 3, but less than 5
D. At least 5, but less than 7
E. At least 7

14.45 (CAS ST, 5/16, Q.3) (2.5 points) For a block of one-year insurance policies you are given:
• Aggregate losses follow a compound Poisson process.
• Claim frequency follows a non-homogenous Poisson process with the following monthly rates:

Month	1	2	3	4	5	6	7	8	9	10	11	12
Rate	5	5	5	2	2	2	2	2	2	3	3	3

• Claim severities follow an exponential distribution with $\theta = 500$.

Calculate the standard deviation of the annual aggregate losses.

A. Less than 4,500
B. At least 4,500 but less than 5,000
C. At least 5,000 but less than 5,500
D. At least 5,500 but less than 6,000
E. More than 6,000

14.46 (CAS S, 5/16, Q.3) (2.2 points) You are given:
• The number of claims, N(t), follows a Poisson process with intensity:

$\lambda(t) = t/2,\ 0 < t < 5$

$\lambda(t) = t/4,\ t \ge 5$

• By time t = 4, 15 claims have occurred.

Calculate the probability that exactly 16 claims will have occurred by time t = 6.

A. Less than 0.075
B. At least 0.075, but less than 0.125
C. At least 0.125, but less than 0.175
D. At least 0.175, but less than 0.225
E. At least 0225

14.47 (CAS S, 11/17, Q.3) (2.2 points) You are given:

- A Poisson process N has a rate function: $\lambda(t) = 3t^2$.
- You've already observed 50 events by time t = 2.1.

Calculate the conditional probability, Pr[N(3)=68 I N(2.1)=50].

A. Less than 5%
B. At least 5%, but less than 10%
C. At least 10%, but less than 15%
D. At least 15%, but less than 20%
E. At least 20%

14.48 (MAS-1, 5/18, Q.3) (2.2 points)
The number of cars passing through the Lexington Tunnel follows a Poisson process with rate:

$$\lambda(t) = \begin{cases} 16 + 2.5t \text{ for } 0 < t \leq 8 \\ 52 - 2t \text{ for } 8 < t \leq 12 \\ -20 + 4t \text{ for } 12 < t \leq 18 \\ 160 - 6t \text{ for } 18 < t \leq 24 \end{cases}$$

Calculate the probability that exactly 50 cars pass through the tunnel between times t = 11 and t = 13.

A. Less than 0.01
B. At least 0.01, but less than 0.02
C. At least 0.02, but less than 0.03
D. At least 0.03, but less than 0.04
E. At least 0.04

Solutions to Problems:

14.1. C. $m(5) = \int_0^5 \lambda(s)\ ds = \int_0^5 \frac{1}{7+s}\ ds = \ln(7+s)\Big]_{s=0}^{s=5} = \ln(12) - \ln(7) = \mathbf{0.5390}$.

14.2. E. $m(2) = \int_0^2 \lambda(s)\ ds = \int_0^2 \frac{1}{7+s}\ ds = \ln(7+s)\Big]_{s=0}^{s=2} = \ln(9/7) = \mathbf{0.2513}$.

14.3. D. m(5) - m(2) = 0.5390 - 0.2513 = **0.2877**.

14.4. D. The number of claims from time 2 to time 5 is Poisson Distributed with mean 0.2877. Thus the chance of observing exactly 2 claims is: $e^{-0.2877}\ 0.2877^2 / 2! =$ **3.10%**.

14.5. A. The number of scientists eaten tomorrow is Poisson.
Integrating λ(t) over the day, the mean number of scientists eaten is:
(3)(0.1) + (6)(0.2) + (3)(0.1) = 1.8.
$f(2) = 1.8^2\ e^{-1.8} / 2! =$ **26.8%**.

14.6. E. The number of claims by time t is Poisson Distributed with mean m(t).
$m(t) = \int_0^t \lambda(s)\ ds = \int_0^t 8s^{2.4}\ ds = (8/3.4)t^{3.4}$. Therefore, $m(5) = (8/3.4)5^{3.4} = 559.9$.
The number of claims by time 5 is Poisson Distributed with mean 559.9. Therefore,
$\text{Prob}(n > 608) = 1 - \text{Prob}(n \le 608) \cong 1 - \Phi[\frac{608.5 - 559.9}{\sqrt{559.9}}] = 1 - \Phi(2.05) =$ **2.02%**.

14.7. C. Mean per day is: (6)(1%) + (3)(10%) + (6)(6%) + (3)(8%) + (3)(4%) + (3)(8%) = 1.32.
Number of accidents per day is Poisson with mean 1.32.
Prob[at least 3 accidents] = $1 - e^{-1.32} - 1.32e^{-1.32} - 1.32^2e^{-1.32}/2 =$ **0.148**.

14.8. B. Mean number of accidents from midnight to noon:
(6)(1%) + (3)(10%) + (3)(6%) = 0.54. Number of such accidents is Poisson.
Prob[2 accidents from midnight to noon] = $0.54^2e^{-0.54}/2 = 0.08496$.
Mean number of accidents from noon to midnight:
(3)(6%) + (3)(8%) + (3)(4%) + (3)(8%) = 0.78. Number of such accidents is Poisson.
Prob[1 accident from noon to midnight] = $0.78e^{-0.78} = 0.3576$.
What happens during disjoint periods of time is independent.
Probability of 2 accidents from midnight to noon and one accident from noon to midnight:
(0.08496)(0.3576) = **3.04%**.

14.9. C. The average number of cars per day is:

$$\int_0^8 12 + 3.5t \, dt + \int_8^{12} 60 - 2.5t \, dt + \int_{12}^{18} -30 + 5t \, dt + \int_{18}^{24} 204 - 8t \, dt$$

= 208 + 140 + 270 + 216 = 834.

Thus the number of cars per day is Poisson with mean 834.

Prob[more than 900 cars] $\cong 1 - \Phi[(900.5 - 834)/\sqrt{834}] = 1 - \Phi(2.30) =$ **1.07%**.

14.10. A. The number of phone calls from his wife is a nonhomogeneous Poisson Process with $\lambda = (0.1)(2) = 0.2$ prior to 3:30, and $\lambda = (0.3)(2) = 0.6$ after 3:30.

The number of calls between 2:00 and 4:30 from his wife is Poisson with mean:

(1.5)(0.2) + (1)(0.6) = 0.9. Prob[2 calls] = $0.9^2 e^{-0.9}/2$ = **16.5%**.

Alternately, the number of calls between 2:00 and 3:30 from his wife is Poisson with mean: (0.1)(1.5)(2) = 0.3. The number between 3:30 and 4:30 is Poisson with mean: (0.3)(1)(2) = 0.6. What happens between 2:00 and 3:30 is independent of what happens between 3:30 and 4:30. Therefore, the number of calls between 3:30 and 4:30 from his wife is Poisson with mean: 0.3 + 0.6 = 0.9. Proceed as before.

14.11. A. The number of withdrawals during a week is Poisson with mean:

1 + 1 + 0.5 + 0.5 + 0.5 + 1.5 = 5.

The mean severity is: (50%)(1) + (30%)(2) + (20%)(5) = 2.1.

The second moment of severity is: (50%)(1^2) + (30%)(2^2) + (20%)(5^2) = 6.7.

Mean aggregate is: (5)(2.1) = 10.5.

Variance of aggregate is: (5)(6.7) = 33.5.

Using the Normal Approximation, the 95th percentile of the aggregate distribution is:

$10.5 + 1.645\sqrt{33.5}$ = **$20.02**.

14.12. A. For Norm, $m(t) = \int \lambda(t) \, dt = t^3/3 + 1.5t^2$.

After playing for 8 minutes, the number of points for Norm is Poisson with mean:

$8^3/3 + 1.5(8^2) = 266.667$.

This Poisson is approximately Normal with mean and variance of 266.667.

For Cliff, $m(t) = \int \lambda(t) \, dt = t^3/6 + 2.5t^2$.

After playing for 8 minutes, the number of points for Cliff is Poisson with mean:

$8^3/6 + 2.5(8^2) = 245.333$.

This Poisson is approximately Normal with mean and variance of 245.333.

Therefore, Norm - Cliff is approximately Normal with mean: 266.667 - 245.333 = 21.334, and variance: (266.667 + 245.333) = 512.

Prob[Norm - Cliff < 0] $\cong \Phi[(-0.5 - 21.334)/\sqrt{512}] = \Phi[-0.96] =$ **17%**.

Comment: Similar to CAS3L, 5/08, Q.11.

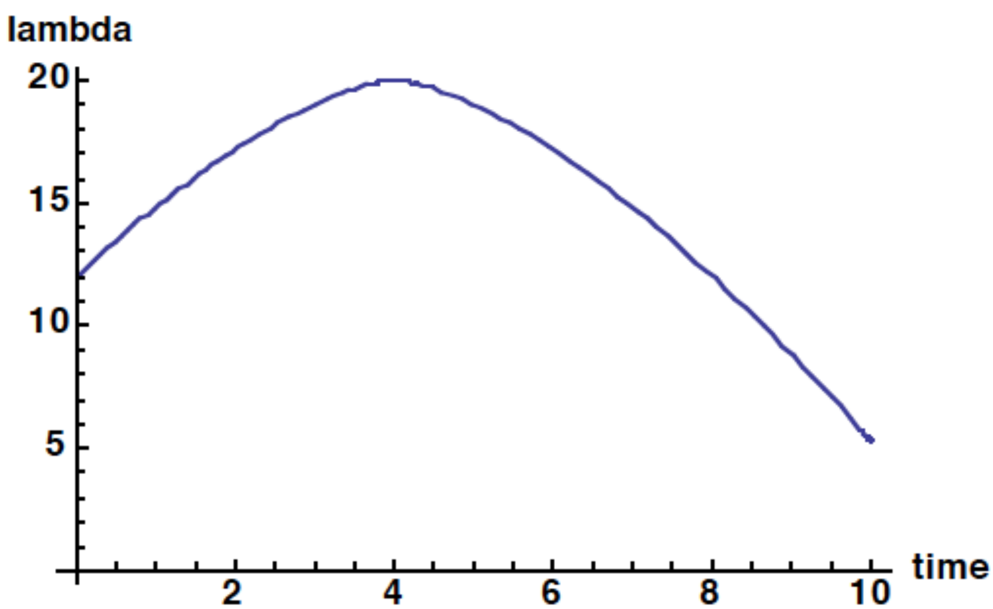
Even though it has little effect here, I have used the "continuity correction"; less than 0 on the discrete distribution of scores corresponds to less than -0.5 on the continuous Normal Distribution.

14.13. B. $m(t) = \int_0^t \lambda(s)\, ds = 20t + (4 - t)^{2.5}/2.5 - 4^{2.5}/2.5$, $0 < t \le 4$.

$m(2) = 29.46$. $m(4) = 67.2$.

$m(t) = m(4) + \int_4^t \lambda(s)\, ds = 67.2 + 20(t - 4) - (t - 4)^{2.5}/2.5$, $4 \le t \le 10$.

$m(9) = 144.84$.
$N(9) - N(2)$ is Poisson with mean: $m(9) - m(2) = 144.84 - 29.46 = 115.38$.
$\text{Var}[N(9) \mid N(2) = 37] = \text{Var}[N(9) - N(2) + 37 \mid N(2) = 37] = \text{Var}[N(9) - N(2) \mid N(2) = 37] =$
$\text{Var}[N(9) - N(2)] =$ **115.38**.
Comment: Similar to CAS 3L, 11/10, Q. 11.
The increment from time 2 to time 9 is independent of N(2).
A graph of lambda as a function of time:

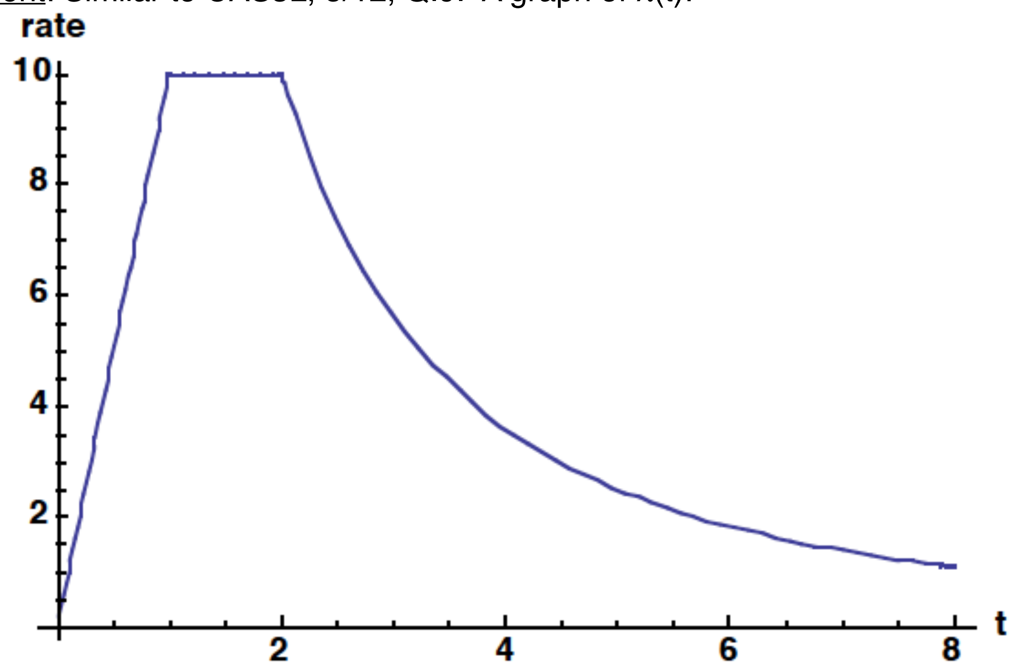


14.14. E. The expected number of claims is:

$$\int_0^1 10\,t\,dt + \int_1^2 10\,dt + \int_2^\infty \frac{90}{(1+t)^2}\,dt = 5 + 10 + 90/3 = 45.$$

The expected number of claims reported during 2015 is: $\int_2^3 \frac{90}{(1+t)^2}\,dt = 90/3 - 90/4 = 7.5.$

The expected percent of claims reported during 2015 is: 7.5 / 45 = **16.67%**.
Comment: Similar to CAS3L, 5/12, Q.9. A graph of $\lambda(t)$:



14.15. E. m(10) = 10. m(15) = 10 + (5)(2) = 20. m(20) = 10 + (10)(2) = 30.
m(26) = 30 + (6)(3) = 48. ⇒ m(26) - m(15) = 28.
⇒ N(26) - N(15) is Poisson with mean 28 and thus variance 28.
Var[N(26) | N(15) = 13] = Var[N(26) - N(15) + N(15) | N(15) = 13] =
Var[N(26) - N(15) + 13 | N(15) =13] = Var[N(26) - N(15) | N(15) = 13].
However what happens from time 15 to 26 is independent of what happens from time 0 to 15.
Thus, Var[N(26) - N(15) | N(15) = 13] = Var[N(26) - N(15)] = 28.
Thus, Var[N(26) | N(15) = 13] = **28**.

13 claims, no variance — Poisson with mean 28

0 ⟶ 15 ⟶ 26

Comment: E[N(26) | N(15) = 13] = 13 + 28 = 41. Var[N(26) | N(15) = 13] = 0 + 28 = 28.

14.16. B. The second moment of the Inverse Gamma Distribution is:

$$\frac{\theta^2}{(\alpha-1)\ (\alpha-2)} = (5000^2) / \{(5)(4)\} = 1.25 \text{ million.}$$

During the first 6 months we have a Compound Poisson Process with variance:
(6)(2)(1.25 million) = 15 million.
During the next 3 months we have a Compound Poisson Process with variance:
(3)(4)(1.25 million) = 15 million.
During the final 3 months we have a Compound Poisson Process with variance:
(3)(3)(1.25 million) = 11.25 million.
What happens during the parts of the year are independent, so their variances add:
15 million + 15 million + 11.25 million = 41.25 million.
Thus the standard deviation is: $\sqrt{41.25 \text{ million}}$ = **6423**.
Comment: Similar to CAS 3L, 11/12, Q.11.

14.17. E. $\frac{0.2\,t + 1}{t + 10} = 0.2 - 1/(t+10)$.

Thus the mean number of failures by time 20 is:

$$(0.2)(20) - \int_0^{20} \frac{1}{t+10}\,dt = 4 - \{\ln(30) - \ln(10)\} = 2.901.$$

Probability of two failures is: $2.901^2\ e^{-2.901} / 2$ = **23.1%**.

14.18. D. $m(7) = \int_0^7 \lambda(t)\ dt = \int_0^7 \frac{2}{3+t}\ dt = 2\,\{\ln(3+7) - \ln(3)\} = 2.408.$

Thus the number of claims by time 7 is Poisson with mean 2.408.

The second moment of severity is: $\int_4^9 (1/5)\ x^2\ dx = (9^3 - 4^3) / 15 = 44.33.$

Thus the variance of the aggregate by time 4 is: (2.408) (44.33) = **106.7**.
Comment: Similar to ST, 5/14, Q.3.

14.19. E. The expected number of accidents over a 24 hour period is Poisson with mean:
(18)(3) + (6)(1) = 60.
The mean number of persons injured per accidents is:
(30%)(0) + (30%)(1) + (20%)(2) + (15%)(3) + (5%)(4) = 1.35.
The second moment number of persons injured per accidents is:
$(30\%)(0^2) + (30\%)(1^2) + (20\%)(2^2) + (15\%)(3^2) + (5\%)(4^2) = 3.25$.
Thus the mean of the total number of persons injured is: (60)(1.35) = 81.
The variance of the total number of persons injured is: (60)(3.25) = 195.
Thus using the Normal approximation with continuity correction:
Prob[more than 90 persons injured] = $1 - \Phi\left[\frac{90.5 - 81}{\sqrt{195}}\right] = 1 - \Phi[0.680] =$ **24.8%**.

Comment: Similar to ST, 11/14, Q.3.

14.20. C. What happens after 4 time is independent of what happened up to time 4.
Thus we need the probability of exactly 37 - 17 = 20 events from time 4 to 6.
The number of events from time 4 to 6 is Poisson with mean:

$$\int_4^6 \lambda(t)\,dt = \int_4^6 t^{1.5}\,dt = t^{2.5}/2.5\Big]_{t=4}^{t=6} = 6^{2.5}/2.5 - 4^{2.5}/2.5 = 22.473.$$

$f(20) = \lambda^{20}\, e^{-\lambda} / 20! = 22.473^{20}\, e^{-22.473} / 20! =$ **7.71%.**
Comment: Similar to Exam S, 11/17, Q.3.

14.21. A. 1. Yes. 2. No. You can get more than one report at a time; Nonhomogeneous Poisson Processes can not have more than one event at a time. The number of reports is a Compound Nonhomogeneous Poisson Process. 3. No. The total dollars of loss are continuous; Nonhomogeneous Poisson Processes are discrete. This is a compound, compound variable.

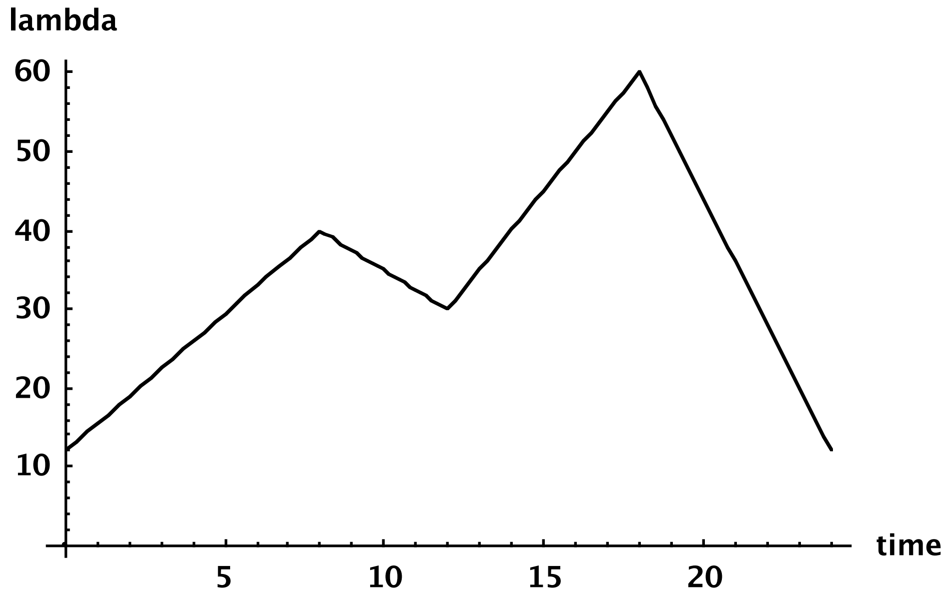
14.22. D. $\int_{11.5}^{12.5} \lambda(t)\ dt + \int_{11.5}^{12} 60 - 2.5t\ dt + \int_{12}^{12.5} -30 + 5t\ dt = 15.3125 + 15.625 = 30.9375.$

Therefore, the number of cars that pass through the tunnel between 11:30 a.m. and 12:30 p.m. is Poisson with mean 30.9375.

$f(25) = 30.9375^{25}\ e^{-30.9375}/25! =$ **0.0432.**

Comment: Better language could have been used than "a Poisson distribution with an increasing hourly rate," in order to describe this Nonhomogeneous Poisson Process.

Here is a graph of $\lambda(t)$:



$m(12.5) - m(11.5) =$ the area under $\lambda(t)$ from 11.5 to 12.5.

14.23. A. $\lambda(t) = 3t + 3.$ $\int_1^2 \lambda(t)\ dt = \int_1^2 3t + 3\ dt = 7.5.$

The number customers that arrive between 1:00 p.m. and 2:00 p.m. is Poisson with mean 7.5.

$f(2) = 7.5^2 e^{-7.5}/2 =$ **0.0156**.

14.24. B. mean = (7)(0.05) + (3)(0.10) + (6)(0.08) + (3)(0.07) + (5)(0.02) = 1.44.

Number of accidents in one day is Poisson with mean 1.44.

Prob[3 or more accidents] = $1 - e^{-1.44}(1 + 1.44 + 1.44^2/2) =$ **17.6%**.

14.25. A. The mean number of injuries from time 23 to 24 is:

$$\int_{23}^{24} \lambda(t)\ dt = \int_{23}^{24} (t/12)^{1/2}\ dt = \{24^{1.5}/1.5 - 23^{1.5}/1.5\}/12^{0.5} = 1.3994.$$

Thus the number of injuries during the last hour is Poisson with mean 1.3994.

Prob[no injuries in last hour] = $e^{-1.3994} =$ **24.7%**.

14.26. C. The mean number of Jackalopes is: $\int_{13}^{17} t^{1/3}/5 \, dt = 1.972$.

The mean number of Snipes is: $\int_{13}^{17} t^{1/2}/10 \, dt = 1.548$.

The number of Snipes is Poisson with mean 1.548.
Prob[at least one J]Prob[at least one S] = $(1 - e^{-1.972})(1 - e^{-1.548})$ = (0.8608)(0.7873) = **0.678**.
Comment: Here is a photograph the student brought back of a Jackalope:

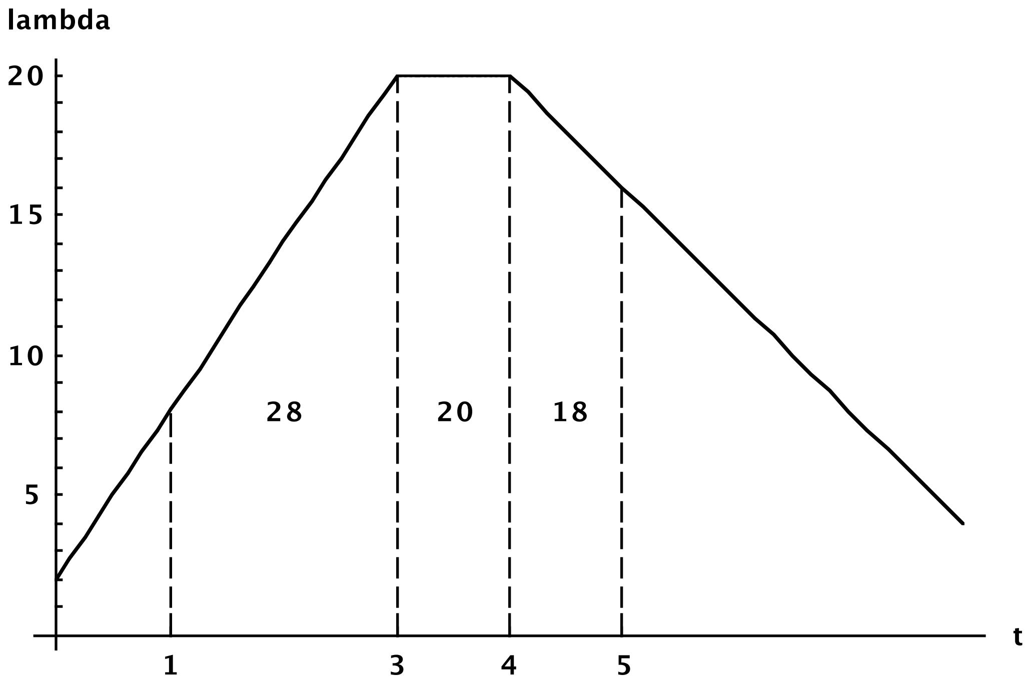
14.27. A. The number of Jackalopes is Poisson with mean 1.972.
The number of Snipes is Poisson with mean 1.548.
Therefore, the total number of photographs is Poisson with mean: 1.972 + 1.548 = 3.520.
Prob[4 photographs] = $3.520^4 e^{-3.520} / 4!$ = **18.9%**.

14.28. C. 9:00 a.m. ⇔ t = 0. 10:00 a.m. ⇔ t = 1. 2:00 p.m. ⇔ t = 5.

$$m(5) - m(1) = \int_1^3 2 + 6t \ dt + \int_3^4 20 \ dt + \int_4^5 36 - 4t \ dt = \{2t + 3t^2 \Big]_{t=1}^{t=3}\} + 20 + \{36t - 2t^2 \Big]_{t=4}^{t=5}\}$$

= 28 + 20 + 18 = **66**.

Comment: Here is a graph of λ(t) and the relevant areas:



14.29. E. 9:00 a.m. ⇔ t = 0. 11:00 a.m. ⇔ t = 2. 4:00 p.m. ⇔ t = 7.

$$m(7) - m(2) = \int_2^3 2+6t \ dt + \int_3^4 20 \ dt + \int_4^7 36-4t \ dt = \{2t+3t^2 \Big]_{t=2}^{t=3}\} + 20 + \{36t-2t^2 \Big]_{t=4}^{t=7}\}$$

= 17 + 20 + 42 = 79.

Thus the number of customers between 11:00 am and 4:00 p.m. is Poisson with mean 79.

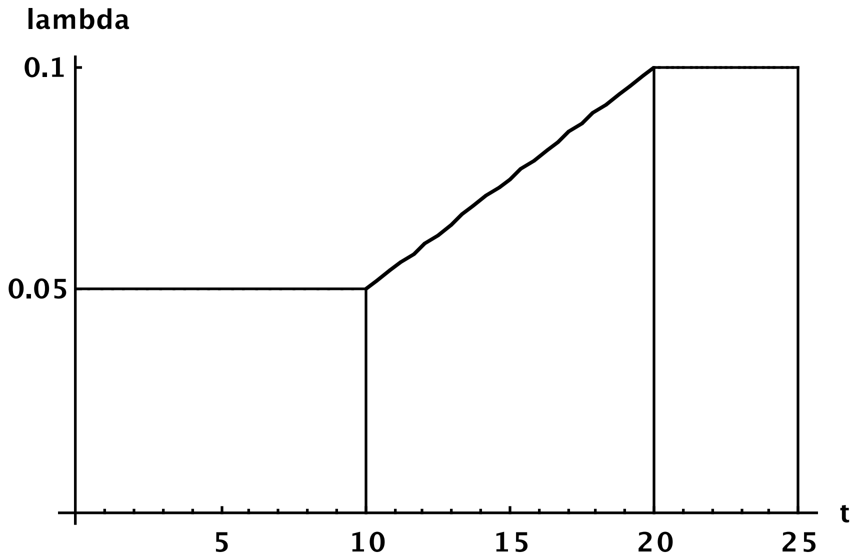
Prob[customers < 70] ≅ $\Phi[(69.5 - 79)/\sqrt{79}] = \Phi[-1.07]$ = **14.23%**.

14.30. B. The number of trains between t = 0 and 25 is Poisson with mean:

$$\int_0^{25} \lambda(t)\,dt = (10)(0.05) + \int_{10}^{20} t/200\ dt + (5)(0.10) = 0.5 + 0.75 + 0.5 = 1.75.$$

$f(4) = 1.75^4\ e^{-1.75} / 4! =$ **0.068**.

Comment: The mean of the Poisson is the area under the graph of λ(t) from 0 to 25:



Real subway trains do not follow a Poisson Process; rather they run on some sort of schedule. Even if they get delayed, there is still some pattern to their arrival, rather than it being a memoryless process.

14.31. D. The number of accidents in a week is Poisson with mean: (2)(3) + (3)(4) + (2)(1) = 20.
$f(18) = 20^{18}\ e^{-20} / 18! =$ **8.44%**.
Comment: A nonhomogeneous Poisson process.

14.32. C. Let X = the number of calls from 9:00 AM to 1:00 PM.
Let Y = the number of calls from 1:00 PM to 5:00 PM.
X is Poisson with mean: (3)(30) + 10 = 100. X has mean of 100 and variance of 100.
Y is Poisson with mean: (2)(25) + (2)(30) = 110. Y has mean of 110 and variance of 110.
X and Y are independent.
Therefore, X - Y has mean of: 100 - 110 = -10, and variance of 100 + 110 = 210.
Using the normal approximation, with continuity correction, greater than zero is equivalent to greater than 0.5:
$\text{Prob}[X > Y] = \text{Prob}[X - Y > 0] \cong 1 - \Phi[\{0.5 - (-10)\}/\sqrt{210}\,] = 1 - \Phi[0.72] =$ **24%**.
Comment: A nonhomogeneous Poisson process.
For X and Y independent, Var[X - Y] = Var[X] + Var[Y].

14.33. D. Expected number of accidents from 3:00 PM to 4:00 PM is:

$$\int_0^1 \lambda(t)\ dt = \int_0^1 4 - (t-2)^2\ dt = 4t - (t-2)^3/3 \Big]_{t=0}^{t=1} = \{4 - (-1)^3/3\} - \{0 - (-2)^3/3\} = 1.667.$$

Expected number of accidents from 4:00 PM to 5:00 PM is:

$$\int_1^2 \lambda(t)\ dt = \int_1^2 4 - (t-2)^2\ dt = 4t - (t-2)^3/3 \Big]_{t=1}^{t=2} = \{8 - (0)^3/3\} - \{4 - (-1)^3/3\} = 3.667.$$

3.667 - 1.667 = **2**.

14.34. D. The number of downloads between times t = 1 and t = 5 is Poisson with mean:

$$m(5) - m(1) = \int_1^5 \lambda(t)\ dt = \int_1^5 e^{-0.25t}\ dt = -4e^{-0.25t} \Big]_{t=1}^{t=5} = (4)(e^{-0.25} - e^{-1.25}) = 1.9692.$$

Prob[more than two downloads between times t = 1 and t = 5] =
$1 - e^{-1.9692} - 1.9692\ e^{-1.9692} - (1.9692^2/2)\ e^{-1.9692}$ = **31.5%**.
Comment: "heterogeneous" should have read "nonhomogeneous."

14.35. B. $m(t) = \int \lambda(t)\ dt = t^3$.

N(3) - N(2.1) is Poisson with mean: $m(3) - m(2.1) = 3^3 - 2.1^3 = 17.739$.
Var[N(3) | N(2.1) = 50] = Var[N(3) - N(2.1) + 50 | N(2.1) = 50] = Var[N(3) - N(2.1) | N(2.1) = 50] = Var[N(3) - N(2.1)] = **17.739**.
Comment: The increment from time 2.1 to time 3 is independent of N(2.1), the number of events by time 2.1.

14.36. B. The expected number of claims reported after January 1, 2011 for this group of policies is: $\int_0^\infty \lambda(t)\ dt = \int_0^\infty \frac{100}{(1+t)^3}\ dt$ = 100/2 = **50**.

Comment: Since the rate at which claims are reported for a fixed group of policies declines over time sufficiently quickly, the mean number is finite.
For example, if the policies provide liability coverage only for automobile accidents that occur during 2011, then a few claims may be reported during 2013, fewer in 2014, etc.
While the given form of $\lambda(t)$ is not realistic, it is not unreasonable for an exam question.

14.37. The expected number of claims reported after January 1, 2013 for this group of policies is:

$$\int_2^\infty \frac{100}{(1+t)^3}\, dt = (100)(1/2)(1/3^2) = 50/9.$$

So **1/9** of claims are expected to be reported more than two years from January 1, 2011.

14.38. C. The second moment of the Pareto Distribution is:

$$\frac{2\theta^2}{(\alpha-1)\ (\alpha-2)} = (2)(2000^2) / \{(4)(3)\} = 666{,}667.$$

During the first 6 months we have a Compound Poisson Process with variance:
(6)(10)(666,667) = 40 million.
During the second 6 months we have a Compound Poisson Process with variance:
(6)(15)(666,667) = 60 million.
What happens during the halves of the year are independent, so their variances add:
40 million + 60 million = 100 million.
Thus the standard deviation is: $\sqrt{100 \text{ million}}$ = **10,000**.

14.39. D. $m(8) = \int_0^8 8t - t^2\, dt = 4t^2 - t^3/3\Big]_{t=0}^{t=8} = 85.33.$

Thus the number of emails is Poisson with mean 85.33, and thus variance **85.33**.

14.40. B. The number of claims between time 0.2 and 0.3 is Poisson with mean:

$$\int_{0.2}^{0.3} 30t^2 + t^3\, dt = 10t^3 + t^4/4\Big]_{t=0.2}^{t=0.3} = 0.191625.$$

Thus the probability of at least two claims between time 0.2 and 0.3 is:
1 - exp[-0.191625] - 0.191625 exp[-0.191625] = **1.62%**.

14.41. E. $m(4) = \int_0^4 \lambda(t)\ dt = \int_0^4 e^{-t}\ dt = 1 - e^{-4} = 0.9817$.

Thus the number of claims by time 4 is Poisson with mean 0.9817.
The mean severity is: 3.14/2 = 1.57.
The variance of severity is: $3.14^2/12 = 0.8216$.
Thus the second moment of severity is: $0.8216 + 1.57^2 = 3.2865$.
The mean aggregate by time 4 is: (0.9817) (1.57) = 1.541.
The variance of the aggregate by time 4 is: (0.9817) (3.2865) = 3.226.
Thus the second moment of the aggregate by time 4 is: $3.226 + 1.541^2$ = **5.60**.

Comment: $m(\infty) = \int_0^\infty \lambda(t)\ dt = \int_0^\infty e^{-t}\ dt = 1$.

14.42. C. The expected number of accidents over a 24 hour period is Poisson with mean:
(12)(5) + (12)(2) = 84.
Thus the mean number of tow trucks needed is: (1.5)(84) = 126.
The variance of the number of tow trucks needed is:
(84) (second moment of the amount distribution) = (84) $(1.5 + 1.5^2)$ = 315.
Thus using the Normal approximation with continuity correction:
Prob[more than 150 tow trucks] $\cong 1 - \Phi[\frac{150.5 - 126}{\sqrt{315}}] = 1 - \Phi[1.38]$ = **8.4%**.

Comment: The last bullet is both redundant and wrong. The number of tow trucks needed for accidents is stated to be a compound Poisson process which means that the number of accidents per hour follows a Poisson distribution. The expected number of accidents per hour does not follow a Poisson Distribution. Rather the number of accidents per hour follows a Poisson Distribution.
We have a nonhomogeneous Poisson Process with an amount attached to each event (accident);
the amounts (the number of tow trucks needed per accident) follow a Poisson Distribution.
The second moment of a Poisson Distribution is: variance + mean^2 = $\lambda + \lambda^2$.

14.43. D. Since the aggregate amount by time 5 is known, the variance of the aggregate amount by time 25 is equal to the variance of the aggregate amount between times 5 and 25.

The mean number of claims from time 5 to 25: $\int_5^{25} \lambda(t)\ dt = \int_5^{25} 3t\ dt = 1.5t^2\Big]_{t=5}^{t=25} = 900$.

The second moment of severity is: $(0.6)(5^2) + (0.3)(10^2) + (0.1)(75^2) = 607.5$.
The variance of the aggregate claim amount at t = 25 is: (900)(607.5) = **546,750**.

14.44. B. $m_1(3) = \int_0^1 2t\ dt + \int_1^3 t^3\ dt = 1 + (3^4 - 1^4)/4 = 21.$

Thus, $N_1(3)$ is Poisson with mean 21, and thus variance 21.

$N_2(3)$ is Poisson with mean 3λ, and thus variance 3λ.

We are given that $\text{Var}[N_1(3)] = 4\ \text{Var}[N_2(3)]$.

$\Rightarrow 21 = (4)(3\lambda). \Rightarrow \lambda = 21/12 =$ **1.75**.

Comment: Since N_2 is homogeneous, its intensity is the same at t = 3 as at any other time.

14.45. A. The number of claims is Poisson with mean:
5 + 5 + 5 + 2 + 2 + 2 + 2 + 2 + 2 + 3 + 3 + 3 = 36.
The second moment of severity is: $2\theta^2 = (2)(500^2) = 500{,}000$.
Thus the variance of the aggregate annual loss is: (36)(500,000) = 18,000,000.
The standard deviation is: $\sqrt{18{,}000{,}000}$ = **4243**.

14.46. B. The number of claims from time 4 to time 6 is Poison with mean:

$$\int_4^5 t/2\ dt + \int_5^6 t/4\ dt = (5^2 - 4^2)/4 + (6^2 - 5^2)/8 = 3.625.$$

Prob[exactly one claim between time 4 and 6] = $3.625\ e^{-3.625}$ = **9.66%**.

14.47. B. What happens after 2.1 time is independent of what happened up to time 2.1.
Thus we need the probability of exactly 68 - 50 = 18 events from time 2.1 to 3.
The number of events from time 2.1 to 3 is Poisson with mean:

$$\int_{2.1}^3 \lambda(t)\ dt = \int_{2.1}^3 3t^2\ dt = t^3 \Big]_{t=2.1}^{t=3} = 3^3 - 2.1^3 = 17.739.$$

$f(18) = \lambda^{18}\ e^{-\lambda} / 18! = 17.739^{18}\ e^{-17.739} / 18! =$ **9.34%**.

14.48. C. $\int_{11}^{13} \lambda(t)\ dt = \int_{11}^{12} (52 - 2t)\ dt + \int_{12}^{13} (-20 + 4t)\ dt = (52t - t^2)\Big]_{t=11}^{t=12} + (-20t + 2t^2)\Big]_{t=12}^{t=13}$

= 29 + 30 = 59.

Therefore, the number of cars between t = 11 and t = 13 is Poisson with mean 59.

The density of this Poisson at 50 is: $e^{-59}\ 59^{50} / 50! =$ **0.0273**.

Section 15, Interevent Times, Nonhomogeneous Poisson Process

The interevent times for a nonhomogeneous Poisson process are somewhat more complicated than in the homogeneous case.

Distribution Function of the Time of the First Event:

As in Life Contingencies, one can write the distribution function of the time of the next claim in terms of the claims intensity or force of mortality, $\lambda(t)$. Starting from time equal to zero,
$F(t)$ = Prob[at least one claim by time t] = 1 - Prob[0 claims by time t] = $1 - e^{-m(t)}$.

$$\mathbf{F(t) = 1 - e^{-m(t)}}.$$

Note that if $\lambda(t)$ is λ, constant, then $m(t) = \lambda t$, and $F(t) = 1 - e^{-\lambda t}$, the homogeneous case.

If for example, let $\lambda(t) = 20t^3$. Then starting at time equals zero:

$$F(t) = 1 - \exp[-\int_0^t 20\, s^3\, ds] = 1 - \exp[-5t^4].$$

This is a Weibull Distribution, $F(x) = 1 - \exp[-(x/\theta)^\tau]$, $x > 0$, with $\theta = 5^{-0.25}$ and $\tau = 4$.

Looked at another way, the probability there will be no claim between time equal zero and time t is: $\exp(-5t^4)$.

In general, starting at time 0, the chance that there will be no claim between time 0 and time t is:

$$S(t) = \exp[-\int_0^t \lambda(s) ds] = e^{-m(t)}.$$

Mean Time Until the First Event:[112]

As discussed above, the waiting time for the first claim is distributed:

$$F(t) = 1 - \exp[- \int_0^t \lambda(s)ds] = 1 - e^{-m(t)}.$$

Thus the **density of the waiting time for the first event** is:

$$\frac{d\,F(t)}{dt} = m'(t)\ e^{-m(t)} = \boldsymbol{\lambda(t)\ e^{-m(t)}}.$$

Exercise: Given a nonhomogeneous Poisson Process with $\lambda(t) = 0.5$ for $0 \le t \le 1$, and $\lambda(t) = 0.8$ for $t > 1$, what is the density of the waiting time until the first event?
[Solution:

$$m(t) = \int_0^t \lambda(s)ds = 0.5t \text{ for } t \le 1, \text{ and } 0.5 + (t - 1)(0.8) = 0.8t - 0.3 \text{ for } t > 1.$$

Thus the density of the waiting time for the first event is:
$\lambda(t)e^{-m(t)} = 0.5\ e^{-0.5t}$ for $t \le 1$, and $e^{0.3}\ 0.8\ e^{-0.8t}$ for $t > 1$.]

One can get the mean time until the first event by integrating t versus this density.

Exercise: Given a nonhomogeneous Poisson Process with $\lambda(t) = 0.5$ for $0 \le t \le 1$, and $\lambda(t) = 0.8$ for $t > 1$, what is the mean waiting time until the first event?
[Solution: The mean waiting time until the first event is:

$$\int_0^1 0.5\ e^{-0.5t}\ t\ dt + e^{0.3} \int_1^\infty 0.8\ e^{-0.8t}\ t\ dt = (-e^{-0.5t}\ t - 2e^{-0.5t})\Big]_{t=0}^{t=1} + e^{0.3}(-e^{-0.8t}\ t - 1.25e^{-0.8t})\Big]_{t=1}^{t=\infty}$$

$$= \{-e^{-0.5} - 2e^{-0.5} - (0 - 2)\} + e^{0.3}(e^{-0.8} + 1.25e^{-0.8}) = 2 - 0.75e^{-0.5} = 1.545.]$$

One can instead compute the mean by integrating the Survival function, $S(t) = 1 - F(t)$, from 0 to ∞.[113]
Therefore, the mean waiting time until the first event is:[114]

$$\int_0^\infty S(t)dt = \int_0^\infty e^{-m(t)}\ dt.$$

[112] See for example, SOA M, 11/06, Q.10.
[113] This result applies to distributions with support starting at zero and can be derived by integration by parts.
[114] As discussed previously, if the claims intensity is constant, λ, $m(t) = \lambda t$, and the mean waiting time until the first claim is $1/\lambda$.

Applying this to the previous exercise, the mean waiting time until the first event is:

$$\int_0^1 e^{-0.5t}\, dt + e^{0.3} \int_1^\infty e^{-0.8t}\, dt = 2 - 2e^{-0.5} + e^{0.3}\, 1.25e^{-0.8} = 2 - 0.75e^{-0.5} = 1.545.$$

Exercise: Given a nonhomogeneous Poisson Process with $\lambda(t) = 2t$, what is the mean waiting time until the first claim? Hint: $\Gamma(1/2) = \sqrt{\pi}$.

[Solution: $m(t) = \int_0^t \lambda(s)ds = \int_0^t 2s ds = t^2$.

Therefore, the mean waiting time until the first claim is:

$$\int_0^\infty e^{-m(t)}\, dt = \int_0^\infty e^{-t^2}\, dt = (1/2) \int_0^\infty e^{-y}\, y^{-1/2}\, dy = \Gamma(1/2) / 2 = \frac{\sqrt{\pi}}{2} = 0.886.$$

Comment: By the change of variables, $y = t^2$, $t = y^{1/2}$, and $dt = (1/2)\, y^{-1/2}\, dy$, the above integral becomes the integral from 0 to ∞ of $e^{-y}\, y^{-1/2}/2$.

You are not responsible for knowing the value of $\Gamma[1/2]$.]

Multiplying the Claim Intensity by a Constant:

In the case of a constant claims intensity, the mean waiting time to the first claim is the inverse of the claims intensity; if one doubles the claims intensity one halves the mean waiting time until the first claim. What happens to the mean waiting time until the first claim for a nonhomogeneous Poisson Process, if we double the claims intensity?

$$m(t) = \int_0^t 2\,\lambda(s)ds = 2 \int_0^t \lambda(s)ds .$$

Therefore m(t), the expected number of claims from time 0 to t, doubles.

Therefore, the mean waiting time until the first claim is now: $\int_0^\infty e^{-2m(t)}\, dt$.

This depends on the form of m(t) and thus the form of the claims intensity.

Exercise: Given a nonhomogeneous Poisson Process with $\lambda(t) = 4t$, what is the mean waiting time until the first claim? Hint: $\Gamma(1/2) = \sqrt{\pi}$.

[Solution: $m(t) = \int_0^t \lambda(s)ds = \int_0^t 4s\ ds = 2t^2$.

Therefore, the mean waiting time until the first claim is:

$$\int_0^\infty e^{-m(t)}\ dt = \int_0^\infty \exp[-2t^2]\ dt = \int_0^\infty \frac{e^{-y}\ y^{-1/2}}{2\sqrt{2}}\ dy = \frac{\Gamma[1/2]}{2\sqrt{2}} = \frac{\sqrt{\pi}}{2\sqrt{2}} = 0.627.$$

Comment: Used the change of variables, $y = 2t^2$.]

As shown in a previous exercise, with $\lambda(t) = 2t$, the mean waiting time until the first claim was $\frac{\sqrt{\pi}}{2} = 0.886$.

Thus in this particular case doubling the claims intensity, resulted in dividing the mean waiting time to the first claim by $\sqrt{2}$. In general, while the waiting time will be smaller for a larger claims intensity, how much smaller depends on the form of the claims intensity for the particular nonhomogeneous Poisson Process.

Interevent Times:

If there is a claim at time s, then the time of the next claim is distributed as:

$$F(t) = 1 - \exp[-\int_s^t \lambda(x)dx\ ].$$

For example, let $\lambda(t) = 20t^3$. If there is a claim at time s, then:

$$F(t) = 1 - \exp[-\int_s^t 20x^3 dx\ ] = 1 - \exp[5s^4 - 5t^4] = 1 - \frac{\exp[-5t^4]}{\exp[-5s^4]}.$$

This is a Weibull Distribution, $F(x) = 1 - \exp[-(x/\theta)^\tau]$, $x > 0$, with $\theta = 5^{-0.25}$ and $\tau = 4$, truncated from below at s.

Exercise: Given a nonhomogeneous Poisson Process with claims intensity $\lambda(t) = 20t^3$, what is the chance of no claims between time 0.4 and 0.5?

[Solution: $\exp[-\int_{0.4}^{0.5} 20\ t^3\ dt\ ] = \exp[\ (5)(0.4^4 - 0.5^4)] = e^{-0.1845} = 83.15\%$.

Comment: We can instead compute the expected number of claims between time 0.4 and 0.5 by integrating $\lambda(t)$ from 0.4 to 0.5, giving 0.1845. Then the number of claims from time 0.4 to 0.5 is Poisson with mean 0.1845. Thus the chance of no claims is $e^{-0.1845}$.]

The **interevent times** are the times between claims. V_1 is the time until the first claim. V_2 is the time from the first claim until the second claim. V_3 is the time from the second claim until the third claim, etc. Therefore, the interevent time for the ith claim, V_i, given we observe claim i-1 at time s is distributed:

$$F(t) = 1 - \exp\left[-\int_s^t \lambda(x)dx\right] = 1 - e^{m(s) - m(t)}.$$

Exercise: Given a nonhomogeneous Poisson Process with claims intensity $\lambda(t) = 20t^3$, if we see a claim at time 0.7, what is the distribution of the time until the next claim?
[Solution: Let V_i be the time until the next claim. Then

$$F(V_i) = 1 - \exp\left[-\int_{0.7}^{0.7+V_i} \lambda(t)dt\right] = 1 - \exp[1.2005 - (5)(0.7+V_i)^4].$$

Comment: This is a Weibull Distribution, $F(x) = 1 - \exp[-(x/\theta)^\tau]$, $x > 0$, with $\theta = 5^{-0.25}$ and $\tau = 4$, truncated and shifted from below at 0.7.]

Unless $\lambda(t)$ is a constant, as it was for a (homogeneous) Poisson Process, the distribution of the interevent time V_i for the ith claim depends on the time we observe the i-1 claim. In general, the interevent times are neither independent nor identically distributed, as they were for the homogeneous Poisson Process.[115] In general, the distribution of the interevent time V_i is not an Exponential Distribution, as it was for the homogeneous Poisson Process.

Exercise: Given a nonhomogeneous Poisson Process with claims intensity $\lambda(t) = 20t^3$, if we see a claim at time 0.7, what is the mean time until the next event?
[Solution: The time until the next event is a Weibull Distribution, as per Appendix A of Loss Models, with $\theta = 5^{-0.25}$ and $\tau = 4$, truncated and shifted from below at 0.7. It has mean equal to e(0.7), the mean excess loss (mean residual life) at 0.7, of a Weibull Distribution, as per Appendix A of Loss Models, with $\theta = 5^{-0.25}$ and $\tau = 4$.

$e(x) = \{E[X] - E[X \wedge x]\} / S(x) = \theta\, \Gamma(1 + 1/\tau)\, \{1 - \Gamma[1 + 1/\tau\,;\, (x/\theta)^\tau]\} \exp[(x/\theta)^\tau] - x.$

$e(0.7) = (5^{-0.25})\, \Gamma(1.25)\, \{1 - \Gamma[1.25\,;\, 5(0.7^4)]\} \exp[5(0.7^4)] - 0.7$

$= 2.2214\, \Gamma(1.25)\, \{1 - \Gamma[1.25\,;\, 1.2005]\} - 0.7 = (2.2214)(0.9064)(1 - 0.6017) - 0.7 = 0.102.$

Comment: Beyond what you will be asked on your exam.
The claims intensity at 0.7 is $20(0.7^3) = 6.86$. Thus a first approximation to the mean waiting time is $1/6.86 = 0.146$. Then a better approximation could be obtained by taking the inverse of the claims intensity at $(0.7 + 0.146/2)$:

$$\frac{1}{\lambda(0.7 + 0.146/2)} = \frac{1}{\lambda(0.773)} = \frac{1}{(20)(0.773)^3} = \frac{1}{9.24} = 0.108.]$$

[115] See Exercise 58 on page 297 of Introduction to Probability Models by Ross, not on the syllabus.

Event Times:

As discussed previously, the number of events from time a to b is given by a Poisson Distribution with mean m(b) - m(a) = the integral the intensity $\lambda(t)$ from a to b.

Exercise: Given a nonhomogeneous Poisson Process with intensity $\lambda(t) = 3t^2$, what is m(t)?
[Solution: The mean number of events from time 0 to t is:

$$m(t) = \int_0^t \lambda(x)dx = \int_0^t 3x^2 dx = t^3.]$$

Exercise: Given a nonhomogeneous Poisson Process with intensity $\lambda(t) = 3t^2$, what is Distribution of the number of events from 0 to t?
[Solution: The number of events from 0 to t is Poisson, with mean $m(t) = t^3$.]

Exercise: Given a nonhomogeneous Poisson Process with intensity $\lambda(t) = 3t^2$, what is probability of having three or more events from time 0 to time t?
[Solution: The number of events from 0 to t is Poisson, with mean $m(t) = t^3$.
Therefore the probability of zero events is $\exp[-t^3]$, the probability of one event is: $t^3\exp[-t^3]$, and the probability of two events is: $t^6\exp[-t^3]/2$.
Thus probability of having three or more events is: $1 - \{\exp[-t^3] + t^3\exp[-t^3] + t^6\exp[-t^3]/2\}$.]

Now if there are three or more events by time t, then the 3rd event time is less than or equal to t. Thus the distribution function for the 3rd event time is the probability of having three or more events from time 0 to time t.

Exercise: Given a nonhomogeneous Poisson Process with intensity $\lambda(t) = 3t^2$, what is distribution function for the 3rd event time?
[Solution: $F(t) = 1 - \{\exp[-t^3] + t^3\exp[-t^3] + t^6\exp[-t^3]/2\}$.]

In general, the 3rd event time is the chance of three or more claims from a Poisson Distribution with mean m(t): $F(t) = 1 - \{\exp[-m(t)] + m(t)\exp[-m(t)] + m(t)^2\exp[-m(t)]/2\}$.

The distribution of the nth event time, is the chance of n or more claims from a Poisson Distribution with mean m(t):

$$F(t) = 1 - \sum_{i=0}^{n-1} e^{-m(t)}\, m(t)^i / i! = \Gamma[n \,;\, m(t)].$$

Where I have made use of the fact discussed previously, to write the sum of Poisson densities in terms of the incomplete Gamma Function.[116]

[116] See Theorem A.1 in Appendix A of Loss Models.

We have shown that for a **nonhomogeneous Poisson Process**, **the distribution of the nth event time is equal to Γ[n ; m(t)]**.[117] [118]

Note that if the intensity is constant, then $m(t) = \lambda t$, and the n^{th} event time is $\Gamma[n ; \lambda t]$, the result previously obtained for a homogeneous Poisson Process.[119]

Therefore, for a nonhomogeneous Poisson Process, the density of the n^{th} event time is equal to:[120]

$$\frac{d\,\Gamma[n ; m(t)]}{dt} = m'(t)\ m(t)^{n-1}\ e^{-m(t)} / \Gamma[n] = \lambda(t)\ m(t)^{n-1}\ e^{-m(t)} / (n-1)!$$

Exercise: If one has an nonhomogeneous Poisson Process with claims intensity $\lambda(t) = ct^b$, with $b > 0$, what is the distribution of the n^{th} event time?
[Solution: $m(t) = ct^{b+1} / (b+1)$. Therefore, the distribution of the n^{th} event time is $F(t) = \Gamma[n ;\ c\, t^{b+1} / (b+1)]$. This is a Transformed Gamma Distribution, as per Loss Models, with parameters $\alpha = n$, $\tau = b+1$, and $\theta = \{(b+1)/c\}^{1/(b+1)}$.]

Exercise: Given a nonhomogeneous Poisson Process with $\lambda(t) = 2.6t^3$, what is the probability that there have been at least 1500 claims by time 7?
[Solution: This is the same as asking for the Distribution at 7 of the 1500th event time.
The distribution of the 1500th event time is a Transformed Gamma Distribution with parameters $\alpha = 1500$, $\tau = 4$, and $\theta = (4/2.6)^{1/4}$.
Therefore, $F(7) = \Gamma[1500; (2.6/4)(7)^4] = \Gamma[1500; 1560.65]$.
One could approximate $\Gamma[1500; 1560.65]$ by using the Normal Approximation.
A Gamma Distribution with $\alpha = 1500$ and θ, has $F(x) = \Gamma[1500; x/\theta]$, with mean of 1500θ and variance $1500\theta^2$.
Thus for a Gamma Distribution with $\alpha = 1500$ and θ, $\Gamma[1500; 1560.65] = F(1560.65\theta) \cong$
$\Phi[(1560.65\theta - 1500\theta)/\sqrt{1500\theta^2}\ ] = \Phi(1.566) = 0.9413$.
Alternately, the number of claims by time 7 is Poisson Distributed with mean:
$m(7)$ = integral from 0 to 7 of $\lambda(t) = (2.6/4)7^4 = 1560.65$.
$\text{Prob}(n \geq 1500) \cong 1 - \Phi[(1499.5 - 1560.65)/\sqrt{1560.65}\ ] = \Phi(1.548) = 0.9392$.
Comment: Well beyond what you expect to asked on your exam!
The exact answer is: $\Gamma[1500; 1560.65] = 0.9399$.]

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[117] Where m(t) is the mean number of claims until time t; m(t) is the integral of the claims intensity $\lambda(t)$ from zero to t.
[118] A Gamma Distribution has $F(x) = \Gamma[n ; x/\theta]$.
For the homogeneous Poisson Process, $F(t) = \Gamma[n ; \lambda t]$, a Gamma Distribution with $\theta = 1/\lambda$.
For the nonhomogeneous Poisson Process, $F(t) = \Gamma[n ; m(t)]$, which is not a Gamma Distribution unless the claims intensity is constant, in which case $m(t) = \lambda t$.
In general, $F(t) = \Gamma[n ; m(t)]$ is a more complicated function of time.
[119] *If one were to redefine the ruler used to measure time so that the claims intensity were constant, then one could directly use the simpler mathematics of the homogeneous Poisson Process.*
See for example, Mathematical Methods of Risk Theory, by Hans Buhlmann.
[120] Using the chain rule and the definition of the Incomplete Gamma Function as an integral.

The mean of a Transformed Gamma Distribution is $\theta\,\Gamma[\alpha+1/\tau]/\Gamma[\alpha]$.
For parameters $\alpha = n$, $\tau = b+1$, *and* $\theta = ((b+1)/c)^{1/(b+1)}$,
the mean is $\{(b+1)/c\}^{1/(b+1)}\ \Gamma[n+1/(b+1)]\ /\ \Gamma[n]$.
Thus for an nonhomogeneous Poisson Process with claims intensity $\lambda(t) = ct^b$, *with* $b > 0$,
the mean n^{th} *event time is:* $\{(b+1)/c\}^{1/(b+1)}\ \Gamma[n+1/(b+1)]/(n-1)!$.

Exercise: Given a nonhomogeneous Poisson Process with $\lambda(t) = 2t$, *what is the mean of the fifth event time? Hint:* $\Gamma(1/2) = \sqrt{\pi}$.
[Solution: Using the above result with $b = 1$, $c = 2$ *and* $n = 5$, $(2/2)^{1/2}\ \Gamma[5+ 1/2]\ /\ 4! =$
$\Gamma[5.5]/ 24 = (4.5)(3.5)(2.5)(1.5)(0.5)\ \Gamma(1/2)\ /\ 24 = 1.2305\ \sqrt{\pi} = 2.181$.
Comment: We had previously calculated that the mean waiting time until the first claim for this nonhomogeneous Poisson Process is $\Gamma(1/2)/2 = \sqrt{\pi}\ /2$. *One can obtain this result by taking* $b =1$, $c = 2$, *and* $n = 1$ *in the above formula, and by noting that* $\Gamma(1.5) = (1/2)\ \Gamma(1/2)$. *]*

We have seen how for a particular form of the claims intensity, the event times have a Transformed Gamma Distribution. This is a generalization of the event times for a homogeneous Poisson, which have a Gamma Distribution. A Transformed Gamma Distribution is obtained from a Gamma Distribution via a power transform. For other forms of the claims intensity, one would get other changes of variables and therefore other, possibly unnamed, generalizations of the Gamma Distribution.

Time Until the Next Event:[121]

Let T(x) = the time from x until the next event.

Since what happened before time x is independent of what happens after time x, we can start a new process at time x.

Therefore, $\Pr[T(x) > t] = \exp[-\{m(x+t) - m(x)\}] = \exp[-\int_x^{x+t} \lambda(y)\,dy]$.

Therefore, $E[T(x)] = \int_{t=x}^{\infty} \exp[-\int_{y=x}^{x+t} \lambda(y)\,dy]\,dt$.

[121] See Example 1.7 in “Poisson Processes” by Daniel.

Problems:

Use the following information for the next six questions:

There is a nonhomogeneous Poisson Process with claims intensity $\lambda(t) = \frac{1}{7 + t}$.

15.1 (2 points) What is the chance that the waiting time until the 1st claim is less than 5?
A. Less than 35%
B. At least 35%, but less than 40%
C. At least 40%, but less than 45%
D. At least 45%, but less than 50%
E. At least 50%

15.2 (3 points) What is the chance that the 4th event time is less than 10?
A. Less than 0.5%
B. At least 0.5%, but less than 1.0%
C. At least 1.0%, but less than 1.5%
D. At least 1.5%, but less than 2.0%
E. At least 2.0%

15.3 (3 points) Let T_4 be the 4th event time.
What is the density of T_4 at time = 10?
A. Less than 0.3%
B. At least 0.3%, but less than 0.4%
C. At least 0.4%, but less than 0.5%
D. At least 0.5%, but less than 0.6%
E. At least 0.6%

15.4 (2 points) What is the mean waiting time until the first claim?
A. Less than 5
B. At least 5, but less than 6
C. At least 6, but less than 7
D. At least 7, but less than 8
E. At least 8

15.5 (2 points) Assume there is a claim at time 2, what is the chance that we will observe the next claim before time 5?
A. 25% B. 27% C. 29% D. 31% E. 33%

15.6 (3 points) What is the probability that the second event occurs between time 3 and 9?
A. 15% B. 17% C. 19% D. 21% E. 23%

Use the following information for the next six questions:
A nonhomogeneous Poisson process has rate function:
$\lambda(t) = 0.1,\ 0 \le t < 5$ $\lambda(t) = 0.2,\ 5 \le t < 10$ $\lambda(t) = 0.3,\ 10 \le t$.

15.7 (1 point) What is the probability that the first event occurs by time 3?
A. 25% B. 30% C. 35% D. 40% E. 45%

15.8 (2 points) What is the probability that the first event occurs between time 3 and time 8?
A. 25% B. 30% C. 35% D. 40% E. 45%

15.9 (2 points) What is the probability that the 2nd event occurs between time 7 and time 12?
A. 25% B. 30% C. 35% D. 40% E. 45%

15.10 (4 points) What is the mean waiting time until the first event?
A. 5.0 B. 5.5 C. 6.0 D. 6.5 E. 7.0

15.11 (2 points) It is time 7. What is the probability that the next event occurs by time 12?
A. 65% B. 70% C. 75% D. 80% E. 85%

15.12 (3 points) It is time 7. What is the mean waiting time until the next event?
A. 4.0 B. 4.5 C. 5.0 D. 5.5 E. 6.0

15.13 (3 points) Each day, traffic passing through the Washington Tunnel increases during the morning and afternoon rush hours, and decreases at other times as follows:
(i) From 12 a.m. to 8 a.m., the numbers of cars follows a Poisson distribution with an increasing hourly rate of $\lambda(t) = 12 + 3.5t$ for $0 \le t \le 8$.
(ii) From 8 a.m. to 12 p.m., the numbers of cars follows a Poisson distribution with a decreasing hourly rate of $\lambda(t) = 60 - 2.5t$ for $8 \le t \le 12$.
(iii) From 12 p.m. to 6 p.m., the numbers of cars follows a Poisson distribution with an increasing hourly rate of $\lambda(t) = -30 + 5t$ for $12 \le t \le 18$.
(iv) From 6 p.m. to 12 a.m., the numbers of cars follows a Poisson distribution with a decreasing hourly rate of $\lambda(t) = 204 - 8t$ for $18 \le t \le 24$.
What is the probability that the time until the 300th car of the day is greater than 10 hours?
A. Less than 65%
B. At least 65%, but less than 70%
C. At least 70%, but less than 75%
D. At least 75%, but less than 80%
E. At least 80%

15.14 (3 points) At 3:45 PM you start waiting for the next bus.
Before 4 PM buses arrive at a Poisson rate of 1 every 30 minutes.
After 4 PM buses arrive at a Poisson rate of 1 every 10 minutes.
What is the mean time you have to wait for the next bus?
A. 16 B. 18 C. 20 D. 22 E. 24

15.15 (6 points) Accidents occur at a Poisson rate on the Silver Inlet Bridge.
From 6 AM to 6 PM the rate is 0.16 per hour.
From 6 PM to 6 AM the rate is 0.08 per hour.
Starting at midnight, calculate your expected time until the next accident.
(A) 8.0 hours (B) 8.5 hours (C) 9.0 hours (D) 9.5 hours (E) 10.0 hours

Use the following information for the next two questions:

- Downloads of a new song from the musical group Jeffster's website follow a nonhomogeneous Poisson process with the following Poisson rate function:

  $\lambda(t) = t$, for $0 \le t \le 20$,
  $\lambda(t) = 20$, for $20 \le t \le 50$,
  $\lambda(t) = 1000 / t$, for $t > 50$.

15.16 (2 points) If the 150^{th} download occurs at time 18.9, determine the probability that the 151^{st} download occurs before time 19.
A. 65% B. 70% C. 75% D. 80% E. 85%

15.17 (3 points) Determine the probability that by time 80 there have been more than 1300 downloads.
A. 14% B. 16% C. 18% D. 20% E. 22%

15.18 (2 points) A Poisson process has the rate function $\lambda(t) = t^{0.6}$.

T_i is the time of the i^{th} event.
Calculate the probability that $T_3 > 4$.
A. Less than 4%
B. At least 4%, but less than 5%
C. At least 5%, but less than 6%
D. At least 6%, but less than 7%
E. At least 7%

15.19 (SOA M, 11/06, Q.10) (2.5 points) You arrive at a subway station at 6:15.
Until 7:00, trains arrive at a Poisson rate of 1 train per 30 minutes.
Starting at 7:00, they arrive at a Poisson rate of 2 trains per 30 minutes.
Calculate your expected waiting time until a train arrives.
(A) 24 minutes (B) 25 minutes (C) 26 minutes (D) 27 minutes (E) 28 minutes

15.20 (MAS-1, 11/18, Q.2) (2.2 points) A Poisson process has the rate function $\lambda(t) = 1.5t$.

T_i is the time of the i^{th} event.

Calculate the probability that $T_2 > 3$.

A. Less than 1%
B. At least 1%, but less than 4%
C. At least 4%, but less than 7%
D. At least 7%, but less than 10%
E. At least 10%

Solutions to Problems:

15.1. C. $m(t) = \int_0^t \lambda(s)\, ds = \int_0^5 \frac{1}{7+s}\, ds = \ln(7+s)\Big]_{s=0}^{s=t} = \ln[(t+7)/7]$.

The number of claims from time 0 to time 5 is Poisson Distributed with mean $m(5) = \ln(12/7) = 0.5390$. Thus the chance of observing zero claims by time 5 is $e^{-0.5390}$.
The chance that the waiting time until the 1st claim is less than 5 is: $1 - e^{-0.5390} =$ **0.417**.

15.2. C. The number of claims from time 0 to time 10 is Poisson Distributed with mean $m(10) = \ln(17/7) = 0.8873$. Thus the chance of observing 4 or more claims is:
$1 - e^{-0.8873}\{1 + 0.8873 + 0.8873^2/ 2 + 0.8873^3/ 6\} = 1 - (0.41177)(2.3974) =$ **1.28%**.
Comment: The 4th event time is less than 10 if and only if there are 4 or more claims by time 10. One can also obtain the answer in terms of an incomplete Gamma Function,
$\Gamma[4; 0.8873] = 0.0128$.

15.3. A. T_4 has a distribution $\Gamma[4; m(t)]$. Therefore, T_4 has a density of: $\lambda(t)\, m(t)^3 e^{-m(t)} / 3!$.
$\lambda(10) = 1/17$. $m(10) = \ln(17/7) = 0.8873$.
$\lambda(10)\, m(10)^3 e^{-m(10)} / 3! = (1/17)(0.8873^3)\, e^{-0.8873} / 6 =$ **0.00282**.
Comment: $\frac{d\,\Gamma[4\,;\,m(t)]}{dt} = m'(t)\,\frac{d\,\Gamma[4\,;\,x]}{dx} = \lambda(t)\,\{x^3 e^{-x} / \Gamma[4]\} = \lambda(t)\, m(t)^3 e^{-m(t)} / 3!$.

15.4. E. $m(t) = \ln[(t+7)/7]$. Thus the survival function of the waiting time until the first claim is: $e^{-m(t)} = 7/(t+7)$. Since we can compute the mean by integrating the Survival function from 0 to ∞, the mean waiting time until the first claim is:

$$\int_0^\infty e^{-m(t)}\, dt = \int_0^\infty \frac{7}{t+7}\, dt = 7\ln(7+t)\Big]_{t=0}^{t=\infty} = \mathbf{\infty}.$$

Comment: The claims intensity decreases sufficiently quickly that the mean waiting time until the first claim is not finite.

15.5. A. The number of events from time 2 to time 5 is Poisson Distributed with mean:
$m(5) - m(2) = \ln(12/7) - \ln(9/7) = \ln(4/3) = 0.2877$.
Thus the chance of observing no events from time 2 to 5 is: $e^{-0.2877} = 0.7500$.
Thus the chance we will observe at least one event is: $1 - 0.7500 =$ **25.00%**.
Comment: The answer would have been the same if we had been asked instead: given we start observing at time equal 2, what is the chance that we will observe the next event before time 5?
The probability that we will observe exactly one claim from time 2 to 5 is:
$0.2877\, e^{-0.2877} = 21.6\%$.

15.6. A. m(t) = ln[(t+7)/7].
The number of events by time 3 is Poisson Distributed with mean: m(3) = ln(10/7) = 0.3567.
The number of events from time 3 to time 9 is Poisson Distributed with mean:
m(9) - m(3) = ln(16/7) - ln(10/7) = ln(1.6) = 0.4700.
The second event occurs between time 3 and 9 if:
There are no events between time 0 and 3, and at least two between time 3 and 9,
or there is one event between time 0 and 3, and at least one between time 3 and 9.
Therefore, the desired probability is:
$e^{-0.3567}$ (1 - $e^{-0.47}$ - 0.47 $e^{-0.47}$) + 0.3567 $e^{-0.3567}$ (1 - $e^{-0.47}$) = **15.05%**.

15.7. A. m(3) = (3)(0.1) = 0.3. Prob[at least one event by time 3] = 1 - $e^{-0.3}$ = **0.259**.

15.8. D. m(3) = (3)(0.1) = 0.3. m(8) = (5)(0.1) + (3)(0.2) = 1.1.
Prob[1st event between 3 and 8] =
Prob[at least one event by time 8] - Prob[at least one event by time 3] =
(1 - $e^{-1.1}$) - (1 - $e^{-0.3}$) = **0.408**.
Alternately, Prob[no events by time 3] Prob[at least one event from time 3 to 8] =
$e^{-0.3}$(1 - $e^{-0.8}$) = $e^{-0.3}$ - $e^{-1.1}$ = **0.408**.

15.9. D. m(7) = (5)(0.1) + (2)(0.2) = 0.9. m(12) = (5)(0.1) + (5)(0.2) + (2)(0.3) = 2.1.
Prob[2nd event between 7 and 12] =
Prob[2nd event by time 12] - Prob[2nd event by time 7] =
Prob[at least two events by time 12] - Prob[at least two events by time 7] =
(1 - $e^{-2.1}$ - 2.1$e^{-2.1}$) - (1 - $e^{-0.9}$ - 0.9$e^{-0.9}$) = 1.9$e^{-0.9}$ - 3.1$e^{-2.1}$ = **0.393**.

15.10. D. $m(t) = 0.1t$, $t < 5$. Therefore $m(5) = 0.5$. $m(t) = 0.5 + (0.2)(t-5) = 0.2t - 0.5$, $5 \le t < 10$.
Therefore, $m(10) = 1.5$. $m(t) = 1.5 + (0.3)(t-10) = 0.3t - 1.5$, $10 \le t$.
Prob[no event by time t] = $e^{-m(t)}$ = Survival Function of the waiting time until first event.
mean waiting time until the first event = $\int S(t)dt = \int e^{-m(t)} dt =$

$$\int_0^5 e^{-0.1t} dt + \int_5^{10} e^{0.5-0.2t} dt + \int_{10}^{\infty} e^{1.5-0.3t} dt =$$

$$-10e^{-0.1t}\Big]_{t=0}^{t=5} - 5e^{0.5-0.2t}\Big]_{t=5}^{t=10} - e^{1.5-0.3t}/0.3\Big]_{t=10}^{t=\infty} =$$

$(10)(1 - e^{-0.5}) + (5)(e^{-0.5} - e^{-1.5}) + e^{-1.5}/0.3 = 3.935 + 1.917 + 0.744 =$ **6.596**.
Alternately, the density of the waiting time for the first event is: $\lambda(t)e^{-m(t)}$.
mean waiting time until the first event = $\int t\, \lambda(t)\, e^{-m(t)} dt =$

$$\int_0^5 t\, 0.1e^{-0.1t} dt + \int_5^{10} t\, 0.2e^{0.5-0.2t} dt + \int_{10}^{\infty} t\, 0.3e^{1.5-0.3t} dt =$$

$$-te^{-0.1t} - 10e^{-0.1t}\Big]_{t=0}^{t=5} - e^{0.5}(te^{-0.2t} + 5e^{-0.2t})\Big]_{t=5}^{t=10} - e^{1.5}(te^{-0.3t} + e^{-0.3t}/0.3)\Big]_{t=10}^{t=\infty}$$

= 0.902 + 2.718 + 2.975 = **6.595**.
Comment: Although it is more work, one can proceed in a similar manner to get the mean n^{th} event time. For example, the 2nd event time has survival function: $e^{-m(t)} + m(t)e^{-m(t)}$.
The mean of the 1st, 2nd, ... , 10th event times are:
6.59546, 11.1168, 14.8406, 18.2921, 21.6573, 24.9981, 28.333, 31.6666, 35, 38.3333.

15.11. B. The number of events from time 7 to time 12 is Poisson with mean:
(3)(0.2) + (2)(0.3) = 1.2.
Prob[next event by time 12] = $1 - e^{-1.2}$ = **69.9%**.

15.12. A. Start a new process at time 7. Let $s = t - 7$.
$\lambda(s) = 0.2$, $0 \le s < 3$ $\lambda(s) = 0.3$, $3 \le s$.
$m(s) = 0.2s$, $s < 3$. Therefore $m(3) = 0.6$. $m(s) = 0.6 + (0.3)(s - 3) = 0.3s - 0.3$, $3 \le s$.
Prob[no event by time s] = $e^{-m(s)}$ = Survival Function of the waiting time until first event.

mean waiting time until the first event = $\int S(s)\, ds = \int e^{-m(s)} ds = \int_0^3 e^{-0.2s} ds + \int_3^{\infty} e^{0.3-0.3s} ds =$

$$-5e^{-0.2s}\Big]_{s=0}^{s=3} - e^{0.3-0.3s}/0.3\Big]_{s=3}^{s=\infty} = (5)(1 - e^{-0.6}) + e^{-0.6}/0.3 = 2.256 + 1.829 = \mathbf{4.085}.$$

15.13. E. The average number of cars in the first 10 hours is:

$$\int_0^8 12 + 3.5t \ dt + \int_8^{10} 60 - 2.5t \ dt = 208 + 75 = 283.$$

Thus the number of cars in the first 10 hours is Poisson with mean 283.

Prob[fewer than 300 cars] $\cong \Phi[(299.5 - 283)/\sqrt{283}\,] = \Phi(0.98)$ = **83.65%**.

Comment: The setup of this question was taken from CAS3, 5/04, Q.27.

15.14. B. $m(t) = t/30$, $t \le 15$, $m(t) = 1/2 + (t-15)/10 = t/10 - 1$, $t \ge 15$.

$S(t) = \exp[-m(t)] = e^{-t/30}$, for $t \le 15$, and $e\, e^{-t/10}$, for $t \ge 15$.

$$\text{mean waiting time} = \int_0^\infty S(t) \ dt = \int_0^{15} e^{-t/30} \ dt + e \int_{15}^\infty e^{-t/10} \ dt = 30(1 - e^{-1/2}) + 10e^{-1/2} =$$

$30 - 20e^{-1/2}$ = **17.87**.

Alternately, the probability of no bus by t = 15 is: $e^{-15/30} = e^{-1/2}$.

If there is no bus by t = 15, then the average additional wait for a bus is 10 minutes.

If there is a bus by t = 15, then the average wait for that bus was:

$$\frac{\int_0^{15} t\, e^{-t/30}/30 \ dt}{\int_0^{15} e^{-t/30}/30 \ dt} = -te^{-t/30} - 30e^{-t/30} \Big]_{t=0}^{t=15} \ / (1 - e^{-1/2}) = (30 - 45e^{-1/2}) / (1 - e^{-1/2}).$$

mean waiting time = $(1 - e^{-1/2})(30 - 45e^{-1/2})/(1 - e^{-1/2}) + e^{-1/2}(15 + 10) = 30 - 20e^{-1/2}$ = **17.87**.

Comment: This is also the complete expectation of life for a constant force of mortality of 30 for $t \le 15$ and a constant force of mortality of 10 for $t > 15$.

15.15. C. Let t = 0 be midnight and measure time in hours.

$m(t) = 0.08t$, $0 \le t \le 6$. $m(t) = 0.16t - 0.48$, $6 \le t \le 18$.
$m(t) = 0.08t + 0.96$, $18 \le t \le 30$. $m(t) = 0.16t - 1.44$, $30 \le t \le 42$.
$m(t) = 0.08t + 1.92$, $42 \le t \le 54$. $m(t) = 0.16t - 2.40$, $54 \le t \le 66$.
$S(t) = \text{Exp}[-m(t)]$.

The mean time until the next accident is the integral of the survival function:

$$\int_0^6 \text{Exp}[-0.08t]\ dt + \int_6^{18} \text{Exp}[-(0.16t - 0.48)]\ dt + \int_{18}^{30} \text{Exp}[-(0.08t + 0.96)]\ dt + \int_{30}^{42} \text{Exp}[-(0.16t - 1.44)]\ dt$$

$$+ \int_{42}^{54} \text{Exp}[-(0.08t + 1.92)]\ dt + \int_{54}^{66} \text{Exp}[-(0.16t - 2.40)]\ dt + \ldots =$$

4.76521 + 3.30041 + 0.69978 + 0.18527 + 0.03928 + 0.01040 + ... = **9.0 hours**.

Comment: Similar to SOA M, 11/06, Q.10.
However, since there is a significant probability that the waiting time is more than one day, this question is more difficult.

15.16. E. The mean number of downloads from time 18.9 to 19 is:

$$\int_{18.9}^{19} \lambda(t)\ dt = \int_{18.9}^{19} t\ dt = 19^2/2 - 18.9^2/2 = 1.895.$$

Thus the chance of at least one event from time 18.9 to 19 is: $1 - e^{-1.895}$ = **0.850**.

15.17. D. The mean number of downloads by time 80 is:

$$\int_0^{80} \lambda(t)\ dt = \int_0^{20} t\ dt + \int_{20}^{50} 20\ dt + \int_{50}^{80} 1000/t\ dt = 20^2/2 + (30)(20) + 1000 \ln(80/50) = 1270.0.$$

Thus the number of downloads by time 80 is Poisson with mean 1270.
Using the continuity correction the probability of more than 1300 downloads is approximately:
$1 - \Phi[(1300.5 - 1270)/\sqrt{1270}\,] = 1 - \Phi[0.856]$ = **19.6%**.

15.18. E. $T_3 > 4$. ⇔ 3rd event after time 4. ⇔ 0, 1, or 2 events in [0, 4].
Number of events by time 4 is N(4).

N(4) is Poisson with mean: $\int_0^4 \lambda(t)\,dt = \int_0^4 t^{0.6}\,dt = t^{1.6}/1.6\Big]_{t=0}^{t=4} = (4^{1.6})/1.6 = 5.7435.$

$\text{Prob}[T_3 > 4] = f(0) + f(1) + f(2) = e^{-5.7435} + 5.7435\, e^{-5.7435} + 5.7435^2\, e^{-5.7435}/2 =$ **7.44%**.
Comment: Similar to MAS-1, 11/18, Q.2.

15.19. D. $\lambda(t) = 1/30$, $t \le 45$. $\lambda(t) = 1/15$, $t > 45$.
$m(t) = t/30$, $t \le 45$. $m(45) = 45/30 = 1.5$.

For $t > 45$, $m(t) = \int_0^{45} \lambda(s)ds + \int_{45}^{t} \lambda(s)ds = 1.5 + (t - 45)/15 = t/15 - 1.5.$

$S(t)$ = Prob[no train by time t] = $\exp[-m(t)]$.

$$\text{mean wait} = \int_0^\infty S(t)dt = \int_0^{45} S(t)dt + \int_{45}^\infty S(t)dt = \int_0^{45} \exp[-t/30]\,dt + \int_{45}^\infty \exp[1.5 - t/15]\,dt =$$

$30(1 - e^{-1.5}) + 15e^{1.5}e^{-3} = 30 - 15e^{-1.5} =$ **26.65 minutes**.
Alternately, the probability that a train fails to arrive by time 45 is: $e^{-45/30} = e^{-1.5}$.
If a train fails to arrive by time 45, then we can start a new Poisson Process with $\lambda = 1/15$, and the average additional wait is 15 minutes.
Thus the mean wait conditional on the train not arriving by time 45 is: 45 + 15 = 60 minutes.

If a train arrives by time 45, then the average wait is: $\dfrac{\int_0^{45} t\, f(t)\,dt}{F(45)} = \dfrac{\int_0^{45} t \exp[-t/30]/30\,dt}{F(45)} =$

$-t\, e^{-t/30} - 30\, e^{-t/30}\Big]_{t=0}^{t=45} / F(45) = (30 - 75\, e^{-1.5}) / F(45) = 13.265 / F(45).$

mean wait =
(mean wait if train arrives by time 45) F(45) + (mean wait if train arrives after time 45) S(45) =
$\{13.265 / F(45)\}\, F(45) + (45 + 15)\, e^{-1.5} =$ **26.65 minutes**.
Comment: I have ignored the vanishingly small probability that the first train does not arrive within 24 hours.
For example, the mean number of trains within 3 hours is: 45 / 30 + 135 / 15 = 10.5.
Probability of no trains within 3 hours is: $e^{-10.5} = 0.003\%$.

15.20. A. $T_2 > 3$. ⇔ 2nd event after time 3. ⇔ 0 or 1 event in [0, 3].

Number of events by time 3 is N(3).

N(3) is Poisson with mean: $\int_0^3 \lambda(t)\ dt = \int_0^3 1.5t\ dt = 0.75t^2\Big]_{t=0}^{t=3} = (0.75)(3^2) = 6.75$.

Prob[$T_2 > 3$] = f(0) + f(1) = $e^{-6.75} + 6.75\ e^{-6.75}$ = **0.907%**.

Section 16, Thinning & Adding Nonhomogeneous Poisson Processes

As discussed previously for homogeneous processes, one can add and thin nonhomogeneous processes in a similar manner.

Adding Poisson Processes:

Assume one has two independent nonhomogeneous Poisson Process on (0,5). Let the first have claims intensity $\lambda_1(t) = t$. Let the second have $\lambda_2(t) = 1 + \sin(2\pi t)$. Then if one adds these two processes, one gets another nonhomogeneous Poisson Process on (0,5), but with claims intensity the sum of the two individual claims intensities: $\lambda(t) = t + 1 + \sin(2\pi t)$.

In general, **if one adds independent nonhomogeneous Poisson Processes,** on the same time interval, **then one gets a new nonhomogeneous Poisson Process, with the sum of the individual claim intensities.**

Exercise: There are three independent nonhomogeneous Poisson Processes
on the interval (0,25).
The first has a claims intensity of $6t^3$.
The second has a claims intensity of $10t^3$.
The third has a claims intensity of $4t^3$.
What is the process of the sum of these three processes?
[Solution: It is a nonhomogeneous Poisson Process on the interval (0,25) with claims intensity:
$6t^3 + 10t^3 + 4t^3 = 20t^3$.
Comment: See below where this sum is divided into these three pieces via thinning.]

Thinning a Poisson Process:

If we select at random a fraction of the claims from a nonhomogeneous Poisson Process, we get a new nonhomogeneous Poisson Process, with smaller claims intensity. This is called **thinning a nonhomogeneous Poisson Process**.

For example, assume we have a nonhomogeneous Poisson Process on (0,5) with $\lambda(t) = 20t^3$.
If one selects at random 1/4 of the claims from this first nonhomogeneous Poisson Process, then one gets a new nonhomogeneous Poisson Process, with $\lambda = 20t^3/4 = 5t^3$.[122]
The remaining 3/4 of the original claims are also a nonhomogeneous Poisson Process, but with $\lambda = 15t^3$. These two nonhomogeneous Poisson Processes are independent.

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[122] The key thing is that the selection process can not depend in any way on the frequency process. For example, if severity is independent of frequency, we may select only the large losses.

Exercise: Assume claims are given by a nonhomogeneous Poisson Process with claims intensity $\lambda(t) = 20t^3$. Assume frequency and severity are independent. 30% of claims of are size less than $10,000, 50% of claims are of size between $10,000 and $25,000, and 20% of the claims are of size greater than $25,000. What are the frequency processes for the claims of different sizes?
[Solution: There are three independent nonhomogeneous Poisson Processes.
Claims of a size less than $10,000, have a claims intensity of $6t^3$.
Claims of size between $10,000 and $25,000 have a claims intensity of $10t^3$.
Claims of size greater than $25,000 have a claims intensity of $4t^3$.]

This is the same result as for a homogeneous Poisson Process. In addition, in the case of a nonhomogeneous Poisson Process, the portion of claims selected can depend on time. For example, assume we have a nonhomogeneous Poisson Process on (0, 5) with $\lambda(t) = 20t^3$.
If a claim occurs at time t, select it at random with probability 5/(5+t). The selected claims follow a nonhomogeneous Poisson Process on (0, 5) with $\lambda(t) = \{5/(5+t)\}\ 20t^3 = 100t^3/(5+t)$.

If claims are from a nonhomogeneous Poisson Process, and one divides these claims into subsets in a manner independent of the frequency process, then the claims in each subset are <u>independent</u> nonhomogeneous Poisson Processes.

If the original nonhomogeneous Poisson Process has claims intensity $\lambda(t)$ and we select at random a claim that occurs at time t with probability p(t), $0 \le p(t) \le 1$, then the selected claims are a nonhomogeneous Poisson Process has claims intensity $\lambda(t)p(t)$. For example,
Prob[large claim at time t] = Prob[claim at time t] Prob[claim is large I there is a claim at time t].

Assume we have a nonhomogeneous Poisson Process with $\lambda(t)$ on (0, T). Assume that $\lambda(t) \le c$ for all t in (0, T). Then if we randomly select claims from a homogeneous Poisson Process with claims intensity c, with a chance of selection $\lambda(t)/c$,[123] we obtain a nonhomogeneous Poisson Process with $\lambda(t)$ on (0,T). Thus, we can thin a homogeneous Poisson Process in order to get a nonhomogeneous Poisson Process.[124]

Exercise: Assume we have a homogeneous Poisson Process with claims intensity 2500 on (0,5).
Assume we select at random claims from this process, with chance of selection: $t^3/125$.
What is the process of the selected claims?
[Solution: The selected claims follow a nonhomogeneous Poisson Process on (0, 5), with claims intensity $(2500)\ (t^3/125) = 20t^3$.
<u>Comment</u>: Note that on (0, 5), $0 \le t^3/125 \le 1$.]

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[123] Note the chance of selection depends on time.
[124] *This could be used to simulate nonhomogeneous Poisson Processes.*

Problems:

16.1 (2 points) For a claims process, you are given:
(i) The number of claims $\{N(t), t \geq 0\}$ is a nonhomogeneous Poisson process with intensity function:
$\lambda(t) = 1, 0 \leq t < 3$ $\lambda(t) = 8, 3 \leq t < 7$ $\lambda(t) = 5, 7 \leq t$.
(ii) Claims amounts Y_i are independently and identically distributed random variables that are also independent of N(t).
(iii) Each Y_i has a Weibull Distribution, $F(x) = 1 - \exp[-(x/\theta)^\tau]$, $x > 0$, with $\tau = 2$ and $\theta = 100$.
(iv) The random variable P is the number of claims with claim amount less than 50 by time t = 10.
(v) The random variable Q is the number of claims with claim amount greater than 50 by time t = 10.
(vi) R is the conditional expected value of Q, given P = 20.
Calculate R.
(A) 30 (B) 33 (C) 36 (D) 39 (E) 42

16.2 (2 points) You are given the following information about Lucky Tom:
- He finds coins on his 60 minute walk to work at a Poisson rate of 2 per minute. 60% of these coins are worth 1 each; 20% are worth 5 each; 20% are worth 10 each. The denominations of the coins found are independent.
- He also finds coins on his 60 minute walk back home at a Poisson rate of 1 per minute. 40% of these coins are worth 1 each; 30% are worth 5 each; 30% are worth 10 each. The denominations of the coins found are independent.

Calculate the conditional expected value of the coins Tom found during his two walks today, given that among the coins he found exactly 57 were worth 5 each.
A. Less than 790
B. At least 790, but less than 800
C. At least 800, but less than 810
D. At least 810, but less than 820
E. At least 830

16.3 (3 points) Customers arrive via a nonhomogeneous Poisson Process with intensity $\lambda(t) = 12 + 8\cos(2\pi t)$, for $0 \leq t \leq 1$.
The portion of customers that are expected to be female at time t is: $\{2 + \sin(2\pi t)\} / 4$.
What is the chance of having exactly 3 female customers arrive between t = 0.25 and t = 0.75?
(A) 13% (B) 15% (C) 17% (D) 19% (E) 21%

16.4 (4 points) For a claims process, you are given:
(i) The number of claims {N(t), t ≥ 0} is a nonhomogeneous Poisson process with intensity function:
$\lambda(t) = 50$, $0 \le t < 2$ $\lambda(t) = 100$, $2 \le t < 4$ $\lambda(t) = 150$, $4 \le t$.
(ii) Claims amounts are independently distributed random variables that are also independent of N(t) .
(iii) Claims amounts are exponentially distributed with mean 1000 / (1 - 0.05t), t ≤ 10.
(iv) The random variable P is the number of claims with claim amount less than 2000 by time t = 6.
(v) The random variable Q is the number of claims with claim amount greater than 2000 by time t = 6.
What is the conditional expected value of P, given Q = 50?
A. 450 B. 460 C. 470 D. 480 E. 490

16.5 (5 points) Each day, traffic passing through the Washington Tunnel increases during the morning and afternoon rush hours, and decreases at other times as follows:
(i) From 12 a.m. to 8 a.m., the numbers of cars follows a Poisson distribution with an increasing hourly rate of $\lambda(t) = 12 + 3.5t$ for $0 \le t \le 8$.
(ii) From 8 a.m. to 12 p.m., the numbers of cars follows a Poisson distribution with a decreasing hourly rate of $\lambda(t) = 60 - 2.5t$ for $8 \le t \le 12$.
(iii) From 12 p.m. to 6 p.m., the numbers of cars follows a Poisson distribution with an increasing hourly rate of $\lambda(t) = -30 + 5t$ for $1\ 2 \le t \le 18$.
(iv) From 6 p.m. to 12 a.m., the numbers of cars follows a Poisson distribution with a decreasing hourly rate of $\lambda(t) = 204 - 8t$ for $18 \le t \le 24$.
From 12 a.m. to 10 a.m. one quarter of the cars are driven by females.
From 10 a.m. to 4 p.m. three quarters of the cars are driven by females.
From 4 p.m. to 12 a.m. one quarter of the cars are driven by females.
What is the probability that at least 300 cars driven by females will pass through the tunnel tomorrow?
A. Less than 75%
B. At least 75%, but less than 80%
C. At least 80%, but less than 85%
D. At least 85%, but less than 90%
E. At least 90%

16.6 *(3 points) Use the following information:*
- *The Concentrated Insurance Company writes life insurance.*
- *The Concentrated Insurance Company only insures people born on January 1, 1940.*
- $q_{70} = 3\%$.
- *Interage deaths follow the hyperbolic assumption.*
  Therefore, ${}_{1-s}q_{x+s} = (1 - s)\, q_x,\ 0 \le s \le 1$.
- *During the year 2010, Concentrated Insurance Company issues new policies according to a Poisson Process with* $\lambda(t) = 500 + 500\sqrt{t}$, $0 \le t \le 1$.

Determine the probability that Concentrated Insurance Company has exactly 9 death claims during 2010 on the new policies it issued during the year 2010.

A. 8% B. 9% C. 10% D. 11% E. 12%

16.7 (3 points) Job offers for a college graduate arrive according to a Poisson process with mean 1.4 per month.
A job offer at time t (in months) is acceptable if the wages are at least 40,000 - 2000t.
Wages offered are mutually independent and follow a Single Parameter Pareto Distribution with $\theta = 20{,}000$ and $\alpha = 4$, $F(x) = 1 - (\theta/x)^{\alpha}$, $x > \theta$.
Calculate the probability that it will take a college graduate more than 5 months to receive an acceptable job offer.

(A) 0.27 (B) 0.39 (C) 0.45 (D) 0.58 (E) 0.61

Use the following information for the next two questions:
- A store is open for 10 hours; $0 \le t \le 10$.
- Customers enter a store at a Poisson rate per minute of:
  $\lambda(t) = 0.3t,\ 0 \le t \le 3$.
  $\lambda(t) = 1.2 - 0.1t,\ 3 \le t \le 10$.
- For $t \le 7$, 30% of the customers are men and 70% are women.
- For $t > 7$, 70% of the customers are men and 30% are women.

16.8 (2 points) On an average day, what portion of the customers are men?

A. 36% B. 38% C. 40% D. 42% E. 44%

16.9 (3 points) Using the Normal Approximation, estimate the probability that more than 175 women customers enter the store tomorrow.

A. 82% B. 84% C. 86% D. 88% E. 90%

16.10 (2 points) The number of accidents on a highway follows a Poisson process, with the rate of accidents varying during the day, as shown below.

Time Interval	Accidents per hour
12 midnight - 7 am	0.05
7 am - 10 am	0.10
10 am - 4 pm	0.08
4 pm - 7 pm	0.07
7 pm - 12 midnight	0.02

40% of accidents involve a single vehicle.
Calculate the probability that tomorrow two accidents involving a single vehicle occur and no accidents involving multiple vehicles occur.
A. 3% B. 4% C. 5% D. 6% E. 7%

16.11 (2 points) Use the following information:
- Losses due to accident are independent of losses due to illness.
- Losses due to accident follow a nonhomogeneous Poisson Process with:
  $\lambda(t) = 3\%$ for $0 \le t < 5$, $\lambda(t) = 5\%$ for $5 \le t < 9$, $\lambda(t) = 2\%$ for $9 \le t \le 12$.
- Losses due to illness follow a nonhomogeneous Poisson Process with:
  $\lambda(t) = 6\%$ for $0 \le t < 3$, $\lambda(t) = 1\%$ for $3 \le t < 10$, $\lambda(t) = 4\%$ for $10 \le t \le 12$.

What is the chance that one observes at least two losses by time 10?
A. 11% B. 12% C. 13% D. 14% E. 15%

16.12 (4 points) Use the following information:
- Employees at the Primatech Paper Company sometimes suffer work related injuries and are temporarily disabled.
- With time measured in months, during 2011, worker's disabilities occur via a Poisson Process with rate $\lambda(t) = 0.7 + 0.1t$, $0 \le t \le 12$.
- 80% of the time, the period a worker remains disabled follows an Exponential Distribution with mean 2 months.
- 20% of the time, the period a worker remains disabled follows an Exponential Distribution with mean 10 months.

Determine the probability that as of March 1, 2012 there are at least four workers whose injuries occurred during 2011 and who are still disabled.
A. 19% B. 21% C. 23% D. 25% E. 27%

16.13 (3 points)
You are given the following information for an insurance policy covering two types of claims:
● Total number of claims is given by a Poisson process with claims intensity $\lambda(t) = 0.4t$, $t > 0$.
● At any time the probability of a claim being from Claim Type A is 0.7 and from Claim Type B is 0.3.
● Frequency and severity of claims are independent.
● For Type A, severities follow a Weibull Distribution with $\theta = 2000$ and $\tau = 2$.
● For Type B, severities follow a Weibull Distribution with $\theta = 1000$ and $\tau = 3$.
Calculate the probability that by time 4 there will be exactly one claim of size between 500 and 1000.
A. Less than 0.40
B. At least 0.40, but less than 0.45
C. At least 0.45, but less than 0.50
D. At least 0.50, but less than 0.55
E. At least 0.55

16.14 (3, 5/01, Q.37) (2.5 points) For a claims process, you are given:
(i) The number of claims $\{N(t), t \geq 0\}$ is a nonhomogeneous Poisson process with intensity function:

$\lambda(t) = 1, 0 \leq t < 1$ $\quad$ $\lambda(t) = 2, 1 \leq t < 2$ $\quad$ $\lambda(t) = 3, 2 \leq t$.

(ii) Claims amounts Y_i are independently and identically distributed random variables that are also independent of N(t).
(iii) Each Y_i is uniformly distributed on [200, 800].
(iv) The random variable P is the number of claims with claim amount less than 500 by time t = 3.
(v) The random variable Q is the number of claims with claim amount greater than 500 by time t = 3.
(vi) R is the conditional expected value of P, given Q = 4.
Calculate R.
(A) 2.0 (B) 2.5 (C) 3.0 (D) 3.5 (E) 4.0

16.15 (CAS3, 5/05, Q.13) (2.5 points) During the hurricane season (August, September, October, and November), hurricanes hit the US coast with a monthly Poisson rate of 1.25 and each hurricane during the period has a 20% chance of being "major." Outside of hurricane season (the other months), hurricanes hit at a Poisson rate of 0.25 per month, and each such hurricane has only a 10% chance of being "major."
Determine the probability that a hurricane selected at random is "major."
A. Less than 14%
B. At least 14%, but less than 15%
C. At least 15%, but less than 16%
D. At least 16%, but less than 17%
E. 17% or more

16.16 (CAS ST, 5/14, Q.1) (2.5 points)
You are given the following information for a policy covering two types of claims:
- Total number of claims is given by a Poisson process with claims intensity $\lambda(t) = 10t$, $t > 0$.
- At any time the probability of a claim being from Claim Type A is 0.6
  and from Claim Type B is 0.4.
- Frequency and severity of claims are independent.
- Claim severities follow the distributions given in the tables below

Claim Type A	
Claim Amount	Probability
< 500	0.3
At least 500, but less than 1000	0.5
At least 1000	0.2

Claim Type B	
Claim Amount	Probability
< 1000	0.1
At least 1000, but less than 2000	0.6
At least 2000	0.3

Calculate the probability that by time 0.5 there will be fewer than two claims with severity at least equal to 1000.

A. Less than 0.55
B. At least 0.55, but less than 0.65
C. At least 0.65, but less than 0.75
D. At least 0.75, but less than 0.85
E. At least 0.85

16.17 (CAS ST, 11/14, Q.2) (2.5 points) You are given:
- The number of tickets sold to a show follows a Poisson process.
- Tickets are sold at a rate of (800 - 45t) per day, where t is the number of days.
- Tickets are on sale for 7 days.
- 90% of the tickets sold will actually be used.

Calculate the expected number of tickets used.
A. Fewer than 3,800
B. At least 3,800, but fewer than 4,100
C. At least 4,100, but fewer than 4,400
D. At least 4,400 but fewer than 4,700
E. At least 4,700

16.18 (2 points) In the previous question, CAS ST, 11/14, Q.2, each ticket has a 90% chance of being used independent of any other ticket. Determine the probability that there are at least 4100 used tickets and at most 420 unused tickets.

16.19 (CAS ST, 5/15, Q.1) (2.5 points) You are given:
- The number of messages sent follows a Poisson process with rate $\lambda(t) = t/4$, $t \geq 0$.
- Each message sent is received with probability 0.80.

Calculate the probability that at least 2 messages will be received by time t = 4.
A. Less than 0.35
B. At least 0.35, but less than 0.45
C. At least 0.45, but less than 0.55
D. At least 0.55, but less than 0.65
E. At least 0.65

Solutions to Problems:

16.1. D. For the whole nonhomogeneous Poisson process, the mean number of claims by t = 10 is: m(10) = (3)(1) + (4)(8) + (3)(5) = 50.
The claims greater than and less than 50 are independent nonhomogeneous Poisson processes. Thus the expected value of Q is independent of P.
For this Weibull Distribution, $S(50) = \exp(-(50/100)^2) = 0.7788$.
Therefore, the mean number of large claims by time 10 is: (0.7788)(50) = **38.9.**
Comment: Similar to 3, 5/01, Q.37.

16.2. C. The finding of different denominations of coins are independent nonhomogeneous Poisson processes.
He expects to find (60)(2)(0.6) = 72 pennies on his way to work, plus (60)(1)(0.4) = 24 pennies on his way home, for a total of 96 pennies.
He expects to find: (60)(2)(0.2) + (60)(1)(0.3) = 42 dimes.
He found 57 nickels.
The conditional expected value of the coins he found is:
(96)(1) + (57)(5) + (42)(10) = **801**.

16.3. **B.** The female customers follow a nonhomogeneous Poisson Process with intensity: (intensity for all customers)(portion of customers that are female).
$\lambda(t) = (12 + 8\cos(2\pi t))(2 + \sin(2\pi t))/4 = 6 + 4\cos(2\pi t) + 3\sin(2\pi t) + 2\cos(2\pi t)\sin(2\pi t)$.

Integrating $\lambda(t)$ from 0 to t: $m(t) = \int_0^t \{6 + 4\cos(2\pi s) + 3\sin(2\pi s) + 2\cos(2\pi s)\sin(2\pi s)\}\, ds =$

$6s + 2\sin(2\pi s)/\pi - 3\cos(2\pi s)/(2\pi) + \sin^2(2\pi s)/(2\pi)\Big]_{s=0}^{s=t} =$

$6t + 2\sin(2\pi t)/\pi - 3\cos(2\pi t)/(2\pi) + \sin^2(2\pi t)/(2\pi) - \{0 + 0 - 3/(2\pi) + 0\} =$
$6t + 2\sin(2\pi t)/\pi - 3\cos(2\pi t)/(2\pi) + \sin^2(2\pi t)/(2\pi) + 1.5/\pi$.
$m(0.25) = 1.5 + 2\sin(\pi/2)/\pi - 3\cos(\pi/2)/(2\pi) + \sin^2(\pi/2)/(2\pi) + 1.5/\pi$
$= 1.5 + 2/\pi + 0 + 1/(2\pi) + 1.5/\pi = 2.773$.
$m(0.75) = 4.5 + 2\sin(3\pi/2)/\pi - 3\cos(3\pi/2)/(2\pi) + \sin^2(3\pi/2)/(2\pi) + 1.5/\pi$
$= 4.5 - 2/\pi + 0 + 1/(2\pi) + 1.5/\pi = 4.5$.
The expected number of female customers arriving between 0.25 and 0.75 is:
m(0.75) - m(0.25) = 4.5 - 2.773 = 1.727. Thus the number of female customers arriving between 0.25 and 0.75 follows a Poisson Distribution, with mean = 1.727. Therefore, the chance of 3 female customers arriving is: $e^{-1.727}\, 1.727^3 / 3! =$ **15.3%**.

16.4. D. We are thinning a nonhomogeneous Poisson process.
P and Q are independent nonhomogeneous Poisson processes.
Thus the conditional expected value of P, given a certain value for Q, is just the unconditional mean of P. For the original process,
Prob[claim < 2000 | claim occurs at time t] = 1 - exp[-2000/(1000/(1 - 0.05t))] =
1 - exp[-2(1 - 0.05t)]. Expected number of claims from time 0 to 2 that are less than 2000:

$$\int_0^2 \lambda(t) \text{ Prob[claim < 2000 | claim occurs at time t] } dt = 50 \int_0^2 1 - \exp[-2(1 - 0.05t)] \ dt =$$

$$50(t - \exp[-2(1 - 0.05t)]/0.1) \Big]_{t=0}^{t=2} = (50)\{(2 - 1.653) - (0 - 1.353)\} = 85.0.$$

Similarly, the expected number of claims from time 2 to 4 that are less than 2000:

$$100(t - \exp[-2(1 - 0.05t)]/0.1) \Big]_{t=2}^{t=4} = (100)\{(4 - 2.019) - (2 - 1.653)\} = 163.4.$$

Similarly, the expected number of claims from time 4 to 6 that are less than 2000:

$$150(t - \exp[-2(1 - 0.05t)]/0.1) \Big]_{t=4}^{t=6} = (150)\{(6 - 2.466) - (4 - 2.019)\} = 233.0.$$

Therefore, the expected value of P is: 85.0 + 163.4 + 233.0 = **481.4**.

16.5. D. The average number of cars driven by females per day is:

$$(1/4)\int_0^8 12 + 3.5t \ dt + (1/4)\int_8^{10} 60 - 2.5t \ dt + (3/4)\int_{10}^{12} 60 - 2.5t \ dt + (3/4)\int_{12}^{16} -30 + 5t \ dt$$

$$+ (1/4)\int_{16}^{18} -30 + 5t \ dt + (1/4)\int_{18}^{24} 204 - 8t \ dt =$$

(1/4)(208) + (1/4)(75) + (3/4)(65) + (3/4)(160) + (1/4)(110) + (1/4)(216) = 321.
Thus the number of cars per day is Poisson with mean 321.
Prob[at least 300 cars] $\cong \Phi[(299.5 - 321)/\sqrt{321}] = 1 - \Phi(-1.20) = \Phi(1.20) =$ **88.49%**.
Comment: Setup of the question taken from CAS3, 5/04, Q.27.

16.6. C. The probability of a claim during 2010 on a policy issued at time t during 2010 is:
${}_{1-t}q_{70+t} = (1 - t)\, q_{70} = (1 - t)\,(0.03)$.
Therefore, expected number of claims on such policies is:

$$\int_0^1 (500 + 500\sqrt{t})\,(1-t)\,(0.03)\,dt = (0.03)(500)\int_0^1 1 - t + t^{1/2} - t^{3/2}\,dt =$$

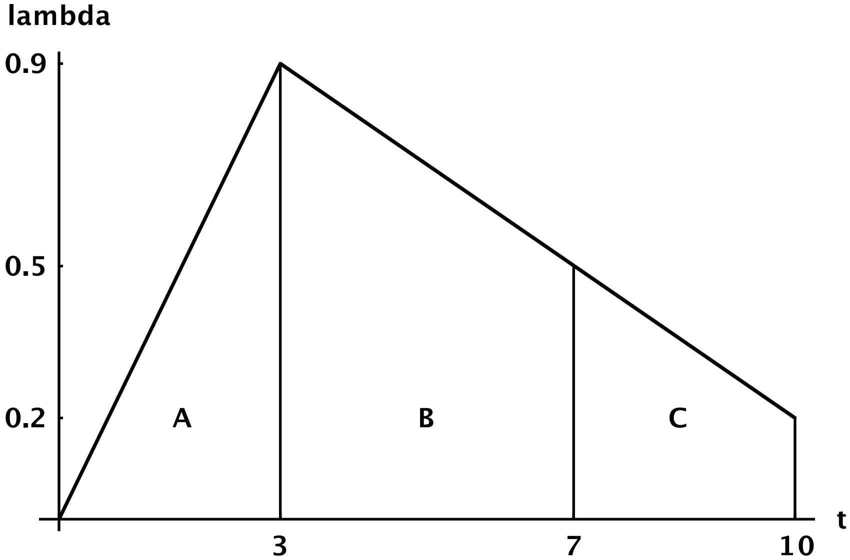
(15) (1 - 1/2 + 2/3 - 2/5) = 11.5.
Thus the number of claims on new policies during 2010 is Poisson with mean 11.5.
The probability of 9 such claims is: $e^{-11.5}\, 11.5^9 / 9!$ = **9.82%**.
Comment: The hyperbolic assumption/distribution is not on the syllabus.

16.7. C. Prob(acceptance at time t) = S(40000 - 2000t) = $((40000 - 2000t)/20000)^{-4}$ =
$(2 - 0.1t)^{-4}$. We are thinning a homogeneous Poisson Process, but since the probability of acceptance is time dependent, we get a nonhomogeneous Poisson Process.
The intensity of acceptable offers is $\lambda(t) = 1.4(2 - 0.1t)^{-4}$.

$$m(5) = 1.4\int_0^5 (2 - 0.1t)^{-4}\,dt = 1.4(2 - 0.1t)^{-3}/0.3\Big]_{t=0}^{t=5} = (14/3)(1.5^{-3} - 2^{-3}) = 0.7994.$$

Thus the number of acceptable offers by time 5 is Poisson with mean 0.7994.
Prob(0 acceptable offers by time 5) = $e^{-0.7994}$ = **45%.**
Comment: Similar to 3, 11/00, Q.29, except that this question involves a time dependent rate of acceptance, which implies acceptable offers are a nonhomogeneous Poisson Process.

16.8. B. & 16.9. E. One can graph $\lambda(t)$ and divide the area under the curve into three portions:



Area A is: (3)(0.9)/2 = 1.35 in hours. (1.35)(60) = 81 customers.
Area B is: (4)(0.9 + 0.5)/2 = 2.8 in hours. (2.8)(60) = 168 customers.
Area C is: (3)(0.5 + 0.2)/2 = 1.05 in hours. (1.05)(60) = 63 customers.
Expected men: (81)(0.3) + (168)(0.3) + (63)(0.7) = 118.8.
Expected portion that are men: 118.8/(81 + 168 + 63) = **38.1%**.
Expected women: (81)(0.7) + (168)(0.7) + (63)(0.3) = 193.2.
Number of women in a day is Poisson with mean 193.2.
Prob[more than 175 women] $\cong 1 - \Phi[(175.5 - 193.2)/\sqrt{193.2}] = 1 - \Phi[-1.27]$ = **89.80%**.

Comment: One can get the relevant areas by integrating $\lambda(t)$ over the appropriate interval of t.

16.10. B. mean = (7)(0.05) + (3)(0.10) + (6)(0.08) + (3)(0.07) + (5)(0.02) = 1.44.
Number of accidents in one day is Poisson with mean 1.44.
Number of accidents involving a single vehicle in one day is Poisson with mean (0.4)(1.44) = 0.576.
Number of accidents involving multiple vehicles in one day is Poisson with mean (0.6)(1.44) = 0.864.
The number of single vehicle accidents is independent of the number of multiple vehicle accidents.
Prob[2 single vehicle accidents] Prob[0 multiple vehicle accidents] = $(e^{-0.576} 0.576^2/2) e^{-0.864}$ = **3.93%**.

16.11. C. Adding up the types of loss, $\lambda(t) = 9\%$ for $0 \le t < 3$, $\lambda(t) = 4\%$ for $3 \le t < 5$, $\lambda(t) = 6\%$ for $5 \le t < 9$, $\lambda(t) = 3\%$ for $9 \le t < 10$.
$m(10) = (9\%)(3) + (4\%)(2) + (6\%)(4) + (3\%)(1) = 62\%$.
Prob[at least 2 losses by time 10] = $1 - e^{-0.62} - 0.62e^{-0.62}$ = **12.85%**.
Alternately, the number of accidents by time 10 is Poisson with mean:
$(5)(3\%) + (4)(5\%) + (1)(2\%) = 37\%$.
The number of illnesses by time 10 is Poisson with mean: $(3)(6\%) + (7)(1\%) = 25\%$.
Therefore, the number of losses by time 10 is Poisson with mean: $37\% + 25\% = 62\%$.
Prob[at least 2 losses by time 10] = $1 - e^{-0.62} - 0.62e^{-0.62}$ = **12.85%**.

16.12. E. The probability that a worker injured at time t is still disabled at time 14 months (March 1, 2012) is: $0.8\, e^{-(14-t)/2} + 0.2\, e^{-(14-t)/10}$.
Therefore, as of March 1, 2012, the expected number of disabled workers whose injuries occurred during 2011 is:

$$\int_0^{12} (0.7 + 0.1t)\,(0.8\, e^{-(14-t)/2} + 0.2 e^{-(14-t)/10})\, dt =$$

$$0.56\, e^{-7} \int_0^{12} e^{t/2}\, dt + 0.08\, e^{-7} \int_0^{12} t\, e^{t/2}\, dt + 0.14\, e^{-1.4} \int_0^{12} e^{t/10}\, dt + 0.02\, e^{-1.4} \int_0^{12} t\, e^{t/10}\, dt$$

$$= (0.56\, e^{-7})\,(2)(e^6 - 1) + (0.08\, e^{-7})\,(2\, t\, e^{t/2} - 4e^{t/2})\Big]_{t=0}^{t=12} +$$

$$+ (0.14e^{-1.4})\,(10)(e^{1.2} - 1) + (0.02\, e^{-1.4})\,(10\, t\, e^{t/2} - 100\, e^{t/2})\Big]_{t=0}^{t=12}$$

$= 0.41100 + 0.58900 + 0.80099 + 0.82069 = 2.622$.
Thus the number of workers whose injuries occurred during 2011 and who are still disabled as of March 1, 2011 is Poisson with mean 2.622.
The probability of at least four such workers is:
$1 - e^{-2.622} - e^{-2.622}\, 2.622 - e^{-2.622}\, 2.622^2 / 2 - e^{-2.622}\, 2.622^3 / 6$ = **26.9%**.
Comment: One can do the relevant integrals by parts.

$$\int t\, e^{t/\theta}\, dt = \theta\, t\, e^{t/\theta} - \theta^2\, e^{t/\theta}.$$

16.13. A. $m(4) = \int_0^4 \lambda(t)\ dt = \int_0^4 0.4t\ dt = (0.2)(4^2) = 3.2.$

The number of claims by time 4 from A is Poisson with mean: (0.7)(3.2) = 2.24.
For Type A, F(1000) - F(500) = S(500) - S(1000) =
$\exp[-(500/2000)^2] - \exp[-(1000/2000)^2] = 0.1606$.
The number of claims of the right size by time 4 from A is Poisson with mean:
(0.1606)(2.24) = 0.3597.
The number of claims by time 4 from B is Poisson with mean: (0.3)(3.2) = 0.96.
For Type B, F(1000) - F(500) = S(500) - S(1000) =
$\exp[-(500/1000)^3] - \exp[-(1000/1000)^3] = 0.5146$.
The number of claims of the right size by time 4 from B is Poisson with mean:
(0.5146)(0.96) = 0.4940.
Thus the total number of claims of the right size by time 4 is Poisson with mean:
0.3597 + 0.4940 = 0.8537.
Probability of exactly one such claim by time 4 is: $0.8537e^{-0.8537}$ = **36.4%**.
Comment: Similar to ST, 5/14, Q.1.

16.14. C. We are thinning a nonhomogeneous Poisson process.
P and Q are independent nonhomogeneous Poisson processes.
Thus the conditional expected value of P, given a certain value for Q, is just the unconditional mean of P. For the original process, the mean number of claims by time 3 is:

$m(3) = \int_0^3 \lambda(t)\ dt = 1 + 2 + 3 = 6$. Of the original claims (500-200)/(800-200) = 1/2 are P.

Therefore the mean of P is: (1/2)(6) = **3**.

16.15. E. The expected number of hurricanes is: (4)(1.25) + (8)(.25) = 7.
The expected number of major hurricanes is: (0.2)(4)(1.25) + (0.1)(8)(.25) = 1.2.
Probability that a hurricane selected at random is major: 1.2/7 = **17.1%**.

16.16. E. $m(0.5) = \int_0^{0.5} \lambda(t)\ dt = \int_0^{0.5} 10t\ dt = (5)(0.5^2) = 1.25.$

The number of claims by time 0.5 from A is Poisson with mean: (0.6)(1.25) = 0.75.
The number of large claims by time 0.5 from A is Poisson with mean: (0.2)(0.75) = 0.15.
The number of claims by time 0.5 from B is Poisson with mean: (0.4)(1.25) = 0.5.
The number of large claims by time 0.5 from B is Poisson with mean: (0.9)(0.5) = 0.45.
Thus the total number of large claims by time 0.5 is Poisson with mean: 0.15 + 0.45 = 0.6.
Probability of 0 or 1 large claims by time 0.5 is: $e^{-0.6} + 0.6e^{-0.6}$ = **87.8%**.
Comment: Note that the severity distributions for A and B differ.

16.17. B. The expected number of tickets sold is:

$$\int_0^7 (800 - 45t)\ dt = (7)(800) - (45/2)(7^2) = 4497.5.$$

Thus the expected number of tickets used is: (0.9)(4497.5) = **4047.75**.
Comment: One can answer this question knowing nothing about Poisson Processes.

16.18. From the previous solution:

- The number of used tickets is Poisson with mean 4047.75.
- The number of unused tickets is Poisson with mean 449.75.
- The numbers of tickets used and unused are independent.

Prob[At least 4100 used tickets] = $1 - \Phi[\frac{4099.5 - 4047.75}{\sqrt{4047.75}}] = 1 - \Phi[0.813] = 20.8\%$.

Prob[At most 420 unused tickets] = $\Phi[\frac{420.5 - 449.75}{\sqrt{449.75}}] = 1 - \Phi[-1.379] = 8.4\%$.

Thus, Prob[At least 4100 used tickets and at most 420 unused tickets]
= (20.8%)(8.4%) = **1.75%**.

16.19. C. The expected number of messages sent by time 4 is:

$$\int_0^4 \lambda(t)\ dt = \int_0^4 t/4\ dt = 4^2/8 = 2.$$

Thinning, the expected number of messages received by time 4 is Poisson with mean:
(0.8)(2) = 1.6.
The probability that at least 2 messages will be received by time 4 is: $1 - e^{-1.6} - 1.6e^{-1.6}$ = **47.5%**.

Section 17, Comparing Nonhomogeneous Poisson Processes[125]

One can compare Nonhomogeneous Poisson Processes. For example, assume you are sitting by the side of a country road watching cars pass in one direction. Assume cars pass with a nonhomogeneous Poisson Process. Assume all of the cars are either red, blue or green.

Assume each color of car has an independent nonhomogeneous Poisson Process. Assume red cars pass with claims intensity $\lambda_r(t)$, and corresponding integral $m_r(t)$. Assume blue cars pass with claims intensity $\lambda_b(t)$, and corresponding integral $m_b(t)$. Assume green cars pass with claims intensity $\lambda_g(t)$, and corresponding integral $m_g(t)$.

Exercise: Assume that $\lambda_r(t) = t/40$, $\lambda_b(t) = \{1 + \sin(2\pi t)\}/10$, and $\lambda_g(t) = \{1 + \cos(2\pi t)\}/5$.
Then what are the mean number of cars of each color that pass by time 10?
[Solution: $m_r(t)$ = integral of $t/40 = t^2/80$. $m_r(10) = 1.25$.

$$m_b(10) = \int_0^{10} \{1+ \sin(2\pi t)\}/10 \ dt = \{t - \cos(2\pi t)/2\pi\}/10 \Big]_{t=0}^{t=10} = (1/10)\ \{(10- 1/4\pi) -(0- 1/4\pi)\} = 1.$$

$$m_g(10) = \int_0^{10} \{1+ \cos(2\pi t)\}/5 \ dt = \{t + \sin(2\pi t)/2\pi\}/5 \Big]_{t=0}^{t=10} = (1/5)\{(10 -0) -(0- 0)\} = 2.]$$

Then if a car has passed in the time interval 0 to 10, then the chance that the car was red is:
1.25 / (1.25 + 1 + 2) = 29.4%.

Exercise:
If a car has passed in the time interval 0 to 10, then the chance that the car was green?
[Solution: 2 / (1.25 + 1 + 2) = 47.1%.
Comment: The chance the car is blue is: 1 / (1.25 + 1 + 2) = 23.5%.]

In general, if an event has occurred in the interval from time s to t, then the probability that it is of type i is: $\frac{m_i(t) - m_i(s)}{m(t) - m(s)}$, where m is sum of the m_i over all types.

With only two types of events, if an event has occurred in the interval from time s to t, then the probability that it is of the first type is: $\frac{m_1(t) - m_1(s)}{\{m_1(t) - m_1(s)\} + \{m_2(t) - m_2(s)\}}$.

[125] Not on the syllabus, but combining ideas that are on the syllabus. I do not expect questions on this material.

Sometimes one is interested in the time until one has at least one event or claim of each type. The waiting time until the first red car has distribution: 1 - exp[- m_r(t)].
Thus the chance that at least one red car has passed by time t, is: 1 - exp[- m_r(t)].
Since the processes are independent, the chance that at least one car of each color (red, blue, and green) has passed by time t is the product of the three individual probabilities: (1 - exp[- m_r(t)]) (1 - exp[- m_b(t)]) (1 - exp[- m_g(t)]).

Exercise: Assume that $\lambda_r(t) = t/40$, $\lambda_b(t) = \{1 + \sin(2\pi t)\}/10$, and $\lambda_g(t) = \{1 + \cos(2\pi t)\}/5$.
What is the chance that at least one car of each color has passed by time = 10?
[Solution: $m_r(10) = 1.25$. $m_b(10) = 1$. $m_g(10) = 2$.
(1 - exp[- m_r(t)]) (1 - exp[- m_b(t)]) (1 - exp[- m_g(t)]) = $(1 - e^{-1.25})\ (1 - e^{-1})\ (1 - e^{-2})$ =
(0.7135) (0.6321) (0.8647) = 0.390.]

If one has two independent nonhomogeneous Poisson Processes with claims intensities $\lambda_1(t)$ and $\lambda_2(t)$, then given that a claim occurs at time t, the chance that this claim is from the first process rather than from the second process is: $\frac{\lambda_1(t)}{\lambda_1(t) + \lambda_2(t)}$.

Exercise: Assume that $\lambda_r(t) = t/40$, $\lambda_b(t) = \{1 + \sin(2\pi t)\}/10$, and $\lambda_g(t) = \{1 + \cos(2\pi t)\}/5$.
If a car passes at time = 5, what is the chance that this car is green?
[Solution: $\lambda_r(5) = 5/40 = 0.125$. $\lambda_b(5) = \{1 + \sin(10\pi)\}/10 = 0.1$. $\lambda_g(5) = \{1 + \cos(10\pi)\}/5 = 0.4$.
The chance the car was green is: $\lambda_g(5) / \{\lambda_r(5) + \lambda_b(5) + \lambda_g(5)\} = 0.4/0.625 = 64\%$.]

If one has two independent nonhomogeneous Poisson Processes with claims intensities $\lambda_1(t)$ and $\lambda_2(t)$, then if the first claim occurs at time s, the chance that the claim is from the first process is $\lambda_1(s) / \{\lambda_1(s) + \lambda_2(s)\} = \lambda_1(s) / \lambda(s)$. The distribution function of the waiting time until the first claim is $1 - e^{-m(s)}$. Therefore by differentiation, the density that the first claim occurs at time s is: $m'(s)\ e^{-m(s)} = \lambda(s) e^{-m(s)}$. Therefore, the chance that the first claim to occur is of the first type is: $\int_0^\infty \{\lambda_1(s) / \lambda(s)\}\ \lambda(s)\ e^{-m(s)}\ ds = \int_0^\infty \lambda_1(s)\ e^{-m(s)}\ ds$. *If we have n different types of claims, each from an independent nonhomogeneous Poisson Process, and if $m(s) = m_1(s) + ... + m_n(s)$, then the chance that the first claim to occur is of the type j is:*

$$\int_0^\infty \lambda_j(s)\ e^{-m(s)}\ ds.$$

Problems:

Use the following information for the next 12 questions:
- Fire losses are independent of hail losses.
- Hail losses follow a nonhomogeneous Poisson Process with:
  $\lambda(t) = 0.1$ for $0 \le t < 5$, $\lambda(t) = 0.6$ for $5 \le t < 9$, $\lambda(t) = 0.1$ for $9 \le t \le 12$.
- Fire losses follow a nonhomogeneous Poisson Process with:
  $\lambda(t) = 0.4$ for $0 \le t < 3$, $\lambda(t) = 0.2$ for $3 \le t < 10$, $\lambda(t) = 0.4$ for $10 \le t \le 12$.

17.1 (1 point) What is the expected number of hail losses between time 0 and 12?
A. Less than 2.4
B. At least 2.4, but less than 2.7
C. At least 2.7, but less than 3.0
D. At least 3.0, but less than 3.3
E. At least 3.3

17.2 (1 point) What is the expected number of fire losses between time 0 and 12?
A. Less than 3.0
B. At least 3.0, but less than 3.3
C. At least 3.3, but less than 3.6
D. At least 3.6, but less than 3.9
E. At least 3.9

17.3 (1 point) Between time 0 and 12, what portion of losses are expected to be from hail?
A. Less than 40%
B. At least 40%, but less than 45%
C. At least 45%, but less than 50%
D. At least 50%, but less than 55%
E. At least 55%

17.4 (1 point) Between time 2 and 8, what portion of losses are expected to be from hail?
A. 45% B. 50% C. 55% D. 60% E. 65%

17.5 (1 point) What is the chance of observing at least one hail loss by time 7?
A. Less than 80%
B. At least 80%, but less than 85%
C. At least 85%, but less than 90%
D. At least 90%, but less than 95%
E. At least 95%

17.6 (1 point) Between time 0 and 6, what portion of losses are expected to be from hail?
A. 34% B. 36% C. 38% D. 40% E. 42%

17.7 (1 point) What is the chance of observing at least one loss (of any type) by time 6?
A. Less than 95%
B. At least 95%, but less than 96%
C. At least 96%, but less than 97%
D. At least 97%, but less than 98%
E. At least 98%

17.8 (3 points) Assuming one observes at least one loss by time 6, what is the chance that the first loss is from hail?
A. 19% B. 21% C. 23% D. 25% E. 27%

17.9 (2 points) What is the chance that between time 1 and 10 one observes 5 hail losses?
A. Less than 6%
B. At least 6%, but less than 7%
C. At least 7%, but less than 8%
D. At least 8%, but less than 9%
E. At least 9%

17.10 (2 points) What is the chance that between time 1 and 10 one observes 5 hail losses and 3 fire losses?
A. Less than 1.2%
B. At least 1.2%, but less than 1.4%
C. At least 1.4%, but less than 1.6%
D. At least 1.6%, but less than 1.8%
E. At least 1.8%

17.11 (3 points) Assuming one observes at least one loss from fire by time 12, what is the mean waiting time until the first fire loss?
A. Less than 2.5
B. At least 2.5, but less than 2.6
C. At least 2.6, but less than 2.7
D. At least 2.7, but less than 2.8
E. At least 2.8

17.12 (1 point) A loss is observed at time 9.5. What is the chance that the loss is from fire?
A. 1/2 B. 4/7 C. 3/5 D. 2/3 E. 7/10

17.13 (3 points) Use the following information:
- Losses due to accident are independent of losses due to illness.
- Losses due to accident follow a nonhomogeneous Poisson Process with: $\lambda(t) = 3\%$ for $0 \le t < 5$, $\lambda(t) = 5\%$ for $5 \le t < 9$, $\lambda(t) = 2\%$ for $9 \le t \le 12$.
- Losses due to illness follow a nonhomogeneous Poisson Process with: $\lambda(t) = 6\%$ for $0 \le t < 3$, $\lambda(t) = 1\%$ for $3 \le t < 10$, $\lambda(t) = 4\%$ for $10 \le t \le 12$.

What is the chance that one observes at least one loss by time 5, and the first loss is from illness?
A. 14% B. 15% C. 16% D. 17% E. 18%

Solutions to Problems:

17.1. D. (0.1)(5) + (0.6)(4) + (0.1)(3) = **3.2**.

17.2. C. (0.4)(3) + (0.2)(7) + (0.4)(2) = **3.4**.

17.3. C. 3.2 / (3.2 + 3.4) = **48.5%**.

17.4. D. We expect (0.1)(3) + (0.6)(3) = 2.1 hail losses.
We expect (0.4)(1) + (0.2)(5) = 1.4 fire losses. 2.1/(2.1 + 1.4) = **60%**.

17.5. B. The chance of seeing at least one hail loss by time t is: 1 - exp[m(t)].
m(7) = (0.1)(5) + (0.6)(2) = 1.7. $1 - e^{-1.7}$ = **0.817**.

17.6. C. The expected number of hail losses is (0.1)(5) + (0.6)(1) = 1.1.
The expected number of fire losses is: (0.4)(3) + (0.2)(3) = 1.8. 1.1 / (1.1+1.8) = **37.9%**.

17.7. A. The expected number of hail losses is: (0.1)(5) + (0.6)(1) = 1.1.
The expected number of fire losses is: (0.4)(3) + (0.2)(3) = 1.8.
So the expected number of losses is 2.9. The chance of no losses is $e^{-2.9}$ = 5.50%.
Therefore, the chance of at least one loss is **94.50%**.

17.8. D. $m(t) = \int \lambda(t)\ dt$. Since $\lambda(t) = 0.5$ for $0 \le t < 3$, $m(t) = 0.5t$ for $0 \le t < 3$.

m(3) = 1.5. m(t) = 1.5 + (t-3)(0.3) = 0.3t + 0.6 for $3 \le t \le 5$.
m(5) = 2.1. m(t) = 2.1 + (t-5)(.08) = 0.8t - 1.9 for $5 \le t \le 9$.
The density of the waiting time until the first loss (regardless of type) is: $\lambda(t)e^{-m(t)}$.
Given a loss occurs at time s, the chance that it is from hail is: $\lambda_h(s) / \lambda(s)$.

$(\lambda(s)e^{-m(s)})\ (\lambda_h(s) / \lambda(s)) = \lambda_h(s)e^{-m(s)}$.

Therefore, the chance that at least one loss occurs by time 6 and the first loss to occur is hail:

$$\int_0^6 \lambda_h(s)\ e^{-m(s)}\ ds = \int_0^3 0.1\ e^{-0.5s}\ ds + \int_3^5 0.1\ e^{-(0.3s+0.6)}\ ds + \int_5^6 0.6\ e^{-(0.8s-1.9)}\ ds =$$

$(0.1)(1/0.5)\{1 - e^{-1.5}\} + (0.1)(e^{-0.6})(1/0.3)(e^{-0.9} - e^{-1.5}) + (0.6)(e^{1.9})(1/0.8)(e^{-4} - e^{-4.8}) =$
0.1554 + 0.0336 + 0.0506 = 0.2396. From the previous solution, the chance that at least one loss occurs by time 6 is 0.9450. Therefore, if at least one loss occurs by time 6, the chance that the first loss is from hail is: 0.2396/0.9450 = **25.4%**.
Comment: Difficult. While 37.9% of the losses are from hail, since the loss intensity for hail is initially small, the chance of the first loss being from hail is smaller than 37.9%.

17.9. E. The expected number of hail losses between time 1 and time 10 is:
(4)(0.1) + (4)(0.6) + (1)(0.1) = 2.9. The number of hail losses between time 1 and 10 is Poisson Distributed with mean 2.9. Therefore, the chance of 5 hail losses is $e^{-2.9}\ 2.9^5 / 5! =$ **9.405%**.

17.10. E. The expected number of fire losses between time 1 and time 10 is:
(2)(0.4) + (7)(0.2) = 2.2. The number of fire losses between time 1 and 10 is Poisson Distributed with mean 2.2. Therefore, the chance of 3 fire losses is: $e^{-2.2}\ 2.2^3 / 3! = 19.664\%$.
From the previous solution the chance of of 5 hail losses is 9.405%.
Since the number of hail losses and fire losses are independent, the chance of 5 hail losses and 3 fire losses is: (0.09405)(0.19664) = **1.85%**.

17.11. C. The density of the waiting time until the first loss is: $\lambda(t)e^{-m(t)}$.
E[waiting to first claim | a claim by time 12] = the integral of t times this density from 0 to 12, divided by the chance of at least one loss by time 12.
The number of fire losses from 0 to 12 is Poisson Distributed with mean 3.4.
Therefore, the chance of at least one fire loss is: $1 - e^{-3.4} = 96.66\%$.
For fire losses, $\lambda(t) = 0.4$ for $0 \le t < 3$, and therefore $m(t) = 0.4t$ for $0 \le t < 3$.
$m(3) = 1.2$ and $\lambda(t) = 0.2$ for $3 \le t \le 10$. $\Rightarrow m(t) = 1.2 + (t-3)(0.2) = 0.2t + 0.6$ for $3 \le t \le 10$.
$m(10) = 2.6$ and $\lambda(t) = 0.4$ for $10 \le t \le 12$. $\Rightarrow m(t) = 2.6 + (t-10)(.4) = 0.4t - 1.4$ for $10 \le t \le 12$.

$$\int_0^{12} t\,\lambda(t)\,e^{-m(t)}\,dt = \int_0^3 0.4t\,e^{-0.4t}\,dt + \int_3^{10} 0.2t\,e^{-(0.2t+0.6)}\,dt + \int_{10}^{12} 0.4t\,e^{-(0.4t-1.4)}\,dt =$$

$$-\{(1/0.4)+t\}\,e^{-0.4t}\Big]_{t=0}^{t=3} + \{(1/0.2)+t\}\,e^{-0.6}\,e^{-0.2t}\Big]_{t=3}^{t=10} + -\{(1/0.4)+t\}\,e^{1.4}\,e^{-0.4t}\Big]_{t=10}^{t=12} =$$

0.843 + 1.295 + 0.445 = 2.583.
Thus if one observes at least one loss from fire by time 12, then the mean waiting time until the first fire loss is 2.583/0.9666 = **2.672**.
Comment: Difficult. We are calculating the mean waiting time conditional on there being a fire loss by time 12, therefore we need to divide by the chance of at least one fire loss by time 12, which is 0.9666. Note that the integral from 0 to 12 of $\lambda(t)e^{-m(t)}$ is 0.9666.
Note that by using integration by parts:

$$\int_0^{12} t\,f(t)\,dt = \int_0^{12} S(t)\,dt - 12S(12) = \int_0^3 e^{-0.4t}\,dt + \int_3^{10} e^{-(0.2t+0.6)}\,dt + \int_{10}^{12} e^{-(0.4t-1.4)}\,dt - 12e^{-3.4} =$$

1.747 + 1.135 + 0.102 - 0.401 = 2.583.

17.12. D. The loss intensity from fire at time = 9.5 is 0.2 .
The loss intensity from hail at time = 9.5 is 0.1.
Thus the chance that the loss is from fire is: 0.2/(0.2+0.1) = **2/3**.

17.13. D. Adding up the types of loss, $\lambda(t) = 9\%$ for $0 \le t < 3$, and $\lambda(t) = 4\%$ for $3 \le t < 5$.
$m(t) = 0.09t$ for $0 \le t \le 3$, and $m(t) = 0.27 + (t-3)(0.04) = 0.04t + 0.15$ for $3 \le t \le 5$.
The density of the waiting time until the first loss (regardless of type) is: $\lambda(t)e^{-m(t)}$.
Given a loss occurs at time t, the chance that it is from illness is: $\lambda_2(t) / \lambda(t)$.
Thus the chance that the first loss occurs at time t and it is from illness is:
$\lambda(t)e^{-m(t)} \lambda_2(t) / \lambda(t) = \lambda_2(t)\ e^{-m(t)}$.
Thus, the chance that at least one loss occurs by time 5 and the first loss to occur is from illness is:

$$\int_0^5 \lambda_2(s)\ e^{-m(s)}\ ds = \int_0^3 (0.06)\ e^{-0.09s}\ ds + \int_3^5 (0.01)\ e^{-(0.04s+0.15)}\ ds =$$

$(0.06)(1/0.09)(1 - e^{-0.27}) + (0.01)(e^{-0.15})(1/0.04)(e^{-0.12} - e^{-0.20}) = 0.1577 + 0.0147 =$ **0.1724**.
Alternatively, $\lambda(t) = 0.09$ for $0 \le t < 3$.
Therefore, Prob(0 losses by time 3) = $e^{-(0.09)(3)} = 0.7633$.
$\lambda(t) = 0.04$ for $3 \le t < 5$.
Therefore, Prob(0 losses for $3 \le t < 5$) = $e^{-(0.04)(2)} = 0.9231$.
Prob(first loss occurs for $0 \le t < 3$) = 1 - 0.7633 = 0.2366.
Prob(first loss occurs for $3 \le t < 5$) = (0.7633)(1 - 0.9231) = 0.0586.
Prob(a loss is from illness | $0 \le t < 3$) = 0.06/(0.06 + 0.03) = 2/3.
Prob(a loss is from illness | $3 \le t < 5$) = 0.01/(0.01 + 0.03) = 1/4.
Therefore, the chance that at least one loss occurs by time 5 and the first loss to occur is from illness is: (2/3)(0.2366) + (1/4)(0.0586) = 0.1577 + 0.0147 = **0.1724**.