Introduction to 2025 Exam ASTAM Study Guides

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Starting in Fall 2022, the SOA replaced Exam STAM.
In Spring 2023, the SOA started to give Exam ASTAM, which covers about half of the material formerly on Exam STAM, plus a little new material.[1]

Each of the released ASTAM exams is 4 hours and consists of 6 questions worth 60 points.[2]
Each question consists of multiple subparts and requires written answers.
You will be able to use Excel during your exam.[3]

You are assumed to know the material on Exam FAM.
Some parts of some of the ASTAM question directly test material from the syllabus of FAM.
Thus you should review those topics on FAM which are continued on ASTAM.

How much detail is needed and how many problems need to be done varies by person and topic. In order to help you to concentrate your efforts:

1. About 1/6 of the many problems are labeled "highly recommended", while another 1/6 are labeled "recommended."
2. Important Sections are listed in bold in the table of contents. Extremely important Sections are listed in larger type and in bold.
3. Important ideas and formulas are in bold.
4. Each Study Guide has a Section of Important Ideas and Formulas.

My Study Aids are a thick stack of paper.[4] However, many students find they do not need to look at the textbooks. **For those who have trouble getting through the material, concentrate on the introductions and sections in bold.**

Highly Recommended problems (about 1/6 of the total) are double underlined.
Recommended problems (about 1/6 of the total) are underlined.
Do at least the Highly Recommended problems your first time through.
It is important that you **do problems when learning a subject and then some more problems a few weeks later.**

I have written some easy and tougher problems.[5] The former exam questions are arranged in chronological order. The more recent exam questions are on average more similar to what you will be asked on your exam, than are less recent questions.

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[1] The SOA added a study note by Hardy "Outstanding Claims Reserving" on Loss Reserving, and a textbook chapter by Hardy and Saunders on Extreme Value Theory.
[2] The SOA has also released sample questions.
The SOA exam and sample questions are in my relevant study guides. See the subsequent chart.
[3] See the SOA webpage for details. I subsequently provide more detail.
[4] The number of pages is not as important as how long it takes you to understand the material. One page in a textbook might take someone as long to understand as ten pages in my Study Guides.
[5] Points are based on 100 points = a 4 hour exam.
Your exam will be 30 questions and 3 hours, equivalent to what I have called 2.5 points per question.

Each of my study guides is divided into sections as shown in its table of contents.
The solutions to the problems in a section of a study guide are at the end of that section.

In the electronic version use the bookmarks / table of contents in the Navigation Panel in order to help you find what you want.

Mahler's Guides for Exam ASTAM have 10 parts,
which are listed below, along with my estimated percent of the exam.[6]

Study Guides for Exam ASTAM

Volume 1	1	13%	Mahler's Guide to Loss Distributions
Volume 1	2	17%	Mahler's Guide to Aggregate Distributions
Volume 2	3	17%	Mahler's Guide to Fitting Distributions
Volume 2	4	9%	Mahler's Guide to Buhlmann Credibility & Bayesian Analysis
Volume 3	5	4%	Mahler's Guide to Conjugate Priors
Volume 3	6	1%	Mahler's Guide to Semiparametric Estimation
Volume 3	7	2%	Mahler's Guide to Nonparametric Credibility
Volume 4	8	8%	Mahler's Guide to P&C Ratemaking
Volume 4	9	16%	Mahler's Guide to P&C Reserving
Volume 4	10	13%	Mahler's Guide to Extreme Value Theory

Some exam questions test ideas from more than one of my study guides.

[6] This is my best estimate, which should be used with appropriate caution, particularly in light of the recent changes to the syllabus and the small number of released exams.
In any case, the number of questions by topic varies from exam to exam.

Besides many past exam questions from the CAS and SOA, my study guides include some past questions from exams given by the Institute of Actuaries and Faculty of Actuaries in Great Britain. These questions are copyright by the Institute of Actuaries and Faculty of Actuaries, and are reproduced here solely to aid students studying for actuarial exams. These IOA questions are somewhat similar in format to those that will appear on your exam.

Your exam will be 3 hours and will consist of several questions, probably 6, for a total of 60 “points”. The questions are of varying values, and each have several subparts.

The examination will be offered via computer-based testing.

The exam will be taken at Prometric testing centers. The written-answer questions will be displayed on the computer and answered in a paper answer booklet to be provided and collected by Prometric. Candidates will be provided with a scratchpad.

An Excel Workbook will be available for candidates. Candidates will be expected to use Excel to calculate probabilities and quantiles from common distributions, including the Normal distribution, the Poisson distribution, the chi-squared distribution, the F distribution, the gamma distribution, the binomial distribution, and the negative binomial distribution.[7]
Candidates may also use Excel for general calculations.

Candidates will also have access to the SOA’s ASTAM Formula Sheet, included subsequently.
Download from the SOA website, a copy of the ASTAM Formula Sheet.[8]
Use this formula sheet when doing problems and practice exams.

I suggest you buy and use the TI-30XS **Multiview** calculator, as well as making sure you know how to use Excel.

While studying, you should do as many problems as possible. Going back and forth between reading and doing problems is the only way to pass this exam. The only way to learn to solve problems is to solve lots of problems. You should not feel satisfied with your study of a subject until you can solve a reasonable number of the problems.

[7] The Student’s t-distribution can also be worked with in Excel.
[8] http://www.soa.org

Note that In some cases, numerical values shown in one of my spreadsheets are unrounded, while the corresponding value in my text may be rounded.

There are two manners in which you should be doing problems. First you can do problems in order to learn the material. Take as long on each problem as you need to fully understand the concepts and the solution. Reread the relevant syllabus material. Carefully go over the solution to see if you really know what to do. Think about what would happen if one or more aspects of the question were revised.[9] This manner of doing problems should be gradually replaced by the following manner as you get closer to the exam.

The second manner is to do a series of problems under exam conditions, with the items you will have when you take the exam. Take in advance a number of points to try based on the time available. For example, if you have an uninterrupted hour, then one might try 60/3 = 20 points of problems. Do problems as you would on an exam in any order, skipping some and coming back to some, until you run out of time. I suggest you leave time to double check your work.

Expose yourself somewhat to everything on the syllabus.
Concentrate on sections and items in bold.
Sections and material in italics is less likely to be needed to directly answer exam questions, and should be skipped on your first time through.

Try not to get bogged down on a single topic. On hard subjects, try to learn at least the simplest important idea. The first time through do enough problems in each section, but leave some problems in each section to do closer to the exam. At least every few weeks review the important ideas and formulas sections of those study guides you have already completed.

Make a schedule and stick to it. Spend a minimum of one hour every day.
I recommend at least two study sessions every day, each of at least 1/2 hour.

Use whatever order to go through the material that works best for you.
Here is a schedule that may work for some people.[10]

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[9] Some may also find it useful to read about a dozen questions on an important subject, thinking about how to set up the solution to each one, but only working out in detail any questions they do not quickly see how to solve.
[10] Just one suggestion. Modify it to meet your own needs, strengths, and weaknesses.

<u>A 12 week Study Schedule for Exam ASTAM</u>:

1. Start of Loss Distributions: sections 1 to 18.

2. Rest of Loss Distributions

3. Aggregate Distributions: sections 1 to 7.

4. Rest of Aggregate Distributions
   Start of Buhlmann Credibility and Bayesian Analysis: sections 1 to 6.

5. Start of Fitting Distributions: sections 1 to 11.

6. Rest of Buhlmann Credibility and Bayesian Analysis

7. Rest of Fitting Distributions

8. P&C Ratemaking

9. First part of of P&C Reserving: sections 1 to 8.

10. Conjugate Priors
    Semiparametric Estimation
    Nonparametric Credibility

11. Remainder of P&C Reserving

12. Extreme Value Theory

Most of you will need to spend a total of 200 or more hours of study time on the entire syllabus; this means an average of at least 1.5 hours a day.

Throughout do Exam Problems and Practice Problems in my study guides. **At least 50% of your time should be spent doing problems.** As you get closer to the Exam, the portion of time spent doing problems should increase.

Review the important formulas and ideas section at the end of each study guide.

Past students helpful suggestions and questions have greatly improved these Study Aids.
I thank them.

Feel free to send me any questions or suggestions:
Howard Mahler, Email: hmahler@mac.com

Please do not copy the Study Aids, except for your own personal use. Giving them to others is unfair to yourself, your fellow students who have paid for them, and myself.[11]
If you found them useful, tell a friend to buy his own.

Please send me any suspected errors by Email.
(Please specify as carefully as possible the page, Study Guide and Exam.)
The errata sheet will be posted on my webpage: www.howardmahler.com/Teaching

[11] These study aids represent thousands of hours of work.

Location of SOA ASTAM Sample Questions in my Study Guides:

Question	Study Guide	Section
1	Fitting Distributions	15
2	P&C Ratemaking	6
3	P&C Reserving	7
4	Buhlmann Credibility	9
5	Extreme Value Theory	5
6	Fitting Distributions	16
7	Aggregate Distributions	7
8	Extreme Value Theory	5
9	Buhlmann Credibility	9
10	Fitting Distributions	15
11	Fitting Distributions	5
12	Fitting Distributions	11
13	Fitting Distributions	17
14	Extreme Value Theory	2
15	Aggregate Distributions	9
16	Aggregate Distributions	11
17	Fitting Distributions	17
18	Fitting Distributions	11
19	Fitting Distributions	17
20	P&C Reserving	8
21	Buhlmann Credibility	9
22	P&C Reserving	6

Each of these questions has multiple parts. No point values are assigned.
While each question is placed in the section of a study guide where it seems to fit, most of these questions cover material in more than one section and/or study guide.

Location of past ASTAM exam questions in my Study Guides:

Exam	Question	Points	Study Guide	Section
Spring 2023	1	12	Aggregate Distributions	11
	2	10	Fitting Distributions	19
	3	11	NonParametric Credibility	5
	4	9	Extreme Value Theory	8
	5	7	P&C Ratemaking	6
	6	11	P&C Reserving	9
Fall 2023	1	11	Aggregate Distributions	9
	2	10	Extreme Value Theory	5
	3	12	Fitting Distributions	12
	4	10	Conjugate Priors	6
	5	9	P&C Reserving	5
	6	8	P&C Reserving	10
Spring 2024	1	9	P&C Reserving	13
	2	8	P&C Ratemaking	6
	3	10	Aggregate Distributions	11
	4	10	Fitting Distributions	17
	5	12	Buhlmann Credibility	9
	6	11	Extreme Value Theory	8
Fall 2024	1	10	Fitting Distributions	8
	2	7	P&C Ratemaking	8
	3	12	Aggregate Distributions	6
	4	10	Extreme Value Theory	5
	5	10	Semiparametric Estimation	4
	6	11	P&C Reserving	10

While each question is placed in the section of a study guide where it seems to fit, most of these questions cover material in more than one section and/or study guide.

Author Biography:

Howard C. Mahler is a Fellow of the Casualty Actuarial Society,
and a Member of the American Academy of Actuaries.

He has published study guides since 1996.
He taught live seminars and/or classes for many different actuarial exams from 1994 to 2017.

He spent over 20 years in the insurance industry, the last 15 as Vice President and Actuary at the Workers' Compensation Rating and Inspection Bureau of Massachusetts.
He has published many major research papers and won the 1987 CAS Dorweiler prize.
He served 12 years on the CAS Examination Committee including three years as head of the whole committee (1990-1993).

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All references in my study guides are to the fifth edition of Loss Models.

Loss Models, Fifth Edition	Mahler Study Guides
Chapter 3.4.2-3.4.6	Loss Distributions: Sections 16, 19, 20.
Chapter 5.1-5.2	Loss Distributions: Sections 15, 24, 25.
Chapter 8	Loss Distributions: Sections 11, 14, 21, 22.
Chapter 9[12]	Aggregate Distributions
Chapter 11.5-11.7	Fitting Distributions: Sections 15-18.
Chapter 12.4	Fitting Distributions: Section 19.
Chapter 13	Buhlmann Credibility, Conjugate Priors.
Chapter 15[13]	Fitting Distributions: Sections 5-14.
Chapter 17	Buhlmann Credibility, Conjugate Priors.
Chapter 18	Nonparametric Estimation, Semiparametric Estimation

Attached is the SOA's ASTAM Formula Sheet.
Download a copy from the SOA webpage.

[12] Sections 3.1-3.2, 4 (Theorem 9.7 and Example 9.9 only), 5, 6 (except 6.1), 7.
[13] Excluding 15.4.2.

ASTAM Formula Sheet

$P(z)$ denotes the probability generating function. $M(z)$ denotes the moment generating function.

Q_α denotes the α-quantile of a distribution, also known as the α-Value at Risk

$\mathrm{ES}_\alpha[X]$ denotes the α-Expected Shortfall of X. This is also known as the α−TailVaR, or the α-CTE.

The distribution function of the standard normal distribution is denoted $\Phi(x)$. The probability density function of the standard normal distribution is denoted $\phi(x)$.

The q-quantile of the standard normal distribution is denoted z_q, that is $\Phi(z_q) = q$.

For counting distributions, p_k denotes the probability function.

Continuous distributions

Pareto(α, θ) Distribution

$$f(x) = \frac{\alpha\theta^\alpha}{(\theta + x)^{\alpha+1}}, \qquad F(x) = 1 - \left(\frac{\theta}{\theta + x}\right)^\alpha,$$

$$\mathrm{E}[X] = \frac{\theta}{\alpha - 1}, \quad \alpha > 1, \qquad \mathrm{Var}[X] = \left(\frac{\theta}{\alpha - 1}\right)^2 \frac{\alpha}{\alpha - 2}, \quad \alpha > 2,$$

$$\mathrm{E}[X \wedge x] = \frac{\theta}{\alpha - 1}\left(1 - \left(\frac{\theta}{x + \theta}\right)^{\alpha - 1}\right), \qquad \alpha > 1,$$

$$\mathrm{E}[X^k] = \frac{k!\,\theta^k}{(\alpha - 1)(\alpha - 2)...(\alpha - k)}, \qquad k = 1, 2, \cdots, \quad \alpha > k,$$

$$\mathrm{E}[X - Q | X > Q] = \frac{\theta + Q}{\alpha - 1}, \quad \alpha > 1.$$

Lognormal(μ, σ) Distribution

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{\log x - \mu}{\sigma}\right)^2\right\}, \qquad F(x) = \Phi\left(\frac{\log x - \mu}{\sigma}\right),$$

$$\mathrm{E}[X] = e^{\mu+\sigma^2/2}, \qquad \mathrm{Var}[X] = e^{2\mu+\sigma^2}\left(e^{\sigma^2} - 1\right),$$

$$\mathrm{E}[X^k] = e^{k\mu+k^2\sigma^2/2},$$

$$\mathrm{E}\left[(X \wedge x)^k\right] = \exp\left(k\mu + \frac{1}{2}k^2\sigma^2\right)\Phi\left(\frac{\log x - \mu - k\sigma^2}{\sigma}\right) + x^k\left(1 - \Phi\left(\frac{\log x - \mu}{\sigma}\right)\right),$$

$$\mathrm{ES}_\alpha[X] = \frac{e^{\mu+\sigma^2/2}}{1-\alpha}\,\Phi\big(z_{1-\alpha} + \sigma\big).$$

Exponential(θ) Distribution

$$f(x) = \frac{e^{-x/\theta}}{\theta}, \qquad F(x) = 1 - e^{-x/\theta},$$

$$\mathrm{E}[X] = \theta, \qquad \mathrm{Var}[X] = \theta^2,$$

$$\mathrm{E}[X^k] = k!\,\theta^k,\ k = 1, 2, \cdots,$$

$$\mathrm{E}[X \wedge x] = \theta\left(1 - e^{-x/\theta}\right),$$

$$\mathrm{E}[X - Q | X > Q] = \theta,$$

$$M_X(t) = (1 - t\theta)^{-1}, \quad t < \frac{1}{\theta}.$$

Gamma(α, θ) Distribution

$$f(x) = \frac{x^{\alpha-1}\,e^{-x/\theta}}{\theta^\alpha\,\Gamma(\alpha)},$$

$$\mathrm{E}[X] = \alpha\,\theta, \qquad \mathrm{Var}[X] = \alpha\,\theta^2,$$

$$\mathrm{E}[X^k] = \alpha(\alpha+1)...(\alpha+k-1)\theta^k, \qquad k = 1, 2, \cdots,$$

$$M_X(t) = (1 - t\theta)^{-\alpha}, \quad t < \frac{1}{\theta}.$$

Chi-squared(ν) Distribution

Gamma distribution with $\alpha = \nu/2$ and $\theta = 2$.
$\nu \in \mathbb{N}^+$ is the degrees of freedom parameter.

$$\mathrm{E}[X] = \nu, \qquad \mathrm{Var}[X] = 2\nu$$

$$M_X(t) = (1-2t)^{-\nu/2}, \quad t < \frac{1}{2}.$$

Beta(a, b) Distribution

$$f(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}, \qquad 0 < x < 1,$$

$$\mathrm{E}[X] = \frac{a}{a+b}, \qquad \mathrm{Var}[X] = \frac{a\,b}{(a+b)^2(a+b+1)}.$$

Normal(μ, σ^2) Distribution

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\}, \qquad F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right),$$

$$\mathrm{E}[X] = \mu, \qquad \mathrm{Var}[X] = \sigma^2,$$

$$\mathrm{ES}_\alpha[X] = \mu + \frac{\sigma}{1-\alpha}\phi\left(z_\alpha\right),$$

$$M_X(t) = e^{t\mu + t^2\sigma^2/2}.$$

Weibull(θ, τ) Distribution

$$f(x) = \frac{\tau x^{\tau-1}}{\theta^\tau} e^{-\left(\frac{x}{\theta}\right)^\tau}, \qquad F(x) = 1 - e^{-(x/\theta)^\tau},$$

$$\mathrm{E}[X] = \theta\,\Gamma\left(1+\frac{1}{\tau}\right), \qquad \mathrm{Var}[X] = \theta^2\left(\Gamma\left(1+\frac{2}{\tau}\right) - \left(\Gamma\left(1+\frac{1}{\tau}\right)\right)^2\right).$$

Counting Distributions

Poisson(λ) Distribution

$$p_k = \frac{\lambda^k e^{-\lambda}}{k!}, \qquad a = 0, \quad b = \lambda,$$

$$\mathrm{E}[N] = \lambda, \qquad \mathrm{Var}[N] = \lambda,$$

$$P_N(z) = \exp\{\lambda(z-1)\}, \qquad M_N(z) = \exp\{\lambda(e^z-1)\}.$$

Binomial(m,q) Distribution

$$p_k = \binom{m}{k} q^k (1-q)^{m-k}, \qquad a = -\frac{q}{1-q}, \quad b = \frac{(m+1)q}{1-q},$$

$$\mathrm{E}[N] = mq, \qquad \mathrm{Var}[N] = mq(1-q)$$

$$P_N(z) = \big(1 + q(z-1)\big)^m, \qquad M_N(z) = \big(1 + q(e^z - 1)\big)^m.$$

Negative Binomial(r,β) Distribution

$$p_0 = \left(\frac{1}{1+\beta}\right)^r, \qquad p_k = \frac{r(r+1)\cdots(r+k-1)}{k!} \left(\frac{\beta}{1+\beta}\right)^k \left(\frac{1}{1+\beta}\right)^r, \quad k = 1, 2, \dots,$$

$$a = \frac{\beta}{1+\beta}, \quad b = \frac{(r-1)\beta}{1+\beta},$$

$$\mathrm{E}[N] = r\beta, \qquad \mathrm{Var}[N] = r\beta(1+\beta),$$

$$P_N(z) = (1 - \beta\,(z-1))^{-r}, \qquad M_N(z) = (1 - \beta\,(e^z - 1))^{-r}.$$

Geometric Distribution

Negative Binomial Distribution with $r = 1$;

Recursions for Compound Distributions

For $N \sim (a, b, 0)$: $$f_S(x) = \frac{\sum_{y=1}^{x \wedge m} \left(a + \frac{by}{x}\right) f_X(y)\, f_S(x-y)}{1 - a f_X(0)}$$

For $N \sim (a, b, 1)$: $$f_S(x) = \frac{\big(p_1 - (a+b)p_0\big) f_X(x) + \sum_{y=1}^{x \wedge m} \left(a + \frac{by}{x}\right) f_X(y)\, f_S(x-y)}{1 - a f_X(0)}$$

Empirical Bayes Credibility

Empirical Bayes parameter estimation for the Bühlmann model:

$$\bar{X}_i = \frac{\sum_{j=1}^{n} X_{ij}}{n}, \qquad \bar{X} = \frac{\sum_{i=1}^{r} \bar{X}_i}{r},$$

$$\hat{v} = \frac{1}{r(n-1)} \sum_{i=1}^{r} \sum_{j=1}^{n} \left(X_{ij} - \bar{X}_i \right)^2, \qquad \hat{a} = \frac{1}{r-1} \sum_{i=1}^{r} \left(\bar{X}_i - \bar{X} \right)^2 - \frac{\hat{v}}{n},$$

$$\hat{\mu} = \bar{X}, \qquad \hat{Z}_i = \frac{n}{n + \hat{v}/\hat{a}}.$$

Empirical Bayes parameter estimation for the Bühlmann-Straub model:

$$m_i = \sum_{j=1}^{n_i} m_{ij}, \qquad m = \sum_{i=1}^{r} m_i, \qquad \bar{X}_i = \frac{\sum_{j=1}^{n_i} m_{ij} X_{ij}}{m_i}, \quad \bar{X} = \frac{\sum_{i=1}^{r} m_i \bar{X}_i}{m},$$

$$\hat{v} = \frac{\sum_{i=1}^{r} \sum_{j=1}^{n_i} m_{ij} \left(X_{ij} - \bar{X}_i \right)^2}{\sum_{i=1}^{r} (n_i - 1)}, \qquad \hat{a} = \frac{\sum_{i=1}^{r} m_i \left(\bar{X}_i - \bar{X} \right)^2 - \hat{v}(r-1)}{m - \frac{1}{m} \sum_{i=1}^{r} m_i^2}$$

$$\hat{Z}_i = \frac{m_i}{m_i + \hat{v}/\hat{a}}, \quad \hat{\mu} = \frac{\sum_{i=1}^{r} \hat{Z}_i \bar{X}_i}{\sum_{i=1}^{r} \hat{Z}_i}.$$

Extreme Value Theory

The Gumbel Distribution

$$F(x) = \exp\left\{-\exp\left(-\frac{x-\mu}{\theta}\right)\right\}, \quad \theta > 0.$$

The Fréchet Distribution

$$F(x) = \exp\left\{-\left(\frac{x-\mu}{\theta}\right)^{-\alpha}\right\}, \quad x > \mu;\ \alpha > 0;\ \theta > 0.$$

The Weibull EV Distribution

$$F(x) = \exp\left\{-\left(\frac{\mu-x}{\theta}\right)^{\tau}\right\}, \quad x < \mu;\ \tau > 0;\ \theta > 0.$$

The Generalized Extreme Value Distribution

The distribution function is $H(x)$ where

$$H_\xi(x) = \begin{cases} \exp\left(-(1+\xi x)^{-\frac{1}{\xi}}\right) & \xi \neq 0,\ \xi x > -1, \\ \exp\left(-e^{-x}\right) & \xi = 0. \end{cases}$$

The GEV can be adjusted for scale and location, to give $H_{\xi,\mu,\theta}$ where

$$H_{\xi,\mu,\theta}(x) = \begin{cases} \exp\left(-(1+\xi(x-\mu)/\theta)^{-\frac{1}{\xi}}\right) & \xi \neq 0,\ (1+\xi\,(x-\mu)/\theta) > 0, \\ \exp\left(-e^{-(x-\mu)/\theta}\right) & \xi = 0, \end{cases}$$

The Generalized Pareto Distribution (GPD)

$$G(x) = \begin{cases} 1-(1+\xi x/\beta)^{-\frac{1}{\xi}} & \xi \neq 0 \\ 1-e^{-\frac{x}{\beta}} & \xi = 0 \end{cases}$$

$$\mathrm{E}[X] = \frac{\beta}{1-\xi} \quad \text{for } 0 < \xi < 1$$

If $X - d | X > d \sim \mathrm{GPD}(\xi,\ \beta)$ then

$$Q_\alpha = d + \frac{\beta}{\xi}\left(\left(\frac{S_X(d)}{1-\alpha}\right)^{\xi} - 1\right)$$

$$\mathrm{ES}_\alpha = \frac{1}{1-\xi}\Big(Q_\alpha + \beta - \xi\, d\Big)$$

The Hill Estimator

$$\hat{\alpha}_j^H = \left(\sum_{k=j}^{n} \frac{\log(x_{(k)})}{n-j+1} - \log(x_{(j)}) \right)^{-1}$$

(Note: the version of the Hill estimator in QERM is incorrect.)

Outstanding Claims Reserves

Functions of development factors

If all claims are settled by the end of DY J, then

for $j = 0, 1, ..., J-1$, $\lambda_j = \prod_{k=j}^{J-1} f_j$; $\lambda_J = 1$

for $j = 0, 1, ..., J$, $\beta_j = \frac{1}{\lambda_j}$; $\gamma_j = \beta_j - \beta_{j-1}$

Tests for correlated development factors

$$T_j = r_j \sqrt{\frac{\nu_j}{1 - r_j^2}}, \qquad T = \frac{\sum_{\nu_j \geq 3} T_j(\nu_j - 2)/\nu_j}{\sum_{\nu_j \geq 3} (\nu_j - 2)/\nu_j}$$

Under the null hypothesis, $T \approx N(0, v)$ where $v = \frac{1}{\sum_{\nu_j \geq 3} ((\nu_j - 2)/\nu_j)}$.

Test for calendar year effects

$Z_k = \min(S_k, L_k)$; $Z = \sum_{k=1}^{I-1} Z_k$.

Under the null hypothesis, approximately:

$$\mathrm{E}[Z_k] = \frac{n_k}{2} - \binom{n_k - 1}{m_k} \frac{n_k}{2^{n_k}}$$

$$\mathrm{Var}[Z_k] = \frac{n_k(n_k - 1)}{4} - \binom{n_k - 1}{m_k} \frac{n_k(n_k - 1)}{2^{n_k}} + \mathrm{E}[Z_k] - \mathrm{E}[Z_k]^2$$

$$\mathrm{E}[Z] = \sum_{k=1}^{I-1} \mathrm{E}[Z_k]; \quad \mathrm{Var}[Z] = \sum_{k=1}^{I-1} \mathrm{Var}[Z_k]; \quad \frac{Z - \mathrm{E}[Z]}{\sqrt{\mathrm{Var}[Z]}} \sim N(0, 1)$$

The Bühlmann-Straub Model of Outstanding Claims

$$m_i = \hat{\beta}_{I-i}; \quad s_i^2 = \frac{1}{I-i}\sum_{j=0}^{I-i} \hat{\gamma}_j \left(\frac{X_{i,j}}{\hat{\gamma}_j} - \widehat{C}_{i,J}\right)^2$$

$$m = \sum_{i=0}^{I} m_i; \qquad \overline{C} = \frac{\sum_{i=0}^{I} C_{i,I-i}}{m}$$

$$\hat{v} = \frac{1}{I}\sum_{i=0}^{I-1} s_i^{;} \qquad \hat{a} = \frac{\sum_{i=0}^{I} m_i \left(\widehat{C}_{i,J} - \overline{C}\right)^2 - I\hat{v}}{m - \frac{1}{m}\sum_{i=0}^{I} m_i^2}$$

$$Z_i = \frac{\hat{\beta}_{I-i}}{\hat{\beta}_{I-i} + \hat{v}/\hat{a}} \qquad \hat{\mu} = \frac{\sum_{i=0}^{I} Z_i \, \widehat{C}_{i,J}}{\sum_{i=0}^{I} Z_i}$$

Mack's Model

$$\hat{\sigma}_j^2 = \frac{1}{I-1-j}\sum_{i=0}^{I-1-j} C_{i,j}\left(f_{i,j} - \hat{f}_j\right)^2 \qquad \text{for } j \le I-2$$

$$\mathrm{Var}\left[C_{i,J} - C_{i,I-i} | C_{i,I-i}\right] \approx \widehat{C}_{i,J}^2 \sum_{j=I-i}^{J-1} \frac{\hat{\sigma}_j^2}{\hat{f}_j^2 \, \widehat{C}_{i,j}}.$$

$$\left(\widehat{C}_{i,J} - \mathrm{E}\left[C_{i,J}|\mathcal{D}_I\right]\right)^2 \approx \widehat{C}_{i,J}^2 \sum_{j=I-i}^{J-1} \frac{\hat{\sigma}_j^2}{\hat{f}_j^2 \, S_j}$$

$$\mathrm{MSEP}\left(\widehat{R}\,|\mathcal{D}_I\right) \approx \sum_{i=1}^{I} \widehat{C}_{i,J}^2 \sum_{j=I-i}^{J-1} \frac{\hat{\sigma}_j^2}{\hat{f}_j^2}\left(\frac{1}{\widehat{C}_{i,j}} + \frac{1}{S_j}\right) + 2\sum_{i=1}^{I-1} \widehat{C}_{i,J}\left(\sum_{j=I-i}^{J-1} \frac{\hat{\sigma}_j^2}{\hat{f}_j^2 \, S_j}\right)\left(\sum_{l=i+1}^{I} \widehat{C}_{l,J}\right)$$

Volume One

Mahler's Guide to

Loss Distributions

Exam ASTAM

prepared by
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Study Aid 2025-STAM-**1**

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Mahler's Guide to Loss Distributions

The Loss Distributions concepts in Loss Models, by Klugman, Panjer, and Willmot, are demonstrated.

SOA Learning Topics 1 and 3 are covered in this Study Guide.[1]

Information in bold or sections whose title is in bold are more important for passing the exam. Larger bold type indicates it is extremely important.
Sections and material in italics is less likely to be needed to directly answer exam questions, and should be skipped on your first time through.

Highly Recommended problems are double underlined.
Recommended problems are underlined.
Solutions to the problems in each section are at the end of that section.

Note that problems include both some written by me and some from past exams.[2] The latter are copyright by the Casualty Actuarial Society and SOA and are reproduced here solely to aid students in studying for exams. The solutions and comments are solely the responsibility of the author; the CAS and SOA bear no responsibility for their accuracy. While some of the comments may seem critical of certain questions, this is intended solely to aid you in studying and in no way is intended as a criticism of the many volunteers who work extremely long and hard to produce quality exams.

Greek letters used in Loss Models:

α = alpha, β = beta, γ= gamma, θ= theta, λ = lambda, μ = mu, σ= sigma, τ = tau
β = beta, used for the Beta and incomplete Beta functions.
Γ = Gamma, used for the Gamma and incomplete Gamma functions.
Φ = Phi, used for the Normal distribution. ϕ = phi, used for the Normal density function.
Π = Pi is used for the continued product just as
$\sum$ = Sigma is used for the continued sum

[1] Learning Objectives 1e and 1f are covered in "Mahler's Guide to Extreme Value Theory".
[2] In some cases I've rewritten these questions in order to match the notation in the current Syllabus.

Loss Distributions as per Loss Models

Distribution Name	Distribution Function	Probability Density Function
Exponential	$1 - e^{-x/\theta}$	$e^{-x/\theta} / \theta$
Single Parameter Pareto	$1 - \left(\frac{\theta}{x}\right)^{\alpha}$, $x > \theta$.	$\frac{\alpha\theta^{\alpha}}{x^{\alpha+1}}$, $x > \theta$
Weibull	$1 - \exp\left[-\left(\frac{x}{\theta}\right)^{\tau}\right]$	$\frac{\tau\left(\frac{x}{\theta}\right)^{\tau}\exp\left[-\left(\frac{x}{\theta}\right)^{\tau}\right]}{x}$
Gamma	$\Gamma[\alpha ; x/\theta]$	$\frac{(x/\theta)^{\alpha}\ e^{-x/\theta}}{x\ \Gamma(\alpha)} = \frac{x^{\alpha-1}\ e^{-x/\theta}}{\theta^{\alpha}\ \Gamma(\alpha)}$
LogNormal	$\Phi\left[\frac{\ln(x) - \mu}{\sigma}\right]$	$\frac{\exp\left[-\frac{(\ln(x) - \mu)^2}{2\sigma^2}\right]}{x\ \sigma\sqrt{2\pi}}$
Pareto	$1 - \left(\frac{\theta}{\theta+x}\right)^{\alpha}$	$\frac{\alpha\theta^{\alpha}}{(\theta+x)^{\alpha+1}}$
Inverse Gamma	$1 - \Gamma[\alpha ; \theta/x]$	$\frac{\theta^{\alpha}\ e^{-\theta/x}}{x^{\alpha+1}\ \Gamma[\alpha]}$

Moments of Loss Distributions as per Loss Models

Distribution Name	Mean	Variance	Moments
Exponential	θ	θ^2	$n!\ \theta^n$
Single Parameter Pareto	$\frac{\alpha\theta}{\alpha - 1}$	$\frac{\alpha\theta^2}{(\alpha-1)^2(\alpha-2)}$	$\frac{\alpha\theta^n}{\alpha - n}$, $\alpha > n$
Weibull	$\theta\ \Gamma[1 + 1/\tau]$	$\theta^2\ \{\Gamma[1 + 2/\tau] - \Gamma[1 + 1/\tau]^2\}$	$\theta^n\ \Gamma[1 + n/\tau]$
Gamma	$\alpha\theta$	$\alpha\theta^2$	$\theta^n \prod_{i=0}^{n-1} (\alpha + i) = \theta^n\ (\alpha)...(\alpha + n\text{-}1)$ $= \theta^n\ \frac{\Gamma[\alpha + n]}{\Gamma[\alpha]}$
LogNormal	$\exp[\mu + \sigma^2/2]$	$\exp[2\mu + \sigma^2]\ (\exp[\sigma^2] - 1)$	$\exp[n\mu + n^2\sigma^2/2]$
Pareto	$\frac{\theta}{\alpha - 1}$	$\frac{\alpha\theta^2}{(\alpha-1)^2(\alpha-2)}$	$\frac{n!\theta^n}{\prod_{i=1}^{n} (\alpha - i)} = \frac{n!\theta^n}{(\alpha-1)...(\alpha-n)}$, $\alpha > n$
Inverse Gamma	$\frac{\theta}{\alpha - 1}$	$\frac{\theta^2}{(\alpha-1)^2(\alpha-2)}$	$\frac{\theta^n}{\prod_{i=1}^{n} (\alpha - i)} = \frac{\theta^n}{(\alpha-1)...(\alpha-n)}$, $\alpha > n$

Section 1, Introduction

This Study Aid covers the material Loss Distributions that is on this exam.
This builds on material covered on Exam FAM.

I have two review sections that list the important ideas covered on this previous exam:
one on Frequency Distributions and one on Loss Distributions.
While this material should not be tested directly, it may help you to answer exam questions.
I would review this material to the extent you need to.

In actuarial work, frequency distributions are applied to the number of losses, the number of claims, the number of accidents, the number of persons injured per accident, etc.
Frequency Distributions are discrete functions on the nonnegative integers: 0, 1, 2, 3, ...

Most of the information you need to know about each of frequency distributions is shown in Appendix B, attached to the exam.

In actuarial work, loss distributions are applied to the size losses, the size of claim payments, etc. Loss Distributions are continuous functions, usually on $x > 0$.

Most of the information you need to know about each of loss distributions is shown in Appendix A, attached to the exam.

Section 2, Review of Important Ideas on Frequency Distributions

These concepts covered on Exam FAM may help you to answer questions on your exam.

Basic Concepts:

The mean is the average or expected value of the random variable.
The mode is the point at which the density function reaches its maximum.
The median, the 50th percentile, is the first value at which the distribution function is ≥ 0.5.
The 100pth percentile as the first value at which the distribution function ≥ p.
Variance = second central moment = $E[(X - E[X])^2] = E[X^2] - E[X]^2$.
Standard Deviation = Square Root of Variance.

Two variables X and Y are independent if and only if
Prob[X = x | Y = y] = Prob[X = x], and Prob[Y = y | X = x] = Prob[Y = y].

Binomial Distribution:

$$f(x) = \binom{m}{x} q^x (1-q)^{m-x} = \frac{m!}{x!\,(m-x)!} q^x (1-q)^{m-x},\ 0 \le x \le m.$$

Mean = mq **Variance = mq(1-q)**

Probability Generating Function: $P(z) = \{1 + q(z-1)\}^m$
The Binomial Distribution for m = 1 is a Bernoulli Distribution.
X is Binomial with parameters q and m_1, and Y is Binomial with parameters q and m_2,
X and Y independent, then X + Y is Binomial with parameters q and $m_1 + m_2$.

Poisson Distribution:

$f(x) = \lambda^x e^{-\lambda} / x!$, $x \ge 0$
Mean = λ **Variance = λ**
Probability Generating Function: $P(z) = e^{\lambda(z-1)}$, $\lambda > 0$.

A Poisson is characterized by a constant independent claim intensity and vice versa.
The sum of two independent variables each of which is Poisson with parameters λ_1 and λ_2 is also Poisson, with parameter $\lambda_1 + \lambda_2$.

If frequency is given by a Poisson and severity is independent of frequency, then the number of claims above a certain amount (in constant dollars) is also a Poisson.

Geometric Distribution:

f(x) $= \frac{\beta^x}{(1+\beta)^{x+1}}$.

Mean = β **Variance = β(1+β)**

Probability Generating Function: $P(z) = \frac{1}{1 - \beta(z-1)}$.

For a Geometric Distribution, for n > 0, the chance of at least n claims is: $\left(\frac{\beta}{1+\beta}\right)^n$.

The Geometric shares with the Exponential distribution, the "memoryless property." If one were to truncate and shift a Geometric Distribution, then one obtains the same Geometric Distribution.

Negative Binomial Distribution:

f(x) $= \frac{r(r+1)...(r+x-1)}{x!} \frac{\beta^x}{(1+\beta)^{x+r}}$. **Mean = rβ** **Variance = r β(1+β)**

Negative Binomial for r = 1 is a Geometric Distribution.

For the Negative Binomial Distribution with parameters β and r, with r integer, can be thought of as the sum of r independent Geometric distributions with parameter β.

If X is Negative Binomial with parameters β and r_1, and Y is Negative Binomial with parameters β and r_2, X and Y independent, then X + Y is Negative Binomial with parameters β and $r_1 + r_2$.

Normal Approximation:

In general, let μ be the mean of the frequency distribution, while σ is the standard deviation of the frequency distribution, then the chance of observing at least i claims and not more than j claims is approximately:

$$\Phi[\frac{(j+0.5)-\mu}{\sigma}] - \Phi[\frac{(i-0.5)-\mu}{\sigma}].$$

Normal Distribution

$F(x) = \Phi[(x-\mu)/\sigma]$

$f(x) = \phi[(x-\mu)/\sigma] / \sigma = \dfrac{\exp[-\frac{(x-\mu)^2}{2\sigma^2}]}{\sigma\sqrt{2\pi}}$, $-\infty < x < \infty$. $\phi(x) = \dfrac{\exp[-x^2/2]}{\sqrt{2\pi}}$, $-\infty < x < \infty$.

Mean = μ **Variance = σ^2**
Skewness = 0 (distribution is symmetric) Kurtosis = 3

<u>Skewness</u>:

Skewness = third central moment /STDDEV^3 = E[(X - E[X])^3]/STDDEV^3
= $\{E[X^3] - 3\bar{X}E[X^2] + 2\bar{X}^3\}$ / Variance$^{3/2}$.

A symmetric distribution has zero skewness.

Binomial Distribution with $q < 1/2 \Leftrightarrow$ positive skewness $\Leftrightarrow$ skewed to the right.
Binomial Distribution $q = 1/2 \Leftrightarrow$ symmetric $\Rightarrow$ zero skewness.
Binomial Distribution $q > 1/2 \Leftrightarrow$ negative skewness $\Leftrightarrow$ skewed to the left.
Poisson and Negative Binomial have positive skewness.

<u>Probability Generating Function</u>:

Probability Generating Function, p.g.f.:

P(z) = Expected Value of z^n = E[z^n] = $\sum_{n=0}^{\infty} f(n)\, z^n$.

The Probability Generating Function of the sum of independent frequencies is the product of the individual Probability Generating Functions.
The distribution determines the probability generating function and vice versa.

$f(n) = \left(\dfrac{d^n P(z)}{dz^n}\right)_{z=0} / n!$. f(0) = P(0). P'(1) = Mean.

Factorial Moments:

nth factorial moment = $\mu_{(n)}$ = E[X(X-1) .. (X+1-n)].

$\mu_{(n)} = \left(\frac{d^n P(z)}{dz^n}\right)_{z=1}$. P'(1) = E[X]. P"(1) = E[X(X-1)].

(a, b, 0) Class of Distributions:

For each of these three frequency distributions: **f(x+1) / f(x) = a + {b / (x+1)}**, x = 0, 1, ...
where a and b depend on the parameters of the distribution:

Distribution	a	b	f(0)
Binomial	-q/(1-q)	(m+1)q/(1-q)	$(1-q)^m$
Poisson	0	λ	$e^{-\lambda}$
Negative Binomial	$\beta/(1+\beta)$	$(r-1)\beta/(1+\beta)$	$1/(1+\beta)^r$

Distribution	Mean	Variance	Variance Over Mean	
Binomial	mq	mq(1-q)	1-q < 1	**Variance < Mean**
Poisson	λ	λ	1	**Variance = Mean**
Negative Binomial	$r\beta$	$r\beta(1+\beta)$	$1+\beta > 1$	**Variance > Mean**

Distribution	Thinning by factor of t	Adding n independent, identical copies
Binomial	$q \to tq$	$m \to nm$
Poisson	$\lambda \to t\lambda$	$\lambda \to n\lambda$
Negative Binomial	$\beta \to t\beta$	$r \to nr$

For X and Y independent:

X	**Y**	**X+Y**
Binomial(q, m_1)	**Binomial(q, m_2)**	**Binomial(q, $m_1 + m_2$)**
Poisson(λ_1)	**Poisson(λ_2)**	**Poisson($\lambda_1 + \lambda_2$)**
Negative Binomial(β, r_1)	**Negative Bin.(β, r_2)**	**Negative Bin.(β, $r_1 + r_2$)**

Accident Profiles:

For the Binomial, Poisson and Negative Binomial Distributions:
$(x+1)\ f(x+1) / f(x) = a(x + 1) + b$, where a and b depend on the parameters of the distribution.
$a < 0$ for the Binomial, $a = 0$ for the Poisson, and $a > 0$ for the Negative Binomial Distribution.

Thus if data is drawn from one of these three distributions, then we expect $(x+1)\ f(x+1) / f(x)$ for this data to be approximately linear with slope a; the sign of the slope, and thus the sign of a, distinguishes between these three distributions of the (a, b, 0) class.

Zero-Truncated Distributions:

In general if f is a distribution on 0, 1, 2, 3,..., then $p_k^T = \frac{f(k)}{1 - f(0)}$ is a distribution on 1, 2, 3, ...

Distribution	Density of the Zero-Truncated Distribution	
Binomial	$\frac{\frac{m!\ q^x\ (1-q)^{m-x}}{x!\ (m-x)!}}{1 - (1-q)^m}$	x = 1, 2, 3,..., m
Poisson	$\frac{e^{-\lambda}\ \lambda^x / x!}{1 - e^{-\lambda}}$	x = 1, 2, 3,...
Negative Binomial	$\frac{\frac{r(r+1)...(r+x-1)}{x!}\ \frac{\beta^x}{(1+\beta)^{x+r}}}{1 - 1/(1+\beta)^r}$	x = 1, 2, 3,...

The moments of a zero-truncated distribution are given in terms of those of the corresponding untruncated distribution, f, by: $E_{\text{Truncated}}[X^n] = \frac{E_f[X^n]}{1 - f(0)}$.

The Logarithmic Distribution has support on the positive integers: $f(x) = \frac{\left(\frac{\beta}{1+\beta}\right)^x}{x\ \ln(1+\beta)}$.

The **(a,b,1) class of frequency distributions** is a generalization of the (a,b,0) class.
As with the (a,b,0) class, the recursion formula applies: $\frac{\text{density at } x+1}{\text{density at } x} = a + \frac{b}{x+1}$.
However, it need only apply now for $x \geq 1$, rather than $x \geq 0$.

Members of the (a,b,1) family include: all the members of the (a,b,0) family, the zero-truncated versions of those distributions: Zero-Truncated Binomial, Zero-Truncated Poisson, Extended Truncated Negative Binomial, and the Logarithmic Distribution.
In addition the (a,b,1) class includes the zero-modified distributions corresponding to these.

Zero-Modified Distributions:

If f is a distribution on 0, 1, 2, 3,..., and $0 < p_0^M < 1$,

then probability at zero is p_0^M, $p_k^M = f(k) \frac{1 - p_0^M}{1 - f(0)}$, k = 1, 2, 3, ... is a distribution on 0, 1, 2, 3, ...

The moments of a zero-modified distribution are given in terms of those of f by:

$E_{Modified}[X^n] = (1 - p_0^M) \frac{E_f[X^n]}{1 - f(0)} = (1 - p_0^M)\ E_{Truncated}[X^n]$.

Compound Frequency Distributions:

A compound frequency distribution has a primary and secondary distribution, each of which is a frequency distribution. The primary distribution determines how many independent random draws from the secondary distribution we sum.

p.g.f. of compound distribution = p.g.f. of primary dist.[p.g.f. of secondary dist.]

$P(z) = P_1[P_2(z)]$.

compound density at 0 = p.g.f. of the primary at the density at 0 of the secondary.

Moments of Compound Distributions:

Mean of Compound Dist. = (Mean of Primary Dist.)(Mean of Sec. Dist.)
Variance of Compound Dist. = (Mean of Primary Dist.)(Var. of Sec. Dist.)
+ (Mean of Secondary Dist.)^2(Variance of Primary Dist.

In the case of a Poisson primary distribution with mean λ, the variance of the compound distribution could be rewritten as: λ(2nd moment of Second. Dist.).
The third central moment of a compound Poisson distribution = λ(3rd moment of Sec. Dist.).

Mixed Frequency Distributions:

The density function of the mixed distribution, is the mixture of the density function for specific values of the parameter that is mixed.

The nth moment of a mixed distribution is the mixture of the nth moments.
First one mixes the moments, and then computes the variance of the mixture from its first and second moments.

The Probability Generating Function of the mixed distribution, is the mixture of the probability generating functions for specific values of the parameter.
For a mixture of Poissons, the variance is always greater than the mean.

Gamma Function:

The (complete) **Gamma Function** is defined as:

$$\boldsymbol{\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}\, dt} = \theta^{-\alpha} \int_0^\infty t^{\alpha-1} e^{-t/\theta}\, dt, \text{ for } \alpha \geq 0, \theta \geq 0.$$

$\Gamma(\alpha) = (a-1)!$ $\Gamma(\alpha) = (\alpha-1)\,\Gamma(\alpha-1)$

$$\boldsymbol{\int_0^\infty t^{\alpha-1} e^{-t/\theta}\, dt = \Gamma(\alpha)\, \theta^\alpha}.$$

The **Incomplete Gamma Function** is defined as:

$$\Gamma(\alpha; x) = \int_0^x t^{\alpha-1} e^{-t}\, dt \;/\; \Gamma(\alpha).$$

Gamma-Poisson Frequency Process:

If one mixes Poissons via a Gamma, then the mixed distribution is in the form of the Negative Binomial distribution with r = α and β = θ.

If one mixes Poissons via a Gamma Distribution with parameters α and θ, then over a period of length Y, the mixed distribution is Negative Binomial with $r = \alpha$ and $\beta = Y\theta$.

For the Gamma-Poisson, the variance of the mixed Negative Binomial is equal to:
mean of the Gamma + variance of the Gamma.

$Var[X] = E[Var[X \mid \lambda]] + Var[E[X \mid \lambda]]$. **Mixing increases the variance**.

Tails of Frequency Distributions:

From lightest to heaviest righthand tail, the frequency distribution in the (a,b,0) class are:
Binomial, Poisson, Negative Binomial r > 1, Geometric, Negative Binomial r < 1.

Section 3, Review of Important Ideas on Loss Distributions

These concepts covered on Exam FAM may help you answer questions on your exam.

Statistics Ungrouped Data :

Average of X = 1st moment = E[X].
Average of X^2 = 2nd moment about the origin = $E[X^2]$.

Mean = **E[X]**.

Mode = the value most likely to occur.
Median = the value at which the distribution function is 50% = 50th percentile.

Variance = second central moment = $E[(X - E[X])^2] = E[X^2] - E[X]^2$.
Standard Deviation = $\sqrt{\text{Variance}}$ **.**

$Var[kX] = k^2 Var[X]$.

For **independent** **random variables the variances add**.

The average of n independent, identically distributed variables has a variance of Var[X] / n.

$Var[X+Y] = Var[X] + Var[Y] + 2Cov[X,Y]$.

$Cov[X,Y] = E[XY] - E[X]E[Y]$. $Corr[X, Y] = Cov[X ,Y] / \sqrt{Var[X]Var[Y]}$.

Sample Mean = $\sum X_i / N = \bar{X}$.

The sample variance is an unbiased estimator of the variance of the distribution from which a data set was drawn: **Sample variance** $\equiv \frac{\sum (X_i - \bar{X})^2}{N - 1}$.

<u>Coefficient of Variation and Skewness</u>:

Coefficient of Variation (CV) = Standard Deviation / Mean.

$\mathbf{1 + CV^2} = \mathbf{E[X^2] / E[X]^2}$ = 2nd moment divided by the square of the mean.

Average of X^3 = 3rd moment about the origin = $E[X^3]$.
Third Central Moment = $E[(X - E[X])^3] = E[X^3] - 3\,E[X]\,E[X^2] + 2\,E[X]^3$.

Skewness = $\gamma_1 = \dfrac{E[(X - E[X])^3]}{\text{STDDEV}^3}$. **A symmetric distribution has zero skewness**.

$$\text{Kurtosis} = \frac{E[(X - E[X])^4]}{\text{Variance}^2} = \frac{E[X^4] - 4\,E[X]\,E[X^3] + 6\,E[X]^2\,E[X^2] - 3\,E[X]^4}{\text{Variance}^2}.$$

When computing the empirical coefficient of variation, skewness, or kurtosis, we use the biased estimate of the variance, with n in the denominator, rather than the sample variance.

<u>Working with Grouped Data</u>:

For Grouped Data, if one is given the dollars of loss for claims in each interval, then one can compute $E[X \wedge x]$, LER(x), R(x), and e(x), provided x is an endpoint of an interval.

<u>Uniform Distribution</u>:

Support: $a \le x \le b$ Parameters: None
D. f. : **F(x) = (x-a) / (b-a)** P. d. f. : **f(x) = 1/ (b-a)**

Moments: $E[X^n] = \dfrac{b^{n+1} - a^{n+1}}{(b-a)\,(n+1)}$

Mean = (b+a)/2 **Variance = $\mathbf{(b-a)^2/12}$**

<u>Statistics of Grouped Data</u>:

One can estimate moments of Grouped Data by assuming the losses are uniformly distributed on each interval and then weighting together the moments for each interval by the number of claims observed in each interval.

<u>Percentiles</u>:

For a continuous distribution, the 100pth percentile is the first value at which F(x) = p.
For a discrete distribution, take the 100pth percentile as the first value at which F(x) ≥ p.

Definitions:

A loss event or claim is an incident in which an insured or group of insureds suffers damages which are potentially covered by their insurance contract.
The loss is the dollar amount of damage suffered by an insured or group of insureds as a result of a loss event. The loss may be zero.

A payment event is an incident in which an insured or group of insureds receives a payment as a result of a loss event covered by their insurance contract.
The amount paid is the actual dollar amount paid to the policyholder(s) as a result of a loss event or a payment event. If it is as the result of a loss event, the amount paid may be zero.
A loss distribution is the probability distribution of either the loss or the amount paid from a loss event or of the amount paid from a payment event.
The severity can be either the loss or amount paid random variable.
The exposure base is the basic unit of measurement upon which premiums are determined.
The frequency is the number of losses or number of payments random variable.

Parameters of Distributions:

For a given type of distribution, in addition to the size of loss x, F(x) depends on what are called parameters. The numerical values of the parameter(s) distinguish among the members of a parametric family of distributions.
It is useful to group families of distributions based on how many parameters they have.
A scale parameter is a parameter which divides x everywhere it appears in the distribution function. A scale parameter will appear to the n^{th} power in the formula for the n^{th} moment of the distribution. A shape parameter affects the shape of the distribution and appears in the coefficient of variation and the skewness.

Exponential Distribution:

Support: $x > 0$ Parameter: $\theta > 0$ (scale parameter)

$F(x) = 1 - e^{-x/\theta}$ **$f(x) = e^{-x/\theta} / \theta$**

Mean = θ **Variance = θ^2** 2nd moment = $2\theta^2$

Coefficient of Variation = 1. **Skewness = 2**.

e(x) = Mean Excess Loss = θ

When an Exponential Distribution is truncated and shifted from below, one gets the same Exponential Distribution, due to its memoryless property.

Single Parameter Pareto Distribution:

Support: **x >** θ Parameter: $\alpha > 0$ (shape parameter)

F(x) = 1 - $\left(\frac{\theta}{x}\right)^{\alpha}$ **f(x) =** $\frac{\alpha\theta^{\alpha}}{x^{\alpha+1}}$ Mean = $\frac{\alpha\theta}{\alpha-1}$, $\alpha > 1$.

Common Two Parameter Distributions:

Pareto: α is a shape parameter and θ is a scale parameter.
The Pareto is a heavy-tailed distribution. **Higher moments may not exist**.

F(x) = **1** - $\left(\frac{\theta}{\theta+x}\right)^{\alpha}$ = 1 - $(1 + x / \theta)^{-\alpha}$ f(x) = $\frac{\alpha\theta^{\alpha}}{(\theta+x)^{\alpha+1}}$ **Mean =** $\frac{\theta}{\alpha-1}$, $\alpha > 1$.

$E[X^n] = \frac{n!\theta^n}{(\alpha-1)...(\alpha-n)}$, $\alpha > n$. Mean Excess Loss = $\frac{\theta + x}{\alpha-1}$, $\alpha > 1$.

If losses prior to any deductible follow a Pareto Distribution with parameters α and θ, then after truncating and shifting from below by a deductible of size d, one gets another Pareto Distribution, but with parameters α and $\theta+d$.

Gamma: α is a shape parameter and θ is a scale parameter. Note the factors of θ in the moments. **For α = 1 you get the Exponential**.
The sum of n independent identically distributed variables which are Gamma with parameters α and θ is a Gamma distribution with parameters $n\alpha$ and θ. For α = a positive integer, the Gamma distribution is the sum of α independent variables each of which follows an Exponential distribution.

$F(x) = \Gamma(\alpha; x/\theta)$ $f(x) = \frac{x^{\alpha-1} e^{-x/\theta}}{\theta^{\alpha}\,\Gamma(\alpha)}$

Mean = $\alpha\theta$ **Variance =** $\alpha\theta^2$ $E[X^n] = \theta^n\,(\alpha)...(\alpha+n-1)$.

The skewness for the Gamma distribution is always twice the coefficient of variation.

LogNormal: If ln(x) follows a Normal, then x itself follows a LogNormal.

$F(x) = \Phi\left[\frac{\ln(x) - \mu}{\sigma}\right]$ $f(x) = \frac{\exp\left[-\frac{(\ln(x) - \mu)^2}{2\sigma^2}\right]}{x\,\sigma\sqrt{2\pi}}$

Mean = $\exp[\mu + 0.5\sigma^2]$ Second Moment = $\exp[2\mu + 2\sigma^2]$

Weibull: τ is a shape parameter, while θ is a scale parameter.

$F(x) = 1 - \exp\left[-\left(\frac{x}{\theta}\right)^{\tau}\right]$ $\qquad f(x) = \frac{\tau}{\theta^{\tau}}\, x^{\tau-1} \exp\left[-\left(\frac{x}{\theta}\right)^{\tau}\right]$

For $\tau = 1$ you get the Exponential Distribution.

Other Two Parameter Distributions:

Inverse Gaussian: Mean = μ Variance = μ^3 / θ

$$F(x) = \Phi\left[\left(\frac{x}{\mu} - 1\right)\sqrt{\frac{\theta}{x}}\right] + e^{2\theta/\mu}\, \Phi\left[-\left(\frac{x}{\mu} + 1\right)\sqrt{\frac{\theta}{x}}\right]. \qquad f(x) = \sqrt{\frac{\theta}{2\pi}}\ \frac{\exp\left[-\frac{\theta\left(\frac{x}{\mu} - 1\right)^2}{2x}\right]}{x^{1.5}}.$$

LogLogistic: $F(x) = \frac{(x/\theta)^{\gamma}}{1 + (x/\theta)^{\gamma}}$. $\qquad f(x) = \frac{\gamma\, x^{\gamma-1}}{\theta^{\gamma}(1 + (x/\theta)^{\gamma})^2}$.

Inverse Gamma : If X follows a Gamma Distribution with parameters α and 1, then θ/x follows an Inverse Gamma Distribution with parameters α and θ. α is the shape parameter and θ is the scale parameter. The Inverse Gamma is heavy-tailed.

$F(x) = 1 - \Gamma(\alpha ; \theta/x)$ $\qquad f(x) = \frac{\theta^{\alpha}\ e^{-\theta/x}}{x^{\alpha+1}\ \Gamma[\alpha]}$

Mean = $\frac{\theta}{\alpha - 1}$, $\alpha > 1$. $\qquad E[X^n] = \frac{\theta^n}{(\alpha-1)...(\alpha-n)}$, $\alpha > n$.

N-Point Mixtures of Models:

Mixing models is a technique that provides a greater variety of loss distributions.

One can take a weighted average of any two Distribution Functions:
G(x) = pA(x) + (1-p)B(x).
This is called a **2-point mixture of models**.

The Distribution Function of the mixture is the mixture of the Distribution Functions.
The Survival Function of the mixture is the mixture of the Survival Functions.
The density of the mixture is the mixture of the densities.

The mean of the mixture is the mixture of the means.
The moment of the mixture is the mixture of the moments:
$\mathbf{E_G[X^n] = p\ E_A[X^n] + (1\text{-}p)\ E_B[X^n]}$.

Limited Moments of the mixed distribution are the weighted average of the limited moments of the individual distributions: $E_G[X \wedge x] = p\ E_A[X \wedge x] + (1-p)\ E_B[X \wedge x]$.

In general, one can weight together any number of distributions, rather than just two.
These are called n-point mixtures.

Sometimes the mixture of models is just a mathematical device with no physical significance.
However, it can also be useful when the data results from different perils.

Variable Mixture ⇔ weighted average of unknown # of distributions of the same family but differing parameters ⇔ $F(x) = \sum w_i\ F_i(x)$, with each F_i of the same family and $\sum w_i$ =

Relationship to Life Contingencies:

${}_{y-x}p_x \equiv$ Prob[Survival past y I Survival past x] = S(y)/S(x).
${}_{y-x}q_x \equiv$ Prob[Not Surviving past y I Survival past x] = {S(x) - S(y)} / S(x) = 1 - ${}_{y-x}p_x$.

$p_x \equiv {}_1p_x$ = Prob[Survival past x+1 I Survival past x] = S(x+1) / S(x).
$q_x \equiv {}_1q_x$ = Prob[Death within one year I Survival past x] = 1 - S(x+1) / S(x).

${}_{t|u}q_x \equiv$ Prob[x+t < time of death ≤ x+t+u I Survival past x] = {S(x+t) - S(x+t+u)} / S(x).

Section 4, Empirical Distribution Function

This section will discuss the Distribution and Survival Functions.

Cumulative Distribution Function:

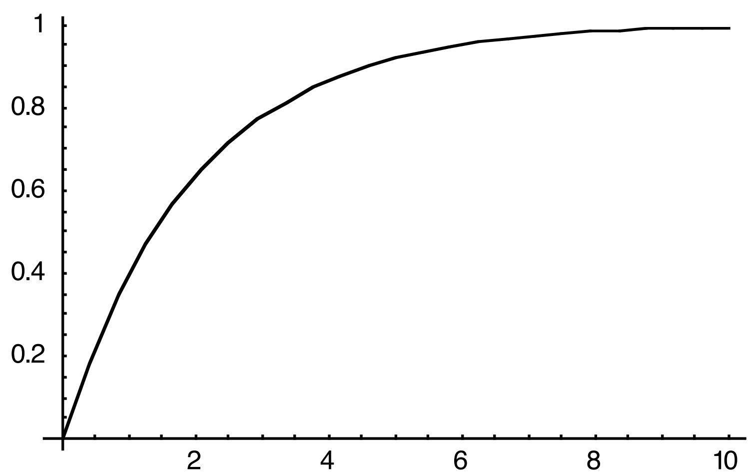
For the **Cumulative Distribution Function**, **F(x) = Prob[X ≤ x]**.

Various Distribution Functions are listed in Appendix A attached to your exam.
For example, for the Exponential Distribution, $F(x) = 1 - e^{-x/\theta}$.

Exercise: What is the value at 3 of an Exponential Distribution with $\theta = 2$.
[Solution: $F(x) = 1 - e^{-x/\theta}$. $F(3) = 1 - e^{-3/2} = 0.777$.]

$F'(x) = f(x) \geq 0$.
$0 \leq F(x) \leq 1$, nondecreasing, right-continuous, starts at 0 and ends at 1.[3]

Here is graph of the Exponential Distribution with $\theta = 2$:



[3] As x approaches y from above, F(x) approaches F(y). F would not be continuous at a jump discontinuity, but would still be right continuous.

Survival Function:

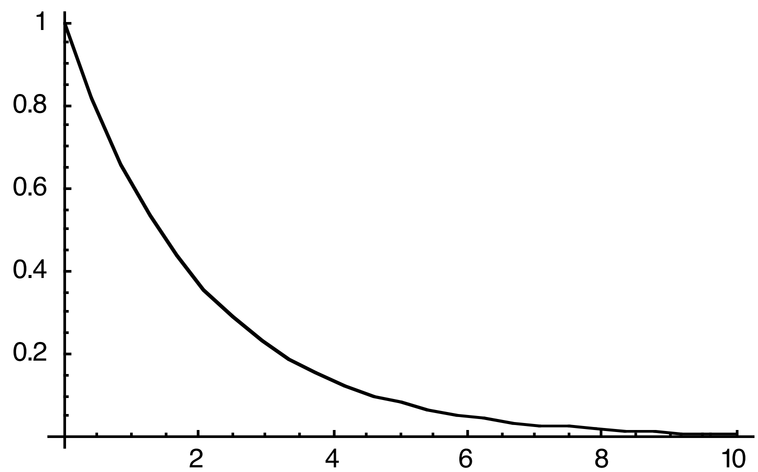
Similarly, we can define the Survival Function, S(x) = 1 - F(x) = Prob[X > x].

S'(x) = -f(x) ≤ 0.
0 ≤ S(x) ≤ 1, nonincreasing, right-continuous, starts at 1 and ends at 0.[4]

S(x) = 1 - F(x) = Prob[X > x] = the Survival Function
= the tail probability of the Distribution Function F.

For example, for the Exponential Distribution, $S(x) = 1 - F(x) = 1 - (1 - e^{-x/\theta}) = e^{-x/\theta}$.

Here is graph of the Survival Function of an Exponential with θ = 2:



Exercise: What is S(5) for a Pareto Distribution with α = 2 and θ = 3?
[Solution: $F(x) = 1 - \{\theta/(x+\theta)\}^{\alpha}$. $S(x) = \{\theta/(x+\theta)\}^{\alpha}$. $S(5) = \{3/(3+5)\}^2 = 9/64 = 14.1\%$.
Comment: See Appendix A in the tables attached to the exam.
The Pareto Distribution will be discussed in a subsequent section.]

In many situations you may find that the survival function is easier for you to use than the distribution function. Whenever a formula has S(x), one can always use 1 - F(x) instead, and vice-versa.

[4] See Definition 2.4 in Loss Models.

Empirical Model:

The **Empirical Model**: probability of 1/(# data points) is assigned to each observed value.[5]

Exercise: The following observations: 17, 16, 16, 19 are taken from a random sample.
What is the probability function (pdf) of the corresponding empirical model?
[Solution: p(16) = 1/2, p(17) = 1/4, p(19) = 1/4.]

Empirical Distribution Function:

The Empirical Model is the density that corresponds to the Empirical Distribution Function:
$F_n(x)$ = (# data points ≤ x)/(total # of data points).

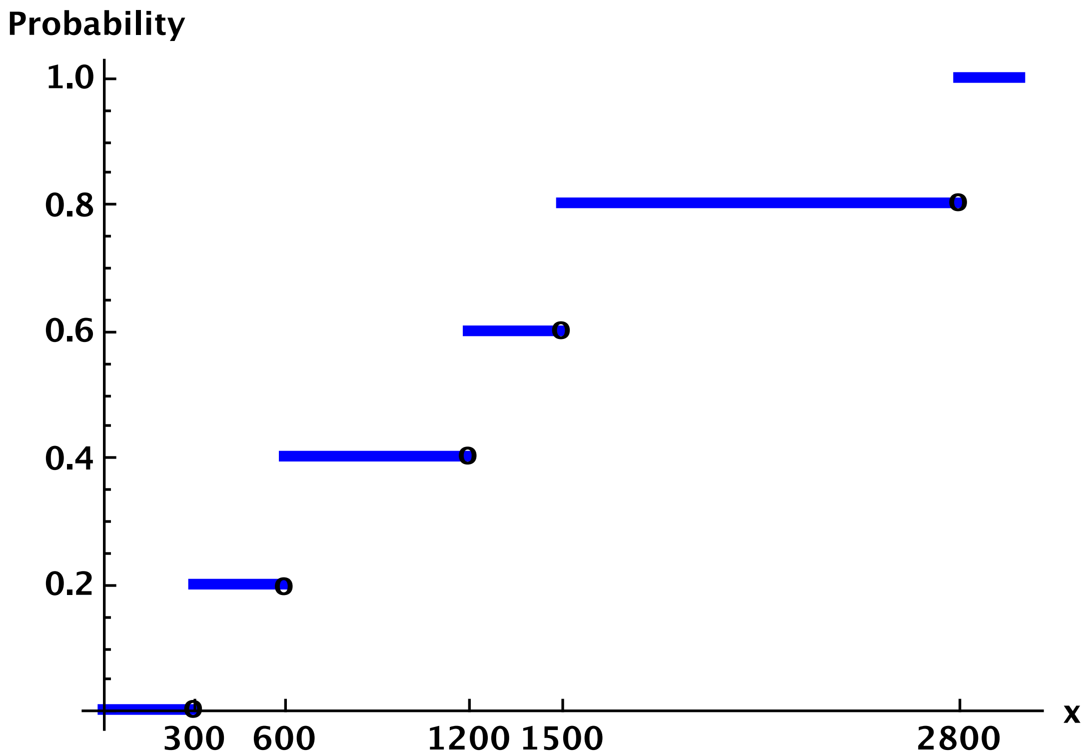
The Empirical Distribution Function at x, is the observed number of claims less than or equal to x divided by the total number of claims observed.

At each observed claim size the Empirical Distribution Function has a jump discontinuity.

Exercise: One observes losses of sizes: $300, $600, $1,200, $1,500, and $2,800.
What is the Empirical Distribution Function?
[Solution: $F_n(x)$ is: 0 for x < 300, 1/5 for 300 ≤ x < 600, 2/5 for 600 ≤ x < 1200,
3/5 for 1200 ≤ x < 1500, 4/5 for 1500 ≤ x < 2800, 1 for x ≥ 2800.]

[5] See Definition 4.8 in Loss Models.

Here is a graph of this Empirical Distribution Function:



The empirical distribution function is constant on intervals, with jumps up of 1/5 at each of the five observed points. For example, it is 1/5 at 599.99999 but 2/5 at 600.

<u>Mean and Variance of the Empirical Distribution Function</u>:

Assume the losses are drawn from a Distribution Function F(x). Then each observed loss has a chance of F(x) of being less than or equal to x. Thus the number of losses observed less than or equal to x is a sum of N independent Bernoulli trials with chance of success F(x). Thus if one has a sample of N losses, the number of losses observed less than or equal to x is Binomially distributed with parameters N and F(x).

Therefore, the Empirical Distribution Function is (1/N) times a Binomial Distribution with parameters N and F(x). Therefore, **the Empirical Distribution Function has mean of F(x) and a variance of: F(x){1-F(x)}/N.**

Exercise: Assume 130 losses are independently drawn from an Exponential Distribution:
$F(x) = 1 - e^{-x/300,000}$.
Then what is the distribution of the number of losses less than or equal to 100,000?
[Solution: The number of losses observed less than or equal to 100,000 is Binomially distributed with parameters 130, $1\text{-}e^{-1/3} = 0.283$.]

Exercise: Assume 130 losses are independently drawn from an Exponential Distribution:
$F(x) = 1 - e^{-x/300,000}$.
Then what is the variance of the number of losses less than or equal to 100,000?
[Solution: The number of losses observed less than or equal to 100,000 is Binomially distributed with parameters 130, $1-e^{-1/3} = 0.283$.
Thus it has a variance of: (130)(0.283)(1 - 0.283) = 26.38.]

Exercise: 130 losses are independently drawn from an Exponential Distribution:
$F(x) = 1 - e^{-x/300,000}$. What is the distribution of the empirical distribution function at 100,000?
[Solution: The number of losses observed less than or equal to 100,000 is Binomially distributed with parameters 130, 0.283. The empirical distribution function at 100,000, $F_n(100,000)$, is the percentage of losses ≤ 100,000. Thus the empirical distribution function at 100,000 is (1/130) times a Binomial with parameters 130 and 0.283.]

Exercise: 130 losses are independently drawn from an Exponential Distribution:
$F(x) = 1 - e^{-x/300,000}$.
What is the variance of the percentage of losses less than or equal to 100,000?
[Solution: $F_n(100,000)$ is (1/130) times a Binomial with parameters 130 and 0.283. Thus it has a variance of $(1/130)^2(130)(0.283)(1 - 0.283) = 0.00156$.]

As the number of losses, N, increases, the variance of the estimate of the distribution decreases as 1/N. All other things being equal, the variance of the empirical distribution function is largest when trying to estimate the middle of the distribution rather than either of the tails.[6]

<u>Empirical Survival Function</u>:

The Empirical Survival Function is: 1 - Empirical Distribution Function.

Empirical Distribution Function at x is: (# losses **≤** x)/(total # of losses).[7]

Empirical Survival Function at x is: (# losses **>** x)/(total # of losses).

Exercise: One observes losses of sizes: $300, $600, $1,200, $1,500, and $2,800.
What are the empirical distribution function and survival function at 1000?
[Solution: $F_n(1000)$ = (# losses ≤ 1000) / (# losses) = 2/5.
$S_n(1000)$ = (# losses > 1000) / (# losses) = 3/5.]

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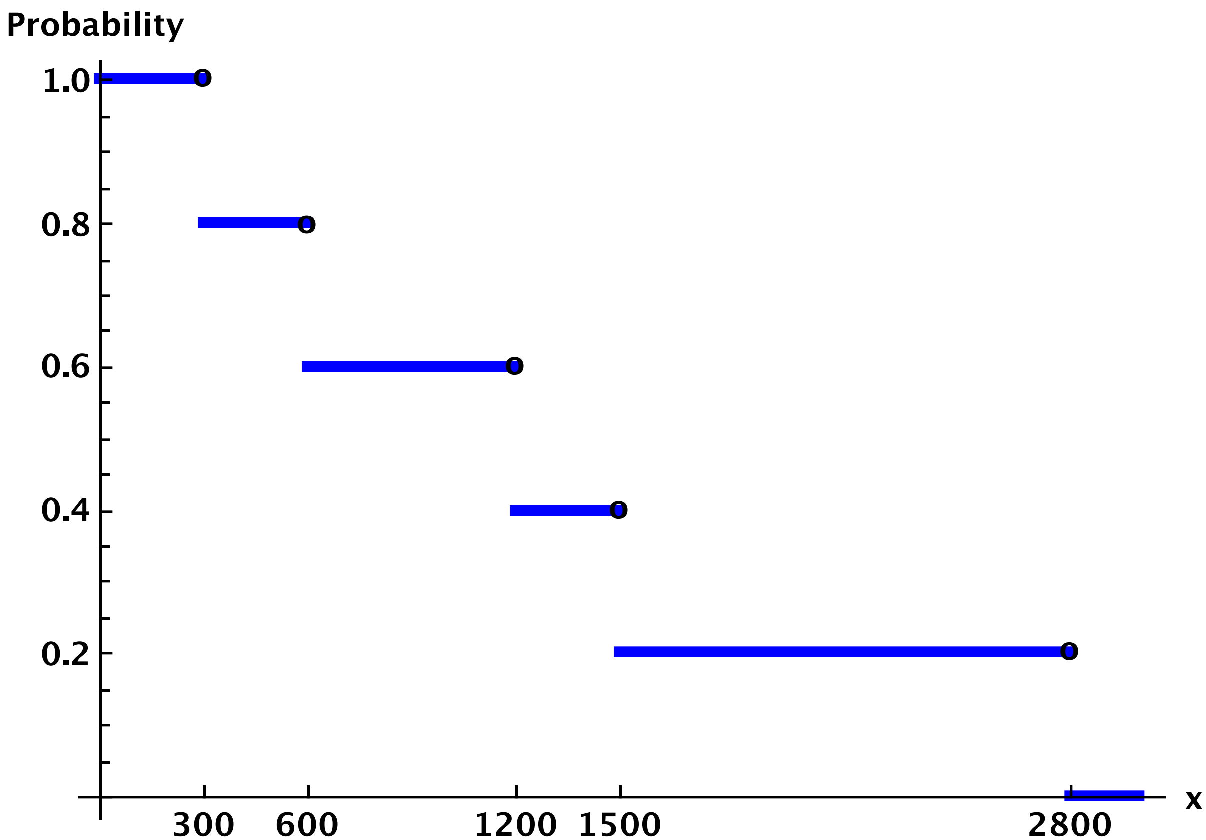
[6] F(x){1-F(x)} is largest for F(x) ≅ 1/2. However, small differences in the tail probabilities can be important.
[7] More generally, the empirical distribution function is: (# observations **≤** x) / (total # of observations).

For 300, 600, 1,200, 1,500, and 2,800, here is a graph of the Empirical Survival Function:



The empirical survival function is constant on intervals, with jumps down of 1/5 at each of the five observed points. For example, it is 4/5 at 599.99999 but 3/5 at 600.

Exercise: Determine the area under this empirical survival function.
[Solution: (1)(300) + (0.8)(300) + (0.6)(600) + (0.4)(300) + (0.2)(1300) = 1280.]

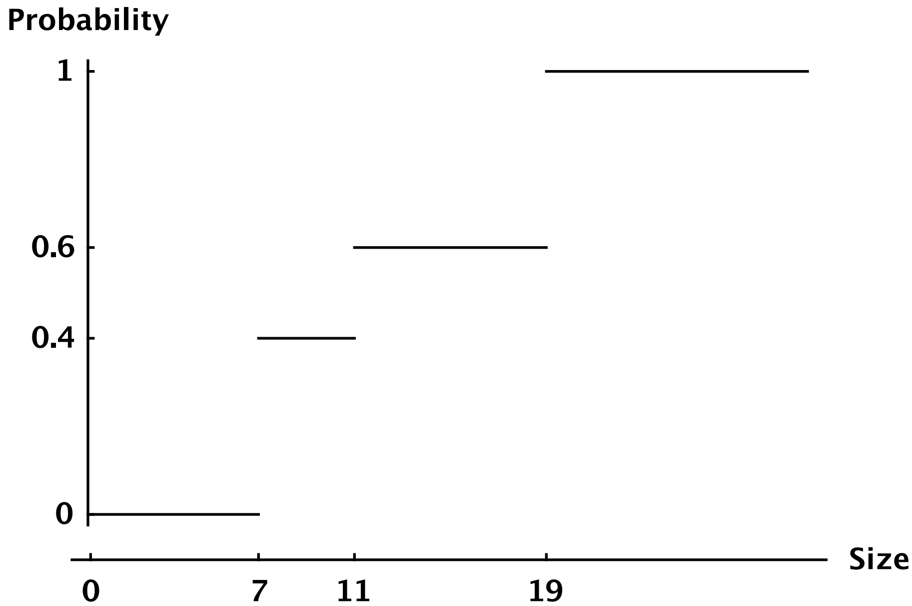
The sample mean, $\bar{X}$ = (300 + 600 + 1200 +1500 + 2800)/5 = 1280.
The sample mean is equal to the integral of the empirical survival function.
As will be discussed in a subsequent section, the mean is equal to the integral of the survival function, for those cases where the support of the survival function starts at zero.

Problems:

4.1 (1 point) Insureds suffer six losses of sizes: 3, 8, 13, 22, 35, 62.
What is the empirical survival function at 30?
A. 1/6 B. 1/3 C. 1/2 D. 2/3 E. 5/6

4.2 (1 point) You observe 5 losses of sizes: 15, 35, 70, 90, 140.
What is the empirical distribution function at 50?
A. 20% B. 30% C. 40% D. 50% E. 60%

Use the following information for the next two questions:
You are given the following graph of an empirical distribution function:



4.3 (1 point) Determine the mean of the data.
(A) Less than 10
(B) At least 10, but less than 11
(C) At least 11, but less than 12
(D) At least 12, but less than 13
(E) At least 13

4.4 (1 point) For this data, determine the biased estimator of the variance, $\frac{\sum (X_i - \bar{X})^2}{N}$.

(A) Less than 26
(B) At least 26, but less than 28
(C) At least 28, but less than 30
(D) At least 30, but less than 32
(E) At least 34

4.5 (2 points) F(200) = 0.9, F(d) = 0.25, and

$$\int_d^{200} f(x)\, dx = 75.$$

$$\int_d^{200} F(x)\, dx + d = 150.$$

Determine d.

A. 60 B. 70 C. 80 D. 90 E. 100

4.6 (CAS9, 11/99, Q.16) *(1 point) Which of the following can cause distortions in a loss claim size distribution derived from empirical data?*

1. Claim values tend to cluster around target values, such as $5,000 or $10,000.

2. Individual clams may come from policies with different policy limits.

3. Final individual claim sizes are not always known.

A. 1 B. 2 C. 3 D. 1, 2 E. 1, 2, 3

4.7 (IOA 101, 9/01, Q.7) (3.75 points) The probability density function of a random variable X is given by $f(x) = kx(1 - ax^2)$, $0 \le x \le 1$, where k and a are positive constants.

(i) (2.25 points) Show that $a \le 1$, and determine the value of k in terms of a.

(ii) (1.5 points) For the case a = 1, determine the mean of X.

Solutions to Problems:

4.1. B. S(30) = (# losses > 30)/(# losses) = 2/6 = **1/3**.

4.2. C. There are 2 losses of size ≤ 50. Empirical distribution function at 50 is: 2/5 = **0.4**.

4.3. D. From the empirical distribution function, 40% of the data is 7, 60% - 40% = 20% of the data is 11, and 100% - 60% = 40% of the data is 19.
The mean is: (40%)(7) + (20%)(11) + (40%)(19) = **12.6**.
Comment: If the data set was of size five, then it was: 7, 7, 11, 19, 19.
The mean is: 63/5 = 12.6.

4.4. C. From the empirical distribution function, 40% of the data is 7,
60% - 40% = 20% of the data is 11, and 100% - 60% = 40% of the data is 19.
The mean is: (40%)(7) + (20%)(11) + (40%)(19) = 12.6.
The second moment is: $(40\%)(7^2) + (20\%)(11^2) + (40\%)(19^2) = 188.2$.

$$\frac{\sum (X_i - \bar{X})^2}{N} = 188.2 - 12.6^2 = \mathbf{29.44}.$$

4.5. A. By integration by parts: $\int_d^{200} F(x)\, dx =$

$$xF(x) \Big]_{x=d}^{x=200} - \int_d^{200} x\, f(x)\, dx = (200)F(200) - d\,F(d) - 75 = (200)(0.9) - 0.25d - 75 = 105 - 0.25d.$$

⇒ 105 - 0.25d + d = 150. ⇒ d = 45/0.75 = **60**.

4.6 . E. All of these are true.
Item #3 is referring to the time between when the insurer knows about a claim and sets up a reserve, and when the claim is paid and closed.

4.7. (i) f(x) ≥ 0. ⇒ $1 - ax^2 \geq 0$, 0 ≤ x ≤ 1. ⇒ a ≤ 1.
Integral from 0 to 1 of $f(x) = k(x - ax^3)$ is: k(1/2 - a/4).
Setting this integral equal to one: k(1/2 - a/4) = 1. ⇒ k = 4/(2 - a).
(ii) k = 4/(2 - a) = 4/(2 - 1) = 4. $f(x) = 4x - 4x^3$.
The integral from zero to one of $xf(x) = 4x^2 - 4x^4$ is: 4/3 - 4/5 = **8/15**.

Section 5, Limited Losses

The next few sections will introduce a number of related ideas: the Limited Loss Variable, Limited Expected Value, Losses Eliminated, Loss Elimination Ratio, Excess Losses, Excess Ratio, Excess Loss Variable, Mean Residual Life/ Mean Excess Loss, and Hazard Rate/ Failure Rate.

$X \wedge 1000 \equiv$ Minimum of x and 1000 = Limited Loss Variable.

Exercise: An insured has losses of sizes: $300, $600, $1,200, $1,500 and $2,800.
What is $X \wedge 1000$?
[Solution: $X \wedge 1000$ = $300, $600, $1000, $1000, $1000.]

If the insured had a policy with a $1000 policy limit (and no deductible), then the insurer would pay $300, $600, $1000, $1000, and $1000, for a total of $3900 for these five losses.

The Limited Loss Variable corresponding to a limit u $\Leftrightarrow$ $\mathbf{X \wedge u}$ $\Leftrightarrow$ X_U $\Leftrightarrow$
censored from above at u $\Leftrightarrow$ right censored at u[8] $\Leftrightarrow$
the payments with a policy limit u (and no deductible) $\Leftrightarrow$ X for $X < u$, u for $X \geq u$.

Limited Expected Value:

Limited Expected Value at 1000 = $E[X \wedge 1000]$ =
an average over all sizes of loss of the minimum of 1000 and the size of loss.

Exercise: An insured has losses of sizes: $300, $600, $1,200, $1,500 and $2,800.
What is the (empirical) limited expected value at $1000?
[Solution: $E[X \wedge 1000] = (300 + 600 + 1000 + 1000 + 1000)/5 = 3900/5 = 780$.]

If the insured had a policy with a $1000 policy limit (and no deductible), then the insurer pays 3900 on 5 losses or an average of 780 per loss.

The mean of the limited loss variable corresponding to L = $\mathbf{E[X \wedge L]}$ =
the average payment per loss with a policy limit of L.

Since $\mathbf{E[X \wedge L]} \cong \mathbf{E[Min[X, L]]}$ = average of numbers each $\leq$ L, $\mathbf{E[X \wedge L] \leq L}$.
Since $E[X \wedge L] \cong E[Min[X, L]]$ = average of numbers each $\leq$ X, $\mathbf{E[X \wedge L] \leq E[X]}$.

[8] Censoring will be discussed in a subsequent section.

The limited expected value can be written as the sum of the contributions of the small losses and the large losses. The (theoretical) Limited Expected Value (LEV),
$E[X \wedge L]$, would be written for a continuous size of loss distribution as two pieces:

$$E[X \wedge L] = \int_0^L x\, f(x)\, dx + L\, S(L)$$

= contribution of small losses + contribution of large losses.

The first piece represents the contribution of losses up to L in size, while the second piece represents the contribution of those losses larger than L. The smaller losses each contribute their size, while the larger losses each contribute L to the average.

For example, for the Exponential Distribution:

$$E[X \wedge L] = \int_0^L x\, e^{-x/\theta}/\theta\, dx + L\, e^{-L/\theta} = (-x\, e^{-x/\theta} - \theta\, e^{-x/\theta})\Big]_{x=0}^{x=L} + L\, e^{-L/\theta} = \theta(1 - e^{-L/\theta}).$$ [9]

[9] See Appendix A of Loss Models and the tables attached to the exam.

Problems:

5.1 (1 point) Insureds suffer six losses of sizes: 3, 8, 13, 22, 35, 62.
What is the Limited Expected Value, for a Limit of 25?
A. 15 B. 16 C. 17 D. 18 E. 19

5.2 (1 point) You observe 5 losses of sizes: 15, 35, 70, 90, 140.
What is the Limited Expected Value at 50?
A. 10 B. 20 C. 30 D. 40 E. 50

Use the following information for the next two questions:
Frequency is Poisson with $\lambda = 20$.
E[X] = \$10,000.
$E[X \wedge 25{,}000]$ = \$8000.

5.3 (1 point) If there is no policy limit, what is the expected aggregate annual loss?

5.4 (1 point) If there is a 25,000 policy limit, what is the expected aggregate annual loss?

5.5 (2 points) For an insurance policy, you are given:
(i) The policy limit is 100,000 per loss, with no deductible.
(ii) Expected aggregate losses are 1,000,000 annually.
(iii) The number of losses follows a Poisson distribution.
(iv) The claim severity distribution has:

$S(50{,}000) = 10\%$.
$S(100{,}000) = 4\%$.
$E[X \wedge 50{,}000] = 28{,}000$.
$E[X \wedge 100{,}000] = 32{,}000$.

(v) Frequency and severity are independent.
Determine the probability that no losses will exceed 50,000 during the next year.
(A) 3.0% (B) 3.5% (C) 4.0% (D) 4.5% (E) 5.0%

5.6 (1 point) $E[X \wedge 5000] = 3200$.

Size	Number of Losses	Dollars of Loss
0 to 5000	170	???
5001 to 25,000	60	700,000
over 25,000	20	???

Determine $E[X \wedge 25{,}000]$.
(A) 5600 (B) 5800 (C) 6000 (D) 6200 (E) 6400

5.7 (4, 11/01, Q.36) (2.5 points) For an insurance policy, you are given:
(i) The policy limit is 1,000,000 per loss, with no deductible.
(ii) Expected aggregate losses are 2,000,000 annually.
(iii) The number of losses exceeding 500,000 follows a Poisson distribution.
(iv) The claim severity distribution has
Pr(Loss > 500,000) = 0.0106
E[min(Loss; 500,000)] = 20,133
E[min(Loss; 1,000,000)] = 23,759
Determine the probability that no losses will exceed 500,000 during 5 years.
(A) 0.01 (B) 0.02 (C) 0.03 (D) 0.04 (E) 0.05

Solutions to Problems:

5.1. B. E[X ∧ 25] = (3 + 8 + 13 + 22 + 25 + 25) / 6 = 96 / 6 = **16**.

5.2. D. E[X ∧ 50] = (15 + 35 + 50 + 50 + 50)/5 = **40**.

5.3. (20)($10,000) = **$200,000**.

5.4. (20)($8000) = **$160,000**.

5.5. D. 1,000,000 = expected annual aggregate loss = (mean frequency)E[X ∧ 100,000] = (mean frequency) (32,000). ⇒ mean frequency = 1 million / 32,000 = 31.25 losses per year.
The expected number of losses exceeding 50,000 is: (31.25)S(50,000) = 3.125.
The large losses are Poisson; the chance of having zero of them is: $e^{-3.125}$ = **4.4%**.
Comment: Similar to 4, 11/01, Q.36.

5.6. E. E[X ∧ 25,000] =
E[X ∧ 5000] + (contribution above 5000 from medium claims)
+ (contribution above 5000 from large claims)
= 3200 + {700,000 - (60)(5000)} / 250 + (20)(25,000 - 5000) / 250 = **6400**.
Alternately, let y be the dollars of loss on losses of size 0 to 5000.
Then, 3200 = E[X ∧ 5000] = {y + (5000) (60 + 20)} / 250. ⇒ y = 400,000.
E[X ∧ 25,000] = {400,000 + 700,000 + (20)(25,000)} / 250 = **6400**.
Comment: Each loss of size more than 25,000, contributes an additional 20,000 to E[X ∧ 25,000], compared to E[X ∧ 5000].
Each loss of size 5001 to 25,000 contributes an additional x - 5000 to E[X ∧ 25,000], compared to E[X ∧ 5000].

5.7. A. 2,000,000 = expected annual aggregate loss = (mean frequency)E[X ∧ 1 million] = (mean frequency) (23,759). Therefore, mean frequency = 2 million/ 23,759 = 84.18 per year.
Assuming frequency and severity are independent, the expected number of losses exceeding 1/2 million is: (84.18)(0.0106) = 0.892 per year.
Over 5 years we expect (5)(0.892) = 4.461 losses > 1/2 million. Since we are told these losses are Poisson Distributed, the chance of having zero of them is: $e^{-4.461}$ = **0.012**.

Section 6, Losses Eliminated

Assume an (ordinary) deductible of $10,000. Then the insurer would pay nothing for the small losses. For a loss of for example size $35,000, the insurer would pay:
$25,300 - $10,000 = $15,000. Similarly, for each large loss $10,000 is eliminated, from the point of view of the insurer.

Assume a deductible amount of d. The total dollars of loss eliminated is computed by summing up the sizes of loss for all losses less than d, and adding to that the sum of d per each loss greater than or equal to d.[10]

Exercise: An insured has losses of sizes: $300, $600, $1,200, $1,500 and $2,800.
How many dollars of loss are eliminated by a deductible of $1000?
[Solution: $300 + $600 + (3)($1000) = $3900.]

Let **N** be the total number of losses.

Then the **Losses Eliminated** by a deductible d would be written for a continuous size of loss distribution as the sum of the same two pieces, the contribution of the small losses plus the contribution of the large losses:

$$N \int_0^d x \, f(x) \, dx + N \, d \, S(d).$$

The first piece is the sum of losses less than d. (We have multiplied by the total number of losses since f(x) is normalized to integrate to unity.) The second piece is the number of losses greater than d times d per such loss. Note that the losses eliminated are just the number of losses times the Limited Expected Value.

Losses Eliminated by deductible d are: $N \, E[X \wedge d]$.

[10] Note that the Empirical Losses Eliminated are a continuous function of the deductible amount; a small increase in the deductible amount produces a corresponding small increase in the empirical losses eliminated.

Loss Elimination Ratio:

Loss Elimination Ratio at d = LER(d) = $\frac{\text{Losses Eliminated by a deductible of size d}}{\text{Total Losses}}$.

Exercise: An insured has losses of sizes: $300, $600, $1,200, $1,500 and $2,800.
What is the (empirical) loss elimination ratio at $1000?
[Solution: $3900 losses are eliminated out of a total of 300 + 600 + 1200 + 1500 + 2800 = 6400.
Therefore, LER(1000) = 3900/6400 = 60.9%.]

In general the LER is the ratio of the losses eliminated to the total losses.[11]

The loss elimination ratio at x can be written as:

$$\text{LER}(x) = \frac{\text{dollars of loss limited by x}}{\text{total losses}} = \frac{\text{(dollars of loss limited by x) / N}}{\text{(total losses) / N}} = \frac{E[X \wedge x]}{\text{Mean}}.$$

$$\mathbf{LER(x) = \frac{E[X \wedge x]}{E[X]}}.$$

For example, for five losses of sizes: $300, $600, $1,200, $1,500 and $2,800,
$E[X \wedge 1000]$ = (300 + 600 + 1000 + 1000 + 1000)/5 = 3900/5 = 780,
and E[X] = (300 + 600 + 1200 + 1500 + 2800)/5 = 6400/5 = 1280.
Then $E[X \wedge 1000]/E[X]$ = 780/1280 = 60.9%. matching the previous computation of LER(1000).

[11] Since its numerator is continuous while its denominator is independent of the deductible amount, the empirical loss elimination ratio is a continuous function of the deductible amount.

Problems:

6.1 (1 point) Insureds suffer six losses of sizes: 3, 8, 13, 22, 35, 62.
What is the Loss Elimination Ratio, for a deductible of 10?
A. less than 0.37
B. at least 0.37 but less than 0.39
C. at least 0.39 but less than 0.41
D. at least 0.41 but less than 0.43
E. at least 0.43

6.2 (1 point) You observe 5 losses of sizes: 15, 35, 70, 90, 140.
What is the Loss Elimination Ratio at 50?
A. 54% B. 57% C. 60% D. 63% E. 66%

6.3 (2 points) You observe the following payments on 6 losses with no deductible applied:
\$200, \$300, \$400, \$800, \$900, \$1,600.
Let A be the loss elimination ratio (LER) for a \$500 deductible.
Let B be the loss elimination ratio (LER) for a \$1000 deductible. Determine B - A.
A. 30% B. 35% C. 40% D. 45% E. 50%

6.4 (4, 5/89, Q.57) (1 point)
Given the following payments on 6 losses, calculate the loss elimination ratio (LER) for a \$300 deductible (assume the paid losses had no deductible applied).
Paid Losses: \$200, \$300, \$400, \$800, \$900, \$1,600.
A. LER ≤ 0.40 B. 0.40 < LER ≤ 0.41 C. 0.41 < LER ≤ 0.42
D. 0.42 < LER ≤ 0.43 E. 0.43 < LER

6.5 (CAS5, 5/03, Q.38) (3 points) Given the information below, calculate the loss elimination ratio for ABC Company's collision coverage in State X at a \$250 deductible. Show all work.
- ABC insures 5,000 cars at a \$250 deductible with the following fully credible data on the collision claims:
  Paid losses are \$1,000,000 per year.
  The average number of claims per year is 500.
- A fully credible study found that in State X:
  The average number of car accidents per year involving collision damage was 10,000.
  The average number of vehicles was 67,000.
- Assume ABC Company's expected ground-up claims frequency is equal to that of State X.
- Assume the average size of accidents that fall below the deductible is \$150.

Solutions to Problems:

6.1. A. LER(10) = Losses Eliminated / Total Losses =
(3 + 8 + 10 + 10 + 10 + 10) / (3 + 8 + 13 + 22 + 35 + 62) = 51 / 143 = **0.357.**

6.2. B. E[X] = (15+35+70+90+140)/ 5 = 70. LER(50) = $E[X \wedge 50]/E[X]$ = 40/70 = **0.571**.

6.3. A. The Losses Eliminated for a \$500 deductible are: 200 + 300 + 400 + (3)(500) = 2400.
The total losses are 4200.
Thus LER(500) = Losses Eliminated / Total Losses = 2400/4200 = 0.571.
Losses Eliminated for a \$1000 deductible are: 200 + 300 + 400 + 800 + 900 + 1000 = 3600.
Thus LER(1000) = Losses Eliminated / Total Losses = 3600/4200 = 0.857.
LER(1000) - LER(500) = 0.857 - 0.571 = **0.286**.

6.4. B. The Losses Eliminated are: 200 + 300 + (4)(300) = 1700. The total losses are 4200.
Thus the LER = Losses Eliminated / Total Losses = 1700/4200 = **0.405**.

6.5. Accident Frequency for State X is: 10,000/67,000 = 14.925%.
For 5000 cars, expect: (14.925%)(5000) = 746.3 accidents.
There were 500 claims, in other words 500 accidents of size greater than the \$250 deductible.
Thus we infer: 746.3 - 500 = 246.3 small accidents.
These small accidents had average size \$150, for a total of: (246.3)(\$150) = \$36,945.
Deductible eliminates \$250 for each large accident, for a total of: (\$250)(500) = \$125,000.
Losses eliminated = \$36,945 + \$125,000 = \$161,945.
Total losses = losses eliminated + losses paid = \$161,945 + \$1,000,000 = \$1,161,945.
LER at \$250 = Losses Eliminated / Total Losses = \$161,945 / \$1,161,945 = **13.9%**.
Alternately, frequency of loss = 10,000/67,000 = 14.925%.
Frequency of claims (accidents of size > 250) = 500/5000 = 10%.
S(250) = 10%/14.925% = 0.6700. F(250) = 1 - S(250) = 0.3300.
Average size of accidents that fall below the deductible = average size of small accidents =
$\$150 = \{E[X \wedge 250] - 250S(250)\}/F(250) = \{E[X \wedge 250] - (\$250)(0.67)\}/0.33$.
$\Rightarrow E[X \wedge 250] = (0.33)(\$150) + (0.67)(\$250) = \$217$.
Average payment per non-zero payment = \$1,000,000/500 =
$\$2000 = (E[X] - E[X \wedge 250])/S(250) = (E[X] - E[X \wedge 250])/0.67$.
$\Rightarrow E[X] - E[X \wedge 250] = \$1340$. $\Rightarrow E[X] = \$1340 + \$217 = \$1557$.
$LER(250) = E[X \wedge 250] / E[X] = \$217/\$1557 =$ **13.9%**.

Section 7, Excess Losses

The dollars of loss excess of \$10,000 per loss are also of interest. These are precisely the dollars of loss not eliminated by a deductible of size \$10,000.

$(X - d)_+ \equiv 0$ when $X \le d$, $X - d$ when $X > d$.[12]

$(X - d)_+$ is the amount paid to an insured with a deductible of d.

The insurer pays nothing if $X \le d$, and pays $X - d$ if $X > d$.

Exercise: An insured has losses of sizes: \$300, \$600, \$1,200, \$1,500 and \$2,800.
What is $(X - 1000)_+$?
[Solution: 0, 0, \$200, \$500, and \$1800.]

$(X - d)_+$ is referred to as the "left censored and shifted variable" at d.[13]

$(X - d)_+ \Leftrightarrow$

0 when $X \le d$, $X - d$ when $X > d$

$\Leftrightarrow$ the amounts paid to insured with a deductible of d

$\Leftrightarrow$ payments per loss, including when the insured is paid nothing due to the deductible of d $\Leftrightarrow$ amount paid per loss.

Exercise: An insured has losses of sizes: \$300, \$600, \$1,200, \$1,500 and \$2,800.
What is $E[(X - 1000)_+]$?
[Solution: (0 + 0 + \$200 + \$500 + \$1800)/ 5 = \$500.]

The expected losses excess of 10,000 per loss would be written for a continuous size of loss distribution as:

$$E[(X - 10{,}000)_+] = \text{Losses Excess of 10,000 per loss} = \int_{10{,}000}^{\infty} (x - 10{,}000)\, f(x)\, dx.$$

Note that we only integrate over those losses greater than \$10,000 in size, since smaller losses contribute nothing to the excess losses. Also larger losses only contribute the amount by which each exceeds \$10,000.

$$E[(X - 10{,}000)_+] = \int_{10{,}000}^{\infty} (x - 10{,}000)\, f(x)\, dx = \int_{10{,}000}^{\infty} x\, f(x)\, dx - 10{,}000 \int_{10{,}000}^{\infty} f(x)\, dx =$$

$$\int_0^{\infty} x\, f(x)\, dx - \left\{ \int_0^{10{,}000} x\, f(x)\, dx + 10{,}000\, S(10{,}000) \right\} = E[X] - E[X \wedge 10{,}000].$$

---

[12] The "+" refers to taking the variable X - d when it is positive, and otherwise setting the result equal to zero.
[13] Censoring will be discussed in a subsequent section. See Definition 3.4 in Loss Models.

Losses Excess of L per loss = $\mathbf{E[(X - L)_+] = E[X] - E[X \wedge L]}$.

Exercise: An insured has losses of sizes: $300, $600, $1,200, $1,500 and $2,800.
Show that $E[(X - 1000)_+] = E[X] - E[X \wedge 1000]$.
[Solution: $E[X] - E[X \wedge 1000] = 1280 - 780 = 500 = E[(X - 1000)_+]$.]

Exercise: For an Exponential Distribution with $\theta = 100$, what is $E[(X - 70)_+]$?
[Solution: $E[(X - 70)_+] = E[X] - E[X \wedge 70] = 100 - 100(1 - e^{-70/100}) = 49.7$.]

Excess Ratio:[14]

The **Excess Ratio** is the losses excess of the given limit divided by the total losses.

Excess Ratio at x = R(x) ≡ (losses excess of x)/(total losses)
$$\mathbf{= E[(X - x)_+] / E[X] = (E[X] - E[X \wedge x]) / E[X].}$$

$$\mathbf{R(x) = 1 - LER(x) = 1 - E[X \wedge x] / E[X].}$$

Exercise: An insured has losses of sizes: $300, $600, $1,200, $1,500 and $2,800.
What is the (empirical) excess ratio at $1000?
[Solution: R(1000) = (200 + 500 + 1800)/6400 = 39.1% = 1 - 60.9% = 1 - LER(1000)
$= 1 - E[X \wedge 1000] / E[X] = 1 - 780/1280$.]

One can also write the Excess Ratio in terms of integrals as:

$$R(L) = \frac{\int_L^\infty (x - L)\ f(x)\ dx}{\int_0^\infty x\ f(x)\ dx} = \frac{\int_L^\infty x\ f(x)\ dx - L\ S(L)}{\int_0^\infty x\ f(x)\ dx}.$$

However, in order to compute the excess ratio or loss elimination ratio, it is usually faster to use the formulas in Appendix A of Loss Models for the Mean and Limited Expected Value.

Exercise: For an Exponential Distribution with $\theta = 100$, what is R(70)?
[Solution: $R(70) = 1 - E[X \wedge 70]/E[X] = 1 - 100(1 - e^{-70/100})/100 = 49.7\%$.]

[14] Loss Models does not use the commonly used term Excess Ratio. However, this important concept may help you to understand and answer questions. Since the Excess Ratio is just one minus the Loss Elimination Ratio, one can always work with the Loss Elimination Ratio instead of the Excess Ratio.

<u>Total Losses = Limited Losses + Excess Losses</u>:

Exercise: For a loss of size 6 and a loss of size 15, list $X \wedge 10$, $(X-10)_+$, and $(X \wedge 10) + (X-10)_+$.
[Solution:

X	$X \wedge 10$	$(X-10)_+$	$(X \wedge 10) + (X-10)_+$
6	6	0	6
15	10	5	15

In general, $\mathbf{X = (X \wedge d) + (X - d)_+}$.

In other words, buying two policies, one with a policy limit of 1000 (and no deductible), and another with a deductible of 1000 (and no policy limit), provides the same coverage as a single policy with no deductible or policy limit.

A deductible of 1000 caps the policyholder's payments at 1000, so from his point of view the 1000 deductible acts as a limit. The policyholder's retained loss is: $X \wedge 1000$. The insurer's payment to the policyholder is: $(X - 1000)_+$. Together they total to the loss, X.

A deductible from one point of view is a policy limit from another point of view.[15] Remember the losses eliminated by a deductible of size 1000 are $E[X \wedge 1000]$, the same expression as the losses paid under a policy with limit of size 1000 (and no deductible).

$X = (X \wedge d) + (X - d)_+$. $\Rightarrow$ $E[X] = E[X \wedge d] + E[(X - d)_+]$. $\Rightarrow$ $\mathbf{E[(X - d)_+] = E[X] - E[X \wedge d]}$.
Expected Excess = Expected Total Losses - Expected Limited Losses.

[15] *An insurer who buys reinsurance with a per claim deductible of 1 million, has capped its retained losses at 1 million per claim. In that sense the 1 million deductible from the point of view of the reinsurer acts as if the insurer had sold policies with a 1 million policy limit from the point of view of the insurer.*

Problems:

7.1 (1 point) Insureds suffer six losses of sizes: 3, 8, 13, 22, 35, 62.
What is the Excess Ratio, excess of 30?
A. less than 0.16
B. at least 0.16 but less than 0.19
C. at least 0.19 but less than 0.22
D. at least 0.22 but less than 0.25
E. at least 0.25

7.2 (2 points) Determine the excess ratio at $200,000.

Frequency of Losses	Dollar Amount
40%	$5,000
20%	$10,000
15%	$25,000
10%	$50,000
5%	$100,000
4%	$250,000
3%	$500,000
2%	$1,000,000
1%	$2,000,000

7.3 (1 point) X is 70 with probability 40% and 700 with probability 60%.
Determine $E[(X - 100)_+]$.
A. less than 345
B. at least 345 but less than 350
C. at least 350 but less than 355
D. at least 355 but less than 360
E. at least 360

7.4 (1 point) X is 5 with probability 80% and 25 with probability 20%.
If $E[(X - d)_+] = 3$, determine d.
A. 4 B. 6 C. 8 D. 10 E. 12

7.5 (CAS9, 11/96, Q.42a) (1 point)
You are given the following empirical data from a closed claim study.

Interval	# of Claims	Losses
0	2,648	0
1 - 50,000	1,500	25,001,000
50,001 - 100,000	400	30,000,000
100,001 - 250,000	274	45,758,000
250,001 - 400,000	100	30,000,000
400,001 - 750,000	22	12,474,000
750,001 - 2,500,000	3	2,621,000
Over 2,500,000	1	2,586,000
Total	4,948	148,440,000

Calculate the losses excess of $250,000 as a percentage of total losses.

Solutions to Problems:

7.1. E. R(30) = (dollars excess of 30) / (total dollars) =
(5 + 32) / (3 + 8 + 13 + 22 + 35 + 62) = 37 / 143 = **0.259.**

7.2. Excess Ratio = expected excess losses / expected total losses = 45,000/82,750 = **54.4%**.

Probability	Amount	Product	Excess of 200,000	Product
0.4	$5,000	$2,000	$0	$0
0.2	$10,000	$2,000	$0	$0
0.15	$25,000	$3,750	$0	$0
0.1	$50,000	$5,000	$0	$0
0.05	$100,000	$5,000	$0	$0
0.04	$250,000	$10,000	$50,000	$2,000
0.03	$500,000	$15,000	$300,000	$9,000
0.02	$1,000,000	$20,000	$800,000	$16,000
0.01	$2,000,000	$20,000	$1,800,000	$18,000
		$82,750		$45,000

7.3. E. $(70 - 100)_+ = 0$. $(700 - 100)_+ = 600$. $E[(X - 100)_+] = (40\%)(0) + (60\%)(600) =$ **360**.

7.4. D. $E[(X - 5)_+] = (0)(80\%) + (25 - 5)(20\%) = 4 > 3$. ⇒ d must be greater than 5.

Therefore, $E[(X - d)_+] = (0.2)(25 - d) = 3$. ⇒ d = **10**.

7.5. # claims of size more than $250,000 is: 100 + 22 + 3 + 1 = 126
$ on claims of size more than $250,000 is:
30,000,000 + 12,474,000 + 2,621,000 + 2,586,000 = 47,681,000
excess losses are: 47,681,000 - (126)(250,000) = 16,181,000
Ratio is: 16,181,000/148,440,000 = **10.9**%.

Section 8, Excess Loss Variable

The **Excess Loss Variable** for d is defined for X > d as X-d and is undefined for X ≤ d.[16]

Exercise: An insured has losses of sizes: \$300, \$600, \$1,200, \$1,500 and \$2,800.
What is the Excess Loss Variable for \$1000?
[Solution: undefined, undefined, \$200, \$500, \$1,800.]

Excess Loss Variable for d ⇔ the nonzero payments excess of a deductible of d
⇔ X - d for X > d ⇔ truncated from below at d and shifted[17] ⇔
amount paid per (non-zero) payment.

The Excess Loss Variable at d, which could be called the left truncated and shifted variable at d, is similar to $(X - d)_+$, the left censored and shifted variable at d. However, the Excess Loss Variable at d is undefined for X ≤ d, while in contrast $(X - d)_+$ is zero for X ≤ d.

Excess Loss Variable ⇔ undefined X ≤ d ⇔ amount paid per (non-zero) payment.

$(X - d)_+$ ⇔ 0 for X ≤ d ⇔ amount paid per loss.

Exercise: An insured has four losses of size: 700, 3500, 16,000 and 40,000.
What are the excess loss variable at 5000, the left censored and shifted variable at 5000, and the limited loss variable at 5000?
[Solution: Excess Loss Variable at 5000: 11,000 and 35,000, corresponding to the last two losses. (It is not defined for the first two losses of size less than 5000.)
Left censored and shifted variable at 5000: 0, 0, 11,000 and 35,000.
Limited Loss Variable at 5000: 700, 3500, 5000, 5000.]

[16] See Definition 3.3 in Loss Models.
[17] Truncation will be discussed in a subsequent section.

Mean Residual Life / Mean Excess Loss:

The mean of the excess loss variable for d =
the mean excess loss, e(d) =
(Losses Excess of d) / (number of losses > d) =
(E[X] - E[$X \wedge d$])/S(d) =
the average payment per (nonzero) payment with a deductible of d.

Exercise: An insured has losses of sizes: \$300, \$600, \$1,200, \$1,500 and \$2,800.
What is the (empirical) mean excess loss at \$1000?
[Solution: e(1000) = (\$200 + \$500 + \$1,800)/3 = \$833.33.]

Note that the first step in computing e(1000) is to ignore the two losses that “died before 1000.” Then one computes the average “lifetime beyond 1000” for the 3 remaining losses.[18]

In this situation, on a policy with a \$1000 deductible, the insurer would make 3 (non-zero) payments totaling \$2500, for an average (non-zero) payment of \$833.33.

The **Mean Residual Life or Mean Excess Loss** at x, e(x), is defined as the average dollars of loss above x on losses of size exceeding x.

$$\mathbf{e(x) = \frac{E[X] - E[X \wedge x]}{S(x)}}.$$

Exercise: An insured has losses of sizes: \$300, \$600, \$1,200, \$1,500 and \$2,800.
Use the above formula to compute the (empirical) mean excess loss at \$1000?
[Solution: E[$X \wedge 1000$] = (300 + 600 + 1000 + 1000 + 1000)/5 = 3900/5 = \$780.
E[X] = (300 + 600 + 1200 + 1500 + 2800)/5 = 6400/5 = \$1280.
S(1000) = 3/5.
e(1000) = (\$1280 - \$780)/(3/5) = \$833.33.]

Note that the empirical mean excess loss is discontinuous. While the excess losses in the numerator are continuous, the empirical survival function in the denominator is discontinuous. The denominator has a jump discontinuity at every observed claim size.

The (theoretical) mean excess loss would be written for a continuous size of loss distribution as:

$$e(L) = \frac{\int_0^\infty x\, f(x)\, dx - \left(\int_0^L x\, f(x)\, dx + L\, S(L)\right)}{S(L)} = \frac{\int_L^\infty x\, f(x)\, dx - L\, S(L)}{S(L)}.$$

The numerator of e(L) is the losses eliminated divided by the total number of losses; this is equal to the excess ratio R(x) times the mean. Thus **e(x) = R(x) mean / S(x)**.

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[18] In Life Contingencies, this is how one computes the mean residual life.

Specifically, for losses of sizes: \$300, \$600, \$1,200, \$1,500 and \$2,800,
mean = \$1280, S(1000) = 3/5, and R(1000) = 39.06%.
e(1000) = (39.06%)(\$1280) / (3/5) ≅ \$833.

One can also write e(L) as:

$$e(L) = \frac{\int_L^\infty x\, f(x)\, dx}{S(L)} - L = \frac{\text{dollars on losses of size} > L}{\#\text{ of losses of size} > L} - L.$$

Thus, **e(x) = (average size of those losses of size greater than x) - x.**

<u>Summary of Related Ideas</u>:

Loss Elimination Ratio at x = LER(x) = $E[X \wedge x] / E[X]$.

Excess Ratio at x = R(x) = $\frac{E[X] - E[X \wedge x]}{E[X]} = 1 - \frac{E[X \wedge x]}{E[X]} = 1 - LER(x)$.

Mean Residual Life at x = Mean Excess Loss at x = e(x) = $\frac{E[X] - E[X \wedge x]}{S(x)}$.

On the exam, one wants to avoid doing integrals if at all possible. Therefore, one should use the formulas for the Limited Expected Value, $E[X \wedge x]$, in Appendix A of <u>Loss Models</u> whenever possible. Those who are graphically oriented may find the Section on Lee Diagrams helps them to understand these concepts.

<u>Hazard Rate/ Failure Rate</u>:[19]

The failure rate, force of mortality, **or hazard rate,** is defined as:

$h(x) = f(x)/S(x), x \geq 0.$

For a given age x, the hazard rate is the density of the deaths, divided by the probability of still being alive at age x.
The hazard rate determines the survival (distribution) function and vice versa:

$$S(x) = \exp\left[-\int_0^x h(t)\, dt\right].$$

$$-\frac{d \ln[S(x)]}{dx} = h(x).$$

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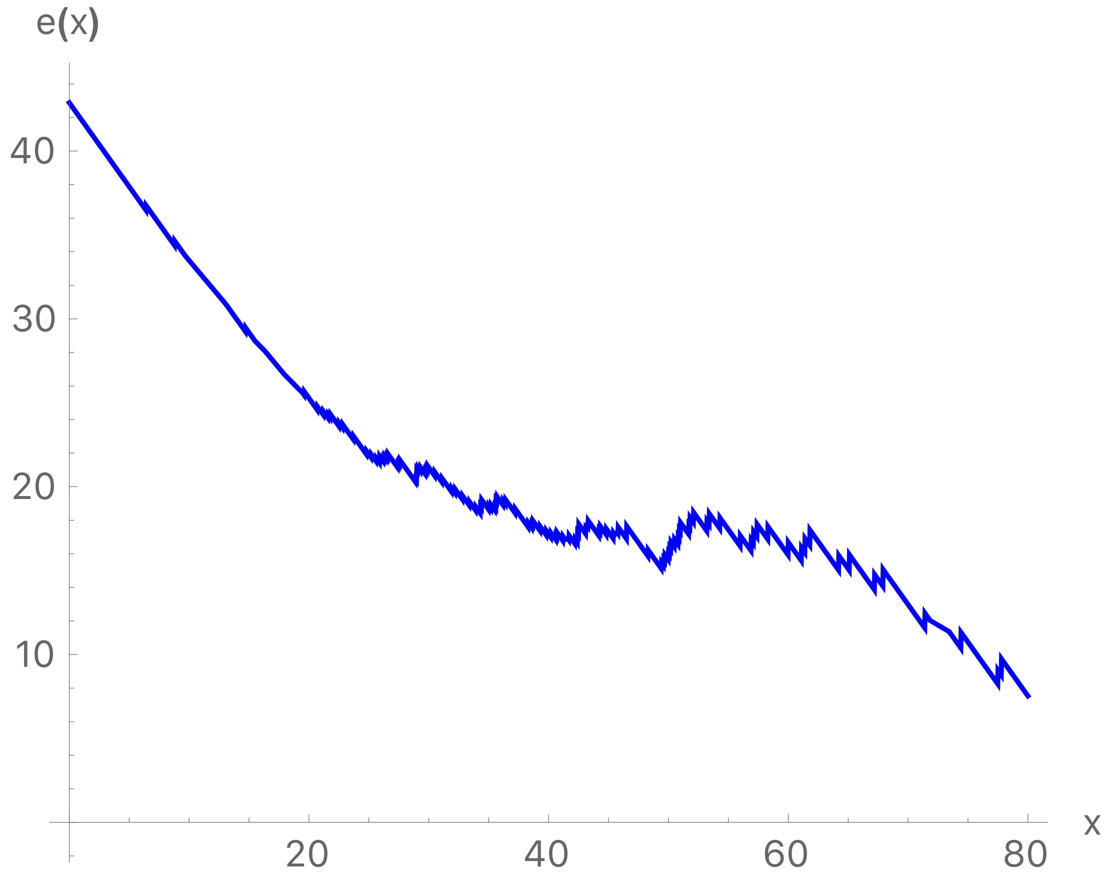
[19] As will be discussed in a subsequent section, the limit as x approaches infinity of e(x) is equal to the limit as x approaches infinity of 1/h(x). These behaviors will be used to distinguish the tails of distributions.

<u>Boston Snowfall Example</u>:

The snow fall in Boston Massachusetts in inches by Calendar Years from 1893 to 2021:[20]
21.8, 21.6, 38.7, 51.0, 50.9, 41.2, 108.6, 50.2, 71.4, 12.2, 60.0, 42.4, 56.0, 49.6, 43.3, 29.0, 87.4, 29.0, 77.8, 24.8, 46.5, 29.0, 35.6, 14.7, 56.9, 86.2, 41.7, 86.4, 85.3, 26.6, 23.7, 29.7, 22.5, 39.8, 52.1, 20.7, 27.4, 44.2, 29.8, 49.7, 31.9, 18.5, 19.6, 89.3, 46.5, 45.4, 41.5, 42.5, 6.4, 40.7, 40.2, 64.2, 61.1, 33.4, 67.2, 51.7, 0.6, 57.4, 42.5, 38.6, 57.1, 51.8, 35.7, 49.4, 36.2, 72.8, 19.3, 33.9, 27.5, 25.9, 35.5, 31.1, 34.3, 67.9, 42.4, 26.2, 80.7, 35.6, 39.3, 30.5, 35.0, 43.2, 38.3, 50.1, 8.8, 34.3, 28.9, 45.8, 50.5, 25.6, 34.4, 26.3, 50.0, 26.5, 17.4, 61.8, 21.2, 27.6, 53.4, 43.1, 32.8, 77.5, 15.9, 44.8, 53.2, 82.6, 25.2, 42.2, 10.8, 37.2, 35.2, 34.4, 29.1, 22.8, 58.3, 50.8, 36.4, 74.4, 29.9, 54.3, 29.3, 25.9, 45.7, 61.4, 48.3, 44.3, 32.2, 65.1, 81.3.

A graph of the empirical mean excess loss:



A case where the mean excess loss, e(x), decreases as x increases.

[20] https://www.extremeweatherwatch.com/cities/boston/most-yearly-snow

Problems:

8.1 (1 point) Insureds suffer six losses of sizes: 3, 8, 13, 22, 35, 62.
What is the empirical mean excess loss at 20?
A. less than 16
B. at least 16 but less than 18
C. at least 18 but less than 20
D. at least 20 but less than 22
E. at least 22

8.2 (2 points) Insureds suffer six losses of sizes: 3, 8, 13, 22, 35, 62.
Match the concepts.

1. Limited Loss Variable at 20	a. 3, 8, 13, 20, 20, 20.
2. Excess Loss Variable at 20	b. 2, 15, 42.
3. $(X - 20)_+$	c. 0, 0, 0, 2, 15, 42.

A. 1a, 2b, 3c B. 1a, 2c, 3b C. 1b, 2a, 3c D. 1b, 2c, 3a E. 1c, 2b, 3a

8.3 (2 points) The random variable for a loss, X, has the following characteristics:

x	F(x)	Limited Expected Value at x
0	0.0	0
500	0.3	360
1000	0.9	670
5000	1.0	770

Calculate the mean excess loss for a deductible of 500.
A. less than 600
B. at least 600 but less than 625
C. at least 625 but less than 650
D. at least 650 but less than 675
E. at least 675

8.4 (2 points) You are given S(60) = 0.50, S(70) = 0.40, and e(70) = 13.
Assuming the survival function is a straight line between ages 60 and 70, estimate e(67).
A. 14.8 B. 15.0 C. 15.2 D. 15.4 E. 15.6

8.5 (1 point) You observe 5 losses of sizes: 15, 35, 70, 90, 140.
What is the Mean Excess Loss at 50?
A. 10 B. 20 C. 30 D. 40 E. 50

8.6 (5 points) The following are the top 50 endowments (in $000) for the colleges and universities in North America in 2013:
1557426, 1593147, 1634685, 1637164, 1678563, 1714876, 1730829, 1735086, 1772394, 1809200, 1823441, 1823748, 1849880, 1925949, 1996681, 2020019, 2023357, 2182171, 2346693, 2381151, 2669948, 2757476, 2949000, 2956803, 2975896, 2987298, 3149169, 3673434, 3733596, 3868355, 4836728, 5166660, 5272228, 5651860, 5816046, 6040973, 6377379, 6668974, 6856301, 7741396, 7883323, 8197880, 8382311, 8732010, 11005932, 18200433, 18688868, 20448313, 20780000, 32334293.
With the aid of a computer, graph e(x) as function of x.

8.7 (5 points) Data was collected on common street names in the United States.
The number of occurrences of the top 76 street names: 2450, 2468, 2487, 2488, 2511, 2533, 2547, 2557, 2578, 2589, 2613, 2615, 2618, 2645, 2652, 2653, 2669, 2682, 2754, 2769, 2780, 2821, 2851, 2856, 2882, 2977, 2994, 3056, 3076, 3082, 3193, 3220, 3268, 3297, 3306, 3309, 3325, 3347, 3402, 3446, 3570, 3656, 3669, 3725, 3778, 3853, 3929, 3975, 4017, 4031, 4044, 4048, 4074, 4165, 4371, 4799, 4877, 4901, 4908, 4974, 5202, 5233, 5524, 5644, 6103, 6170, 6377, 6946, 7283, 7664, 8186, 8926, 9190, 9898, 10131, 10866.
With the aid of a computer, graph e(x) as function of x.

8.8 (4, 5/88, Q.60) (1 point) What is the empirical mean residual life at x = 4 given the following sample of total lifetimes:

3, 2, 5, 8, 10, 1, 6, 9.

A. Less than 1.5
B. At least 1.5, but less than 2.5
C. At least 2.5, but less than 3.5
D. 3.5 or more
E. Cannot be determined from the information given

8.9 (4B, 5/93, Q.25) (2 points) The following random sample has been observed:

2.0, 10.3, 4.8, 16.4, 21.6, 3.7, 21.4, 34.4

Calculate the value of the empirical mean excess loss function for x = 8.
A. less than 7.00
B. at least 7.00 but less than 9.00
C. at least 9.00 but less than 11.00
D. at least 11.00 but less than 13.00
E. at least 13.00

8.10 (4B, 11/94, Q.16) (1 point) A random sample of auto glass claims has yielded the following five observed claim amounts:

100, 125, 200, 250, 300.

What is the value of the empirical mean excess loss function at x = 150?
A. 75 B. 100 C. 200 D. 225 E. 250

8.11 (3, 11/01, Q.35 & 2009 Sample Q.101) (2.5 points)
The random variable for a loss, X, has the following characteristics:

x	F(x)	Limited Expected Value at x
0	0.0	0
100	0.2	91
200	0.6	153
1000	1.0	331

Calculate the mean excess loss for a deductible of 100.
(A) 250 (B) 300 (C) 350 (D) 400 (E) 450

Solutions to Problems:

8.1. C. e(20) = (dollars excess of 20) / (# claims greater than 20) = (2 + 15 + 42) / 3 = 59/3 = **19.7**.

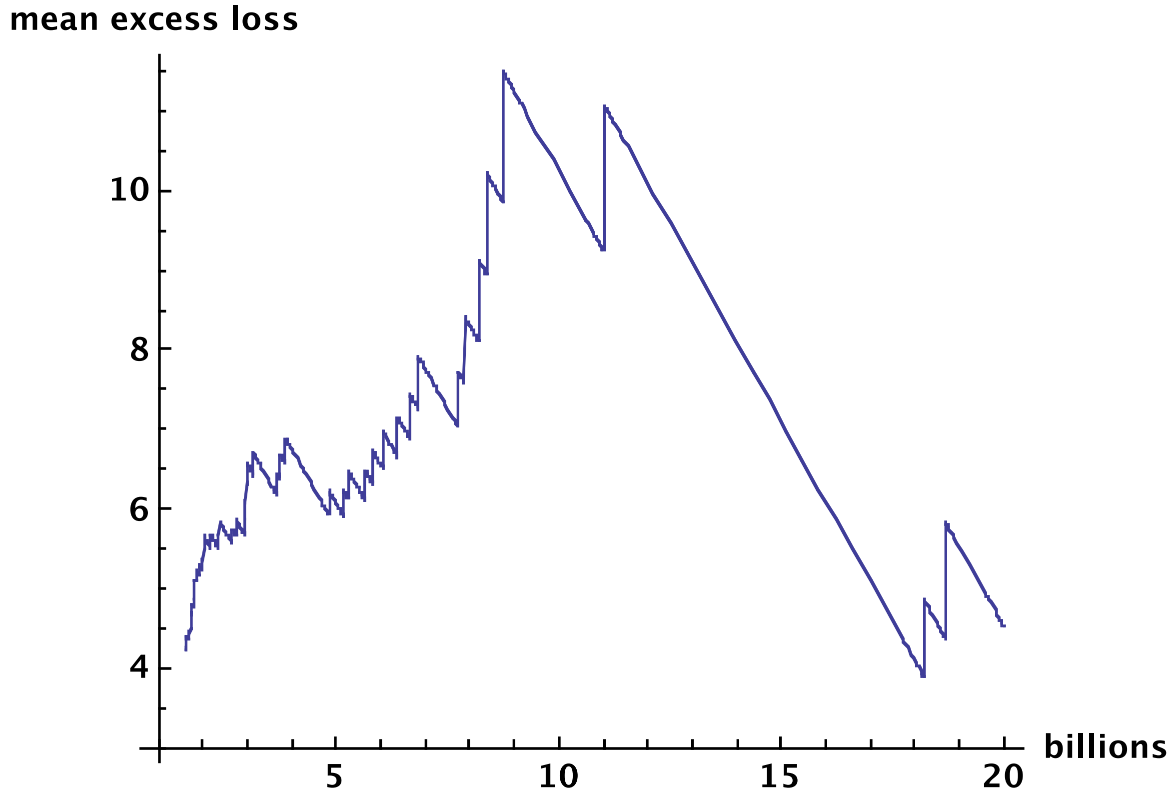
8.2. A. Limited Loss Variable at 20, limit each large loss to 20: 3, 8, 13, 20, 20, 20.
Excess Loss Variable at 20: 2, 15, 42, corresponding to 20 subtracted from each of the last 3 losses. It is not defined for the first 3 losses, each of size less than 20.
$(X - 20)_+$ is 0 for $X \le 20$, and X - 20 for X > 20: 0, 0, 0, 2, 15, 42.

8.3. A. F(5000) = 1. ⇒ $E[X] = E[X \wedge 5000] = 770$.
$e(500) = (E[X] - E[X \wedge 500])/S(500) = (770 - 360)/(1 - 0.3) =$ **586**.
Comment: Similar to 3, 11/01, Q.35 (2009 Sample Q.101).

8.4. B. Years excess of 70 = S(70)e(70) = (0.4)(13) = 5.2.
S(70) = 0.40, S(69) ≅ 0.41, S(68) ≅ 0.42, S(67) ≅ 0.43.
Years lived between ages 67 and 70 ≅ 0.425 + 0.415 + 0.405 = 1.245.
e(67) = (years excess of 67)/S(67) ≅ (5.2 + 1.245)/0.43 = **15.0**.
Comment: Assume we started with 1000 people.
Then 430 are alive at age 67, and 420 are alive at exact age 68.
So between ages 67 and 68, we would have 420 years lived from those who survived until 68; plus the ten people who died in the interval would contribute about 5 years lived in the interval.
420 + 5 = 425. 425/1000 = 0.425.
See Section 33, where I write e(x) in terms of the Survival Function.

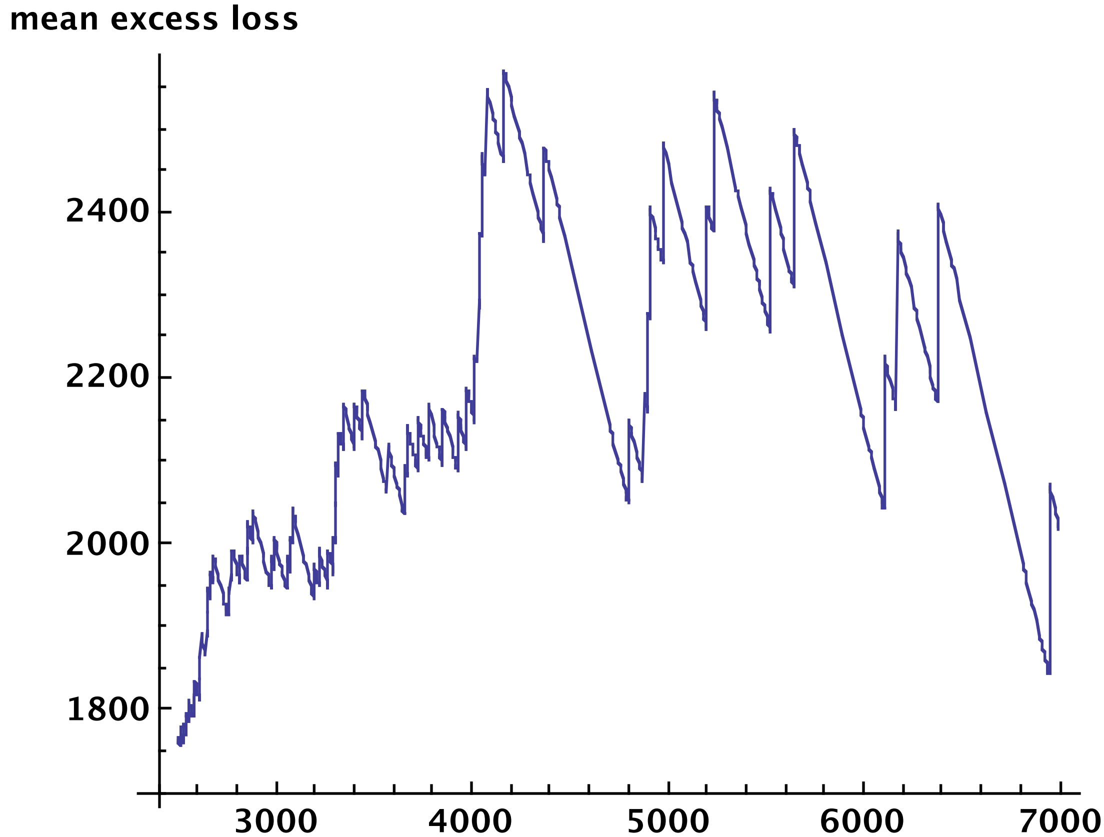
8.5. E. e(50) = (20 + 40 + 90)/3 = **50**.
Alternately, $e(50) = (E[X] - E[X \wedge 50])/S(50) = (70 - 40)/(1 - 0.4) =$ **50**.

8.6. For example, e($2 billion) = $5.36446 billion.
Here is a graph of the mean excess losses in billions of dollars:



<u>Comment</u>: The mean excess losses increase a lot up to about 10 billion. After that the mean excess losses decrease; however, only 6 endowments are larger than 10 billion.
The righthand tail seems relatively heavy.

8.7. For example, e(5000) = 2459.
Here is a graph of the mean excess losses:



Comment: The mean excess losses increase slightly from 2500 to about 4500. After that the mean excess losses seems to stay about the same, taking into account the limited amount of data. The righthand tail seems relatively light.

8.8. D. We ignore all claims of size 4 or less. Each of the 5 claims greater than 4 contributes the amount by which it exceeds 4. The empirical mean excess loss at x=4 is:
{(5-4) + (8-4) + (10-4) + (6-4) + (9-4)} / 5 = 18/5 = **3.6**.

8.9. D. To compute the mean excess loss at 8, we only look at accidents greater than 8.
There are 5 such accidents, and we compute the average amount by which they exceed 8:
e(8) = (2.3 + 8.4 + 13.6 + 13.4 + 26.4) / 5 = 64.1 / 5 = **12.82**.

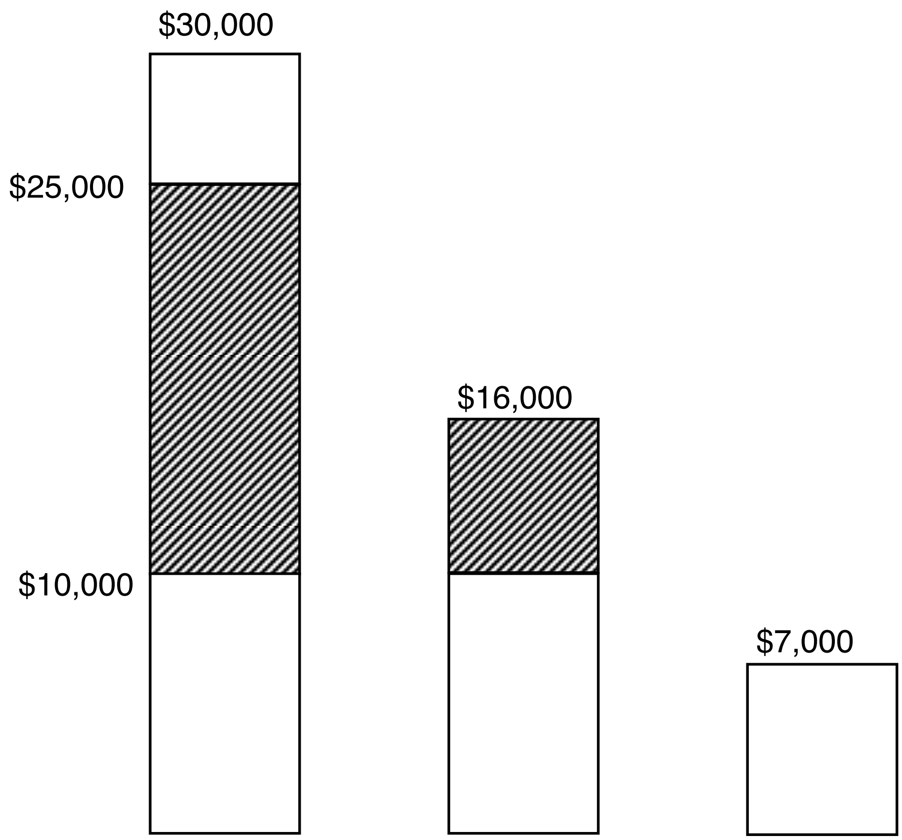
8.10. B. Add up the dollars excess of 150 and divide by the 3 claims of size exceeding 150.
e(150) = (50 + 100 + 150) / 3 = **100**.

8.11. B. F(1000) = 1. ⇒ E[X] = E[X ∧ 1000] = 331.
e(100) = (E[X] - E[X ∧ 100])/S(100) = (331 - 91)/(1 - 0.2) = **300**.

Section 9, Layers of Loss

Actuaries, particularly those working with reinsurance, often look at the losses in a **layer**.
The following diagram shows how the Layer of Loss between $10,000 and $25,000 relates to three specific claims of size: $30,000, $16,000 and $7,000.

The claim of size $30,000 contributes to the layer $15,000, the width of the layer, since it is larger than the upper boundary of the layer. The claim of size $16,000 contributes to the layer $16,000 - $10,000 = $6,000; since it is between the two boundaries of the layer it contributes its size minus the lower boundary of the layer. The claim of size $7,000 contributes nothing to the layer, since it is smaller than the lower boundary of the layer.



In general, the losses in the layer between $10,000 and $25,000 would be calculated in three pieces. The losses smaller than $10,000 contribute nothing to this layer. The losses between $10,000 and $25,000 each contribute their value minus $10,000. The remaining losses which are bigger than the upper limit of the interval at $25,000, each contribute the width of the interval, $25,000 - $10,000 = $15,000.

For a continuous size of loss distribution, the **percentage of losses in the layer** from $10,000 to $25,000 would be written as:

$$\frac{\int_{10,000}^{25,000} (x - 10,000)\, f(x)\, dx + S(25,000)\,(25,000-10,000)}{\int_0^{\infty} x\, f(x)\, dx}.$$

The percentage of losses in a layer can be rewritten in terms of limited expected values.
The percentage of losses in the layer from $10,000 to $25,000 is:
$(E[X \wedge 25,000] - E[X \wedge 10,000])$ / mean.

This can also be written in terms of the Loss Elimination Ratios:
LER(25,000) - LER(10,000).

This can also be written in terms of the Excess Ratios (with the order reversed):
R(10,000) - R(25,000).

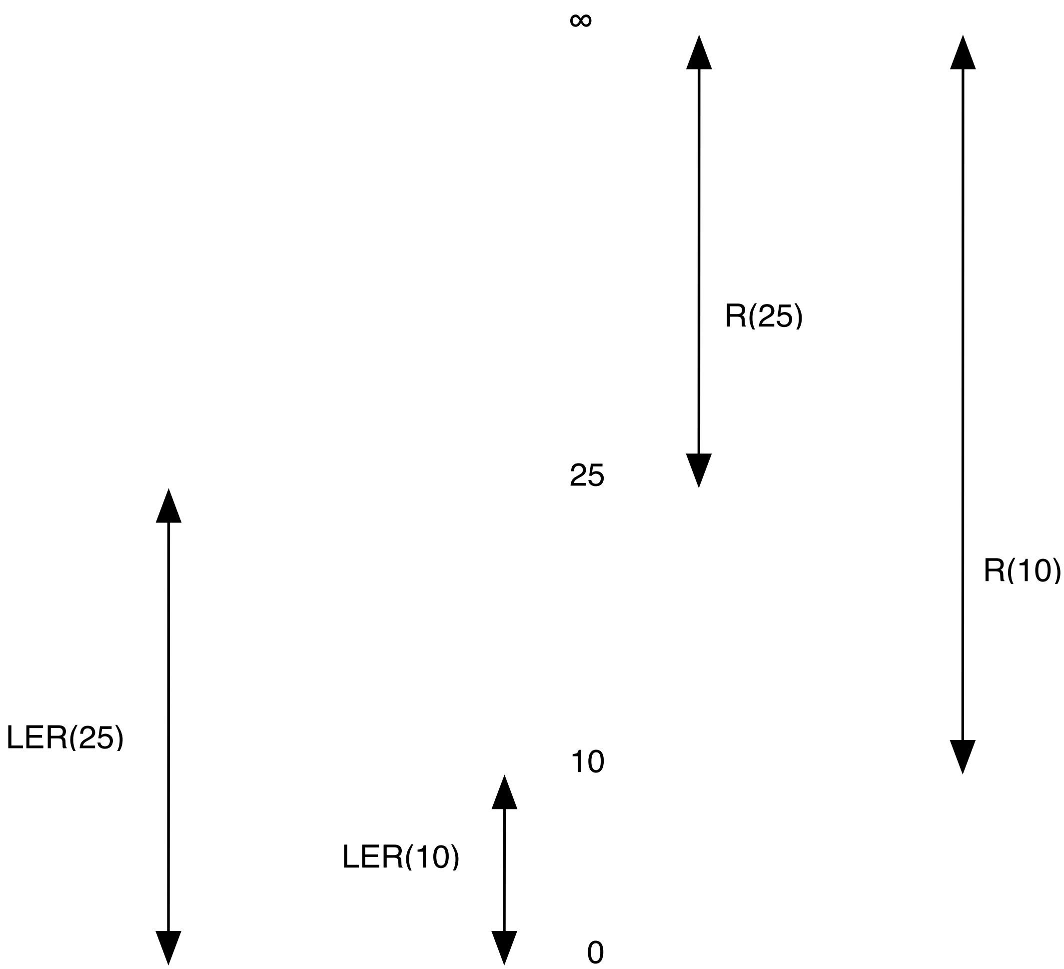
The percentage of losses in the layer from d to u =

$$\frac{\int_d^u (x - d)\, f(x)\, dx + S(u)\,(u - d)}{\int_0^{\infty} x\, f(x)\, dx} = \frac{E[X \wedge u] - E[X \wedge d]}{E[X]} = \text{LER}(u) - \text{LER}(d) = R(d) - R(u).$$

Layer Average Severity for the layer from d to u =
The mean losses in the layer from d to u = $E[X \wedge u] - E[X \wedge d]$ =
{LER(u) - LER(d)} E[X] = {R(d) - R(u)} E[X].

The Layer from d to u can be thought of as either:
(Layer from 0 to u) - (Layer from 0 to d)
or (Layer from d to ∞) - (Layer from u to ∞).

For example, the Layer from 10 to 25 can be thought of as either:
(Layer from 0 to 25) - (Layer from 0 to 10)
or (Layer from 10 to ∞) - (Layer from 25 to ∞):



The percentage of losses in the layer from 10 to 25 is: LER(25) - LER(10) = R(10) - R(25).

Those who are graphically oriented may find that my Section on Lee Diagrams helps them to understand these concepts.

Problems:

9.1 (1 point) Insureds suffer six losses of sizes: 3, 8, 13, 22, 35, 62.
What is the percentage of total losses in the layer from 10 to 25?
A. less than 29%
B. at least 29% but less than 30%
C. at least 30% but less than 31%
D. at least 31% but less than 32%
E. at least 32%

9.2 (1 point) Four accidents occur of sizes: \$230,000, \$810,000, \$1,170,000, and \$2,570,000.
A reinsurer is responsible for the layer of loss from \$500,000 to \$1,500,000
(\$1 million excess of 1/2 million.)
How much does the reinsurer pay as a result of these four accidents?
A. \$1.7 million B. \$1.8 million C. \$1.9 million D. \$2.0 million E. \$2.1 million

Use the following information for the next two questions:
- A reinsurer expects 50 accidents per year from a certain book of business.
- Limited Expected Values for this book of business are estimated to be:
  $E[X \wedge \$1 \text{ million}] = \$300,000$
  $E[X \wedge \$4 \text{ million}] = \$375,000$
  $E[X \wedge \$5 \text{ million}] = \$390,000$
  $E[X \wedge \$9 \text{ million}] = \$420,000$
  $E[X \wedge \$10 \text{ million}] = \$425,000$

9.3 (1 point) If the reinsurer were responsible for the layer of loss from \$1 million to \$5 million (\$4 million excess of \$1 million), how much does the reinsurer expect to pay per year as a result of accidents from this book of business?
A. \$4.0 million B. \$4.5 million C. \$5.0 million D. \$5.5 million E. \$6.0 million

9.4 (1 point) Let A be the amount the reinsurer would expect to pay per year as a result of accidents from this book of business, if the reinsurer were responsible for the layer of loss from \$1 million to \$5 million (\$4 million excess of \$1 million). Let B be the amount the reinsurer would expect to pay per year as a result of accidents from this book of business, if the reinsurer were instead responsible for the layer of loss from \$1 million to \$10 million
(\$9 million excess of \$1 million).
What is the ratio of B/A?
A. 1.30 B. 1.35 C. 1.40 D. 1.45 E. 1.50

Use the following information for the next four questions:
- A reinsurer expects 50 accidents per year from a certain book of business.
- The average size of accident from this book of business is estimated as $450,000.
- Excess Ratios (Unity minus the Loss Elimination Ratio) for this book of business are:

R($1 million) = 0.100
R($4 million) = 0.025
R($5 million) = 0.015
R($9 million) = 0.006
R($10 million) = 0.005

9.5 (1 point) What is the percentage of total losses in the layer from $1 million to $5 million?
A. less than 6%
B. at least 6% but less than 7%
C. at least 7% but less than 8%
D. at least 8% but less than 9%
E. at least 9%

9.6 (1 point) If the reinsurer were responsible for the layer of loss from $1 million to $5 million ($4 million excess of $1 million), how much does the reinsurer expect to pay per year as a result of accidents from this book of business?
A. less than $1 million
B. at least $1 million but less than $2 million
C. at least $2 million but less than $3 million
D. at least $3 million but less than $4 million
E. at least $4 million

9.7 (1 point) What is the percentage of total losses in the layer from $1 million to $10 million?
A. less than 6%
B. at least 6% but less than 7%
C. at least 7% but less than 8%
D. at least 8% but less than 9%
E. at least 9%

9.8 (1 point) Let A be the amount the reinsurer would expect to pay per year as a result of accidents from this book of business, if the reinsurer were responsible for the layer of loss from $1 million to $5 million ($4 million excess of $1 million). Let B be the amount the reinsurer would expect to pay per year as a result of accidents from this book of business, if the reinsurer were instead responsible for the layer of loss from $1 million to $10 million ($9 million excess of $1 million). What is the ratio of B/A?
A. 1.1 B. 1.2 C. 1.3 D. 1.4 E. 1.5

9.9 (2 points) Assume you have a Pareto distribution with $\alpha = 5$ and $\theta = \$1000$.
What percentage of total losses are represented by the layer from \$500 to \$2000?
A. less than 16%
B. at least 16% but less than 17%
C. at least 17% but less than 18%
D. at least 18% but less than 19%
E. at least 19%

9.10 (1 point) There are seven losses of sizes: 2, 5, 8, 11, 13, 21, 32.
What is the percentage of total losses in the layer from 5 to 15?
A. 35% B. 40% C. 45% D. 50% E. 55%

9.11 (2 points) Use the following information:
● Limited Expected Values for Security Blanket Insurance are estimated to be:

$E[X \wedge 100{,}000] = 40{,}000$
$E[X \wedge 200{,}000] = 50{,}000$
$E[X \wedge 300{,}000] = 57{,}000$
$E[X \wedge 400{,}000] = 61{,}000$
$E[X \wedge 500{,}000] = 63{,}000$

● Security Blanket Insurance buys reinsurance from Plantagenet Reinsurance.
Let A be the amount Plantagenet would expect to pay per year as a result of accidents from Security Blanket, if the reinsurance had a deductible of 100,000, maximum covered loss of 300,000, and a coinsurance factor of 90%. Let B be the amount Plantagenet would expect to pay per year as a result of accidents from Security Blanket, if the reinsurance had a deductible of 100,000, a maximum covered loss of 400,000, and a coinsurance factor of 80%.
What is the ratio of B/A?
(A) 1.05 (B) 1.10 (C) 1.15 (D) 1.20 (E) 1.25

9.12 (CAS5, 5/07, Q.9) (1 point)
Using the table below, what is the formula for the loss elimination ratio at deductible D?

Loss Limit	Number of Losses	Total Loss Amount
D and Below	N1	L1
Over D	N2	L2
Total	N1+N2	L1+L2

B. 1 - [L1 + (N2)(D)] / [L1 + L2]
C. 1 - [L2 - (N2)(D)] / [L1 + L2]
D. [L2 + (N2)(D)] / [L1 +(N2)(D)]
E. [L1 + (N1)(D)] / [L1]

Solutions to Problems:

9.1. D. (losses in layer from 10 to 25) / total losses =
(0+0+3+12+15+15) / (3+8+13+22+35+62) = 45 / 143 = **0.315**.
Alternately, average contribution to the layer is: 45 / 6. Mean is: 143 / 6.
$\frac{45/6}{143/6}$ = **0.315**.

9.2. D. The accidents of sizes \$230,000, \$810,000, \$1,170,000, and \$2,570,000 contribute to the layer of loss from \$500,000 to \$1,500,000:
0 + 310,000 + 670,000 + 1,000,000 = **\$1,980,000**.

9.3. B. (50)(E[X∧ \$5 million] - E[X∧ \$1 million]) = (50)(390,000 - 300,000) = **\$4.5 million**.

9.4. C. A = (50)(E[X∧ \$5 million] - E[X∧ \$1 million]) = (50)(390,000 - 300,000) =
\$4.5 million. B = (50)(E[X∧ \$10 million] - E[X∧ \$1 million]) = (50)(425,000 - 300,000) =
\$6.25 million. B/A = 6.25 / 4.5 = **1.389**.
Comment: One can solve this problem without knowing that 50 accidents are expected per year, since 50 multiplies both the numerator and denominator. The ratio between two layers of loss depends on the severity distribution, not the frequency distribution.

9.5. D. R(\$1 million) - R(\$5 million) = 0.100 - 0.015 = **0.085**.

9.6. B. The annual losses from the layer from \$1 million to \$5 million =
(number of accidents per year)(mean accident){R(\$1 million) - R(\$5 million)} =
(50)(\$450,000){R(\$1 million) - R(\$5 million)} = (\$22.5 million){0.100 - 0.015} = **\$1,912,500**.
Alternately, the total expected losses are:
(# of accidents per year)(mean accident) = (50)(\$450,000) = \$22,500,000.
(0.085)(\$22,500,000) = **\$1,912,500**.

9.7. E. R(\$1 million) - R(\$10 million) = 0.100 - 0.005 = **0.095**.

9.8. A. B/A = {R(\$1 million) - R(\$10 million)} / {R(\$1 million) - R(\$5 million)}
= (0.100 - 0.005) / (0.100 - 0.015) = 0.095 / 0.085 = **1.12**.
Comment: A = (number of accidents per year)(mean accident){R(\$1 million) - R(\$5 million)}.
B = (number of accidents per year)(mean accident){R(\$1 million) - R(\$10 million)}.

9.9. D. Use the formula given in Appendix A of Loss Models for the Limited Expected Value of the Pareto, $E[X \wedge x] = \{\theta/(\alpha-1)\}\{1-(\theta/(\theta+x))^{\alpha-1}\}$.
Percentage of Losses in the Layer \$500 to \$2000 = ($E[X \wedge 2000]$ - $E[X \wedge 500]$) / mean =
(246.9 - 200.6)/250 = **18.5%**.
Alternately, use the formula given in a subsequent section for the Excess Ratio of the Pareto, $R(x) = (\theta/(\theta+x))^{\alpha-1}$. Percentage of Losses in the Layer \$500 to \$2000 = R(500) - R(2000) = 19.75% - 1.23% = 18.5%.

9.10. B.

Loss:	2	5	8	11	13	21	32
Contribution to Layer from 5 to 15:	0	0	3	6	8	10	10

(0 + 0 + 3 + 6 + 8 + 10 + 10) / (2 + 5 + 8 + 11 + 13 + 21 + 32) = 37/92 = **40.2%**.

9.11. B. A = (0.9)($E[X \wedge 300{,}000]$ - $E[X \wedge 100{,}000]$) = (0.9)(57,000 - 40,000) = 15,300.
B = (0.8)($E[X \wedge 400{,}000]$ - $E[X \wedge 100{,}000]$) = (0.8)(61,000 - 40,000) = 16,800.
B/A = 16,800 / 15,300 = **1.098**.
Comment: Both A and B have been calculated per accident. Their ratio does not depend on the expected number of accidents.

9.12. C. The losses eliminated are: L1 + (N2)(D).
Loss Elimination Ratio is: {L1 + (N2)(D)} / (L1 + L2) = **1 - {L2 - (N2)(D)}/(L1 + L2)**.
Alternately, each loss of size less than D contributes nothing to the excess losses.
Each loss of size x > D, contributes x - D to the excess losses.
Therefore, the excess losses = L2 - (N2)(D).
Excess Ratio = (Excess Losses)/(Total Losses) = {L2 - (N2)(D)}/(L1 + L2).
Loss Elimination Ratio = 1 - Excess Ratio = **1 - {L2 - (N2)(D)}/(L1 + L2)**.

Section 10, Average Size of Losses in an Interval

Exercise: An insured has losses of sizes: $300, $600, $1200, $1500, and $2800.
Determine the losses in the layer from $500 to $2500.
[Solution: The loss of size 300 contributes nothing. The loss of size 600 contributes 100.
The loss size 1200 contributes 700. The loss of 1500 contributes 1000.
The loss of 2800 contributes the width of the layer or 2000.
0 + 100 + 700 + 1000 + 2000 = 3800.]

Exercise: An insured has losses of sizes: $300, $600, $1200, $1500, and $2800.
Determine the sum of those losses of size from $500 to $2500.
[Solution: 600 + 1200 + 1500 = 3300.
Comment: The average size of these three losses is: 3300/3 = 1100.]

Note the average size of losses in an interval differs from a layer of loss.
Here we are ignoring all losses other than those in a certain size category.
In contrast, losses of all sizes contribute to each layer.

For a discrete size of loss distribution, the dollars from those losses of size ≤ 10,000 is:

$$\sum_{x_i \le 10,000} x_i \, \text{Prob}[X = x_i] .$$

For a continuous size of loss distribution, the dollars from those losses of size ≤ 10,000 is:[21]

$$\int_0^{10,000} x \, f(x) \, dx = E[X \wedge 10,000] - 10,000 \, S(10,000).$$

For a continuous size of loss distribution, the average size of loss for those losses of size less than or equal to 10,000 is:

$$\frac{\int_0^{10,000} x \, f(x) \, dx}{F(10,000)} = \frac{E[X \wedge 10,000] - 10,000 \, S(10,000)}{F(10,000)} .$$

Exercise: For an Exponential Distribution with mean = 50,000, what is the average size of those losses of size less than or equal to 10,000?
[Solution: $E[X \wedge x] = \theta (1 - e^{-x/\theta})$. $E[X \wedge 10,000] = 50000 (1 - e^{-1/5}) = 9063$.
$S(x) = e^{-x/\theta}$. $S(10,000) = e^{-1/5} = 0.8187$. $\{E[X \wedge 10,000] - 10,000 S(10,000)\}/F(10,000) = 4832$.]

[21] The limited expected value = contribution of small losses + contribution of large losses.
Therefore, contribution of small losses = limited expected value - contribution of large losses.

For a continuous size of loss distribution the average size of loss for those losses of size between 10,000 and 25,000 would be written as:

$$\frac{\int_{10,000}^{25,000} x\, f(x)\, dx}{F(25,000) - F(10,000)} = \frac{\int_0^{25,000} x\, f(x)\, dx - \int_0^{10,000} x\, f(x)\, dx}{F(25,000) - F(10,000)} =$$

$$\frac{\{E[X \wedge 25000] - 25000\ S(25000)\} - \{E[X \wedge 10000] - 10000\ S(10000)\}}{F(25000) - F(10000)}.$$

Exercise: For an Exponential Distribution with mean = 50,000, what is the average size of those losses of size between 10,000 and 25,000?

[Solution: $E[X \wedge x] = \theta\ (1 - e^{-x/\theta})$. $E[X \wedge 10,000] = 50000\ (1 - e^{-1/5}) = 9063$.

$E[X \wedge 25,000] = 50,000\ (1 - e^{-1/2}) = 19,673$.

$S(x) = e^{-x/\theta}$. $S(10,000) = e^{-1/5} = 0.8187$. $S(25,000) = e^{-1/2} = 0.6065$

$(\{E[X \wedge 25,000] - 25,000 S(25,000)\} - \{E[X \wedge 10,000] - 10,000 S(10,000)\}) / \{F(25,000) - F(10,000)\}$

$= (\{19,673 - (25,000)(0.6065)\} - \{9063 - (10,000)(0.8187)\}) / \{0.3935 - 0.1813\} = 17,127$.]

In general, **the average size of loss for those losses of size between a and b is:**

$$\frac{\{E[X \wedge b] - b\ S(b)\} - \{E[X \wedge a] - a\ S(a)\}}{F(b) - F(a)}.$$

The numerator is the dollars per loss contributed by the losses of size a to b =
(contribution of losses of size 0 to b) minus (contribution of losses of size 0 to a).
The denominator is the percent of losses of size a to b =
(percent of losses of size 0 to b) minus (percent of losses of size 0 to a).

For an Exponential with θ = 50,000, here are the average sizes for various size intervals:

Bottom	Top	E[X ∧ Top]	S(Top)	Average Size
0	10,000	9,063	81.9%	4,833
10,000	25,000	19,673	60.7%	17,126
25,000	50,000	31,606	36.8%	36,463
50,000	100,000	43,233	13.5%	70,901
100,000	250,000	49,663	0.7%	142,141
250,000	Infinity	50,000	0.0%	300,000

For a Pareto Distribution, $S(x) = (\theta/(\theta+x))^{\alpha}$, and $E[X \wedge x] = \{\theta/(\alpha-1)\}\{1 - (\theta/(\theta+x))^{\alpha-1}\}$.
A Pareto Distribution with α = 3 and θ = 100,000, has a mean of: $\theta/(\alpha-1)$ = 50,000.

For this Pareto Distribution, here are the average sizes for various size intervals:

Bottom	Top	E[X ∧ Top]	S(Top)	Average Size
0	10,000	8,678	75.1%	4,683
10,000	25,000	18,000	51.2%	16,863
25,000	50,000	27,778	29.6%	35,989
50,000	100,000	37,500	12.5%	70,270
100,000	250,000	45,918	2.3%	148,387
250,000	Infinity	50,000	0.0%	425,000

Notice the difference between the results for the Pareto and the Exponential Distributions.

Proportion of Dollars of Loss From Losses of a Given Size:

Another quantity of interest, is the percentage of the total losses from losses in a certain size interval.

Proportional of Total Losses from Losses in the Interval [a, b] is:

$$\frac{\int_a^b x\, f(x)\, dx}{E[X]} = \frac{\{E[X \wedge b] - b\, S(b)\} - \{E[X \wedge a] - a\, S(a)\}}{E[X]}.$$

Exercise: For an Exponential Distribution with mean = 50,000, what percentage of the total dollars of those losses come from losses of size between 10,000 and 25,000?

[Solution: $E[X \wedge x] = \theta\,(1 - e^{-x/\theta})$. $E[X \wedge 10{,}000] = 50{,}000\,(1 - e^{-1/5}) = 9063$.

$E[X \wedge 25{,}000] = 50{,}000\,(1 - e^{-1/2}) = 19{,}673$.

$S(x) = e^{-x/\theta}$. $S(10{,}000) = e^{-1/5} = 0.8187$. $S(25{,}000) = e^{-1/2} = 0.6065$

$(\{E[X \wedge 25{,}000] - 25{,}000 S(25{,}000)\} - \{E[X \wedge 10{,}000] - 10{,}000 S(10{,}000)\}) / E[X] =$

$(\{19{,}673 - (25{,}000)(0.6065)\} - \{9063 - (10{,}000)(0.8187)\}) / 50{,}000 = 7.3\%$.]

For an Exponential with θ = 50,000, here are the percentages for various size intervals:

Bottom	Top	E[X ∧ Top]	S(Top)	Percentage of Total Losses
0	10,000	9,063	81.9%	1.8%
10,000	25,000	19,673	60.7%	7.3%
25,000	50,000	31,606	36.8%	17.4%
50,000	100,000	43,233	13.5%	33.0%
100,000	250,000	49,663	0.7%	36.6%
250,000	Infinity	50,000	0.0%	4.0%

A Pareto Distribution with $\alpha = 3$ and $\theta = 100,000$, here are the percentages for various size intervals:

Bottom	Top	$E[X \wedge \text{Top}]$	S(Top)	Percentage of Total Losses
0	10,000	8,678	75.1%	2.3%
10,000	25,000	18,000	51.2%	8.1%
25,000	50,000	27,778	29.6%	15.5%
50,000	100,000	37,500	12.5%	24.1%
100,000	250,000	45,918	2.3%	30.2%
250,000	Infinity	50,000	0.0%	19.8%

Notice the difference between the results for the Pareto and the Exponential Distributions.

Problems:

For each of the following three problems, assume you have a Pareto distribution with parameters $\alpha = 5$ and $\theta = \$1000$.

10.1 (2 points) What is the average size of those losses less than $500 in size?
A. less than $160
B. at least $160 but less than $170
C. at least $170 but less than $180
D. at least $180 but less than $190
E. at least $190

10.2 (2 points) What is the average size of those losses greater than $500 in size but less than $2000?
A. less than $800
B. at least $800 but less than $825
C. at least $825 but less than $850
D. at least $850 but less than $875
E. at least $875

10.3 (2 points) Assume you expect 100 losses per year. What is the expected dollars of loss paid on those losses greater than $500 in size but less than $2000?
A. less than $10,500
B. at least $10,500 but less than $11,000
C. at least $11,000 but less than $11,500
D. at least $11,500 but less than $12,000
E. at least $12,000

10.4 (2 points) You are given the following:
- A sample of 5,000 losses contains 1800 that are no greater than $100, 2500 that are greater than $100 but no greater than $1000, and 700 that are greater than $1000.
- The empirical limited expected value function for this sample evaluated at $100 is $73.
- The empirical limited expected value function for this sample evaluated at $1000 is $450.

Determine the total amount of the 2500 losses that are greater than $100 but no greater than $1000.
A. Less than $1.50 million
B. At least $1.50 million, but less than $1.52 million
C. At least $1.52 million, but less than $1.54 million
D. At least $1.54 million, but less than $1.56 million
E. At least $1.56 million

10.5 (3 points) Severity is LogNormal with $\mu = 5$ and $\sigma = 3$.
What is the average size of those losses greater than 20,000 in size but less than 35,000?
A. less than 25,000
B. at least 25,000 but less than 27,000
C. at least 27,000 but less than 29,000
D. at least 29,000 but less than 31,000
E. at least 31,000

10.6 (2 points) You are given the following:

x	F(x)	$E[X \wedge x]$
$20,000	0.75	$7050
$30,000	0.80	$9340

Determine the average size of those losses of size between $20,000 and $30,000.
A. Less than $23,500
B. At least $23,500, but less than $24,500
C. At least $24,500, but less than $25,500
D. At least $25,500, but less than $26,500
E. At least $26,500

10.7 (2 points) You are given the following:
- A sample of 3,000 losses contains 2100 that are no greater than $1,000, 830 that are greater than $1,000 but no greater than $5,000, and 70 that are greater than $5,000.
- The total amount of the 830 losses that are greater than $1,000 but no greater than $5,000 is $1,600,000.
- The empirical limited expected value function for this sample evaluated at $1,000 is $560.

Determine the empirical limited expected value function for this sample evaluated at $5,000.
A. Less than $905
B. At least $905, but less than $915
C. At least $915, but less than $925
D. At least $925, but less than $935
E. At least $935

10.8 (2 points) The random variable for a loss, X, has the following characteristics:

x	F(x)	Limited Expected Value at x
0	0.0	0
100	0.2	91
200	0.6	153
1000	1.0	331

Calculate the average size of those losses of size greater than 100 but less than 200.
(A) 140 (B) 145 (C) 150 (D) 155 (E) 160

10.9 (2 points) Aggregate Losses for an insurer follow a LogNormal Distribution with $\mu = 15.3$ and $\sigma = 0.8$, for which:

Limit ($ million)	Limited Expected Value ($ million)	Distribution Function
4	3.277	0.4512
7	4.476	0.7180

Determine the average size of aggregate losses in the interval 4 to 7 million.
A. 5.3 million B. 5.4 million C. 5.5 million D. 5.6 million E. 5.7 million

10.10 (3 points) An insurance pays the following for a loss of size X:

$$\begin{cases} 0, \ X < 5000 \\ 0.75\, X - 3750, \ 5000 \le X \le 50{,}000 \\ 0.9X - 11{,}250, \ 50{,}000 < X \end{cases}$$

You are given the following values:
$E[X] = 40{,}000$. $E[X \wedge 3750] = 3730$. $E[X \wedge 5000] = 4954$.
$E[X \wedge 11{,}250] = 10{,}799$. $E[X \wedge 12{,}500] = 11{,}899$. $E[X \wedge 50{,}000] = 32{,}612$.
Determine the average payment per loss for this insurance.

10.11 (160, 5/88, Q.5) (2.1 points) A population experiences mortality consistent with an exponential distribution with $\theta = 10$. Calculate the average fraction of the interval (x, x+3] lived by those who die during the interval.
(A) $(1 + e^{-0.1} + e^{-0.2} - 3e^{-0.3})\, e / \{6(1 - e^{-0.3})\}$
(B) $(1 + e^{-0.1} + e^{-0.2} - 3e^{-0.3}) / \{3(1 - e^{-0.3})\}$
(C) 1/3
(D) $(13 - 10e^{-0.3}) / \{3(1 - e^{-0.3})\}$
(E) $(10 - 13e^{-0.3}) / \{3(1 - e^{-0.3})\}$

10.12 (4B, 5/92, Q.23) (2 points) You are given the following information:
A large risk has a lognormal claim size distribution with parameters $\mu = 8.443$ and $\sigma = 1.239$.
The insurance agent for the risk settles all claims under $5,000.
(Claims of $5,000 or more are settled by the insurer, not the agent.)
Determine the expected value of a claim settled by the insurance agent.
A. Less than 500
B. At least 500 but less than 1,000
C. At least 1,000 but less than 1,500
D. At least 1,500 but less than 2,000
E. At least 2,000

10.13 (4B, 5/93, Q.33) (3 points) The distribution for claim severity follows a Single Parameter Pareto distribution of the following form:

$f(x) = (3/1000)(x/1000)^{-4}$, $x > 1000$

Determine the average size of a claim between \$10,000 and \$100,000, given that the claim is between \$10,000 and \$100,000.
A. Less than \$18,000
B. At least \$18,000 but less than \$28,000
C. At least \$28,000 but less than \$38,000
D. At least \$38,000 but less than \$48,000
E. At least \$48,000

10.14 (4B, 5/99, Q.10) (2 points) You are given the following:
● One hundred claims greater than 3,000 have been recorded as follows:

Interval	Number of Claims
(3,000, 5,000]	6
(5,000, 10,000]	29
(10,000, 25,000]	39
(25,000, ∞)	26

● Claims of 3,000 or less have not been recorded.
● The null hypothesis, H_0, is that claim sizes follow a Pareto distribution, with parameters $\alpha = 2$ and $\theta = 25,000$.
If H_0 is true, determine the expected claim size for claims in the interval (25,000, ∞).
A. 12,500 B. 25,000 C. 50,000 D. 75,000 E. 100,000

10.15 (4B, 11/99, Q.1) (2 points) You are given the following:
● Losses follow a distribution (prior to the application of any deductible) with mean 2,000.
● The loss elimination ratio (LER) at a deductible of 1,000 is 0.30.
● 60 percent of the losses (in number) are less than the deductible of 1,000.
Determine the average size of a loss that is less than the deductible of 1,000.
A. Less than 350
B. At least 350, but less than 550
C. At least 550, but less than 750
D. At least 750, but less than 950
E. At least 950

Solutions to Problems:

10.1. A. The Limited Expected Value of the Pareto, $E[X \wedge x] = \{\theta/(\alpha-1)\}\{1 - (\theta/\theta+x)^{\alpha-1}\}$.

$$\int_0^{500} x\, f(x)\, dx = E[X \wedge 500] - 500S(500) = 200.6 - 500\,(1000/1500)^5 = 200.6 - 65.8 = 134.8.$$

Average size of claim = 134.8 / F(500) = 134.8 / 0.869 = **$155**.

10.2. B. $\int_{500}^{2000} x\, f(x)\, dx = \int_0^{2000} x\, f(x)\, dx - \int_0^{500} x\, f(x)\, dx =$

$E[X \wedge 2000] - 2000S(2000) - \{E[X \wedge 500] - 500S(500)\} =$

$\{246.9 - 2000\,(1000/3000)^5\} - \{200.6 - 500\,(1000/1500)^5\} = 238.7 - 134.8 = 103.9.$

Average Size of Claim = 103.9 / (F(2000)-F(500)) = 103.9 / (0.996 - 0.869) = **$818**.

10.3. A. $100\int_{500}^{2000} x\, f(x)\, dx = 100\int_0^{2000} x\, f(x)\, dx - 100\int_0^{500} x\, f(x)\, dx =$

$100\{\{E[X \wedge 2000] - 2000S(2000)\} - \{E[X \wedge 500] - 500S(500)\}\} =$

$100\{\{246.9 - 2000\,(1000/3000)^5\} - \{200.6 - 500\,(1000/1500)^5\}\} = 100\{238.7 - 134.8\} =$ **$10,390.**

Alternately, one expects 100{F(2000)-F(500)} = 100{.996 - 0.869} = 12.7 such claims per year, with an average size of $818, based on the previous problem. Thus the expected dollars of loss on these claims = (12.7)($818) = $10,389.

10.4. B. The average size of those claims of size between 100 and 1,000 equals:
$(\{E[X \wedge 1000] - 1000S(1000)\} - \{E[X \wedge 100] - 100S(100)\})/\{F(1000)-F(100)\}$
$= \{(450 - (1000)(700/5000)) - (73 - (100)(3200/5000)\}/\{(4300/5000)-(1800/5000)\} =$
$(310 - 9) / 0.5 = \$602$. Thus these 2500 claims total: (2500)($602) = **$1,505,000**.
Alternately, (Losses Limited to $100) / (Number of Claims) = $E[X \wedge 100]$ = $73.
Since there are 5000 claims, Losses Limited to $100 = ($73)(5000) = $365,000.
Now there are: 2500 + 700 = 3200 claims greater than $100 in size.
Since these claims contribute $100 each to the losses limited to $100, they contribute a total of: (3200)($100) = $320,000.
Losses limited to $100 = (losses on Claims ≤ $100) + (contribution of claims > $100).
Thus losses on Claims ≤ $100 is: $365,000 - $320,000 = $45,000.
(Losses Limited to $1000) / (Number of Claims) = $E[X \wedge 1000]$ = $450.
Since there are 5000 claims, Losses Limited to $1000 = ($450)(5000) = $2,250,000.
Now there are 700 claims greater than $1000 in size.
Since these claims contribute $1000 each to the losses limited to $1000, they contribute a total of: (700)($1000) = $700,000.
Losses limited to $1000 = (losses on Claims ≤$1000)+(contribution of Claims > $1000).
Thus losses on Claims ≤$1000 = $2,250,000 - $700,000 = $1,550,000.
The total amount of the claims that are greater than $100 but no greater than $1000 is:
(losses on Claims ≤ $1000) - (losses on Claims ≤ $100) =
$1,550,000 - $45,000 = **$1,505,000**.

10.5. B. $F(x) = \Phi[(\ln x - \mu)/\sigma]$. $F(20{,}000) = \Phi[1.63]$. $F(35000) = \Phi[1.82]$.
$E[X \wedge x] = \exp(\mu + \sigma^2/2)F[(\ln x - \mu - \sigma^2)/\sigma] + x\{1 - \Phi[(\ln x - \mu)/\sigma]\}$.
$E[X \wedge x] - xS(x) = \exp(\mu + \sigma^2/2)\,\Phi[(\ln x - \mu - \sigma^2)/\sigma]$.
$E[X \wedge 20{,}000] - (20{,}000)S(20{,}000) = (e^{9.5})\ \Phi[(\ln 20{,}000 - 5 - 3^2)/3]) = 13{,}360\ \Phi[-1.37]$.
$E[X \wedge 35{,}000] - (35{,}000)S(35{,}000) = (e^{9.5})\ \Phi[(\ln 35{,}000 - 5 - 3^2)/3]) = 13{,}360\ \Phi[-1.18]$.
The average size of claims of size between $20,000 and $25,000 is:
$(E[X \wedge 35{,}000] - 35{,}000S(35{,}000) - \{E[X \wedge 20{,}000] - 20{,}000S(20{,}000)\}) / \{F(35{,}000)-F(20{,}000)\} =$
$13{,}360\ \{\Phi[-1.18] - \Phi[-1.37]\} / \{\Phi[1.82] - \Phi[1.63]\} =$
13,360 (0.1190 - 0.0853) / (0.9656 - 0.9484) = **26,176**.

10.6. D. The average size of those claims of size between 20,000 and 30,000 equals:
$(\{E[X \wedge 30{,}000] - 30{,}000S(30{,}000)\} - \{E[X \wedge 20{,}000] - 20{,}000S(20{,}000)\}) / \{F(30{,}000)-F(20{,}000)\}$
= {(9340-(0.2)(30,000)) - (7050-(0.25)(20,000))} / (0.80-0.75) = (3340-2050)/0.05 = **$25,800**.

10.7. B. (Losses Limited to \$1000) / (Number of Claims) = $E[X \wedge 1000]$ = \$560.
Since there are 3000 claims, Losses Limited to \$1000 = (\$560)(3000) = \$1,680,000.
Now there are 830+70 = 900 claims greater than \$1000 in size.
Since these claims contribute \$1000 each to the losses limited to \$1000, they contribute a total of (900)(\$1000) = \$900,000.
Losses limited to \$1000 = (losses on Claims ≤ \$1000) + (contribution of claims > \$1000).
Thus the losses on claims ≤ \$1000 = \$1,680,000 - \$900,000 = \$780,000.
Now the losses on claims ≤ \$5000 =
(losses on claims ≤ \$1000) + (losses on claims > \$1000 and ≤ \$5000) =
\$780,000 + (\$1,600,000) = \$2,380,000.
Finally, the losses limited to \$5000 =
(the losses on claims ≤ \$5000) + (Number of Claims > \$5000)(\$5000) =
\$2,380,000 + (70)(\$5000) = \$2,730,000.
$E[X \wedge 5000]$ = (Losses limited to \$5000)/(Total Number of Claims) =
\$2,730,000 / 3000 = **\$910**.
Alternately, the average size of those claims of size between 1,000 and 5,000 equals:
$(\{E[X \wedge 5000] - 5000S(5000)\} - \{E[X \wedge 1000] - 1000S(1000)\})/\{F(5000)-F(1000)\}$.
We are given that: S(1000) = 900/3000 = 0.30, S(5000) = 70/3000 = 0.0233, $E[X \wedge 1000]$ = 560.
The observed average size of those claims of size 1000 to 5000 is: 1,600,000/ 830 = 1927.7
Setting the observed average size of those claims of size 1000 to 5000 equal to the above formula for the same quantity:
$1927.7 = (\{E[X \wedge 5000] - 5000S(5000)\} - \{E[X \wedge 1000] - 1000S(1000)\})/\{F(5000)-F(1000)\}$
$= (\{E[X \wedge 5000] - 5000(0.0233)\} - \{560 - 1000(0.30))/\{0.9767 - 0.70\}$.
Solving, $E[X \wedge 5000] = (1927.7)(0.2767) + 116.5 + 560 - 300 = \$910$.

10.8. D. Average size of losses between 100 and 200 is:
$(\{E[X \wedge 200] - 200S(200)\} - \{E[X \wedge 100] - 100S(100)\}) / (F(200) - F(100)) =$
$(\{153 - (200)(1 - 0.6)\} - \{91 - (100)(1 - 0.2)\}) / (0.6 - 0.2) = (73 - 11)/0.4 =$ **155**.
Comment: Same information as in 3, 11/01, Q.35.

10.9. A. The average size of aggregate losses in the interval 4 to 7 million is:

$$\frac{E[X \wedge 7m] - (7m)\,S(7m) - \{E[X \wedge 4m] - (4m)\,S(4m)\}}{F(7m) - F(4m)} =$$

$$\frac{4.476 - (7)(1 - 0.7180) - \{3.277 - (4)(1 - 0.4512)\}}{0.7180 - 0.4512} = \textbf{5.32 million}.$$

Comment: The LogNormal Distribution and how to determine limited expected values for named distributions are discussed in subsequent sections. Here we just use the given values.

10.10. We can divide the payments into two pieces: $(0.75)(X - 5000)_+$,
plus for X > 50,000: 0.9X - 11,250 - (0.75)(X - 5000) = 0.15 X - 7500 = (0.15)(X - 50,000).
So the payment is: $(0.75)(X - 5000)_+ + (0.15)(X - 50{,}000)_+$
Thus the average payment per loss is:
$(0.75)(E[X] - E[X \wedge 5000]) + (0.15)(E[X] - E[X \wedge 50{,}000]) =$
$0.9\ E[X] - 0.75\ E[X \wedge 5000] - 0.15\ E[X \wedge 50{,}000] =$
(0.9)(40,000) - (0.75)(4954) - (0.15)(32,612) = **27,393**.
Alternately, the average loss per payment is:

$$\int_{5000}^{50{,}000} (0.75x - 3750)\ f(x)\ dx + \int_{50{,}000}^{\infty} (0.9x - 11{,}250)\ f(x)\ dx =$$

$$0.75 \int_{5000}^{50{,}000} x\ f(x)\ dx - 3750\ \{S(5000) - S(50{,}000)\} + 0.9 \int_{50{,}000}^{\infty} x\ f(x)\ dx - 11{,}250\ S(50{,}000) =$$

$0.75\ \{E[X \wedge 50{,}000] - 50{,}000\ S(50{,}000) - E[X \wedge 5000] + 5000\ S(5000)\} - 3750\ S(5000)$
$- 7500\ S(50{,}000) + 0.9\ \{E[X] - E[X \wedge 50{,}000] + 50{,}000\ S(50{,}000)\} =$
$0.9\ E[X] - 0.75\ E[X \wedge 5000] - 0.15\ E[X \wedge 50{,}000] =$
(0.9)(40,000) - (0.75)(4954) - (0.15)(32,612) = **27,393**.

10.11. E. The average size for losses of size between x and x + 3 is:
$\{E[X\wedge (x+3)] - (x+3)S(x+3)\} - E[X\wedge x] + xS(x)\} / \{F(x+3) - F(x)\} =$
$\{10(1 - e^{-(x+3)/10} - (x+3)e^{-(x+3)/10} - 10(1 - e^{-x/10}) + xe^{-x/10}\}/(e^{-x/10} - e^{-(x+3)/10}) =$
$\{10e^{-x/10} - 13e^{-(x+3)/10} - xe^{-(x+3)/10} + xe^{-x/10}\}/\{e^{-x/10}(1 - e^{-.3})\} =$
$(10 - 13e^{-.3} - xe^{-.3} + x)/(1 - e^{-.3}) = (10 - 13e^{-.3})/(1 - e^{-.3}) + x.$
The average fraction of the interval (x, x+3] lived by those who die during the interval =
{(The average size for losses of size between x and x + 3) - x} / (x + 3 - x) =
$\mathbf{(10 - 13e^{-0.3}) / \{3(1 - e^{-0.3})\}}$.
Alternately, the fraction for someone who dies at age x+ t is: t/3. Average fraction is:

$$\int_{t=0}^{3} (t/3)\ e^{(x+t)/10}/10\ dt\ /\ \int_{t=0}^{3} e^{(x+t)/10}/10\ dt\ =$$

$$\int_{t=0}^{3} (t/3)\ e^{(x+t)/10}/10\ dt\ = (e^{-x/10}/3)\,(-te^{-t/10} - 10e^{-t/10})\Big]_{t=0}^{t=3} = 10 - 13e^{-0.3}.$$

$$\int_{t=0}^{3} e^{(x+t)/10}/10\ dt\ = (e^{-x/10})\,(-e^{-t/10})\Big]_{t=0}^{t=3} = 3(1 - e^{-0.3}).$$

Thus the answer is: $\mathbf{(10 - 13e^{-0.3}) / \{3(1 - e^{-0.3})\}}$.

10.12. E. One is asked for the average size of those claims of size less than 5000. This is:

$$\int_0^{5000} x\ f(x)\ dx\ / F(5000) = \{E[X\wedge 5000] - 5000(1 - F(5000)\} / F(5000).$$

For this LogNormal Distribution:
$F(5000) = \Phi[\{\ln(x) - \mu\} / \sigma] = \Phi[\{\ln(5000) - 8.443\} / 1.239\ ] = \Phi[0.06] = .5239.$
$E[X\wedge 5000] = \exp(\mu + \sigma^2/2)\ \Phi[(\ln x - \mu - \sigma^2)/\sigma] + x\ \{1 - \Phi[(\ln x - \mu)/\sigma]\} =$
$\exp(8.443 + 1.239^2/2)\Phi[(\ln 5000 - 8.443 - 1.239^2)/1.239] + 5000\ \{1\text{-}0.5238\} =$
$10{,}002\ \Phi[-1.18] + 2381 = (10002)(0.1190) + 2381 = 3571.$
$\Rightarrow \{E[X\wedge 5000] - 5000(1 - F(5000)\} / F(5000) = (3571 - 2381)/0.5239 =$ **2271**.

10.13. A. $F(x) = 1 - (x/1000)^{-3}$, x > 1000. S(10,000) = 0.001. S(100,000) = 0.000001.
The average size of claim between 10,000 and 100,000 is the ratio of the dollars of loss on such claims to the number of such claims:

$$\int_{10,000}^{100,000} x\, f(x)\, dx \;/\{F(10,000) - F(100,000)\} = (3 \times 10^9) \int_{10,000}^{100,000} x^3\, dx \;/ 0.000999 =$$

$(3.003 \times 10^{12})\,(1/2)\,(10{,}000^{-2} - 100{,}000^{-2}) = (1.5015)(10{,}000 - 100) =$ **14,865**.
Comment: One can get the Distribution Function either by integrating the density function from 1000 to x or by recognizing that this is a Single Parameter Pareto Distribution. Note that in this case, as is common for a distribution skewed to the right, the average size of claim is near the left end of the interval rather than near the middle.

10.14. D. $F(x) = 1 - (\theta/(\theta+x))^{\alpha}$. S(25,000) = 1-F(25,000) = $\{25{,}000/(25{,}000+25{,}000)\}^2$ = 1/4.
$E[X\wedge x] = \{\theta/(\alpha-1)\}\{1-(\theta/(\theta+x))^{\alpha-1}\}$.
$E[X\wedge 25{,}000] = \{25{,}000/(2-1)\}\,\{1-(25{,}000/(25{,}000+25{,}000))^{2-1}\} = 25{,}000(1/2) = 12{,}500$.
The expected claim size for claims in the interval (25,000, ∞) =
(E[X] - {E[X∧25,000] - 25,000S(25,000)})/S(25,000) = (25,000-(12,500-(1/4)(25,000))/(1/4) = (18,750)(4) = **75,000**.
Alternately, the average payment the insurer would make excess of 25,000, per non-zero such payment, is {E[X] - E[X∧25,000]}/S(25,000) = 50,000. Then the expected claim size for claims in the interval (25,000, ∞) is this portion excess of 25,000 plus an additional 25,000 per large claim; 50,000 + 25,000 = **75,000**.

10.15. A. LER(1000) = 0.30. E[X] = 2000. F(1000) = 0.60.
E[X ∧ 1000] = LER(1000)E[X] = (0.30)(2000) = 600.
The average size of those losses less than 1000 is:
{E[X ∧ 1000] - 1000S(1000)} / F(1000) = {600-(1000)(1-0.6)} / 0.6 = (600-400)/0.6 = **333.33**.

Section 11, Policy Provisions

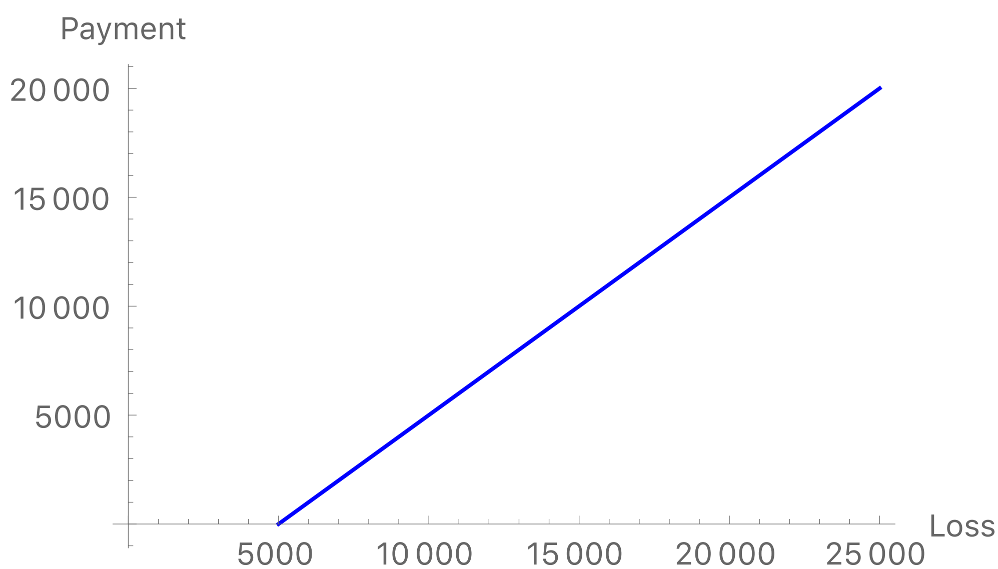
Insurance policies may have various provisions which determine the amount paid, such as deductibles, maximum covered losses, and coinsurance clauses.

(Ordinary) Deductible:

An **ordinary deductible** is a provision which states that when the loss is less than or equal to the deductible there is no payment, and when the loss exceeds the deductible the amount paid is the loss less the deductible.[22] Unless specifically stated otherwise, assume a deductible is ordinary. Unless stated otherwise assume the deductible operates per loss.
In actual applications, deductibles can apply per claim, per person, per accident, per occurrence, per event, per location, per annual aggregate, etc.[23]

Exercise: An insured suffers losses of size: $3000, $8000 and $17,000.
If the insured has a $5000 (ordinary) deductible, what does the insurer pay for each loss?
[Solution: Nothing, $8000 - $5000 = $3000, and $17,000 - $5000 = $12,000.]

Here is a graph of the payment under an ordinary deductible of 5000:



[22] See Definition 8.1 in Loss Models.
[23] An annual aggregate deductible is discussed in the section on Stop Loss Premiums in "Mahler's Guide to Aggregate Losses"

Maximum Covered Loss:[24]

Maximum Covered Loss ⇔ u
⇔ **size of loss above which no additional payments are made**
⇔ censorship point from above**.**

Exercise: An insured suffers losses of size: $2,000, $13,000, $38,000.
If the insured has a $25,000 maximum covered loss, what does the insurer pay for each loss?
[Solution: $2,000, $13,000, $25,000.]

Most insurance policies have a maximum covered loss or equivalent. *For example, a liability policy with a $100,000 per occurrence limit would pay at most $100,000 in losses from any single occurrence, regardless of the total losses suffered by any claimants.*
An automobile collision policy will never pay more than the total value of the covered automobile minus any deductible, thus it has an implicit maximum covered loss.
An exception is a Workers' Compensation policy, which provides unlimited medical coverage to injured workers.[25]

Coinsurance:

A **coinsurance factor** is the proportion of any loss that is paid by the insurer after any other modifications (such as deductibles or maximum covered losses) have been applied.

A **coinsurance** is a provision which states that a coinsurance factor is to be applied.

For example, a policy might have a 80% coinsurance factor. Then the insurer pays 80% of what it would have paid in the absence of the coinsurance factor.

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[24] See Section 8.5 of Loss Models. Professor Klugman made up the term "maximum covered loss."
[25] *While benefits for lost wages are frequently also unlimited, since they are based on a formula in the specific workers' compensation law, which includes a maximum weekly benefit, there is an implicit maximum benefit for lost wages, assuming a maximum possible lifetime.*

<u>Policy Limit</u>:[26]

Policy Limit ⇔ maximum possible payment on a single claim.

Policy Limit = c (u - d),
where c = coinsurance factor, u = maximum covered loss, and d = deductible.

If c = 90%, d = 1000, and u = 5000, then the policy limit = (90%)(5000 - 1000) = 3600;
if a loss is of size 5000 or greater, the insurer pays 3600.

With a coinsurance factor, deductible, and Policy Limit L:
u = d + L/c.

In the above example, 1000 + 3600/0.9 = 5000.

With no deductible and no coinsurance, the policy limit is the same as the maximum covered loss.

Exercise: An insured has a policy with a $25,000 maximum covered loss, $5000 deductible, and a 80% coinsurance factor. The insured suffers a losses of: $5000, $15,000, $38,000.
How much does the insurer pay?
[Solution: Nothing for the loss of $5000. (0.8)(15000 - 5000) = $8000 for the loss of $15,000.
For the loss of $38,000, first the insurer limits the loss to $25,000. Then it reduces the loss by the $5,000 deductible, $25,000 - $5,000 = $20,000. Then the 80% coinsurance factor is applied: (80%)($20,000) = $16,000.
<u>Comment</u>: The maximum possible amount paid for any loss, $16,000, is the policy limit.]

If an insured with a policy with a $25,000 maximum covered loss, $5000 deductible, and a coinsurance factor of 80%, suffers a loss of size x, then the insurer pays:

0, if x ≤ $5000
0.8(x-5000), if $5000 < x ≤ $25,000
$16,000, if x ≥ $25,000

More generally, if an insured has a policy with a maximum covered loss of u, a deductible of d, and a coinsurance factor of c, suffers a loss of size x, then the insurer pays:

0, if x ≤ d
c(x-d), if d < x ≤ u
c(u-d), if x ≥ u

[26] See Section 8.5 of <u>Loss Models</u>. This definition of a policy limit differs from that used by many actuaries.

If an insured has a policy with a policy limit of L, a deductible of d, and a coinsurance factor of c, suffers a loss of size x, then the insurer pays:

0, if $x \le d$
c(x-d), if $d < x \le d + L/c$
L, if $x \ge d + L/c$

Exercise: There is a deductible of $10,000, policy limit of $100,000, and a coinsurance factor of 90%. Let X_i be the individual loss amount of the i^{th} claim and Y_i be the claim payment of the i^{th} claim. What is the relationship between X_i and Y_i?
[Solution: The maximum covered loss, u = 10,000 + 100,000/0.9 = $121,111.

$$Y_i = \begin{cases} 0 & X_i \le 10,000 \\ 0.90(X_i - 10,000) & 10,000 < X_i \le 121,111 \\ 100,000 & X_i > 121,111. \end{cases}$$

$Y_i = 0.90(X_i - 10,000)$]

<u>Order of Operations</u>:

If one has a deductible, maximum covered loss, and a coinsurance, then on this exam unless stated otherwise, in order to determine the amount paid on a loss, the order to operations is:

1. Limit the size of loss to the maximum covered loss.
2. Subtract the deductible. If the result is negative, set the payment equal to zero.
3. Multiply by the coinsurance factor.

<u>Franchise Deductible</u>:[27]

Besides an ordinary deductible, there is the “franchise deductible.” Unless specifically stated otherwise, assume a deductible is ordinary.

Under a **franchise deductible** the insurer pays nothing if the loss is less than the deductible amount, but <u>ignores</u> the deductible if the loss is greater than the deductible amount.

Exercise: An insured suffers losses of size: $3000, $8000 and $17,000. If the insured has a $5000 franchise deductible, what does the insurer pay for each loss?
[Solution: Nothing, $8000, and $17,000.]

[27] In Definition 8.2 in <u>Loss Models</u>.

Under a franchise deductible with deductible amount is d, if the insured has a loss of size x, then the insurer pays:

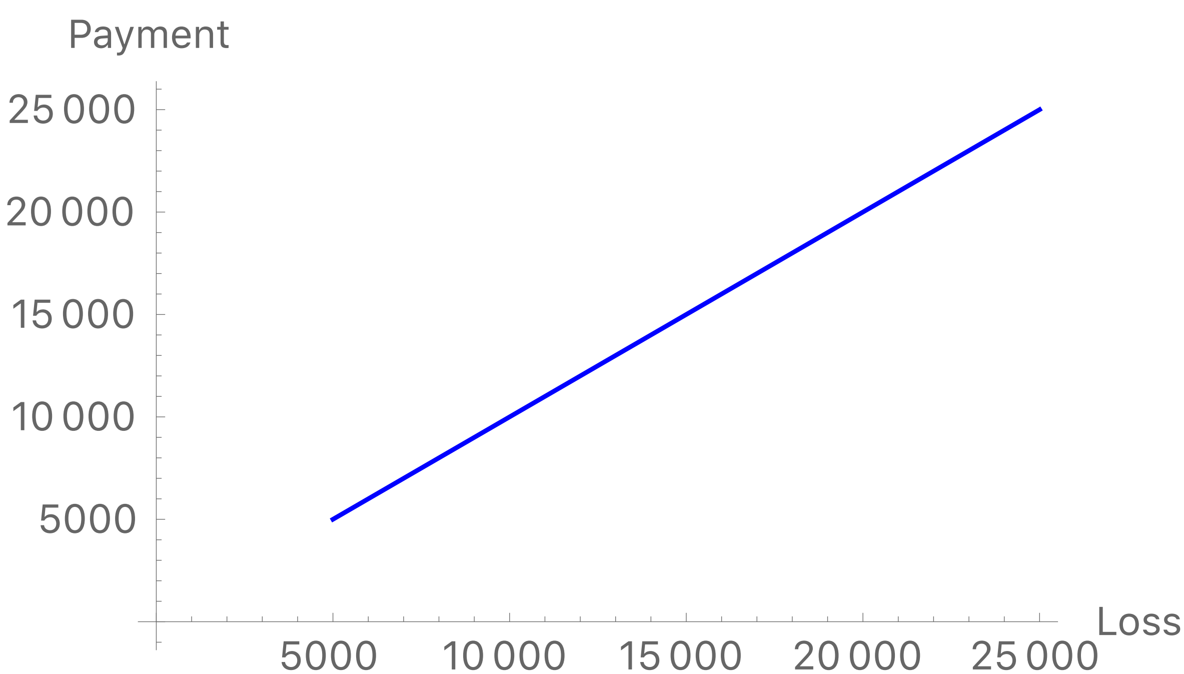
0	$x \le d$
x	$x > d$

Thus **data from a policy with a franchise deductible is truncated from below** at the deductible amount.[28]

Therefore under a franchise deductible, the average nonzero payment is:
$e(d) + d = \{E[X] - E[X \wedge d]\}/S(d) + d$.[29]

The average cost per loss is: (average nonzero payment)(chance of nonzero payment) =
$\{(E[X] - E[X \wedge d])/S(d) + d\}S(d) = (E[X] - E[X \wedge d]) + dS(d)$.[30]

Here is a graph of the payment under a franchise deductible of 5000:



[28] See the next section for a discussion of truncation from below (truncation from the right.)
[29] See Theorem 8.3 in Loss Models.
[30] See Theorem 8.3 in Loss Models.

Definitions of Loss and Payment Random Variables:[31] [32]

Name	Description
ground-up loss	Losses prior to the impact of any deductible or maximum covered loss; the full economic value of the loss suffered by the insured regardless of how much the insurer is required to pay in light of any deductible, maximum covered loss, coinsurance, etc.
amount paid per payment	Undefined when there is no payment due to a deductible or other policy provision. Otherwise it is the amount paid by the insurer. Thus for example, data truncated and shifted from below consists of the amounts paid per payment.
amount paid per loss	Defined as zero when the insured suffers a loss but there is no payment due to a deductible or other policy provision. Otherwise it is the amount paid by the insurer.

The per loss variable is: 0 if $X \le d$, X if $X > d$.

The per payment variable is: undefined if $X \le d$, X if $X > d$.

Loss Models uses the notation Y^L for the per loss variable and Y^P for the per payment variable.

Unless stated otherwise, assume a distribution from Appendix A of Loss Models will be used to model ground-up losses, prior to the effects of any coverage modifications. The effects on distributions of coverage modifications will be discussed in subsequent sections.

[31] See Section 8.2 of Loss Models.
[32] Ground-up losses are sometimes referred to as ground-up unlimited losses.

Problems:

Use the following information for the next 3 questions:
The ABC Bookstore has an insurance policy with a $100,000 maximum covered loss, $20,000 per loss deductible, and a 90% coinsurance factor.
During the year, ABC Bookstore suffers three losses of sizes: $17,000, $60,000 and $234,000.

11.1 (1 point) How much does the insurer pay in total?
A. less than $95,000
B. at least $95,000 but less than $100,000
C. at least $100,000 but less than $105,000
D. at least $105,000 but less than $110,000
E. at least $110,000

11.2 (1 point) What is the amount paid per loss?
A. less than $35,000
B. at least $35,000 but less than $40,000
C. at least $40,000 but less than $45,000
D. at least $45,000 but less than $50,000
E. at least $50,000

11.3 (1 point) What is the amount paid per payment?
A. less than $35,000
B. at least $35,000 but less than $40,000
C. at least $40,000 but less than $45,000
D. at least $45,000 but less than $50,000
E. at least $50,000

11.4 (2 points) The size of loss is uniform on [0, 400].
Policy A has an ordinary deductible of 100.
Policy B has a franchise deductible of 100.
What is the ratio of the expected losses paid under Policy B to the expected losses paid under Policy A?
A. 7/6 B. 5/4 C. 4/3 D. 3/2 E. 5/3

11.5 (1 point) An insured suffers 4 losses of size: $2500, $7700, $10,100, and $23,200.
The insured has a $10,000 franchise deductible. How much does the insurer pay in total?
A. less than 32,500
B. at least 32,500 but less than 33,000
C. at least 33,000 but less than 33,500
D. at least 33,500 but less than 34,000
E. at least 34,000

Use the following information for the next 3 questions:
An insurance policy has a deductible of 10,000, policy limit of 100,000, and a coinsurance factor of 80%. (The policy limit is the maximum possible payment by the insurer on a single loss.)
During the year, the insured suffers six losses of sizes: 3000, 8000, 14,000, 80,000, 120,000, and 200,000.

11.6 (2 points) How much does the insurer pay in total?
A. less than 235,000
B. at least 235,000 but less than 240,000
C. at least 240,000 but less than 245,000
D. at least 245,000 but less than 250,000
E. at least 250,000

11.7 (1 point) What is the amount paid per loss?
A. less than 45,000
B. at least 45,000 but less than 50,000
C. at least 50,000 but less than 55,000
D. at least 55,000 but less than 60,000
E. at least 60,000

11.8 (1 point) What is the amount paid per payment?
A. less than 45,000
B. at least 45,000 but less than 50,000
C. at least 50,000 but less than 55,000
D. at least 55,000 but less than 60,000
E. at least 60,000

Use the following size of loss distribution for the next 2 questions:

Size of Loss	Probability
100	70%
1000	20%
10,000	10%

11.9 (2 points) If there is an ordinary deductible of 500, what is the coefficient of variation of the nonzero payments?
A. less than 1.0
B. at least 1.0 but less than 1.1
C. at least 1.1 but less than 1.2
D. at least 1.2 but less than 1.3
E. at least 1.3

11.10 (2 points) If there is a franchise deductible of 500, what is the coefficient of variation of the nonzero payments?
A. less than 1.0
B. at least 1.0 but less than 1.1
C. at least 1.1 but less than 1.2
D. at least 1.2 but less than 1.3
E. at least 1.3

Use the following information for the next four questions:
The Mockingbird Tequila Company buys insurance from the Atticus Insurance Company, with a deductible of $5000, maximum covered loss of $250,000, and coinsurance factor of 90%.
Atticus Insurance Company buys reinsurance from the Finch Reinsurance Company.
Finch will pay Atticus for the portion of any payment in excess of $100,000.
Let X be an individual loss amount suffered by the Mockingbird Tequila Company.

11.11 (2 points) Let Y be the amount retained by the Mockingbird Tequila Company.
What is the relationship between X and Y?

11.12 (2 points) Let Y be the amount paid by the Atticus Insurance Company to the Mockingbird Tequila Company, prior to the impact of reinsurance. What is the relationship between X and Y?

11.13 (2 points) Let Y be the payment made by the Finch Reinsurance Company to the Atticus Insurance Company. What is the relationship between X and Y?

11.14 (2 points) Let Y be the net amount paid by the Atticus Insurance Company after the impact of reinsurance. What is the relationship between X and Y?

11.15 (2 points) Assume a loss of size x.
Policy A calculates the payment based on limiting to a maximum of 10,000, then subtracting a deductible of 1000, and then applying a coinsurance factor of 90%.
Policy B instead calculates the payment based on subtracting a deductible of 1000, then limiting it to a maximum of 10,000, and then applying a coinsurance factor of 90%.
What is the difference in payments between that under Policy A and Policy B?

11.16 (1 point) Calculate the percentage reduction in the loss costs for a $500 ordinary deductible compared to full coverage.

Full Coverage Loss Data		
Loss Size	Number of Claims	Amount of Loss
0 - 250	1,500	375,000
250 - 500	1,000	450,000
500 - 750	750	487,500
750-1,000	500	400,000
1,000-1,500	250	312,500
1,500 or more	100	300,000
Total	4,100	2,325,000

11.17 (**CAS6, 5/94, Q.21)** (1 point) Last year, an insured in a group medical plan incurred charges of $600. This year, the same medical care resulted in a charge of $660. The group comprehensive medical care plan provides 80% payment after a $100 deductible. Determine the increase in the insured's retention under his or her comprehensive medical care plan.
A. Less than 7.0%
B. At least 7.0% but less than 9.0%
C. At least 9.0% but less than 11.0%
D. At least 11.0% but less than 13.0%
E. 13.0% or more

11.18 (**CAS9, 11/94, Q.11)** (1 point) Given the following:
W = The amount paid by the insurer.
d = The insured's deductible amount.
X = The total amount of loss
Which of the following describes W, if X > d, and the insured has a franchise deductible?
A. W = X B. W = X - d C. W = (X - d) / d D. W = d E. None of A, B, C, or D

11.19 (**CAS6, 5/96, Q.41)** (2 points)
You are given the following full coverage experience:

Loss Size	Number of Claims	Amount of Loss
$ 0-99	1,400	$76,000
$100-249	400	$80,000
$250-499	200	$84,000
$500-999	100	$85,000
$1,000 or more	50	$125,000
Total	2,150	$450,000

(a) (1 point)
Calculate the expected percentage reduction in losses for a $250 ordinary deductible.
(b) (1 point)
Calculate the expected percentage reduction in losses for a $250 franchise deductible.

11.20 (**CAS9 11/97, Q.36a)** (2 points)
An insured is trying to decide which type of policy to purchase:

- A policy with a franchise deductible of $50 will cost her $8 more than a policy with a straight deductible of $50.
- A policy with a franchise deductible of $100 will cost her $10 more than a policy with a straight deductible of $100.

An expected ground-up claim frequency of 1.000 is assumed for each of the policies described above.
Calculate the probability that the insured will suffer a loss between $50 and $100.
Show all work.

Use the following information for the next two questions:

Loss Size	Number of Claims	Total Amount of Loss
$0-249	5,000	$1,125,000
250-499	2,250	765,000
500-999	950	640,000
1,000-2,499	575	610,000
2500 or more	200	890,000
Total	8,975	$4,030,000

11.21 (CAS6, 5/98, Q.7) (1 point) Calculate the percentage reduction in loss costs caused by the introduction of a $500 franchise deductible. Assume there is no adverse selection or padding of claims to reach the deductible.
A. Less than 25.0%
B. At least 25.0%, but less than 40.0%
C. At least 40.0%, but less than 55.0%
D. At least 55.0%, but less than 70.0%
E. 70.0% or more

11.22 (1 point) Calculate the percentage reduction in loss costs caused by the introduction of a $500 ordinary deductible. Assume there is no adverse selection or padding of claims to reach the deductible.

11.23 (1, 5/03, Q.25) (2.5 points) An insurance policy pays for a random loss X subject to a deductible of C, where $0 < C < 1$. The loss amount is modeled as a continuous random variable with density function $f(x) = 2x$ for $0 < x < 1$.
Given a random loss X, the probability that the insurance payment is less than 0.5 is equal to 0.64. Calculate C.
(A) 0.1 (B) 0.3 (C) 0.4 (D) 0.6 (E) 0.8

11.24 (CAS5, 5/03, Q.9) (1 point) An insured has a catastrophic health insurance policy with a $1,500 deductible and a 75% coinsurance clause. The policy has a $3,000 maximum retention. If the insured incurs a $10,000 loss, what amount of the loss must the insurer pay?
Note: I have rewritten this past exam question in order to match the current syllabus.

11.25 (CAS3, 5/04, Q.35) (2.5 points) The XYZ Insurance Company sells property insurance policies with a deductible of $5,000, policy limit of $500,000, and a coinsurance factor of 80%. Let X_i be the individual loss amount of the i^{th} claim and Y_i be the claim payment of the i^{th} claim. Which of the following represents the relationship between X_i and Y_i?

A. $\begin{cases} 0 & X_i \leq 5{,}000 \\ Y_i = 0.80(X_i - 5{,}000) & 5{,}000 < X_i \leq 625{,}000 \\ 500{,}000 & X_i > 625{,}000 \end{cases}$

B. $\begin{cases} 0 & X_i \leq 4{,}000 \\ Y_i = 0.80(X_i - 4{,}000) & 4{,}000 < X_i \leq 500{,}000 \\ 500{,}000 & X_i > 500{,}000 \end{cases}$

C. $\begin{cases} 0 & X_i \leq 5{,}000 \\ Y_i = 0.80(X_i - 5{,}000) & 5{,}000 < X_i \leq 630{,}000 \\ 500{,}000 & X_i > 630{,}000 \end{cases}$

D. $\begin{cases} 0 & X_i \leq 6{,}250 \\ Y_i = 0.80(X_i - 6{,}250) & 6{,}250 < X_i \leq 631{,}250 \\ 500{,}000 & X_i > 631{,}250 \end{cases}$

500,000 $X_i > 631{,}250$

E. $\begin{cases} 0 & X_i \leq 5{,}000 \\ Y_i = 0.80(X_i - 5{,}000) & 5{,}000 < X_i \leq 505{,}000 \\ 500{,}000 & X_i > 505{,}000 \end{cases}$

11.26 (SOA M, 5/05, Q.32 & 2009 Sample Q.168) (2.5 points) For an insurance:
(i) Losses can be 100, 200, or 300 with respective probabilities 0.2, 0.2, and 0.6.
(ii) The insurance has an ordinary deductible of 150 per loss.
(iii) Y^P is the claim payment per payment random variable.
Calculate Var(Y^P).
(A) 1500 (B) 1875 (C) 2250 (D) 2625 (E) 3000

Solutions to Problems:

11.1. D. First the insurer limits each loss to \$100,000: 17, 60, 100. Then it reduces each loss by the \$20,000 deductible: 0, 40, 80. Then the 90% coinsurance factor is applied: 0, 36, 72.
The insurer pays a total of 0 + 36 + 72 = **\$108 thousand.**

11.2. B. There are three losses and 108,000 in total is paid: \$108,000/3 = **\$36,000**.

11.3. E. There are two (non-zero) payments and 108,000 in total is paid: \$108,000/2 = **\$54,000**.

11.4. E. Under Policy A one pays x - 100 for x > 100. 3/4 of the losses are greater than 100, and those losses have average size (100 + 400)/2 = 250.
Thus under Policy A the expected payment per loss is: (3/4)(250 -100) = 112.5
Under Policy B, one pays x for x > 100.
Thus the expected payment per loss is: (3/4)(250) = 187.5. Ratio is: 187.5/112.5 = **5/3**.

11.5. C. The insurer pays: 0 + 0 + \$10,100 + \$23,200 = **\$33,300**.

11.6. D. Subtract the deductible: 0, 0, 4000, 70,000, 110,000, 190,000.
Multiply by the coinsurance factor: 0, 0, 3200, 56,000, 88,000, 152,000.
Limit each payment to 100,000: 0, 0, 3200, 56,000, 88,000, 100,000.
0 + 0 + 3200 + 56,000 + 88,000 + 100,000 = **247,200**.
Alternately, the maximum covered loss is: 10,000 + 100,000/0.8 = 135,000.
Limit each loss to the maximum covered loss: 3000, 8000, 14,000, 80,000, 120,000, 135,000.
Subtract the deductible: 0, 0, 4000, 70,000, 110,000, and 125,000.
Multiply by the coinsurance factor: 0, 0, 3200, 56,000, 88,000, and 100,000.
0 + 0 + 3200 + 56,000 + 88,000 + 100,000 = **247,200**.

11.7. A. 247,200/6 = **41,200**.

11.8. E. 247,200/4 = **61,800**.

11.9. D. The nonzero payments are: 500@2/3 and 9500@1/3.
Mean = (2/3)(500) + (1/3)(9500) = 3500.
2nd moment = $(2/3)(500^2) + (1/3)(9500^2)$ = 30,250,000.
variance = 30,250,000 - 3500^2 = 18,000,000.
CV = $\sqrt{18{,}000{,}000}$ / 3500 = **1.212**.

11.10. B. The nonzero payments are: 1000@2/3 and 10,000@1/3.
Mean = (2/3)(1000) + (1/3)(10,000) = 4000.
2nd moment = $(2/3)(1000^2) + (1/3)(10{,}000^2)$ = 34,000,000.
variance = 34,000,000 - 4000^2 = 18,000,000. CV = $\sqrt{18{,}000{,}000}$ / 4000 = **1.061**.

11.11. Mockingbird retains all of any loss less than $5000.
For a loss of size greater than $5000, it retains $5000 plus 10% of the portion above $5000.
Mockingbird retains the portion of any loss above the maximum covered loss of $250,000.
Y = X, for X ≤ 5000.
Y = 5000 + (0.1)(X - 5000) = 4500 + 0.1X, for 5000 ≤ X ≤ 250,000.
Y = 4500 + (0.1)(250,000) + (X - 250,000) = X - 220,500, for 250,000 ≤ X.
Comment: The maximum amount that Atticus Insurance Company retains on any loss is: (0.9)(250,000 - 5000) = 220,500. Therefore, for a loss X of size greater than 250,000, Mockingbird retains X - 220,500.

11.12. Atticus Insurance pays nothing for a loss less than $5000. For a loss of size greater than $5000, Atticus Insurance pays 90% of the portion above $5000.
For a loss of size 250,000, Atticus Insurance pays: (0.9)(250,000 - 5000) = 220,500.
Atticus Insurance pays no more for a loss larger than the maximum covered loss of $250,000.
Y = 0, for X ≤ 5000.
Y = (0.9)(X - 5000) = 0.9X - 4500, for 5000 ≤ X ≤ 250,000.
Y = 220,500, for 250,000 ≤ X.
Comment: The amount retained by Mockingbird, plus the amount paid by Atticus to Mockingbird, equals the total loss.

11.13. Finch Reinsurance pays something when the loss results in a payment by Atticus of more than $100,000. Solve for the loss that results in a payment of $100,000:
100,000 = (0.9)(X - 5000). ⇒ x = 116,111.
Y = 0, for X ≤ 116,111.
Y = (0.9)(X - 116,111) = 0.9X - 104,500, for 116,111 < X ≤ 250,000.
Y = 120,500, for 250,000 ≤ X.

11.14. For a loss greater than 116,111, Atticus pays 100,000 net of reinsurance.
Y = 0, for X ≤ 5000.
Y = (0.9)(X - 5000) = 0.9X - 4500, for 5000 ≤ X ≤ 116,111.
Y = 100,000, for 116,111 < X.

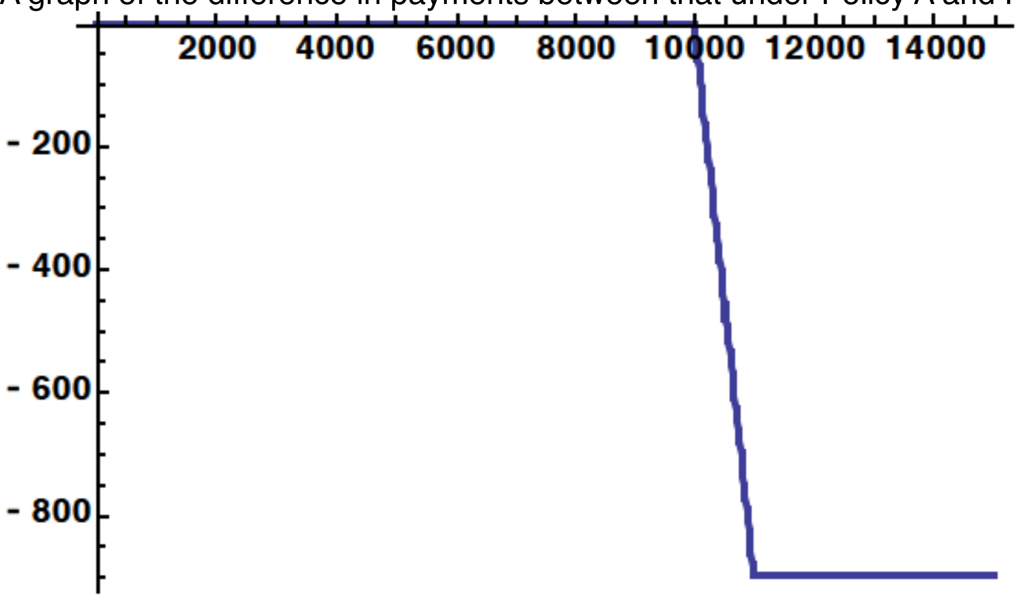
11.15. Policy A: (0.9) $(\text{Min}[x, 10{,}000] - 1000)_+ = (0.9)\ (\text{Min}[x - 1000, 9000])_+$
$= (\text{Min}[0.9x - 900, 8100])_+$

Policy B: (0.9) $\text{Min}[(x - 1000)_+, 10{,}000] = \text{Min}[(0.9x - 900)_+, 9000]$
$= (\text{Min}[0.9x - 900, 9000])_+$.

Policy A - Policy B = $(\text{Min}[0.9x - 900, 8100])_+ - (\text{Min}[0.9x - 900, 9000])_+$.

If x ≥ 11,000, then this difference is: 8100 - 9000 = -900.
If 11,000 > x > 10,000, then this difference is: 8100 - (0.9x - 900) = 9000 - 0.9x.
If 10,000 ≥ x > 1000, then this difference is: (0.9x - 900) - (0.9x - 900) = 0.
If x ≤ 1000, then this difference is: 0 - 0 = 0.

Comment: Policy A follows the order of operations you should follow on your exam, unless specifically stated otherwise.
A graph of the difference in payments between that under Policy A and Policy B:



I would attack this type of problem by just trying various values for x.
Here are some examples:

x	Payment Under A	Payment Under B	Difference
12,000	8100	9000	-900
10,300	8100	8370	-270
8000	6300	6300	0
700	0	0	0

11.16. {375,000 + 450,000 + (500)(750 + 500 + 250 + 100)} / 2,325,000 = **69.9%**.

11.17. A. Last year, the insured gets paid: (80%)(600 - 100) = 400.
Insured retains: 600 - 400 = 200.
This year, the insured gets paid: (80%)(660 - 100) = 448.
Insured retains: 660 - 448 = 212.
Increase in the insured's retention is: 212 / 200 - 1 = **6.0%**.

11.18. A. For a loss greater than the deductible amount, the franchise deducible pays the full loss.

11.19. (a) Losses eliminated: 76,000 + 80,000 + (250)(200 + 100 + 50) = \$243,500.
\$243,500/\$450,000 = 0.541 = **54.1% reduction in expected losses**.
(b) Under the franchise deductible, we pay the whole loss for a loss of size greater than 250.
Losses eliminated: 76,000 + 80,000 = \$156,000.
\$156,000/\$450,000 = 0.347 = **34.7% reduction in expected losses**.

11.20. When a payment is made, the \$50 franchise deductible pays \$50 more than the \$50 straight deductible. Therefore, \$8 = (1.000) S(50) (\$50). ⇒ S(50) = 16%.
When a payment is made, the \$100 franchise deductible pays \$100 more than the \$100 straight deductible. Therefore, \$10 = (1.000) S(100) (\$100). ⇒ S(100) = 10%.
The probability that the insured will suffer a loss between \$50 and \$100 is:
S(50) - S(100) = 16% - 10% = **6%**.

11.21. C. (1,125,000 + 765,000)/4,030,000 = **46.9%**.

11.22. {1,125,000 + 765,000 + (500)(950 + 575 + 200)} / 4,030,000 = **68.3%**.

11.23. B. $F(x) = x^2$.
$0.64 = \text{Prob}[\text{payment} < 0.5] = \text{Prob}[X - C < 0.5] = \text{Prob}[X < 0.5 + C] = (0.5 + C)^2$.
⇒ C = 0.8 - 0.5 = **0.3**.

11.24. Without the maximum retention the insurer would pay: (10,000 - 1500)(0.75) = 6375.
In that case the insured would retain: 10,000 - 6375 = 3625.
However, the insured retains at most 3000, so the insurer pays: 10,000 - 3000 = **7000**.

11.25. C. The policy limit is: c(u-d), where u is the maximum covered loss.
Therefore, u = d + (policy limit)/c = 5000 + 500,000/0.8 = 630,000.
Therefore the payment is: 0 if X ≤ 5,000, 0.80(X - 5,000) if 5,000 < X ≤ 630,000, and 500,000 if X > 630,000.
Comment: For a loss of size 3000 nothing is paid. For a loss of size 100,000, the payment is: (0.8)(100,000 - 5000) = 76,000. For a loss of size 700,000, the payment would be: (0.8)(700,000 - 5000) = 556,000, except that the maximum payment is the policy limit of 500,000. Increasing the size of loss above the maximum covered loss of 630,000, results in no increase in payment beyond 500,000.

11.26. B. $\text{Prob}[Y^P = 50] = 0.2/(0.2 + 0.6) = 1/4$. $\text{Prob}[Y^P = 150] = 0.6/(0.2 + 0.6) = 3/4$.
$E[Y^P] = (50)(1/4) + (150)(3/4) = 125$. $E[(Y^P)^2] = (50^2)(1/4) + (150^2)(3/4) = 17,500$.
$\text{Var}[Y^P] = 17,500 - 125^2 =$ **1875**.

Section 12, Truncated Data

Unless stated otherwise, data is assumed to be ground-up (first dollar) unlimited losses, on all loss events that occurred. By "first dollar", I mean that we start counting from the first dollar of economic loss, in other words as if there were no deductible. By "unlimited" I mean we count every dollar of economic loss, as if there were no maximum covered loss.

Sometimes some of this information is not reported, most commonly due to a deductible and/or maximum covered loss.
There are four such situations likely to come up on your exam, each of which has two names:
left truncation ⇔ truncation from below
left truncation and shifting ⇔ truncation and shifting from below
left censoring and shifting ⇔ censoring and shifting from below
right censoring ⇔ censoring from above.

In the following, ground-up, unlimited losses are assumed to have distribution function F(x).
G(x) is what one would see after the effects of either a deductible or a maximum covered loss.

Left Truncation / Truncation from Below:[33]

Left Truncation ⇔ Truncation from Below at d ⇔
deductible d and record size of loss for size > d.

For example, data **left truncated** or **truncated from below** at $10,000, would have no information on any loss which resulted in less than $10,000 of loss. The actuary would not even know how many such small losses existed.[34] The same information would be reported on the remaining large losses.

When data is truncated from below at the value d, losses of size less than d are not in the reported data base.[35] This generally occurs when there is an (ordinary) deductible of size d, and the insurer records the amount of loss to the insured.

Exercise: An insured has losses of sizes: $300, $600, $1,200, $1,500 and $2,800.
How is this data reported, if it is left truncated / truncated from below at $1000?
[Solution: $1,200, $1,500 and $2,800 .
Comment: The two smaller losses are never reported to the insurer.]

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[33] The terms "left truncation" and "truncation from below" are synonymous.
[34] This would commonly occur in the case of a $10,000 deductible.
[35] Note that the Mean Excess Loss, e(x), is unaffected by truncation from below at d, provided x > d.

The distribution function and the probability density functions are revised as follows:

$G(x) = \frac{F(x) - F(d)}{S(d)}$, x > d

g(x) = f(x) / S(d), x > d
x ⇔ the size of loss.

Thus the data truncated from below has a distribution function which is zero at d and 1 at infinity. The revised probability density function has been divided by the original chance of having a loss of size greater than d. Thus for the revised p.d.f. the probability from d to infinity integrates to unity as it should.

Note that G(x) = {F(x) - F(d)} / S(d) = (S(d) - S(x))/S(d) = 1 - S(x)/S(d), x > d.
1 - G(x) = S(x)/S(d). The revised survival function after truncation from below is the survival function prior to truncation divided by the survival function at the truncation point.

Both data truncated from below and the mean excess loss exclude the smaller losses. In order to compute the mean excess loss, we would take the average of the losses greater than d, and then subtract d. Therefore, the average size of the data truncated from below at d, is d plus the mean excess loss at d, e(d) + d.

Exercise: An insured has losses of sizes: $300, $600, $1,200, $1,500 and $2,800.
What is the average size of the data reported, if it is truncated from below at $1000?
[Solution: ($1,200+ $1,500 + $2,800) /3 = 1833.33 = 1000 + 833.33 = 1000 + e(1000).]

Franchise Deductible:

Under a franchise deductible the insurer pays nothing if the loss is less than the deductible amount, but ignores the deductible if the loss is greater than the deductible amount. If the deductible amount is d and the insured has a loss of size x, then the insurer pays:

0	x ≤ d
x	x > d

Thus **data from a policy with a franchise deductible is truncated from below** at the deductible amount.

Left Truncation and Shifting / Truncation and Shifting from Below:

Left Truncation & Shifting at d ⇔ Truncation & Shifting from Below at d
⇔ Excess Loss Variable ⇔ deductible d and record non-zero payment.

If data were truncated and shifted from below at \$10,000, the data on the remaining large losses would have each amount reduced by \$10,000. When data is truncated and shifted from below at the value d, losses of size less than d are not in the reported data base, and larger losses have their reported values reduced by d. This generally occurs when there is an (ordinary) deductible of size d, and the insurer records the amount of payment to the insured.

Exercise: An insured has losses of sizes: \$300, \$600, \$1,200, \$1,500 and \$2,800.
How is this data reported, if it is truncated and shifted from below at \$1000?
[Solution: \$200, \$500 and \$1,800.]

The distribution, survival, and the probability density functions are revised as follows:

$G(x) = \frac{F(x+d) - F(d)}{S(d)}$, x > 0. New survival function = 1 - G(x) = S(x+d)/S(d), x > 0

g(x) = f(x+d) / S(d), x > 0

x ⇔ the size of (non-zero) payment. x+d ⇔ the size of loss.

As discussed previously, the Excess Loss Variable for d is defined for X > d as X-d and is undefined for X ≤ d, which is the same as the effect of truncating and shifting from below at d.

Exercise: Prior to a deductible, losses are Weibull with $\tau = 2$ and $\theta = 1000$.
What is the probability density function of the excess loss variable corresponding to d = 500?
[Solution: For the Weibull, $F(x) = 1 - \exp(-(x/\theta)^\tau)$ and $f(x) = \tau x^{\tau-1} \exp(-(x/\theta)^\tau) / \theta^\tau$.
$S(500) = \exp[-(500/1000)^2] = 0.7788$. Let Y be the truncated and shifted variable.
Then, $g(y) = f(500 + y)/S(500) = (y+500) \exp[-((y+500)/1000)^2] / 389{,}400$, y > 0.]

The average size of the data truncated and shifted from below at d, is the mean excess loss (mean residual life) at d, e(d).

Exercise: An insured has losses of sizes: \$300, \$600, \$1,200, \$1,500 and \$2,800.
What is the average size of the data reported, if it is truncated and shifted from below at \$1000?
[Solution: (\$200+ \$500 + \$1,800) / 3 = 833.33 = e(1000).]

Complete Expectation of Life:

These ideas are mathematically equivalent to ideas discussed in Life Contingencies.

The expected future lifetime for an age x is $\mathring{e}_x$, the complete expectation of life.

$\mathring{e}_x$ is the mean residual life (mean excess loss) at x, e(x).

$\mathring{e}_x$ is the mean of the lives truncated and shifted from below at x.

Exercise: Three people die at ages: 55, 70, 80. Calculate $\mathring{e}_{65}$.

[Solution: Truncate and shift the data at 65; eliminate any ages ≤ 65 and subtract 65:
70 - 65 = 5, 80 - 65 = 15.

Average the truncated and shifted data: $\mathring{e}_{65}$ = (5 + 15)/2 = 10.]

The survival function at age x + t for the data truncated and shifted at x is: S(x+t)/S(x) = ${}_tp_x$.

As will be discussed in a subsequent section, one can get the mean by integrating the survival function.

$\mathring{e}_x$ = mean of the data truncated and shifted at x = $\int_0^\infty {}_tp_x \, dt$.[36]

[36] See equation 3.5.2 in Actuarial Mathematics.

Right Truncation / Truncation from Above:[37]

In the case of the data **right truncated** or **truncated from above** at $25,000, there would be no information on any losses larger than $25,000. Truncation from above contrasts with data censored from above at $25,000, which would have each of the large losses reported as being $25,000 or more.[38]

When data is right truncated or truncated from above at the value L, losses of size greater than L are not in the reported data base.[39]

Exercise: An insured has losses of sizes: $300, $600, $1,200, $1,500 and $2,800.
How is this data reported, if it is truncated from above at $1000?
[Solution: $300 and $600.]

The distribution function and the probability density functions are revised as follows:

$G(x) = F(x) / F(L), x \le L$
$g(x) = f(x) / F(L), x \le L$

The average size of the data truncated from above at L,
is the average size of losses from 0 to L: $\frac{E[X \wedge L] - L\,S(L)}{F(L)}$.

Exercise: An insured has losses of sizes: $300, $600, $1,200, $1,500 and $2,800.
What is the average size of the data reported, if it is truncated from above at $1000?
[Solution: ($300 + $600)/2 = $450.]

Truncation from above would be appropriate where one thought there was a maximum possible loss.[40]

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[37] See Definition 14.10 in Loss Models.
[38] Under censoring from above, one would not know the total size of loss for each of these large losses. This is quite common for data reporting when there are maximum covered losses.
[39] Right truncation can happen when insufficient time has elapsed to receive all of the data. For example, one might be doing a mortality study based on death records, which would exclude from the data anyone who has yet to die. Right truncation can also occur when looking at claim count development. *One might not have data beyond a given "report" and fit a distribution function (truncated from above) to the available claim counts by report. See for example, "Estimation of the Distribution of Report Lags by the Method of Maximum Likelihood," by Edward W. Weisner, PCAS 1978.*
[40] See for example, "A Note on the Upper-Truncated Pareto Distribution," by David R. Clark, Winter 2013 CAS E-Forum. The size of loss from any type of event has some upper limit, even if very big.

Truncation from Both Above and Below:

When data is both truncated from above at L and truncated from below at the value d, losses of size greater than L or less than or equal to d are not in the reported data base.

Exercise: Events occur at times: 3, 6, 12, 15, 24, and 28.
How is this data reported, if it is truncated from below at 10 and truncated from above at 20?
[Solution: 12 and 15.
Comment: One starts observing at time 10 and stops observing at time 20.]

The distribution function and the probability density functions are revised as follows:

$$G(x) = \frac{F(x) - F(d)}{F(L) - F(d)}, \quad d < x \le L$$

$$g(x) = \frac{f(x)}{F(L) - F(d)}, \quad d < x \le L$$

Note that whenever we have truncation, the probability remaining after truncation is the denominator of the altered density and distribution functions.

The average size of the data truncated from below at d and truncated from above at L, is the average size of losses from d to L, $\frac{\{E[X \wedge L] - L\ S(L)\} - \{E[X \wedge d] - d\ S(d)\}}{F(L) - F(d)}$.

Exercise: Events occur at times: 3, 6, 12, 15, 24, and 28. What is the average size of the data reported, if it is truncated from below at 10 and truncated from above at 20?
[Solution: (12 + 15)/2 = 13.5.]

Problems:

Use the following information for the next 4 questions:
There are 6 losses: 100, 400, 700, 800, 1200, and 2300.

12.1 (1 point) If these losses are truncated from below (left truncated) at 500, what appears in the data base?

12.2 (1 point) If these losses are truncated and shifted from below (left truncated and shifted) at 500, what appears in the data base?

12.3 (1 point) If these losses are truncated from above (right truncated) at 1000, what appears in the data base?

12.4 (1 point) If these losses are truncated from below (left truncated) at 500 and truncated from above (right truncated) at 1000, what appears in the data base.

Use the following information for the next 2 questions:
Losses are uniformly distributed from 0 to 1000.

12.5 (1 point) What is the distribution function for these losses left truncated at 100?

12.6 (1 point) What is the distribution function for these losses left truncated and shifted at 100?

12.7 (1 point) Assume that claims follow a distribution, $F(x) = 1 - \frac{1}{\{1 + (x/\theta)^\gamma\}^\alpha}$.

Which of the following represents the distribution function for the data truncated from below at d?

A. $\left(\frac{\theta\gamma + d\gamma}{\theta\gamma + x\gamma}\right)^\alpha$ $x > d$

B. $1 - \left(\frac{\theta\gamma + d\gamma}{\theta\gamma + x\gamma}\right)^\alpha$ $x > d$

C. $\left(\frac{\theta\gamma}{\theta\gamma + x\gamma}\right)^\alpha$ $x > d$

D. $1 - \left(\frac{\theta\gamma}{\theta\gamma + x\gamma}\right)^\alpha$ $x > d$

E. None of the above.

12.8 (1 point) The report lag for claims is assumed to be exponentially distributed:
$F(x) = 1 - e^{-\lambda x}$, where x is the delay in reporting.
What is the probability density function for data truncated from above at 3?

A. $3e^{-3x}$ B. $\lambda e^{-3\lambda}$ C. $e^{-\lambda x} / (1 - e^{-3\lambda})$ D. $\lambda e^{-\lambda x} / (1 - e^{-3\lambda})$ E. $\lambda e^{-\lambda(x-3)}$

Use the following information for the next three questions:

- Losses follow a Distribution Function $F(x) = 1 - \{2000 / (2000+x)\}^3$.

12.9 (1 point) If the reported data is truncated from below at 400, what is the Density Function at 1000?
A. less than 0.0004
B. at least 0.0004 but less than 0.0005
C. at least 0.0005 but less than 0.0006
D. at least 0.0006 but less than 0.0007
E. at least 0.0007

12.10 (1 point) If the reported data is truncated from above at 2000, what is the Density Function at 1000?
A. less than 0.0004
B. at least 0.0004 but less than 0.0005
C. at least 0.0005 but less than 0.0006
D. at least 0.0006 but less than 0.0007
E. at least 0.0007

12.11 (1 point) If the reported data is truncated from below at 400 and truncated from above at 2000, what is the Density Function at 1000?
A. less than 0.0004
B. at least 0.0004 but less than 0.0005
C. at least 0.0005 but less than 0.0006
D. at least 0.0006 but less than 0.0007
E. at least 0.0007

Use the following information for the next 2 questions:
There are 3 losses: 800, 2500, 7000.

12.12 (1 point) If these losses are truncated from below (left truncated) at 1000, what appears in the data base?

12.13 (1 point) If these losses are truncated and shifted from below (left truncated and shifted) at 1000, what appears in the data base?

Use the following information for the next 3 questions:
The probability density function is: f(x) = x/50, 0 ≤ x ≤ 10.

12.14 (2 points) Determine the mean of this distribution left truncated at 4.
A. 7.3 B. 7.4 C. 7.5 D. 7.6 E. 7.7

12.15 (2 points) Determine the median of this distribution left truncated at 4
A. 7.3 B. 7.4 C. 7.5 D. 7.6 E. 7.7

12.16 (2 points) Determine the variance of this distribution left truncated at 4
A. 2.8 B. 3.0 C. 3.2 D. 3.4 E. 3.6

12.17 (4, 5/85, Q.56) (2 points) Let f be the probability density function of x, and let F be the distribution function of x. Which of the following expressions represent the probability density function of x truncated and shifted from below at d?

A. $\begin{cases} 0, & x \le d \\ f(x) / \{1 - F(d)\}, & d < x \end{cases}$

B. $\begin{cases} 0, & x \le 0 \\ f(x) / \{1 - F(d)\}, & 0 < x \end{cases}$

C. $\begin{cases} 0, & x \le d \\ f(x-d) / \{1 - F(d)\}, & d < x \end{cases}$

D. $\begin{cases} 0, & x \le -d \\ f(x+d) / \{1 - F(d)\}, & -d < x \end{cases}$

E. $\begin{cases} 0, & x \le 0 \\ f(x+d) / \{1 - F(d)\}, & 0 < x \end{cases}$

12.18 (4B, 11/92, Q.3) (1 point) You are given the following:
- Based on observed data truncated from above at $10,000, the probability of a claim exceeding $3,000 is 0.30.
- Based on the underlying distribution of losses, the probability of a claim exceeding $10,000 is 0.02.

Determine the probability that a claim exceeds $3,000.
A. Less than 0.28
B. At least 0.28 but less than 0.30
C. At least 0.30 but less than 0.32
D. At least 0.32 but less than 0.34
E. At least 0.34

Solutions to Problems:

12.1. If these losses are truncated from below (left truncated) at 500, the two small losses do not appear: **700, 800, 1200, 2300**.

12.2. If these losses are truncated and shifted from below at 500, the two small losses do not appear and the other losses have 500 subtracted from them: **200, 300, 700, 1800**.
Comment: The (non-zero) payments with a $500 deductible.

12.3. If these losses are truncated from above (right truncated) at 1000, the two large losses do not appear: **100, 400, 700, 800**.

12.4. Neither the two small losses nor the two large losses appear: **700, 800**.

12.5. Losses of size less than 100 do not appear.
G(x) = (x - 100)/900 for 100 < x < 1000.
Alternately, F(x) = x/1000, 0 < x < 1000 and G(x) = {F(x) - F(100)} / S(100) =
{(x/1000) - 100/1000)} / (1 - 100/1000) **= (x - 100)/900 for 100 < x < 1000.**

12.6. Losses of size less than 100 do not appear. We record the payment amount with a $100 deductible. **G(x) = x/900 for 0 < x < 900.**
Alternately, F(x) = x/1000, 0 < x < 1000 and G(x) = {F(x + 100) - F(100)}/S(100) =
{((x + 100)/1000) - 100/1000)}/(1 - 100/1000) **= x/900 for 0 < x < 900.**
Comment: A uniform distribution from 0 to 900.

12.7. B. The new distribution function is for x > d: {F(x) - F(d)} / {1 - F(d)} =
$[\{\theta^\gamma / (\theta^\gamma + d^\gamma)\}^\alpha - \{\theta^\gamma / (\theta^\gamma + x^\gamma)\}^\alpha] / \{\theta^\gamma / (\theta^\gamma + d^\gamma)\}^\alpha = \mathbf{1 - \{(\theta^\gamma + d^\gamma) / (\theta^\gamma + x^\gamma)\}^\alpha}$.
Comment: A Burr Distribution.

12.8. D. The Distribution Function of the data truncated from above at 3 is G(x) = F(x)/F(3) =
$(1 - e^{-\lambda x}) / (1 - e^{-3\lambda})$. The density function is g(x) = G'(x) $\mathbf{= \lambda e^{-\lambda x} / (1 - e^{-3\lambda})}$.

12.9. C. Prior to truncation the density function is: $f(x) = (3)(2000)^3/ (2000+x)^4$.
After truncation from below at 400, the density function is: g(x) = f(x) / {1 - F(400)} =
$f(x) / \{2000 / (2000+400)\}^3 = f(x) / 0.5787$.
$f(1000) = (3)(2000)^3/ (2000+1000)^4 = 0.000296$.
g(1000) = 0.000296 / 0.5787 = **0.00051**.

12.10. A. Prior to truncation the density function is: $f(x) = (3)(2000)^3/ (2000+x)^4$.
After truncation from above at 2000, the density function is:
$g(x) = f(x) / F(2000) = f(x) / \{1 - (2000 / (2000+2000))^3\} = f(x) / 0.875$.
$f(1000) = (3)(2000)^3 / (2000 +1000)^4 = 0.000296$.
g(1000) = 0.000296 / 0.875 = **0.00034**.

12.11. D. Prior to truncation the density function is: $f(x) = (3)(2000)^3 / (2000+x)^4$.
After truncation from below at 400 and from above at 2000, the density function is:
$g(x) = f(x) / \{F(2000)-F(400)\}$. $F(2000) = 0.875$. $F(400) = 0.4213$.
$f(1000) = (3)(2000^3) / (2000+1000)^4 = 0.000296$.
$g(1000) = 0.000296 / (0.875 - 0.4213) =$ **0.00065**.

12.12. If these losses are truncated from below (left truncated) at 1000, then losses of size less than or equal to 1000 do not appear. The data base is: **2500, 7000.**

12.13. If these losses are truncated and shifted from below at 1000, then losses of size less than or equal to 1000 do not appear, and the other losses have 1000 subtracted from them. The data base is: **1500, 6000.**
Comment: The (non-zero) payments with a $1000 deductible.

12.14. B. & 12.15. D. & 12.16. A. For the original distribution: $F(4) = 4^2/100 = 0.16$.
Therefore, the density left truncated at 4 is: $(x/50)/(1 - 0.16) = x/42$, $4 \le x \le 10$.

The mean of the truncated distribution is: $\int_4^{10} (x/42)\, x \, dx = (10^3 - 4^3)/126 =$ **7.429**.

For $x > 4$, by integrating the truncated density, the truncated distribution is: $(x^2 - 4^2) / 84$.
Set the truncated distribution equal to 50%: $0.5 = (x^2 - 4^2) / 84$. $\Rightarrow x =$ **7.616**.

The second moment of the truncated distribution is: $\int_4^{10} (x/42)\, x^2 \, dx = (10^4 - 4^4)/168 = 58$.

The variance of the truncated distribution is: $58 - 7.429^2 =$ **2.81**.
Comment: The density right truncated at 4 is: $(x/50)/0.16 = x/8$, $0 \le x \le 4$.

12.17. E. The p.d.f. is: 0 for $x \le 0$, and $f(x+d)/[1-F(d)]$ for $0 < x$
Comment: Choice A is the p.d.f for the data truncated from below at d, but not shifted.

12.18. C. $P(x \le 3000 \mid x \le 10{,}000) = P(x \le 3000) / P(x \le 10{,}000)$.
Thus, $1 - 0.3 = P(x \le 3000) / (1 - 0.02)$.
$P(x \le 3000) = (1 - 0.3)(0.98) = 0.686$.
$P(x > 3000) = 1 - 0.686 =$ **0.314.**
Alternately, let F(x) be the distribution of the untruncated losses.
Let G(x) be the distribution of the losses truncated from above at 10,000.
Then $G(x) = F(x) / F(10{,}000)$, for $x \le 10{,}000$.
We are given that $1 - G(3000) = 0.3$, and that $1 - F(10{,}000) = 0.02$.
Thus $F(10{,}000) = 0.98$.
Also, $0.7 = G(3000) = F(3000) / F(10{,}000) = F(3000) / 0.98$.
Thus $F(3000) = (0.7)(0.98) = 0.686$.
$1 - F(3000) = 1 - 0.686 =$ **0.314**.

Section 13, Censored Data

Censoring is somewhat different than truncation. With truncation we do not know of the existence of certain losses. With censoring we do not know the size of certain losses. The most important example of censoring is due to the effect of a maximum covered loss.

Right Censored / Censored from Above:[41]

Right Censored at u ⇔ Censored from Above at u ⇔ $X \wedge u$ ⇔ Min[X, u] ⇔
Maximum Covered Loss u and don't know exact size of loss, when ≥ u.

When data is **right censored** or **censored from above** at the value u, losses of size more than u are recorded in the data base as u. This generally occurs when there is a maximum covered loss of size u. When a loss (covered by insurance) is larger than the maximum covered loss, the insurer pays the maximum covered loss (if there is no deductible) and may neither know nor care how much bigger the loss is than the maximum covered loss.

Exercise: An insured has losses of sizes: $300, $600, $1,200, $1,500 and $2,800.
How is this data reported, if it is censored from above at $1000?
[Solution: $300, $600, $1000, $1000, $1000.
Comment: The values recorded as $1000 are $1000 or more. They may be shown as 1000^+.]

The revised Distribution Function under censoring from above at u is:

$$G(x) = \begin{cases} F(x) & x < u \\ 1 & x = u \end{cases}$$

$$g(x) = \begin{cases} f(x) & x < u \\ \text{point mass of probability } S(u) & x = u \end{cases}$$

The data censored from above at u is the limited loss variable, $X \wedge u \cong$ Min[X, u], discussed previously. The average size of the data censored from above at u, is the Limited Expected Value at u, $E[X \wedge u]$.

Exercise: An insured has losses of sizes: $300, $600, $1,200, $1,500 and $2,800.
What is the average size of the data reported, if it is censored from above at $1000?
[Solution: ($300 + $600 + $1000 + $1000 + $1000) / 5 = $780 = $E[X \wedge 1000]$.]

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[41] The terms "right censored" and "censored from above" are synonymous. "From the right" refers to a graph with the size of loss along the x-axis with the large values on the righthand side. "From above" uses similar terminology as "higher layers of loss." "From above" is how the effect of a maximum covered loss looks in a Lee Diagram.

Truncation from Below and Censoring from Above:

When data is subject to both a maximum covered loss and a deductible, and one records the loss by the insured, then the data is censored from above and truncated from below.

For example, with a deductible of $1000 and a maximum covered loss of $25,000:

Loss Size	As Recorded after Truncation from Below at 1000 and Censoring from Above at 25,000
600	Not recorded
4500	4500
37000	25,000

For truncation from below at d and censoring from above at u,
the data are recorded as follows:[42]

Loss by Insured	Amount Recorded by Insurer
$x \le d$	Not Recorded
$d < x \le L$	x
$u \le x$	u

Exercise: An insured has losses of sizes: $300, $600, $1,200, $1,500 and $2,800.
How is this data reported, if it is truncated from below at $1000 and censored from above at $2000?
[Solution: $1200, $1500, $2000.]

The revised Distribution Function under censoring from above at u and truncation from below at d is:

$$G(x) = \begin{cases} \dfrac{F(x) - F(d)}{S(d)} & d < x < u \\ 1 & x = u \end{cases}$$

$$g(x) = \begin{cases} f(x)/S(d) & d < x < u \\ \text{point mass of probability } S(u)/S(d) & x = u \end{cases}$$

[42] For the example u = $25,000 and d = $1000.

The total losses of the data censored from above at u and truncated from below at d, is the losses in the layer from d to u, plus d times the number of losses in the data set. The number of losses in the data set is S(d). Therefore, the average size of the data censored from above at u and truncated from below at d, is:

$$\frac{(E[X \wedge u] - E[X \wedge d]) + d\, S(d)}{S(d)} = \frac{E[X \wedge u] - E[X \wedge d]}{S(d)} + d.$$

Exercise: An insured has losses of sizes: $300, $600, $1,200, $1,500 and $2,800.
What is the average size of the data reported, if it is truncated from below at $1000 and censored from above at $2000?
[Solution: ($1200+ $1500 + $2000) / 3 = $1567.
Comment: $(E[X \wedge 2000] - E[X \wedge 1000]) / S(1000) + 1000 = (1120 - 780)/0.6 + 1000 =$ $1567.]

Truncation and Shifting from Below and Censoring from Above:

When data is subject to both a maximum covered loss and a deductible, and one records the amount paid by the insurer, then the data is censored from above and truncated and shifted from below.

For example, with a deductible of $1000 and a maximum covered loss of $25,000:

Loss Size	As Recorded after Truncation and Shifting from Below at 1000 and Censoring from Above at 25,000
600	Not recorded
4500	3500
37000	24000

For truncation and shifting from below at d and censoring from above at u, the data are recorded as follows:

Loss by Insured	Amount Recorded by Insurer
$x \le d$	Not Recorded
$d < x \le u$	x-d
$u \le x$	u-d

The revised Distribution Function under censoring from above at u and truncation and shifting from below at d is:

$$G(x) = \begin{cases} \dfrac{F(x+d) - F(d)}{S(d)} & 0 < x < u - d \\ 1 & x = u - d \end{cases}$$

$$g(x) = \begin{cases} f(x+d) / S(d) & 0 < x < u - d \\ \text{point mass of probability } S(u)/S(d) & x = u - d \end{cases}$$

x ⇔ the size of (non-zero) payment. x+d ⇔ the size of loss.

Exercise: An insured has losses of sizes: $300, $600, $1,200, $1,500 and $2,800.
How is this data reported, if it is truncated and shifted from below at $1000 and censored from above at $2000?
[Solution: $200, $500, $1000.]

The total losses of the data censored from above at u and truncated and shifted from below at d, is the losses in the layer from d to u. The number of losses in the data base is S(d). Therefore the average size of the data censored from above at u and truncated and shifted from below at d, is: $\dfrac{E[X \wedge u] - E[X \wedge d]}{S(d)}$.

Exercise: An insured has losses of sizes: $300, $600, $1,200, $1,500 and $2,800.
What is the average size of the data reported, if it is truncated and shifted from below at $1000 and censored from above at $2000?
[Solution: ($200+ $500 + $1000) / 3 = $567.
<u>Comment</u>: (E[X ∧ 2000] - E[X ∧ 1000])/S(1000) = (1120 - 780)/0.6 = $567.]

<u>Left Censored and Shifted / Censored and Shifted from Below</u>:

left censored and shifted variable at d ⇔ $\mathbf{(X - d)_+}$ ⇔

0 when X ≤ d, X - d when X > d ⇔ the amounts paid to insured with a deductible of d

⇔ payments per loss, including when the insured is paid nothing due to the deductible of d

⇔ amount paid per <u>loss</u>.

When data is left censored and shifted at the value d, losses of size less than d are recorded in the data base as 0. Losses of size x > d, are recorded as x - d.
What appears in the data base is $(X - d)_+$.

The revised Distribution Function under left censoring and shifting at d is:

$G(x) = F(x + d) \qquad x \geq 0$

$$g(x) = \begin{cases} \text{point mass of probability } F(d) & x = 0 \\ f(x+d) & x > 0 \end{cases}$$

x ⇔ the size of payment. x+d ⇔ the size of loss.

Exercise: An insured has losses of sizes: $300, $600, $1,200, $1,500 and $2,800.
How is this data reported, if it is left censored and shifted at $1000?
[Solution: $0, $0, $200, $500 and $1,800.]

The mean of the left censored at d and shifted variable =
the average payment per loss with a deductible of d = E[X] - E[X ∧ d] ⇔ Layer from d to ∞.

$$E[(X - d)_+] = E[X] - E[X \wedge d] = \int_d^\infty (x-d)\, f(x)\, dx\,.$$

$$E[(X-d)_+^n] = \int_d^\infty (x-d)^n\, f(x)\, dx\,.$$

Exercise: An insured has losses of sizes: $300, $600, $1,200, $1,500 and $2,800.
What is the average of the data reported, if it is left censored and shifted at $1000?
[Solution: (0 + 0 + $200 + $500 + $1800)/5 = $500.
E[X] - E[X ∧ 1000] = $1280 - $780 = $500 = average payment per <u>loss</u>. In contrast, the average payment per non-zero <u>payment</u> is: ($200 + $500 + $1800)/3 = $833.33.]

Left Censored / Censored from Below:

Sometimes data is censored from below, so that one only knows how many small values there are, but does not know their exact sizes.[43]

For example, an actuary might have access to detailed information on all Workers' Compensation losses of size greater than $2000, including the size of each such loss, but might only know how many losses there were of size less than or equal to $2000. Such data has been censored from below at $2000.

When data is censored from below at the value d, losses of size less than d are recorded in the data base as d.

The revised Distribution Function under censoring from below at d is:

$$G(x) = \begin{cases} 0 & x < d \\ F(x) & x \geq d \end{cases}$$

$$g(x) = \begin{cases} \text{point mass of probability } F(d) & x < d \\ f(x) & x \geq d \end{cases}$$

Exercise: An insured has losses of sizes: $300, $600, $1,200, $1,500 and $2,800.
How is this data reported, if it is censored from below at $1000?
[Solution: $1000, $1000, $1,200, $1,500 and $2,800.
Comment: The values recorded as $1000 are actually $1000 or less.]

The average size of the data censored from below at d, is $(E[X] - E[X \wedge d]) + d$.
The losses are those in the layer from d to ∞, plus d per loss.

Exercise: An insured has losses of sizes: $300, $600, $1,200, $1,500 and $2,800.
What is the average of the data reported, if it is censored from below at $1000?
[Solution: ($1000 + $1000 + $1200 + $1500 + $2800) / 5 = $1500.
Comment: $(E[X] - E[X \wedge 1000]) + 1000 = (1280-780) + 1000 = 1500$.]

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[43] See for example, 4, 11/06, Q.5.

Problems:

Use the following information for the next 4 questions:
Losses are uniformly distributed from 0 to 1000.

13.1 (1 point) What is the distribution function for these losses left censored at 100?

13.2 (1 point) What is the distribution function for these losses left censored and shifted at 100?

13.3 (1 point) What is the distribution function for these losses right censored at 800?

13.4 (1 point) What is the distribution function for these losses left truncated and shifted at 100 and right censored at 800?

Use the following information for the next 4 questions:
There are 6 losses: 100, 400, 700, 800, 1200, and 2300.

13.5 (1 point) If these losses are left censored (censored from below) at 500, what appears in the data base?

13.6 (1 point) If these losses are left censored and shifted at 500, what appears in the data base?

13.7 (1 point) If these losses are censored from above (right censored) at 2000, what appears in the data base?

13.8 (1 point) If these losses are left truncated and shifted at 500 and right censored at 2000, what appears in the data base?

13.9 (1 point) There are five accidents with losses equal to:
\$500, \$2500, \$4000, \$6000, and \$8000.
Which of the following statements are true regarding the reporting of this data?
1. If the data is censored from above at \$5000, then the data is reported as:
\$500, \$2500, \$4000.
2. If the data is truncated from below at \$1000, then the data is reported as:
\$2500, \$4000, \$6000, \$8000.
3. If the data is truncated and shifted at \$1000, then the data is reported as:
\$1500, \$3000, \$5000, \$7000.

A. 1, 2 B. 1, 3 C. 2, 3 D. 1, 2, 3 E. None of A, B, C or D

13.10 (2 points) It can take many years for a Medical Malpractice claim to be reported to an insurer and can take many more years to be closed, in other words resolved.
You are studying how long it takes Medical Malpractice claims to be reported to your insurer. You have data on incidents that occurred two years ago and how long they took to be reported. You are also studying how long it takes for Medical Malpractice claims to be closed once they are reported. You have data on all incidents that were reported two years ago and how long it took to close those that are not still open. For each of these two sets of data, state whether it is truncated and/or censored and briefly explain why.

Solutions to Problems:

13.1. We only know that small losses are of size at most 100.
G(100) = 0.1; G(x) = x/1000 for 100 < x < 1000.

13.2. G(0) = 0.1; G(x) = (x + 100)/1000 for 0 < x < 900.

13.3. All losses are limited to 800. **G(x) = x/1000 for 0 < x < 800; G(800) = 1.**

13.4. Losses less than 100 do not appear. Other losses are limited to 800 and then have 100 subtracted. **G(x) = x/900 for x < 700; G(700) = 1.**
Alternately, F(x) = x/1000, 0 < x < 1000 and G(x) = {F(x + 100) - F(100)}/S(100) =
{((x + 100)/1000) - 100/1000)}/(1 - 100/1000) **=**
x/900 for 0 < x < 800 - 100 = 700; G(700) = 1.

13.5. The two smaller losses appear as 500: **500, 500, 700, 800, 1200, 2300**.

13.6. $(X - 500)_+$ = **0, 0, 200, 300, 700, 1800**.
Comment: The amounts the insured receives with a $500 deductible.

13.7. The large loss is limited to 2000: **100, 400, 700, 800, 1200, 2000**.
Comment: Payments with a 2000 maximum covered loss.
Right censored observations might be indicated with a plus as follows:
100, 400, 700, 800, 1200, 2000^+. The 2000 corresponds to a loss of 2000 or more.

13.8. The two small losses do not appear; the other losses are limited to 2000 and then have 500 subtracted: **200, 300, 700, 1500**.
Comment: Payments with a 500 deductible and 2000 maximum covered loss.
Apply the maximum covered loss first and then the deductible; therefore, apply the censorship first and then the truncation.

13.9. C. If the data is censored by a $5000 limit, then the data is reported as: $500, $2500, $4000, $5000, $5000. Statement 1 would be true if it referred to truncation from above rather than censoring. Under censoring the size of large accidents is limited in the reported data to the maximum covered loss. Under truncation from above, the large accidents do not even make it into the reported data. Statements 2 and 3 are each true.

13.10. The data on incidents that occurred two years ago is **truncated from above** at two years. Those incidents, if any, that will take more than 2 years to be reported will not be in our data base yet. We don't know how many of them there may be nor how long they will take to be reported. The data on claims that were reported two years ago is **censored from above** at two years. Those claims that are still opened, we know will be closed eventually. However, while we know it will take more than 2 years to close each of them, we don't know exactly how long it will take.

Section 14, Average Sizes

For each of the different types of data there are corresponding average sizes. The most important cases involve a deductible and/or a maximum covered loss; one should know well the average payment per loss and the average payment per (non-zero) payment.

Average Amount Paid per Loss:

Exercise: When there is a deductible of size 1000, a maximum covered loss of 25,000, and thus a policy limit of 25,000 - 1000 = 24,000, what is the average amount paid per loss?
[Solution: The average amount paid per loss are the average losses in the layer from 1000 to 25,000: $E[X \wedge 25{,}000] - E[X \wedge 1000]$.]

Situation	Average Amount Paid per Loss
No Maximum Covered Loss, No Deductible	$E[X]$
Maximum Covered Loss u, No Deductible	$E[X \wedge u]$
No Maximum Covered Loss, (ordinary) Deductible d	$E[X] - E[X \wedge d]$
Maximum Covered Loss u, (ordinary) Deductible d	$E[X \wedge u] - E[X \wedge d]$

Recalling that $E[X \wedge \infty] = E[X]$ and $E[X \wedge 0] = 0$, we have a single formula that covers all four situations:

Assuming the usual order of operations, with Maximum Covered Loss of u and an (ordinary) deductible of d, the average amount paid by the insurer per loss is: $E[X \wedge u] - E[X \wedge d]$.

Note that the average payment per loss is just the layer from d to u. As discussed previously, this layer can also be expressed as: (layer from d to ∞) - (layer from u to ∞) =
$E[(X - d)_+] - E[(X - u)_+] = \{E[X] - E[X \wedge d]\} - \{E[X] - E[X \wedge u]\} = E[X \wedge u] - E[X \wedge d]$.

Average Amount Paid per Non-Zero Payment:

Exercise: What is the average non-zero payment when there is a deductible of size 1000 and no maximum covered loss?
[Solution: The average non-zero payment when there is a deductible of size 1000 is the ratio of the losses excess of 1000, $E[X] - E[X \wedge 1000]$, to the probability of a loss greater than 1000, S(1000). Thus the expected non-zero payment is: $(E[X] - E[X \wedge 1000]) / S(1000)$.]

With a deductible, some losses to the insured are too small to result in a payment by the insurer. Thus there are fewer non-zero payments than losses. In order to convert the average amount paid per loss to the average amount paid per non-zero payment, one needs to divide by S(d).

Assuming the usual order of operations, **with Maximum Covered Loss of u and an (ordinary) deductible of d, the average amount paid by the insurer per <u>non-zero payment</u> to the insured is:** $\frac{E[X \wedge u] - E[X \wedge d]}{S(d)}$.

If u = ∞, in other words there is no maximum covered loss, then this is e(d).

<u>Coinsurance Factor</u>:

For example, an insurance policy might have a 80% coinsurance factor.
Then the insurer pays 80% of what it would have paid in the absence of the coinsurance factor.
Thus the average payment, either per loss or per non-zero payment would be multiplied by 80%.
In general, a coinsurance factor of c, multiplies the average payment, either per loss or per non-zero payment by c.

With Maximum Covered Loss of u, an (ordinary) deductible of d, and a coinsurance factor of c, the average amount paid by the insurer per <u>loss</u> by the insured is: c (E[X ∧ u] - E[X ∧ d]).

With Maximum Covered Loss of u, an (ordinary) deductible of d, and a coinsurance factor of c, the average amount paid by the insurer per non-zero <u>payment</u> to the insured is:

$$c \, \frac{E[X \wedge u] - E[X \wedge d]}{S(d)}.$$

<u>Exercise:</u> Prior to the application of any coverage modifications, losses follow a Pareto Distribution, as per <u>Loss Models</u>, with parameters α = 3 and θ = 20,000.
An insured has a policy with a \$100,000 maximum covered loss, a \$5000 deductible, and a 90% coinsurance factor. Thus the policy limit is: (0.9)(100,000 - 5000) = 85,500.
Determine the average amount per non-zero payment.
[Solution: For the Pareto Distribution, as shown in Appendix A of <u>Loss Models</u>,

$S(x) = \left(\frac{\theta}{\theta + x}\right)^{\alpha}$. $E[X \wedge x] = \frac{\theta}{\alpha - 1}\left\{1 - \left(\frac{\theta}{\theta + x}\right)^{\alpha - 1}\right\}$.

S(5000) = $(20/25)^3$ = 0.512. E[X∧5000] = 10,000 {1 - $(20/25)^2$} = 3600.
E[X∧100,000] = 10,000 {1 - $(20/120)^2$} = 9722.

$$90\% \, \frac{E[X \wedge 100{,}000] - E[X \wedge 5000]}{S(5000)} = \frac{(90\%)(9722 - 3600)}{0.512} = \$10{,}761.$$]

Average Sizes for the Different Types of Data Sets:

Type of Data	Average Size
Ground-up, Total Limits	$E[X]$
Censored from Above at u	$E[X \wedge u]$
Truncated from Below at d	$e(d) + d = \frac{E[X] - E[X \wedge d]}{S(d)} + d$
Truncated and Shifted from Below at d	$e(d) = \frac{E[X] - E[X \wedge d]}{S(d)}$
Truncated from Above at L	$\frac{E[X \wedge L] - L\, S(L)}{F(L)}$
Censored from Below at d	$(E[X] - E[X \wedge d]) + d$
Left Censored and Shifted	$E[(X - d)_+] = E[X] - E[X \wedge d]$
Censored from Above at u and Truncated from Below at d	$\frac{E[X \wedge u] - E[X \wedge d]}{S(d)} + d$
Censored from Above at u and Truncated and Shifted from Below at d	$\frac{E[X \wedge u] - E[X \wedge d]}{S(d)}$
Truncated from Above at L and Truncated from Below at d	$\frac{\{E[X \wedge L] - L\, S(L)\} - \{E[X \wedge d] - d\, S(d)\}}{F(L) - F(d)}$
Truncated from Above at L and Truncated and Shifted from Below at d	$\frac{\{E[X \wedge L] - L\, S(L)\} - \{E[X \wedge d] - d\, S(d)\}}{F(L) - F(d)} - d$

Problems:

Use the following information for the next 5 questions:
There are five losses of size: $2000, $6000, $12,000, $27,000, and $48,000.

14.1 (1 point) With a $5000 deductible, what is the average payment per loss?
A. less than 12,000
B. at least 12,000 but less than 13,000
C. at least 13,000 but less than 14,000
D. at least 14,000 but less than 15,000
E. at least 15,000

14.2 (1 point) With a $5000 deductible, what is the average payment per (non-zero) payment?
A. less than 17,000
B. at least 17,000 but less than 18,000
C. at least 18,000 but less than 19,000
D. at least 19,000 but less than 20,000
E. at least 20,000

14.3 (1 point) With a $25,000 policy limit, what is the average payment?
A. less than 14,000
B. at least 14,000 but less than 15,000
C. at least 15,000 but less than 16,000
D. at least 16,000 but less than 17,000
E. at least 17,000

14.4 (1 point) With a $5000 deductible and a $25,000 maximum covered loss,
what is the average payment per loss?
A. less than 10,000
B. at least 10,000 but less than 11,000
C. at least 11,000 but less than 12,000
D. at least 12,000 but less than 13,000
E. at least 13,000

14.5 (1 point) With a $5000 deductible and a $25,000 maximum covered loss,
what is the average payment per (non-zero) payment?
A. less than 11,000
B. at least 11,000 but less than 12,000
C. at least 12,000 but less than 13,000
D. at least 13,000 but less than 14,000
E. at least 14,000

Use the following information for the next 5 questions:

Loss Size (x)	F(x)	$E[X \wedge x]$
10,000	0.325	8,418
25,000	0.599	16,198
50,000	0.784	23,544
100,000	0.906	30,668
250,000	0.978	37,507
∞	1.000	41,982

14.6 (1 point) With a 25,000 deductible, determine $E[Y^L]$.
A. less than 22,000
B. at least 22,000 but less than 23,000
C. at least 23,000 but less than 24,000
D. at least 24,000 but less than 25,000
E. at least 25,000

14.7 (1 point) With a 25,000 deductible, determine $E[Y^P]$.
A. less than 62,000
B. at least 62,000 but less than 63,000
C. at least 63,000 but less than 64,000
D. at least 64,000 but less than 65,000
E. at least 65,000

14.8 (1 point) With a 100,000 policy limit, what is the average payment?
A. less than 27,000
B. at least 27,000 but less than 28,000
C. at least 28,000 but less than 29,000
D. at least 29,000 but less than 30,000
E. at least 30,000

14.9 (1 point) With a 25,000 deductible and a 100,000 maximum covered loss, what is the average payment per loss?
A. less than 15,000
B. at least 15,000 but less than 16,000
C. at least 16,000 but less than 17,000
D. at least 17,000 but less than 18,000
E. at least 18,000

14.10 (1 point) With a 25,000 deductible and a 100,000 maximum covered loss, determine $E[Y^P]$.
A. 32,000 B. 33,000 C. 34,000 D. 35,000 E. 36,000

Use the following information for the next five questions:
There are six accidents of size: $800, $2100, $3300, $4600, $6100, and $8900.

14.11 (1 point) If the reported data is truncated from below at $1000,
what is the average size of claim in the reported data?
A. 4800 B. 4900 C. 5000 D. 5100 E. 5200

14.12 (1 point) If the reported data is truncated and shifted from below at $1000,
what is the average size of claim in the reported data?
A. less than 3900
B. at least 3900 but less than 4000
C. at least 4000 but less than 4100
D. at least 4100 but less than 4200
E. at least 4200

14.13 (1 point) If the reported data is left censored and shifted at $1000,
what is the average size of claim in the reported data?
A. less than 3700
B. at least 3700 but less than 3800
C. at least 3800 but less than 3900
D. at least 3900 but less than 4000
E. at least 4000

14.14 (1 point) If the reported data is censored from above at $5000,
what is the average size of claim in the reported data?
A. less than 3400
B. at least 3400 but less than 3500
C. at least 3500 but less than 3600
D. at least 3600 but less than 3700
E. at least 3700.

14.15 (1 point) If the reported data is truncated from above at $5000,
what is the average size of claim in the reported data?
A. less than 2300
B. at least 2300 but less than 2400
C. at least 2400 but less than 2500
D. at least 2500 but less than 2600
E. at least 2600

Use the following information for the next four questions:
Losses follow a uniform distribution from 0 to 20,000.

14.16 (2 points) If there is a deductible of 1000,
what is the average payment by the insurer per loss?
A. less than 8800
B. at least 8800 but less than 8900
C. at least 8900 but less than 9000
D. at least 9000 but less than 9100
E. at least 9100

14.17 (2 points) If there is a policy limit of 15,000,
what is the average payment by the insurer per loss?
A. less than 9300
B. at least 9300 but less than 9400
C. at least 9400 but less than 9500
D. at least 9500 but less than 9600
E. at least 9600

14.18 (2 points) There is a maximum covered loss of 15,000 and a deductible of 1000.
What is the average payment by the insurer per loss?
(Include situations where the insurer pays nothing.)
A. less than 8200
B. at least 8200 but less than 8300
C. at least 8300 but less than 8400
D. at least 8400 but less than 8500
E. at least 8500

14.19 (2 points) There is a maximum covered loss of 15,000 and a deductible of 1000.
What is the average value of a <u>non-zero</u> payment by the insurer?
A. less than 8700
B. at least 8700 but less than 8800
C. at least 8800 but less than 8900
D. at least 8900 but less than 9000
E. at least 9000

14.20 (2 points) An insurance policy has a maximum covered loss of 2000 and a deductible of 100. For the ground up unlimited losses: F(100) = 0.20, F(2000) = 0.97, and

$$\int_{100}^{2000} x\ f(x)\ dx = 400.$$

What is the average payment per loss?
A. 360 B. 380 C. 400 D. 420 E. 440

14.21 (1 point) A policy has a policy limit of 50,000 and deductible of 1000.
What is the expected payment per loss?
A. $E[X \wedge 49{,}000] - E[X \wedge 1000]$
B. $E[X \wedge 50{,}000] - E[X \wedge 1000]$
C. $E[X \wedge 51{,}000] - E[X \wedge 1000]$
D. $E[X \wedge 49{,}000] - E[X \wedge 1000] + 1000$
E. $E[X \wedge 51{,}000] - 1000$

14.22 (2 points) You are given:
● In the absence of a deductible the average loss is 15,900.
● With a 10,000 deductible, the average amount paid per loss is 7,800.
● With a 10,000 deductible, the average amount paid per nonzero payment is 13,300.
What is the average of those losses of size less than 10,000?
(A) 5000 (B) 5200 (C) 5400 (D) 5600 (E) 5800

14.23 (1 point) $E[(X - 1000)_+] = 3500$. $E[(X - 25{,}000)_+] = 500$.
There is a 1000 deductible and a 25,000 maximum covered loss.
Determine the average payment per loss.

14.24 (2 points) Use the following information:
● The average payment per loss with a deductible of d is 450.
● The average payment per payment with a deductible of d is 1000.
● The average payment per loss with a franchise deductible of d is 1080.
Determine d.
A. 1000 B. 1100 C. 1200 D. 1300 E. 1400

Use the following information for the next two questions:

- Flushing Reinsurance reinsures a certain book of business.
- Limited Expected Values for this book of business are estimated to be:
  - E[X∧ $1 million] = $300,000
  - E[X∧ $4 million] = $375,000
  - E[X∧ $5 million] = $390,000
  - E[X∧ $9 million] = $420,000
  - E[X∧ $10 million] = $425,000
- The survival functions, S(x) = 1 - F(x), for this book of business are estimated to be:
  - S($1 million) = 3.50%
  - S($4 million) = 1.70%
  - S($5 million) = 1.30%
  - S($9 million) = 0.55%
  - S($10 million) = 0.45%
- Flushing Reinsurance makes a nonzero payment, y, on this book of business.

14.25 (1 point) If Flushing Reinsurance were responsible for the layer of loss from $1 million to $5 million ($4 million excess of $1 million), what is the expected value of y?
A. less than $1 million
B. at least $1 million but less than $2 million
C. at least $2 million but less than $3 million
D. at least $3 million but less than $4 million
E. at least $4 million

14.26 (1 point) If Flushing Reinsurance were responsible for the layer of loss from $1 million to $10 million ($9 million excess of $1 million), what is the expected value of y?
A. less than $1 million
B. at least $1 million but less than $2 million
C. at least $2 million but less than $3 million
D. at least $3 million but less than $4 million
E. at least $4 million

14.27 (3 points) Losses are distributed uniformly from 0 to w.
There is a deductible of size d < w.
Determine the variance of the payment per loss.

Use the following information for the next 12 questions:

- The distribution of losses suffered by insureds is estimated to have the following limited expected values:
  $E[X \wedge 5{,}000] = 3{,}600$
  $E[X \wedge 20{,}000] = 7{,}500$
  $E[X \wedge 25{,}000] = 8{,}025$
  $E[X \wedge \infty] = 10{,}000$
- The survival functions, S(x), for the distribution of losses suffered by insureds is estimated to have the following values:
  S(5,000) = 51.2%
  S(20,000) = 12.5%
  S(25,000) = 8.8%

14.28 (1 point) What is average loss suffered by the insureds?
A. 9,600 B. 9,700 C. 9,800 D. 9,900 E. 10,000

14.29 (1 point) What is the average size of data truncated from above at 25,000?
A. less than 6,300
B. at least 6,300 but less than 6,400
C. at least 6,400 but less than 6,500
D. at least 6,500 but less than 6,600
E. at least 6,600

14.30 (1 point) What is the average size of data truncated and shifted from below at 5000?
A. 12,500 B. 12,600 C. 12,700 D. 12,800 E. 12,900

14.31 (1 point) What is the average size of data censored from above at 25,000?
A. 7800 B. 7900 C. 8000 D. 8100 E. 8200

14.32 (1 point) What is the average size of data censored from below at 5,000?
A. less than 10,700
B. at least 10,700 but less than 10,800
C. at least 10,800 but less than 10,900
D. at least 10,900 but less than 11,000
E. at least 11,000

14.33 (1 point) What is the average size of data left censored and shifted at 5,000?
A. 6200 B. 6300 C. 6400 D. 6500 E. 6600

14.34 (2 points) What is the average size of data truncated from below at 5,000 and truncated from above at 25,000?
A. less than 11,200
B. at least 11,200 but less than 11,300
C. at least 11,300 but less than 11,400
D. at least 11,400 but less than 11,500
E. at least 11,500

14.35 (1 point) What is the average size of data truncated from below at 5,000 and censored from above at 25,000?
A. less than 13,500
B. at least 13,500 but less than 13,600
C. at least 13,600 but less than 13,700
D. at least 13,700 but less than 13,800
E. at least 13,800

14.36 (2 points) What is the average size of data censored from below at 5,000 and censored from above at 25,000?
A. 9100 B. 9200 C. 9300 D. 9400 E. 9500

14.37 (2 points) What is the average size of data truncated and shifted from below at 5,000 and truncated from above at 25,000?
A. less than 6,100
B. at least 6,100 but less than 6,200
C. at least 6,200 but less than 6,300
D. at least 6,300 but less than 6,400
E. at least 6,400

14.38 (2 points) What is the average size of data truncated and shifted from below at 5,000 and censored from above at 25,000?
A. less than 8,700
B. at least 8,700 but less than 8,800
C. at least 8,800 but less than 8,900
D. at least 8,900 but less than 9,000
E. at least 9,000

14.39 (1 point) What is the average size of data truncated from below at 5000?
A. 17,000 B. 17,500 C. 18,000 D. 18,500 E. 19,000

14.40 (2 points) The size of loss distribution has the following characteristics:
(i) S(100) = 0.65.
(ii) E[X | X > 100] = 345.
There is an ordinary deductible of 100 per loss.
Determine the average payment per loss.
(A) 160 (B) 165 (C)170 (D) 175 (E) 180

14.41 (3 points) A business has obtained two separate insurance policies that together provide full coverage. You are given:
(i) The average ground-up loss is 27,000.
(ii) Policy B has no deductible and a maximum covered loss of 25,000.
(iii) Policy A has an ordinary deductible of 25,000 with no maximum covered loss.
(iv) Under policy A, the expected amount paid per loss is 10,000.
(v) Under policy A, the expected amount paid per payment is 22,000.
Given that a loss less than or equal to 25,000 has occurred, what is the expected payment under policy B?
A. Less than 11,000
B. At least 11,000, but less than 12,000
C. At least 12,000, but less than 13,000
D. At least 13,000, but less than 14,000
E. At least 14,000

14.42 (2 points) X is the size of loss prior to the effects of any policy provisions.
Given the following information, calculate the average payment per loss under a policy with a 1000 deductible and a 25,000 maximum covered loss.

x	e(x)	F(x)
1000	30,000	72.7%
25,000	980,000	99.7%

A. 4250 B. 4500 C. 4750 D. 5000 E. 5250

<u>**14.43**</u> (2 points) For a certain policy, in order to determine the payment on a claim, first the deductible of 500 is applied, and then the payment is capped at 10,000.
What is the expected payment per loss?
A. $E[X \wedge 10,000] - E[X \wedge 500]$
B. $E[X \wedge 10,500] - E[X \wedge 500]$
C. $E[X \wedge 10,000] - E[X \wedge 500] + 500$
D. $E[X \wedge 10,500] - E[X \wedge 500] + 500$
E. None of A, B, C, or D

14.44 (2 points) If an ordinary deductible of 5000 is applied, then the average payment per payment is 40,000.
If a franchise deductible of 5000 is applied, then the average payment per loss is 33,750.
If an ordinary deductible of 5000 is applied, determine the average payment per loss.
A. 28,000 B. 29,000 C. 30,000 D. 31,000 E. 32,000

14.45 (2 points) The size of losses has the following density:

$$f(x) = \begin{cases} 0.007, & 0 < x \le 100 \\ 0.001, & 100 < x \le 300 \\ 0.0002, & 300 < x \le 800 \end{cases}.$$

If there is an ordinary deductible of 200, determine the expected payment per loss.
A. 40 B. 45 C. 50 D. 55 E. 60

14.46 (4B, 5/92, Q.20) (1 point) Accidents for a coverage are uniformly distributed on the interval 0 to $5,000. An insurer sells a policy for the coverage which has a $500 deductible.
Determine the insurer's expected payment per loss.
A. $1,575 B. $2,000 C. $2,025 D. $2,475 E. $2,500

14.47 (4B, 5/95, Q.22) (2 points) You are given the following:
- Losses follow a Pareto distribution, with parameters θ = 1000 and α = 2.
- 10 losses are expected each year.
- The number of losses and the individual loss amounts are independent.
- For each loss that occurs, the insurer's payment is equal to the **entire** amount of the loss if the loss is greater than 100.

The insurer makes no payment if the loss is less than or equal to 100.
Determine the insurer's expected annual payments.
A. Less than 8,000
B. At least 8,000, but less than 9,000
C. At least 9,000, but less than 9,500
D. At least 9,500, but less than 9,900
E. At least 9,900

14.48 (4B, 11/95, Q.13 & 4B, 5/98 Q.9) (3 points) You are given the following:
- Losses follow a uniform distribution on the interval from 0 to 50,000.
- There is a maximum covered loss of 25,000 per loss and a deductible of 5,000 per loss.
- The insurer applies the maximum covered loss prior to applying the deductible (i.e., the insurer's maximum payment is 20,000 per loss).
- The insurer makes a nonzero payment p.

Determine the expected value of p.
A. Less than 15,000
B. At least 15,000, but less than 17,000
C. At least 17,000, but less than 19,000
D. At least 19,000, but less than 21,000
E. At least 21,000

14.49 (**CAS9, 11/97, Q.40a**) (2.5 points) You are the large accounts actuary for Pacific International Group, and you have a risk with a $1 million limit.
The facultative underwriters from AnyRe have indicated that they are willing to reinsure the following layers:
from $100,000 to $200,000 ($100,000 excess of $100,000)
from $200,000 to $500,000 ($300,000 excess of $200,000)
from $500,000 to $1 million ($500,000 excess of $500,000).
You have gathered the following information:

Limit	$E[X \wedge x]$	F(x)
100,000	58,175	0.603
200,000	89,629	0.748
500,000	139,699	0.885
1,000,000	179,602	0.943

Expected frequency = 100 claims.
Calculate the frequency, severity, and expected losses for each of the facultative layers.
Show all work.

14.50 (4B, 11/98, Q.12) (2 points) You are given the following:
- Losses follow a distribution (prior to the application of any deductible) with cumulative distribution function and limited expected values as follows:

Loss Size (x)	F(x)	$E[X \wedge x]$
10,000	0.60	6,000
15,000	0.70	7,700
22,500	0.80	9,500
∞	1.00	20,000

- There is a deductible of 15,000 per loss and no policy limit.
- The insurer makes a nonzero payment p.

Determine the expected value of p.
A. Less than 15,000
B. At least 15,000, but less than 30,000
C. At least 30,000, but less than 45,000
D. At least 45,000, but less than 60,000
E. At least 60,000

14.51 (4B, 5/99, Q.7) (2 points) You are given the following:
- Losses follow a distribution (prior to the application of any deductible) with cumulative distribution function and limited expected values as follows:

Loss Size (x)	F(x)	E[X∧ x]
10,000	0.60	6,000
15,000	0.70	7,700
22,500	0.80	9,500
32,500	0.90	11,000
∞	1.00	20,000

- There is a deductible of 10,000 per loss and no policy limit.
- The insurer makes a payment on a loss only if the loss exceeds the deductible.

The deductible is raised so that half the number of losses exceed the new deductible compared to the old deductible of 10,000.
Determine the percentage change in the expected size of a nonzero payment made by the insurer.
A. Less than -37.5%
B. At least -37.5%, but less than -12.5%
C. At least -12.5%, but less than 12.5%
D. At least 12.5%, but less than 37.5%
E. At least 37.5

14.52 (Course 3 Sample Exam, Q.5) You are given the following:
- The probability density function for the amount of a single loss is
  $f(x) = 0.01(1 - q + 0.01qx)e^{-0.01x}$, x >0.
- If an ordinary deductible of 100 is imposed, the expected payment
  (given that a payment is made) is 125.

Determine the expected payment (given that a payment is made) if the deductible is increased to 200.

14.53 (4, 5/00, Q.6) (2.5 points) A jewelry store has obtained two separate insurance policies that together provide full coverage. You are given:
(i) The average ground-up loss is 11,100.
(ii) Policy A has an ordinary deductible of 5,000 with no maximum covered loss
(iii) Under policy A, the expected amount paid per loss is 6,500.
(iv) Under policy A, the expected amount paid per payment is 10,000.
(v) Policy B has no deductible and a maximum covered loss of 5,000.
Given that a loss less than or equal to 5,000 has occurred, what is the expected payment under policy B?
(A) Less than 2,500
(B) At least 2,500, but less than 3,000
(C) At least 3,000, but less than 3,500
(D) At least 3,500, but less than 4,000
(E) At least 4,000

14.54 (4, 11/00, Q.18) (2.5 points) A jewelry store has obtained two separate insurance policies that together provide full coverage.
You are given:
(i) The average ground-up loss is 11,100.
(ii) Policy A has an ordinary deductible of 5,000 with no maximum covered loss.
(iii) Under policy A, the expected amount paid per loss is 6,500.
(iv) Under policy A, the expected amount paid per payment is 10,000.
(v) Policy B has no deductible and a maximum covered loss of 5,000.
Given that a loss has occurred, determine the probability that the payment under policy B is 5,000.
(A) Less than 0.3
(B) At least 0.3, but less than 0.4
(C) At least 0.4, but less than 0.5
(D) At least 0.5, but less than 0.6
(E) At least 0.6

14.55 (CAS3, 11/03, Q.22) (2.5 points) The severity distribution function of claims data for automobile property damage coverage for Le Behemoth Insurance Company is given by an exponential distribution, F(x).
F(x) = 1 - exp(-x/5000).
To improve the profitability of this portfolio of policies, Le Behemoth institutes the following policy modifications:
i) It imposes a per-claim deductible of 500.
ii) It imposes a per-claim limit of 25,000.
(The maximum paid per claim is 25,000 - 500 = 24,500.)
Previously, there was no deductible and no limit.
Calculate the average savings per (old) claim if the new deductible and policy limit had been in place.
A. 490 B. 500 C. 510 D. 520 E. 530

14.56 (SOA M, 11/05, Q.26 & 2009 Sample Q.207) (2.5 points) For an insurance:
(i) Losses have density function

$$f_X(x) = \begin{cases} 0.02x & 0 < x < 10 \\ 0 & \text{elsewhere} \end{cases}$$

(ii) The insurance has an ordinary deductible of 4 per loss.
(iii) Y^P is the claim payment per payment random variable.
Calculate $E[Y^P]$.
(A) 2.9 (B) 3.0 (C) 3.2 (D) 3.3 (E) 3.4

14.57 (SOA M, 11/06, Q.6 & 2009 Sample Q.279) (2.5 points)
Loss amounts have the distribution function

$$F(x) = \begin{cases} (x/100)^2, & 0 \le x \le 100 \\ 1, & 100 < x \end{cases}$$

An insurance pays 80% of the amount of the loss in excess of an ordinary deductible of 20, subject to a maximum payment of 60 per loss.
Calculate the conditional expected claim payment, given that a payment has been made.
(A) 37 (B) 39 (C) 43 (D) 47 (E) 49

14.58 (CAS5, 5/07, Q.47) (2.0 points) You are given the following information:

Claim	Ground-up Uncensored Loss Amount
A	$250,000
B	$300,000
C	$450,000
D	$750,000
E	$1,200,000
F	$2,500,000
G	$4,000,000
H	$7,500,000
I	$9,000,000
J	$15,000,000

a. (1.25 points) Calculate the ratio of the limited expected value at $5 million to the limited expected value at $1 million
b. (0.75 points) Calculate the average payment per payment with a deductible of $1 million and a maximum covered loss of $5 million.
Comment: I have reworded this exam question in order to match the syllabus of your exam.

Solutions to Problems:

14.1. D. The payments are: 0, 1000, 7000, 22,000 and 43,000.
Average payment per loss is: (0 + 1000 + 7000 + 22,000 + 43,000)/5 = **14,600**.

14.2. C. Average payment per payment is: (1000 + 7000 + 22,000 + 43,000)/4 = **18,250**.

14.3. B. The payments are: \$2000, \$6000, \$12,000, \$25,000, and \$25,000.
Average payment is: (2000 + 6000 + 12,000 + 25,000 + 25,000)/5 = **14,000**.

14.4. A. The payments are: 0, 1000, 7000, 20,000 and 20,000.
Average payment per loss is: (0 + 1000 + 7000 + 20,000 + 20,000)/5 = **9,600**.
Alternately, $E[X \wedge 5000]$ = (2000 + 5000 + 5000 + 5000 + 5000 + 5000)/6 = 4400.
$E[X \wedge 25,000]$ = (2000 + 6000 + 12,000 + 25,000 + 25,000)/6 = 14,000.
$E[X \wedge 25,000] - E[X \wedge 5000]$ = 14,000 - 4400 = **9,600**.
Comment: The layer from 5000 to 25,000.

14.5. C. Average payment per payment is: (1000 + 7000 + 20,000 + 20,000)/4 = **12,000**.
Alternately, $(E[X \wedge 25,000] - E[X \wedge 5000])/S(5000)$ = (14,000 - 4400)/0.8 = **12,000**.

14.6. E. $E[X] - E[X \wedge 25,000]$ = 41,982 - 16,198 = **25,784**.
Comment: Based on a LogNormal distribution with $\mu = 9.8$ and $\sigma = 1.3$.

14.7. D. Average payment per payment is:
$(E[X] - E[X \wedge 25,000]) / S(25,000)$ = (41,982 - 16,198) / (1 - 0.599) = **64,299**.

14.8. E. $E[X \wedge 100,000]$ = **30,668**

14.9. A. $E[X \wedge 100,000] - E[X \wedge 25,000]$ = 30,668 - 16,198 = **14,470**.
Comment: The layer from 25,000 to 100,000.

14.10. E. $(E[X \wedge 100,000] - E[X \wedge 25,000])/S(25,000)$ = (30,668 - 16,198)/(1 - 0.599) = **36,085**.

14.11. C. (\$2100 + \$3300 + \$4600 + \$6100 + \$8900) / 5 = **5000**.

14.12. C. (\$1100 + \$2300 + \$3600 + \$5100 + \$7900) / 5 = **4000**.
Alternately, one can subtract 1000 from the solution to the previous question.

14.13. A. (\$0 + \$1100 + \$2300 + \$3600 + \$5100 + \$7900) / 6 = **3333**.

14.14. B. (\$800 + \$2100 + \$3300 + \$4600 + \$5000 + \$5000) / 6 = **3467**.

14.15. E. (\$800 + \$2100 + \$3300 + \$4600) / 4 = **2700**.

14.16. D. For this uniform distribution, f(x) = 1/20,000 for 0 ≤ x ≤ 20,000.
The payment by the insurer depends as follows on the size of loss x:

Insurer's Payment	
0	x ≤ 1000
x - 1000	1000 ≤ x ≤ 20,000

We need to compute the average dollars paid by the insurer per loss:

$$\int_{1000}^{20,000} (x-1000)\ f(x)\ dx = \int_{1000}^{20,000} \{(x-1000)/20,000\}\ dx = (x-1000)^2/40,000 \Big]_{x=1000}^{x=20,000} = \mathbf{9025}.$$

14.17. B. f(x) = 1/20,000 for 0 ≤ x ≤ 20,000.

Insurer's Payment	
x	x ≤ 15,000
15,000	15,000 ≤ x ≤ 20,000

We need to compute the average dollars paid by the insurer per loss, the sum of two terms corresponding to 0 ≤ x ≤ 15000 and 15000 ≤ x ≤ 20,000:

$$\int_{0}^{15,000} x\ f(x)\ dx + 15,000\ S(15,000)\} = \int_{0}^{15,000} x/20000\ dx + (15,000)(1 - 0.75) =$$

$$x^2/40,000 \Big]_{x=0}^{x=15,000} + 3750 = 5625 + 3750 = \mathbf{9375}.$$

14.18. D. For this uniform distribution, f(x) = 1/20,000 for 0≤x≤20,000. The payment by the insurer depends as follows on the size of loss x:

Insurer's Payment	
0	x ≤ 1000
x - 1000	1000 ≤ x ≤ 15,000
14,000	x ≥ 15,000

We need to compute the average dollars paid by the insurer per loss and the probability that a loss as the sum of two terms corresponding to 1000 ≤ x ≤ 15,000 and 20,000 ≥x ≥ 15,000:

$$\int_{1000}^{15,000} (x-1000)\ f(x)\ dx + 140,00\ S(15,000) = \int_{1000}^{15,000} \{(x-1000)/20,000\}\ dx + 14,000(1 - 0.75) =$$

$$\{(x-1000)^2/40,000\} \Big]_{x=1000}^{x=20,000} + 3500 = 4900 + 3500 = \mathbf{8400}.$$

14.19. C. We need to compute the ratio of two quantities, the average dollars paid by the insurer per loss and the probability that a loss will result in a non-zero payment. The latter is the chance that x>1000, which is for the uniform distribution: 1 - (1000/20,000) = 0.95.
The former is the solution to the previous question: 8400.
Therefore, the average non-zero payment is 8400 / 0.95 = **8842**.
Comment: Similar to 4B, 11/95, Q.13.

14.20. B. Average payment per loss = $E[X \wedge u] - E[X \wedge d]$ =

$$E[X \wedge 2000] - E[X \wedge 100] = \int_0^{2000} x\, f(x)\, dx + 2000S(2000) - \{\int_0^{100} x\, f(x)\, dx + 100S(100)\} =$$

$$\int_{100}^{2000} x\, f(x)\, dx + 2000S(2000) - 100S(100) = 400 + (2000)(1 - 0.97) - (100)(1 - 0..20) = \mathbf{380}.$$

Comment: *Can be done via a Lee Diagram.*

14.21. C. Policy limit = maximum covered loss - deductible.
Thus the maximum covered loss = 51,000.
Expected payment per loss = $E[X \wedge u] - E[X \wedge d]$ = **E[X ∧ 51,000] - E[X ∧ 1000]**.

14.22. C. E[X] = 15,900.
With a 10,000 deductible, the average amount paid per loss = $E[X] - E[X \wedge 10{,}000] = 7800$.
Therefore, $E[X \wedge 10{,}000] = 15{,}900 - 7800 = 8{,}100$.
With a 10,000 deductible, the average amount paid per nonzero payment =
$(E[X] - E[X \wedge 10{,}000])/S(10{,}000) = 13{,}300$.
Therefore, S(10,000) = 7800/ 13,300 = 0.5865.
Average loss of size between 0 and 10,000 =

$$\int_0^{10{,}000} x\, f(x)\, dx \;/\; F(10{,}000) = \{E[X \wedge 10{,}000] - 10{,}000S(10{,}000)\} / F(10{,}000) =$$

{8100 - (10,000)(0.5865)} / (1 - 0.5865) = 2235/0.4135 = **5405**.

14.23. $E[(X - 1000)_+] - E[(X - 25{,}000)_+] = \{E[X] - E[X \wedge 1000]\} - \{E[X] - E[X \wedge 25{,}000]\} =$
$E[X \wedge 25{,}000] - E[X \wedge 1000]$ = average payment per loss.
Thus, the average payment per loss is: 3500 - 500 = **3000**.

14.24. E. $450 = E[X] - E[X \wedge d]$.
$1000 = (E[X] - E[X \wedge d]) / S(d). \Rightarrow S(d) = 450/1000 = 0.45$.
The average payment per payment with a franchise deductible is d more than for an ordinary deductible d: $d + (E[X] - E[X \wedge d]) / S(d)$.
Thus the average payment per loss with a franchise deductible is:
$S(d) \{d + (E[X] - E[X \wedge d]) / S(d)\} = d\, S(d) + (E[X] - E[X \wedge d])$.
Thus, $1080 = d\, 0.45 + 450. \Rightarrow d = (1080 - 450) / 0.45 =$ **1400**.

14.25. C. The reinsurer pays the dollars in the layer of loss from \$1 million to \$5 million, which are: $E[X \wedge \$5 \text{ million}] - E[X \wedge \$1 \text{ million}]$. The number of nonzero payments is 1 - F(1 million) = S(1 million). Thus the average nonzero payment is:
$(E[X \wedge \$5 \text{ million}] - E[X \wedge \$1 \text{ million}]) / S(\$1 \text{ million}) = (390{,}000 - 300{,}000)/0.035 =$ **2,571,429**.

14.26. D. The reinsurer pays the dollars in the layer of loss from \$1 million to \$10 million, which are: $E[X \wedge \$10 \text{ million}] - E[X \wedge \$1 \text{ million}]$. The number of nonzero payments is
1 - F(1 million) = S(1 million). Thus the average nonzero payment is:
$(E[X \wedge \$10 \text{ million}] - E[X \wedge \$1 \text{ million}]) / S(\$1 \text{ million}) = (425{,}000 - 300{,}000)/0.035 =$ **3,571,429**.

14.27. The payment per loss of size x is: 0 for $x \le d$, and $x - d$ for $x > d$.

Mean payment per loss is: $\int_d^{\omega} (x-d)\,(1/\omega)\,dx = (\omega - d)^2 / (2\omega)$.

Second Moment of the payment per loss is: $\int_d^{\omega} (x-d)^2\,(1/\omega)\,dx = (\omega - d)^3 / (3\omega)$.

Thus, the variance of the payment per loss is:
$(\omega - d)^3 / (3\omega) - (\omega - d)^4 / (2\omega)^2 =$ $\mathbf{(\omega - d)^3 (\omega + 3d) / (12\omega^2)}$.

14.28. E. $E[X] = E[X \wedge \infty] =$ **10,000**.

14.29. B. $\{E[X \wedge 25{,}000] - 25{,}000 S(25{,}000)\} / F(25{,}000) =$
$\{8025 - (25{,}000)(0.088)\} / (1-0.088) =$ **6387**.

14.30. A. $e(5000) = \{E[X] - E[X \wedge 5000]\}/S(5000) = (10{,}000-3600)/0.512 =$ **12,500**.

14.31. C. $E[X \wedge 25{,}000] =$ **8025**.

14.32. E. The small losses are each recorded at 5000. Subtracting 5000 from every recorded loss, we would get the layer from 5000 to ∞. Thus the average loss is:
$(E[X] - E[X \wedge 5000]) + 5000 = (10{,}000 - 3600) + 5000 =$ **11,400**.

14.33. C. $E[(X - 5000)_+] = E[X] - E[X \wedge 5000] = 10{,}000 - 3600 =$ **6,400**.

14.34. B. This is the average size of losses in the interval from 5000 to 25,000:
$(\{E[X \wedge 25{,}000] - 25{,}000S(25{,}000)\} - \{E[X \wedge 5000] - 5000S(5000)\})/(F(25{,}000)-F(5000)) =$
$$\frac{8025-(25000)(0.088) - \{3600-(5000)(0.512)\}}{0.512 - 0.088} = (5825 - 1040)/0.424 = \mathbf{11{,}285}.$$

14.35. C. $\{E[X \wedge 25{,}000] - E[X \wedge 5000]\}/S(5000) + 5000 = (8025-3600)/0.512 + 5000 =$ **13,643**.

14.36. D. The losses are 5000 per loss plus the layer of losses from 5000 to 25,000.
Thus the average loss is: $(E[X \wedge 25{,}000] - E[X \wedge 5000]) + 5000 = (8025 - 3600) + 5000 =$ **9425**.
Alternately, the average size of loss is reduced compared to data just censored from below, by $E[X] - E[X \wedge 25{,}000] = 10{,}000 - 8025 = 1975$. Since from a previous solution, the average size of data censored from below at 5,000 is 11,400, the solution to this question is:
11,400 - 1975 = 9425.

14.37. C. Using a previous solution, where there was no shifting, this is 5000 less than the average size of data truncated from below at 5,000 and truncated from above at 25,000 = 11,285 - 5000 = **6,285**. Alternately, the dollars of loss are those for the layer from 5,000 to 25,000, less the width of the layer times the number of losses greater than 25,000:
$\{E[X \wedge 25{,}000] - E[X \wedge 5000]\} - (20{,}000)S(25{,}000)$. The number of losses in the database is:
$F(25{,}000) - F(5000) = S(5000) - S(25{,}000)$. Thus the average size is:
$\{(8025-3600) - (20{,}000)(0.088)\} / (0.512 - 0.088) =$ **6,285**.

14.38. A. $\{E[X \wedge 25{,}000] - E[X \wedge 5000]\} / S(5000) = (8025-3600)/0.512 =$ **8,643**.

14.39. B. $e(5000) + 5000 = \{E[X] - E[X \wedge 5000]\}/S(5000) + 5000 =$
$(10{,}000-3600)/0.512 + 5000 =$ **17,500**.

14.40. A. $E[X \mid X > 100] = \int_{100}^{\infty} x\, f(x)\, dx \ / \ S(100). \Rightarrow$

$$\int_{100}^{\infty} x\, f(x)\, dx = S(100)\ E[X \mid X > 100] = (0.65)(345) = 224.25.$$

With a deductible of 100 per loss, the average payment per loss is:

$$\int_{100}^{\infty} (x - 100)\, f(x)\, dx = \int_{100}^{\infty} x\, f(x)\, dx - 100 \int_{100}^{\infty} f(x)\, dx = 224.25 - (100)(0.65) = \mathbf{159.25}.$$

Alternately, average payment per loss = S(100) (average payment per payment) =
$S(100)\ E[X - 100 \mid X > 100] = S(100)\ \{E[X \mid X > 100] - 100\} = (0.65)(345 - 100) =$ **159.25**.

14.41. A. Average ground-up loss = E[X] = 27,000.
Under policy A, average amount paid per loss = $E[X] - E[X \wedge 25{,}000] = 10{,}000$.
Therefore, $E[X \wedge 25{,}000] = 27{,}000 - 10{,}000 = 17{,}000$.
Under policy A, average amount paid per payment = $(E[X] - E[X \wedge 25{,}000]) / S(25{,}000) = 22{,}000$.
Therefore, $S(25{,}000) = 10{,}000/22{,}000 = 0.4545$.
Given that a loss less than or equal to 25,000 has occurred, the expected payment under policy B = average loss of size between 0 and 25,000 = $\int_0^{25{,}000} x\, f(x)\, dx \ / \ F(25{,}000) =$

$\{E[X \wedge 25{,}000] - 25{,}000 S(25{,}000)\} / F(25{,}000) =$
$\{17{,}000 - (25{,}000)(0.4545)\} / (1 - 0.4545) =$ **10,335**.

14.42. E. $E[(X - 1000)_+]$ = e(1000) S(1000) = (30,000)(1 - 0.727) = 8190.
$E[(X - 25{,}000)_+]$ = e(25,000) S(25,000) = (980,000)(1 - 0.997) = 2940.
The average payment per loss is:
$E[X \wedge 25{,}000] - E[X \wedge 1000] = E[(X - 1000)_+] - E[(X - 25{,}000)_+]$ = 8190 - 2940 = **5250**.

14.43. B. 10,000 = maximum payment = Policy limit = maximum covered loss - deductible.
Thus the maximum covered loss = 10,500.
Expected payment per loss = $E[X \wedge u] - E[X \wedge d]$ = **$E[X \wedge 10{,}500] - E[X \wedge 500]$**.
Alternately, let X be the size of loss.
If for example x = 11,000, then the payment is: Min[10,500, 10,000] = 10,000.
If for example x = 10,200, then the payment is: Min[9700, 10,000] = 9700.
If for example x = 7000, then the payment is: Min[6500, 10,000] = 6500.
If for example x = 300, then the payment is: Min[0, 10,000] = 0.

$$\text{payment} = \begin{cases} 10{,}000, & x \geq 10{,}500 \\ x - 500, & 500 < x < 10{,}500 \\ 0, & x \leq 500 \end{cases}.$$

These are the same payments as if there were a 10,500 maximum covered loss (applied first) and a 500 deductible (applied second). Proceed as before.
Alternately, we can compute the average payment per loss:

$$\int_{500}^{10{,}500} (x - 500)\, f(x)\, dx + 10{,}000\, S(10{,}500) =$$

$$\int_{500}^{10{,}500} x\, f(x)\, dx - 500 \int_{500}^{10{,}500} f(x)\, dx + 10{,}000\, S(10{,}500) =$$

$E[X \wedge 10{,}500] - 10{,}500\, S(10{,}500) - \{E[X \wedge 500] - 500\, S(500)\}$
$- 500\{F(10{,}500) - F(500)\} + 10{,}000\, S(10{,}500) =$
$E[X \wedge 10{,}500] - 500\, S(10{,}500) - E[X \wedge 500] + 500\, S(500) - 500\, F(10{,}500) + 500\, F(500) =$
$E[X \wedge 10{,}500] - E[X \wedge 500] + 500\{F(500) + S(500) - F(10{,}500) - S(10{,}500)\} =$
$E[X \wedge 10{,}500] - E[X \wedge 500] + (500)(1 - 1) =$ **$E[X \wedge 10{,}500] - E[X \wedge 500]$**.

Comment: As mentioned in the section on Policy Provisions, the default on the exam is to apply the maximum covered loss first and then apply the deductible.
What is done in this question is mathematically the same as first applying a maximum covered loss of 10,500 and then applying a deductible of 500.

14.44. C. 40,000 = (E[X] - E[X ∧ 5000]) / S(5000). ⇒ E[X] - E[X ∧ 5000] = 40,000 S(5000).
When there is a positive payment with a franchise deductible then it is 5000 more than that when there is an ordinary deductible. The probability of a positive payment if S(5000).
Thus, 33,750 = E[X] - E[X ∧ 5000] + 5000 S(5000) = 40,000 S(5000) + 5000 S(5000).
⇒ S(5000) = 33,750 / 45,000 = 0.75.
With an ordinary deductible of 5000, the average payment per loss is:
E[X] - E[X ∧ 5000] = 40,000 S(5000) = (40,000)(0.75) = **30,000**.

14.45. A. Expected payment per loss is:

$$\int_{200}^{300} (x - 200)\ 0.001\ dx + \int_{300}^{800} (x - 200)\ 0.0002\ dx = 5 + 35 = \mathbf{40}.$$

<u>Comment</u>: The mean size of loss is 130; the loss elimination ratio is: 1 - 40/130 = 0.692.

14.46 C. For an accident that does not exceed $500 the insurer pays nothing.
For an accident of size x > 500, the insurer pays x - 500.
The density function for x is f(x) = 1/5000 for 0≤x≤5000.
Thus the insurer's average payment per accident is:

$$\int_{500}^{5000} (x\text{-}500)\ f(x)\ dx = \int_{500}^{5000} (x\text{-}500)\ (1/5000)\ dx = (x\text{-}500)^2\ (1/10{,}000) \Big]_{x=500}^{x=5000} = \mathbf{2025}.$$

14.47. E.

Expected amount paid per loss = $\int_{100}^{\infty} x\, f(x)\, dx = \int_0^{\infty} x\, f(x)\, dx - \int_0^{100} x\, f(x)\, dx =$

Mean - {E[X∧100] - 100S(100)}.

$S(100) = \{\theta/(\theta+100)\}^2 = (1000/1100)^2 = 0.8264.$

$E[X\wedge 100] = \{\theta/(\alpha-1)\}\{1-(\theta/(\theta+100))^{\alpha-1}\} = \{1000/(2-1)\}\{1-(1000/1100)^{2-1}\} = 90.90.$

Mean = $\theta/(\alpha-1)$ = 1000.

Therefore, expected amount paid per loss is: 1000 - {90.90 - 82.64} = 991.74.

Expect 10 losses per year, so the average cost per year is: (10)(991.7) = **$9917**.

Alternately the expected cost per year of 10 losses is:

$$10 \int_{100}^{\infty} x\, f(x)\, dx = (10)(2)(1000^2) \int_{100}^{\infty} x\, (1000+x)^{-3}\, dx =$$

$$10^7 \{-x\, (1000+x)^{-2}\}\Big]_{x=100}^{x=\infty} + 10^7 \int_{100}^{\infty} (1000+x)^{-2}\, dx = 10^7\, (100/1100^2 + 1/1100) = 9917.$$

Alternately, the average severity per loss > $100 is:

100 + e(100) = 100 + $(\theta+100)/(\alpha - 1)$ = 1100 + 100 = $1200.

Expected number of losses > $100 = 10S(100) = 8.2645.

Expected annual payment = $1200(8.2645) = $9917.

<u>Comment</u>: Almost all questions involve the ordinary deductible, in which for a loss X larger than d, X - d is paid. For these situations the average payment per loss is: E[X] - E[X∧d].

Instead, here for a large loss the whole amount is paid. This is a franchise deductible, as discussed in the section on Policy Provisions. In this case, the average payment per loss is d S(d) more than for the ordinary deductible or: E[X] - E[X∧ d] + d S(d).

One can either compute the expected total amount paid per year by an insurer either as (average payment insured receives per loss)(expected losses the insured has per year) or as (average payment insurer makes per non-zero payment)(expected non-zero payments the insurer makes per year). The former is ($991.7)(10) = $9917; the latter is ($1200)(8.2645) = $9917. Thus whether one looks at it from the point of view of the insurer or the insured, one gets the same result.

14.48. B. For this uniform distribution, f(x) = 1/50,000 for 0 ≤ x ≤ 50,000. The payment by the insurer depends as follows on the size of loss x:

<u>Insurer's Payment</u>	
0	x ≤ 5000
x - 5000	5000 ≤ x ≤ 25,000
20,000	x ≥ 25,000

We need to compute the ratio of two quantities, the average dollars paid by the insurer per loss and the probability that a loss will result in a nonzero payment. The latter is the chance that x > 5000, which is: 1 - (5000/50,000) = 0.9. The former is the sum of two terms corresponding to 5000 ≤ x ≤ 25,000 and x > 25,000:

$$\int_{5000}^{25,000} (x-5000)\ f(x)\ dx\ + 20,000\ S(25,000) = \int_{5000}^{25,000} (x-5000)\ (1/50,000)\ dx\,x + 20,000(1 - 0.5) =$$

$$(x-5000)^2/100,000 \Big]_{x=5000}^{x=25,000} + 10,000 = 4000 + 10,000 = 14,000.$$

Thus the average nonzero payment by the insurer is: 14,000 / 0.9 = **15,556**.
Alternately, S(x) = 1 - x/50,000, x < 50,000.
The average payment per (nonzero) payment is:
(E[X ∧ L] - E[X ∧ d]) / S(d) = (E[X ∧ 25,000] - E[X ∧ 5000]) / S(5000) =

$$\int_{5000}^{25,000} S(x)\ dx\ \mathbf{/}\ S(5000) = \int_{5000}^{25,000} (1 - x/50,000)\ dx\ / 0.9 = (20,000 - 6250 + 250)/0.9 = \mathbf{15,556}.$$

14.49. For the layer from \$100,000 to \$200,000, the expected number of payments is:
100 S(100,000) = 39.7.
The expected losses are: (100) (E[X ∧ 200,000] - E[X ∧ 100,000]) = \$3,145,400.
The average payment per payment in the layer is: 3,145,400/39.7 = \$79,229.
For the layer from \$200,000 to \$500,000, the expected number of payments is:
100 S(200,000) = 25.2.
The expected losses are: (100) (E[X ∧ 500,000] - E[X ∧ 200,000]) = \$5,007,000.
The average payment per payment in the layer is: 5,007,000/25.2 = \$198,690.
For the layer from \$500,000 to \$1,000,000, the expected number of payments is:
100 S(500,000) = 11.5.
The expected losses are: (100) (E[X ∧ 1,000,000] - E[X ∧ 500,000]) = \$3,990,300.
The average payment per payment in the layer is: 5,007,000/11.5 = \$346,983.

14.50. C. The insurer pays the dollars of loss excess of $15,000, which are:
$E[X] - E[X \wedge 15{,}000] = E[X \wedge \infty] - E[X \wedge 15{,}000]$. The number of non-zero payments is
$1 - F(15{,}000)$. Thus the average nonzero payment is:
$(E[X] - E[X \wedge 15{,}000])/ (1 - F(15{,}000)) = (20{,}000 - 7700)/(1-0.7) = 12{,}300 / 0.3 =$ **41,000**.

14.51. E. Since 40% of the losses exceed a deductible of 10,000 and half of 40% is 20%, the new deductible is 22,500 which is exceeded by 20% of the losses.
In other words, $S(22{,}500) = 20\% = 40\%/ 2 = S(10{,}000) / 2$.
For a deductible of size d, the expected size of a nonzero payment made by the insurer is
$(E[X] - E[X \wedge d]) / S(d) = e(d)$ = the mean excess loss at d.
$e(10{,}000) = (20{,}000 - 6000) / (1-0.6) = 35{,}000$.
$e(22{,}500) = (20{,}000 - 9500) / (1-0.8) = 52{,}500$.
52,500 / 35,000 = 1.5 or a **50% increase**.
Comment: One can do the problem without using the specific numbers in the Loss Size column.

14.52. For this density, the survival function is:

$$S(x) = \int_x^\infty 0.01(1 - q + 0.01qt)\, e^{-0.01t}\, dt = -e^{-0.01t} - 0.01qt\, e^{-0.01t} \Big]_{t=x}^{t=\infty}$$

$= e^{-0.01x} + 0.01qxe^{-0.01x}$. We will also need integrals of xf(x):

$$\int_x^\infty t\, f(t)\, dt = 0.01 \int_x^\infty (1 - q)t\, e^{-0.01t} + 0.01q\, t^2\, e^{-0.01t}\, dt =$$

$$-0.01e^{-0.01t} \{(1-q)(100t+100^2) + q(t^2 + 2t100 + (2)100^2)\} \Big]_{t=x}^{t=\infty} =$$

$e^{-0.01x}\{(1-q)(x+100) + q(0.01x^2 + 2x + 200)\} = e^{-0.01x}\{((x+100) + q(0.01x^2 + x + 100)\}$.
First, given q, calculate the average value of a non-zero payment, given a deductible of 100.

We need to compute: $\int_{100}^\infty (t-100)\, f(t)\, dt\ /\ S(100) = \int_{100}^\infty t\, f(t)\, dt\ /\ S(100) - 100 =$

$\{e^{-1}\{200 + q(300)\}\} / \{(1 + q)e^{-1}\} - 100 = \{200 + 300q\} / (1 + q) - 100$.
Setting this equal to 125, one can solve for q: $\{200 + 300q\} / (1 + q) - 100 = 125$.
$225(1 + q) = 200 + 300q$. ⇒ $q = 25/75 = 1/3$.
Now the average non-zero payment, given a deductible of 200 is:

$$\int_{200}^\infty t\, f(t)\, dt\ /\ S(200) - 200 = \{e^{-2}\{300 + (700/3)\}\} / \{(1 + 2/3)e^{-2}\} - 200 = (1600/3)/(5/3) - 200 =$$

= 320 - 200 = **120.**
Alternately, the given density is a mixture of an Exponential with $\theta = 100$, given weight 1-q, and a Gamma Distribution with parameters $\alpha = 2$ and $\theta = 100$, given weight q.
$\Gamma(2; 0.01x) = 1 - e^{-0.01x}\{1 + 0.01x\}$. Therefore, for this Gamma Distribution,
$E[X \wedge x] = 200 - e^{-0.01x}\{200 + 2x + 0.01x^2\} + e^{-0.01x}\{x + 0.01x^2\} = 200 - e^{-0.01x}(200 + x)$.
Thus, for the mixed distribution, $E[X \wedge x] = q\{200 - e^{-0.01x}(200 + x)\} + (1-q)100(1- e^{-0.01x}) =$
$100(1-e^{-0.01x}) + q(100 - 100e^{-0.01x} - xe^{-0.01x})$.
For this Exponential Distribution, $S(x) = e^{-0.01x}$.
For this Gamma Distribution, $S(x) = 1 - \Gamma(2; 0.01x) = e^{-0.01x}\{1 + 0.01x\}$.
For the mixed distribution, $S(x) = q\{e^{-0.01x}(1 + 0.01x)\} + (1-q)\, e^{-0.01x} = e^{-0.01x} + q.01x\, e^{-0.01x}$.
The expected non-zero payment given a deductible of size x is:
$(E[X] - E[X \wedge x])/S(x) = \{100e^{-0.01x} + q(100+x)e^{-0.01x}\}/\{e^{-0.01x} + q.01x\, e^{-0.01x}\} =$
$\{100 + q(100+x)\} / \{1 + 0.01xq\}$.
Thus for a deductible of 100, the average non-zero payment is:
$(100+200q)/(1+q)$. Setting this equal to 125 and solving for q,
$125 = (100+200q)/(1+q)$. ⇒ $q = 25/75 = 1/3$.
Thus for a deductible of 200, the average non-zero payment is:
$\{100+(300/3)\} / (1+2/3) = 200/(5/3) =$ **120.**

14.53. D. Average ground-up loss = E[X] = 11,100.
Under policy A, average amount paid per loss = E[X] - E[X $\wedge$ 5000] = 6500.
Therefore, E[X $\wedge$ 5000] = 11,100 - 6500 = 4600.
Under policy A, average amount paid per payment =
(E[X] - E[X $\wedge$ 5000])/S(5000) = 10,000.
Therefore, S(5000) = 6500/ 10,000 = 0.65.
Given that a loss less than or equal to 5,000 has occurred, the expected payment under policy B = average loss of size between 0 and 5000 =

$$\int_0^{5000} x\ f(x)\ dx \ / \ F(5000) = \{E[X \wedge 5000] - 5000S(5000)\}/F(5000) =$$

{4600 - (5000)(0.65)} / (1 - 0.65) = 1350/0.35 = **3857**.
Comment: F, S, f, the mean, and the Limited Expected Value, are all for the ground-up unlimited losses of the jewelry store, whether or not it has insurance.

14.54. E. Under policy A, with an ordinary deductible of 5,000 with no maximum covered loss, the expected amount paid per loss is: E[X] - E[X $\wedge$ 5000] = 6,500. Under policy A, the the expected amount paid per payment is: (E[X] - E[X $\wedge$ 5000])/S(5000) = 10,000.
Therefore, S(5000) = 6500/10,000 = 0.65. Given that a loss has occurred, the payment under policy B, with no deductible and a policy limit of 5,000, is 5,000 if and only if the original loss is 5000 or more. The probability of this is S(5000) = **0.65**.

14.55. C. An Exponential Distribution with θ = 5000. E[X $\wedge$ x] = $\theta(1 - e^{-x/\theta}) = 5000(1 - e^{-x/5000})$.
E[X $\wedge$ 500] = $5000(1 - e^{-0.1})$ = 475.8. E[X $\wedge$ 25,000] = $5000(1 - e^{-5})$ = 4966.3. E[X] = θ = 5000.
Average payment per loss before: E[X] = 5000.
Average payment per loss after: E[X $\wedge$ 25,000] - E[X $\wedge$ 500] = 4966.3 - 475.8 = 4490.5.
Average savings per loss: 5000 - 4490.5 = **509.5**.

14.56. E. By integrating f(x), F(x) = $0.01x^2$, 0 < x < 10. S(4) = 1 - (0.01)(4^2) = 0.84.

$$E[X] = \int_0^{10} S(x)\ dx = \int_0^{10} (1 - 0.01x^2)\ dx = x - 0.01x^3/3 \Big]_{x=0}^{x=10} = 6.667$$

$$E[X \wedge 4] = \int_0^{4} S(x)\ dx = \int_0^{4} (1 - 0.01x^2)\ dx = x - 0.01x^3/3 \Big]_{x=0}^{x=4} = 3.787$$

$E[Y^P]$ = (E[X] - E[X $\wedge$ 4])/S(4) = (6.667 - 3.787)/0.84 = **3.43**.

14.57. B. Let m be the maximum covered loss. 60 = 0.8(m - 20). ⇒ m = 95.
The insurance pays 80% of the layer from 20 to 95.

Expected payment per loss is: $0.8 \int_{20}^{95} S(x)\ dx = 0.8 \int_{20}^{95} 1 - x^2/10000\ dx = 0.8(75 - 28.31) = 37.35$.

Expected payment per payment is: $37.35/S(20) = 37.35/(1 - 0.2^2) =$ **38.91**.
Alternately, $f(x) = x/5000$, $0 \le x \le 100$.

$E[X \wedge 95] = \int_0^{95} x\ f(x)\ dx + 95\ S(95) = 57.16 + (95)(1 - 0.95^2) = 66.42$.

$E[X \wedge 20] = \int_0^{20} x\ f(x)\ dx + 20\ S(20) = 0.533 + (20)(1 - 0.20^2) = 19.73$.

$E[Y^P] = 0.8\ \{E[X \wedge 95] - E[X \wedge 20]\} / S(20) = 0.8\ (66.42 - 19.73) / 0.96 =$ **38.91**.

14.58. a. E[X ∧ 1 million] = 7,750,000/10 = $775,000.
E[X ∧ 5 million] = 24,450,000/10 = $2,445,000. $2,445,000/$775,000 = **3.155**.

Claim	Loss	Limited to 1 million	Limited to 5 million
A	$250,000	$250,000	$250,000
B	$300,000	$300,000	$300,000
C	$450,000	$450,000	$450,000
D	$750,000	$750,000	$750,000
E	$1,200,000	$1,000,000	$1,200,000
F	$2,500,000	$1,000,000	$2,500,000
G	$4,000,000	$1,000,000	$4,000,000
H	$7,500,000	$1,000,000	$5,000,000
I	$9,000,000	$1,000,000	$5,000,000
J	$15,000,000	$1,000,000	$5,000,000
Sum	$40,950,000	$7,750,000	$24,450,000

b. With a deductible of $1 million there are 6 non-zero payments out of 10 losses.
Average payment per payment is: ($2,445,000 - $775,000)/0.6 = **$2,783,333**.
Alternately, the six non-zero payments are in millions: 0.2, 1.5, 3, 4, 4, 4.
(0.2 + 1.5 + 3 + 4 + 4 + 4)/6 = 16.7 million/6 = **$2,783,333**.
<u>Comment</u>: *The solution to part a is one way to determine the $5 million increased limit factor for a basic limit of $1 million.*

Section 15, Producing Additional Distributions

Given a light-tailed distribution, one can produce a more heavy-tailed distribution by looking at the inverse of x. Let G(x) = 1 - F(1/x).[44] For example, if F is a Gamma Distribution, then G is an Inverse Gamma Distribution. For a Gamma Distribution with $\theta = 1$, F(x) = $\Gamma[\alpha; x]$.
Letting y = θ/x, G(y) = 1 - F(x) = 1 - F(θ/y) = $\Gamma[\alpha; \theta/y]$, the Inverse Gamma Distribution.

Given a Distribution, one can produce another distribution by adding up independent identical copies. For example, adding up α independent Exponential Distributions gives a Gamma Distribution. As α approaches infinity one approaches a very light-tailed Normal Distribution.

One can get a more heavy-tailed distribution by the change of variables y = ln(x).
Let G(x) = F[ln(x)]. For example, if F is the Normal Distribution, then G is the heavier-tailed LogNormal Distribution. Loss Models refers to this as "**exponentiating**", since if y = ln(x), then x = e^y.

One can get new distributions by the change of variables y = $x^{1/\tau}$. Loss Models refers to this as "**raising to a power**". Let G(x) = F[$x^{1/\tau}$], $\tau > 0$. For example, if F is the Exponential Distribution with mean θ, then G is a Weibull Distribution, with scale parameter θ^τ. For $\tau > 1$ the Weibull Distribution has a lighter tail than the Exponential Distribution. For $\tau < 1$ the Weibull Distribution has a heavier tail than the Exponential Distribution.

For $\tau > 0$, Loss Models refers to the new distribution as **transformed**, for example Transformed Gamma versus Gamma.[45]

If $\tau < 0$, then G(x) = 1 - F[$x^{1/\tau}$].
For $\tau < 0$, this is called the **inverse transformed** distribution, such as the Inverse Transformed Gamma versus the Gamma. This can be usefully thought of as two separate changes of variables: raising to a positive power and inverting.
For the special case, $\tau = -1$, this is the **inverse** distribution as discussed previously, such as the Inverse Gamma versus the Gamma.

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[44] We need to subtract from one, so that G(0) = 0 and G(∞) = 1.
[45] However, some distribution retain their special names. For example the Weibull is not called the transformed Exponential, nor is the Burr called the transformed Pareto.

<u>Exercise</u>: X is Exponential with mean 10. Determine the form of the distribution of $Y = X^3$.
[Solution: F(x) = 1 - exp[-x/10]. $Y = X^3$. $X = Y^{1/3}$.
$F_Y(y) = F_X[x] = F_X[y^{1/3}] = 1 - \exp[-y^{1/3}/10] = 1 - \exp[-(y/1000)^{1/3}]$.
A Weibull Distribution with θ = 1000 and τ = 1/3.
Alternately, $F_Y(y) = \text{Prob}[Y \le y] = \text{Prob}[X^3 \le y] = \text{Prob}[X \le y^{1/3}] = 1 - \exp[-y^{1/3}/10]$.
$F_Y(y) = 1 - \exp[-(y/1000)^{1/3}]$. A Weibull Distribution with θ = 1000 and τ = 1/3.
Alternately, f(x) = exp[-x/10]/10.
$f_Y(y) = f_X[y^{1/3}] \, |dx/dy| = \exp[-y^{1/3}/10]/10 \; (1/3)y^{-2/3} = (1/3) \; (y/1000)^{1/3} \exp[-(y/1000)^{1/3}] \,/\, y$.
A Weibull Distribution with θ = 1000 and τ = 1/3.
<u>Comment</u>: A change of variables as in calculus class.]

One can get additional distributions as a mixture of distributions. As will be discussed in a subsequent section, the Pareto can be obtained as a mixture of an Exponential by an Inverse Gamma.[46] Usually such mixing produces a heavier tail; the Pareto has a heavier tail than an Exponential. The Negative Binomial which can be obtained as a mixture of Poissons via a Gamma has a heavier tail than the Poisson. <u>Loss Models</u> refers to this as **Mixing**.

Another method of getting new distributions is to weight together two or more existing distributions. Such mixed distributions, referred to by <u>Loss Models</u> as n-point or two-point mixtures, are discussed in a subsequent section.[47]

One can get additional distributions as a ratio of independent variables each of which follows a known distribution. For example an F-distribution is a ratio of Chi-Squares.[48] As a special case, the Pareto can be obtained as a ratio of an Exponential variable and a Gamma variable.[49] The Beta Distribution can be obtained as a combination of two Gammas.[50] Generally the distributions so obtained have heavier tails.

Finally, one can introduce a scale parameter. If one had the distribution $F(x) = 1 - e^{-x}$, one can create a family of distributions by substituting x/θ everywhere x appears. θ is now a scale parameter. $F(x) = 1 - e^{-x/\theta}$. Introducing a scale parameter does not affect either the tail behavior or the shape of the distribution.

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[46] <u>Loss Models</u> Example 5.4 shows that mixing an exponential via an inverse exponential yields a Pareto Distribution. This is just a special case of the Inverse Gamma-Exponential, with mixed distribution a Pareto. *Example 5.6 shows that mixing an Inverse Weibull via a Transformed Gamma with the same τ parameter, gives an Inverse Burr Distribution.*
[47] <u>Loss Models</u>, Section 4.2.3.
[48] The F-Distribution from Statistics is related to the Generalized Pareto Distribution.
[49] See p. 47, <u>Loss Distributions</u> by Hogg & Klugman.
[50] If X is a random draw from a Gamma Distribution with shape parameter α and scale parameter θ, and Y is a random draw from a Gamma Distribution with shape parameter β and scale parameter θ,
then Z = X / (X+Y) is a random draw from a Beta Distribution with parameters α and β.

Exercise: X is uniform from 0 to 0.1. Y = $\sqrt{\frac{10}{x}}$ - 10. Determine the distribution of Y.

[Solution: F(x) = 10x, 0 ≤ x ≤ 0.1. X = 10/(Y + 10)^2.
x = 0 ⇔ y = ∞. x = 0.1 ⇔ y = 0. Small Y ⇔ Large X. ⇒ Need to take 1 - F(x).
$F_Y(y) = 1 - F_X[10/(y + 10)^2] = 1 - 10^2/(y + 10)^2$. A Pareto Distribution with α = 2 and θ = 10.
Alternately, $F_Y(y)$ = Prob[Y ≤ y] = Prob[$\sqrt{\frac{10}{x}}$ - 10 ≤ y] = Prob[$\sqrt{\frac{10}{x}}$ ≤ y + 10] =
Prob[10/X ≤ (y + 10)^2] = Prob[X ≥ 10/(y + 10)^2] = 1 - Prob[X ≤ 10/(y + 10)^2] =
1 - (10){10/(y + 10)^2} = 1 - 10^2/(y + 10)^2.
$F_Y(y) = 1 - 10^2/(y + 10)^2$, a Pareto Distribution with α = 2 and θ = 10.
Alternately, f(x) = 10, 0 ≤ x ≤ 0.1.
$f_Y(y) = f_X[10/(y + 10)^2] \left|\frac{dx}{dy}\right| = (10)\ 20/(Y + 10)^3 = (2)\ 10^2/(Y + 10)^3$.
A Pareto Distribution with α = 2 and θ = 10.]

Percentiles:

A one-to-one monotone transformation, such as ln(x), e^x, or x^2, preserves the percentiles, including the median. For example, the median of a Normal Distribution is μ, which implies that the median of a LogNormal Distribution is e^μ.

Problems:

15.1 (4 points) X follows a Standard Normal Distribution, with mean zero and standard deviation of 1. Y = 1/X.
(a) (1 point) What is the density of y?
(b) (2 points) Graph the density of y.
(c) (1 point) What is E[Y]?

15.2 (1 point) X follows an Exponential Distribution with mean 1. Let Y = θ exp[X/α].
What is the distribution of Y?

15.3 (3 points) X follows a Weibull Distribution with parameters θ and τ. Let Y = -ln[X].
What are the algebraic forms of the distribution and density functions of Y?

15.4 (3 points) ln(X) follows a LogNormal Distribution with μ = 1.3 and σ = 0.4.
Determine the density function of X.

15.5 (3 points) X follows a Standard Normal Distribution, with mean 0 and standard deviation of 1. Let $Y^\tau = X^2$, for $\tau > 0$. Determine the form of the Distribution of Y.

15.6 (2 points) Let X follows the density $f(x) = e^{-x} / (1 + e^{-x})^2$, $-\infty < x < \infty$.
Let $Y = \theta\ e^{X/\gamma}$, for $\theta > 0$, $\gamma > 0$. Determine the form of the distribution of Y.

15.7 (2 points) Let X be a uniform distribution from 0 to 1.
Let $Y = \theta \ln[\frac{1 - p}{1 - p^X}]$, for $\theta > 0$, $1 > p > 0$. Determine the form of the distribution function of Y.

15.8 (2 points) X follows an Exponential Distribution with hazard rate λ. Let Y = exp[-δX].
What is the distribution of Y?

15.9 (3 points) X is uniform on (0, 1). Y is uniform on (0, $\sqrt{x}$).
(a) What is the distribution of Y?
(b) Determine E[Y].

15.10 (1 point) X follows an Exponential Distribution with mean 10. Let Y = 1/X.
What is the distribution of Y?

15.11 (4 points) You are given the following:
- The random variable X has a Normal Distribution, with mean zero and standard deviation σ.
- The random variable Y also has a Normal Distribution, with mean zero and standard deviation σ.
- X and Y are independent.
- $R^2 = X^2 + Y^2$.

Determine the form of the distribution of R.
Hint: The sum of squares of n independent Standard Normals is a Ch-Square Distribution with ν degrees a freedom, which is a Gamma Distribution with $\alpha = \nu/2$ and $\theta = 2$.

15.12 (CAS Part 2 Exam, 1965, Q. 44) (1.5 points)
Given the probability density function $f(x) = x/2$, $0 \le x \le 2$.
Find the probability density function for y, where $y = x^2/2$.

<u>15.13</u> **(2, 5/90, Q.36)** (1.7 points) If Y it uniformly distributed on the interval (0, 1) and if $Z = -a \ln(1 - Y)$ for some $a > 0$, then to which of the following families of distributions does Z belong?
A. Pareto B. LogNormal C. Normal D. Exponential E. Uniform

15.14 (4B, 11/97, Q.11) (2 points) You are given the following:
- The random variable X has a Pareto distribution, with parameters θ and α.
- Y is defined to be $\ln(1 + X/\theta)$.

Determine the form of the distribution of Y.
A. Negative Binomial B. Exponential C. Pareto D. Lognormal E. Normal

<u>15.15</u> **(IOA 101, 4/00, Q.13)** (3.75 points) Suppose that the distribution of a physical coefficient, X, can be modeled using a uniform distribution on (0, 1).
A researcher is interested in the distribution of Y, an adjusted form of the reciprocal of the coefficient, where $Y = (1/X) - 1$.
(i) (2.25 points) Determine the probability density function of Y.
(ii) (1.5 points) Determine the mean of Y.

15.16 (1, 11/01, Q.13) (1.9 points) An actuary models the lifetime of a device using the random variable $Y = 10X^{0.8}$, where X is an exponential random variable with mean 1 year.
Determine the probability density function f(y), for y > 0, of the random variable Y.
(A) $10\ y^{0.8} \exp[-8y^{-0.2}]$
(B) $8\ y^{-0.2} \exp[-10y^{0.8}]$
(C) $8\ y^{-0.2} \exp[-(0.1y)^{1.25}]$
(D) $(0.1y)^{1.25} \exp[-1.25(0.1y)^{.25}]$
(E) $0.125\ (0.1y)^{0.25} \exp[-(0.1y)^{1.25}]$

15.17 (CAS3, 11/05, Q.19) (2.5 points)
Claim size, X, follows a Pareto distribution with parameters α and θ.
A transformed distribution, Y, is created such that $Y = X^{1/\tau}$.
Which of the following is the probability density function of Y?
A. $\tau\theta y^{\tau-1} / (y + \theta)^{\tau+1}$
B. $\alpha\theta^{\alpha}\tau y^{\tau-1} / (y^{\tau} + \theta)^{\alpha+1}$
C. $\theta\alpha^{\theta} / (y + \alpha)^{\theta+1}$
D. $\alpha\tau(y/\theta)^{\tau} / \{y[1 + (y/\theta)^{\tau}]^{\alpha+1}\}$
E. $\alpha\theta^{\alpha} / (y^{\tau} + \theta)^{\alpha+1}$

15.18 (CAS3, 5/06, Q.27) (2.5 points)
The following information is available regarding the random variables X and Y:
- X follows a Pareto Distribution with $\alpha = 2$ and $\theta = 100$.
- $Y = \ln[1 + (X/\theta)]$

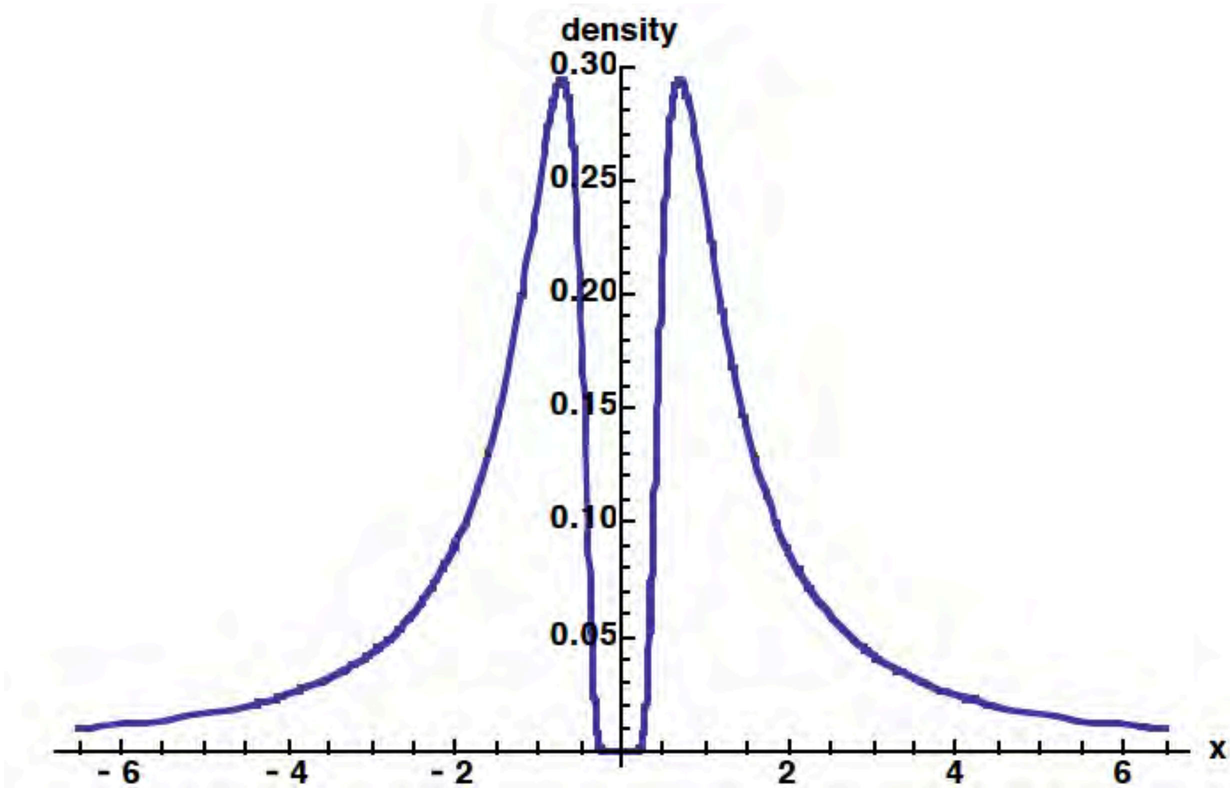
Calculate the variance of Y.
A. Less than 0.10
B. At least 0.10, but less than 0.20
C. At least 0.20, but less than 0.30
D. At least 0.30, but less than 0.40
E. At least 0.40

Solutions to Problems:

15.1. (a) $f(x) = \exp[-0.5\ x^2]/\sqrt{2\pi}$. $x = 1/y$. $\frac{dx}{dy} = -1/y^2$.

$g(y) = f(x) \left|\frac{dx}{dy}\right| =$ **$(\exp[-0.5/y^2]/y^2)/\sqrt{2\pi}$** , $-\infty < y < \infty$.

(b) This density is zero at zero, is symmetric, and has maximums at $\pm 1/\sqrt{2}$:



(c) $E[Y] = 2 \int_0^\infty (\exp[-0.5/y^2]/y)\ dy\ \ /\sqrt{2\pi}$. Now as $y \to \infty$, $\exp[-0.5/y^2] \to e^0 = 1$.

Therefore, for large values of y, the integrand is basically 1/y, which has no finite integral since $\ln[\infty] = \infty$. Therefore, **the first moment of Y does not exist.**
Comment: This is an Inverse Normal Distribution, none of whose positive moments exist.

15.2. $F(x) = 1 - \exp[-x]$. $y = \theta \exp[x/\alpha]$. $\Rightarrow \exp[x/\alpha] = y/\theta$. $\Rightarrow \exp[x] = (y/\theta)^\alpha$.
Substituting into F(x), $F(y) = 1 - (\theta/y)^\alpha$, a Single Parameter Pareto Distribution.
Comment: While x goes from 0 to ∞, y goes from $\theta \exp[0/\alpha] = \theta$ to ∞.
In general, if ln[Y] follows a Gamma, then Y follows what is called a LogGamma.
A special case is when ln[Y] is Exponential with mean α. Then Y follows a LogExponential, which is just a Single Parameter Pareto Distribution with θ = 1.

15.3. $F(x) = 1 - \exp[-(x/\theta)^\tau]$, $x > 0$. $y = -\ln[x]$. $\Rightarrow x = e^{-y}$. $x = 0 \Leftrightarrow y = \infty$, and $x = \infty \Leftrightarrow y = -\infty$.
Since large x corresponds to small y, we need to substitute into S(x) rather than F(x).
Substituting into S(x), $F(y) = \exp[-(e^{-y}/\theta)^\tau] = \exp[-e^{-\tau y}/\theta^\tau]$, $-\infty < y < \infty$.
Differentiating, $f(y) = \exp[-e^{-\tau y}/\theta^\tau]\ \tau\ e^{-\tau y}/\theta^\tau$, $-\infty < y < \infty$.
Alternately, for the Weibull Distribution, $f(x) = \tau x^{\tau-1} \exp(-(x/\theta)^\tau) / \theta^\tau$.
$f(y) = f(x)\ |dx/dy| = \{\tau e^{-y(\tau-1)} \exp(-(e^{-y}/\theta)^\tau) / \theta^\tau\}\ e^{-y} = \exp[-e^{-\tau y}/\theta^\tau]\ \tau\ e^{-\tau y}/\theta^\tau$, $-\infty < y < \infty$.
Comment: *This distribution is sometimes called the Gumbel Distribution.*
For $\tau = 1$ and $\theta = 1$, $F(y) = \exp[-e^{-y}]$, and $f(x) = \exp[-y - e^{-y}]$, $-\infty < y < \infty$, which is a form of what is called the Extreme Value Distribution, the Fisher-Tippet Type I Distribution, or the Doubly Exponential Distribution.

15.4. The LogNormal Distribution has support starting at 0, so we want $\ln(x) > 0$. $\Rightarrow x > 1$.
F(x) = LogNormal Distribution at ln(x): $\Phi[\{\ln(\ln(x)) - 1.3\}/0.4]$.
$f(x) = \phi[\{\ln(\ln(x)) - 1.3\}/0.4]\ d\ln(\ln(x))/dx =$
$\{\exp[-\{\ln(\ln(x)) - 1.3\}^2/(2\ 0.4^2)]/ (0.4\ \sqrt{2\pi}\ )\ \} / \{x \ln(x)\} =$
$\mathbf{\exp[-3.125\{\ln(\ln(x)) - 1.3\}^2] / \{0.4\ x \ln(x)\ \sqrt{2\pi}\ \},\ x > 1}$.
Comment: Beyond what you are likely to be asked on your exam. *Just as the LogNormal Distribution has a much heavier righthand tail than the Normal Distribution, the "LogLogNormal" Distribution has a much heavier righthand tail than the LogNormal Distribution.*

15.5. $f(x) = \exp[-x^2/2]/\sqrt{2\pi}$. $X = Y^{\tau/2}$.
Since x is symmetric around zero, but $x^2 \ge 0$, we need to double the density of x.
$g(y) = 2\ f(x)\ |dx/dy| = 2\{\exp[-y^\tau/2]/\sqrt{2\pi}\ \}(\tau/2)y^{\tau/2-1} = \tau y^{\tau/2-1} \exp[-y^\tau/2] / \sqrt{2\pi}$.
Comment: $\theta^{\tau\alpha}\Gamma(\alpha) = (\theta^\tau)^\alpha\ \Gamma(1/2) = 2^{1/2}\ \sqrt{\pi} = \sqrt{2\pi}$.
If $\tau = 1$, then Y has a Gamma Distribution with $\alpha = 1/2$ and $\theta = 2$,
which is a Chi-Square Distribution with one degree of freedom.
The density of a Transformed Gamma Distribution, not on the syllabus, is:
$f(x) = \tau x^{\tau\alpha-1} \exp[-x^\tau/ \theta^\tau] / \{\theta^{\tau\alpha}\Gamma(\alpha)\}$.
Matching parameters, $\tau\alpha = \tau/2$, and $2 = \theta^\tau$. The density of y is a Transformed Gamma Distribution with parameters $\alpha = 1/2$, τ, and $\theta = 2^{1/\tau}$.

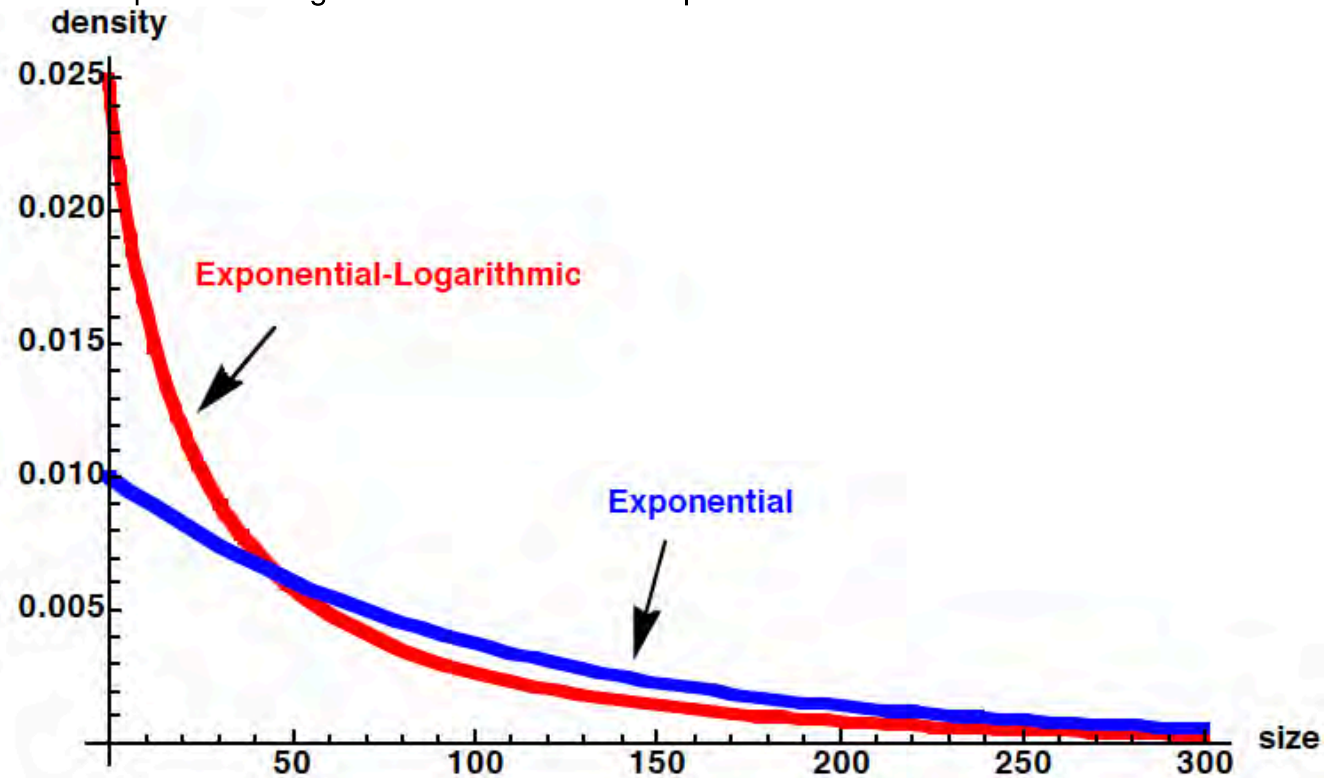
15.6. By integration, $F(x) = 1/(1 + e^{-x}) = e^x / (1 + e^x)$, $-\infty < x < \infty$.
$e^x = (y/\theta)^\gamma$. Therefore, $F_Y(y) = \dfrac{(y/\theta)^\gamma}{1 + (y/\theta)^\gamma}$, $y > 0$. This is a Loglogistic Distribution.
Comment: The original distribution is called a Logistic Distribution. The Loglogistic has a similar relationship to the Logistic Distribution, as the LogNormal has to the Normal.

15.7. $y/\theta = \ln[\frac{1 - p}{1 - p^x}]$. ⇒ $e^{y/\theta} = \frac{1 - p}{1 - p^x}$. ⇒ $(1-p)\ e^{-y/\theta} = 1 - p^x$. ⇒ $p^x = 1 - (1-p)\ e^{-y/\theta}$. ⇒

$x \ln(p) = \ln[1 - (1-p)\ e^{-y/\theta}]$. ⇒ $x = \ln[1 - (1-p)\ e^{-y/\theta}] / \ln(p)$.
For x = 1, y = 0, while as x approaches zero, y approaches infinity.
Since X is uniform, $F_X(x) = x$. ⇒ $\mathbf{F_Y(y) = 1 - \ln[1 - (1-p)\ e^{-y/\theta}] / \ln(p)}$, y > 0.

<u>Comment</u>: The density is $f_Y(y) = \frac{-e^{-y/\theta}\ (1-p)}{\{1 - (1-p)e^{-y/\theta}\}\ \theta \ln[p]}$, y > 0.

The distribution of Y is called an Exponential-Logarithmic Distribution.
As p approaches 1, the distribution of Y approaches an Exponential Distribution.
The Exponential-Logarithmic Distribution has a declining hazard rate.
If frequency follows a Logarithmic Distribution, and severity is Exponential, then the minimum of the claim sizes follows an Exponential-Logarithmic Distribution.
Here is a graph comparing the density of an Exponential with mean 100
and an Exponential-Logarithmic Distribution with p = 0.2 and θ = 100:



15.8. $F(x) = 1 - \exp[-\lambda x]$. $X = -\ln[Y]/\delta$.
When x is big y is small and vice-versa. As x goes from zero to infinity, y goes from 1 to 0.
Therefore, we get the distribution function of Y by plugging into the survival function of X:
$F(y) = \exp[-\lambda (-\ln[y]/\delta)] = y^{\lambda/\delta}$, $0 < y < 1$. $\Rightarrow f(y) = (\lambda/\delta)\, y^{\lambda/\delta - 1}$, $0 < y < 1$.
Y follows a Beta Distribution, with parameters $\theta = 1$, $a = \lambda/\delta$, and $b = 1$.
Comment: If X is the future lifetime, and δ is the force of interest, then Y is the present value of a life insurance that pays 1. The actuarial present value of this insurance is:

$$E[Y] = \theta \frac{a}{a + b} = \frac{\lambda/\delta}{\lambda/\delta + 1} = \frac{\lambda}{\lambda + \delta}.$$

As discussed in Life Contingencies, if the distribution of future lifetimes is Exponential with hazard rate λ, then $\frac{\lambda}{\lambda + \delta}$ is the actuarial present value of a life insurance that pays 1.

15.9. a. $F[y \mid x] = y/\sqrt{x}$ for $0 \le y \le \sqrt{x}$. $F[y \mid x] = 1$ for $y > \sqrt{x}$.

In other words, $F[y \mid x] = \begin{cases} 1 \text{ for } 0 \le x \le y^2 \\ y/\sqrt{x} \text{ for } y^2 \le x \le 1 \end{cases}$.

Thus, $$F[y] = \int_0^{y^2} 1\, dx + \int_{y^2}^{1} y/\sqrt{x}\, dx = y^2 + 2y\sqrt{x}\Big]_{x=y^2}^{x=1} = y^2 + 2y - 2y^2 = \mathbf{2y - y^2,\ 0 \le y \le 1}.$$

b. $S(y) = 1 + y^2 - 2y$. $E[Y] = \int_0^1 1 + y^2 - 2y\, dy = 1 + 1/3 - 1 =$ **1/3**.

Alternately, $f(x) = 2 - 2y$, $0 \le y \le 1$.
This is a Beta Distribution with $a = 1$, $b = 2$, and $\theta = 1$.
$E[Y] = \theta\, a / (a + b) =$ **1/3**.

15.10. $F(x) = 1 - e^{-x/10}$. Let G be the distribution function of Y.
$G(y) = 1 - F(x) = 1 - F(1/y) = \exp[-0.1/y]$.
This is an **Inverse Exponential Distribution with θ = 0.1**.
Comment: We need to subtract from one, so that $G(0) = 0$ and $G(\infty) = 1$.

15.11. For $\sigma = 1$, R^2 is the sum of two unit Normals, and thus a Chi-Square with 2 degrees of freedom, which is an Exponential Distribution with $\theta = 2$.
Now $R = (R^2)^{1/2}$, so we have a power transformation, and thus R is Weibull with $\tau = 2$.
Specifically, the survival function of R is: S(r) = survival function of $R^2 = \exp[-r^2/2]$.
Now if $\sigma \neq 1$, we just have a scale transformation, and r is divided by σ wherever r appears in the survival function:

$$S(r) = \exp[-\frac{r^2}{2\,\sigma^2}] = \exp[-\left(\frac{r}{\sigma\,\sqrt{2}}\right)^2].$$

R follows a Weibull Distribution with $\tau = 2$ and $\theta = \sigma\sqrt{2}$.
Alternately, X/σ and Y/σ are each a unit Normal. $(R/\sigma)^2$ is the sum of two unit Normals, and thus follows a Chi-Square with 2 degrees of freedom, which is an Exponential Distribution with $\theta = 2$.

$$S(r) = \exp[-(r/\sigma)^2/2] = \exp[-\frac{r^2}{2\,\sigma^2}] = \exp[-\left(\frac{r}{\sigma\,\sqrt{2}}\right)^2].$$

<u>Comment</u>: This is called a Rayleigh Distribution.
In general, if X is Exponential with mean 1, then $X^{1/\tau}$ is Weibull with $\theta = 1$ and τ.

15.12. $F(x) = x^2/4$. Thus $F(y) = (2y) / 4 = y/2$, $0 \le y \le 2$. $f(y) = 1/2$, $0 \le y \le 2$.
Alternately, $f(y) = f(x) / (dy/dx) = (x/2) / x = 1/2$.

15.13. D. $F(z) = \text{Prob}[Z \le z] = \text{Prob}[-a \ln(1 - Y) \le z] = \text{Prob}[\ln(1 - Y) \ge -z/a] =$
$\text{Prob}[1 - Y \ge e^{-z/a}] = \text{Prob}[1 - e^{-z/a} \ge Y] = 1 - e^{-z/a}$. An Exponential Distribution with $\theta = a$.
<u>Comment</u>: For Y uniform on [0, 1], $\text{Prob}[Y \le y] = y$.
This is the basis of one way to simulate an Exponential Distribution.
$Z = -a \ln(Y)$, also follows an Exponential Distribution with $\theta = a$, *which is the basis of another way to simulate an Exponential Distribution.*

15.14. B. If $y = \ln(1+ x/\theta)$, then $dy/dx = (1/\theta) / (1+x/\theta) = 1/(\theta+x)$. Note that $e^y = 1+ x/\theta$.
$g(y) = f(x) / |dy/dx| = \{(\alpha\theta^\alpha)(\theta + x)^{-(\alpha+1)}\} / (1/(\theta+x)) = \alpha(1 + x/\theta)^{-\alpha} = \alpha(e^y)^{-\alpha} = \alpha e^{-\alpha y}$.
Thus y is distributed as per an **Exponential**.
<u>Comment</u>: See for example page 107 of <u>Insurance Risk Models</u> by Panjer and Willmot, <u>not</u> on the Syllabus.

15.15. (i) $F(x) = x$, $0 < x < 1$. $y = 1/x - 1$. $\Rightarrow x = 1/(1 + y)$. $\Rightarrow F(y) = 1 - 1/(1+y)$, $0 < y < \infty$.
$\Rightarrow$ **$f(y) = 1/(1+y)^2$, $0 < y < \infty$.**
Alternately, $f(x) = 1$, $0 < x < 1$. $dy/dx = -1/x^2$. $f(y) = f(x)/(|dy/dx|) = 1/(1+y)^2$.
When x is 0, y is ∞, and when x = 1, y is 0. $\Rightarrow$ **$f(y) = 1/(1+y)^2$, $0 < y < \infty$.**
(ii) Y follows a Pareto Distribution with $\alpha = 1$ (and $\theta = 1$), and therefore the mean does not exist.
Alternately, E[Y] is the integral from 0 to ∞ of $y/(1+y)^2$, which does not exist, since for large y the integrand acts like 1/y.

15.16. E. $S(x) = \exp[-x]$. $y = 10x^{0.8}$. $\Rightarrow x = (y/10)^{1.25}$. $\Rightarrow S(y) = \exp[-(y/10)^{1.25}]$.
$f(y) = 1.25\ y^{0.25} \exp[-(y/10)^{1.25}] / 10^{1.25} =$ **$0.125\ (0.1y)^{0.25} \exp[-(0.1y)^{1.25}]$** .
Comment: Y follows a Weibull Distribution with $\tau = 1.25$ and $\theta = 10$.

15.17. B. $Y = X^{1/\tau}$. $x = y^\tau$. $dx/dy = \tau\ y^{\tau-1}$.
$f(x) = \alpha\theta^\alpha/(x + \theta)^{\alpha+1}$.
$f(y) = dF/dy = dF/dx\ dx/dy = \{\alpha\theta^\alpha/(x + \theta)^{\alpha+1}\}\tau\ y^{\tau-1} = \{\alpha\theta^\alpha/(y^\tau + \theta)^{\alpha+1}\}\tau\ y^{\tau-1}$
$=$ **$\alpha\theta^\alpha\tau y^{\tau - 1} / (y^\tau + \theta)^{\alpha + 1}$**.
Alternately, $F(x) = 1 - \{\theta/(x + \theta)\}^\alpha$. $x = y^\tau$. $\Rightarrow F(y) = 1 - \{\theta/(y^\tau + \theta)\}^\alpha$.
Differentiating with respect to y, $f(y) =$ **$\alpha\theta^\alpha\tau y^{\tau - 1} / (y^\tau + \theta)^{\alpha + 1}$**.
Comment: Basically, just a change of variables from calculus. The result is a Burr Distribution, not on the syllabus, but with a somewhat different treatment of the scale parameter than in Loss Models.
If $\tau = 1$, one should just get the density of the original Pareto. This is not the case for choices A and C, eliminating them. While it is not obvious, choice D does pass this test.

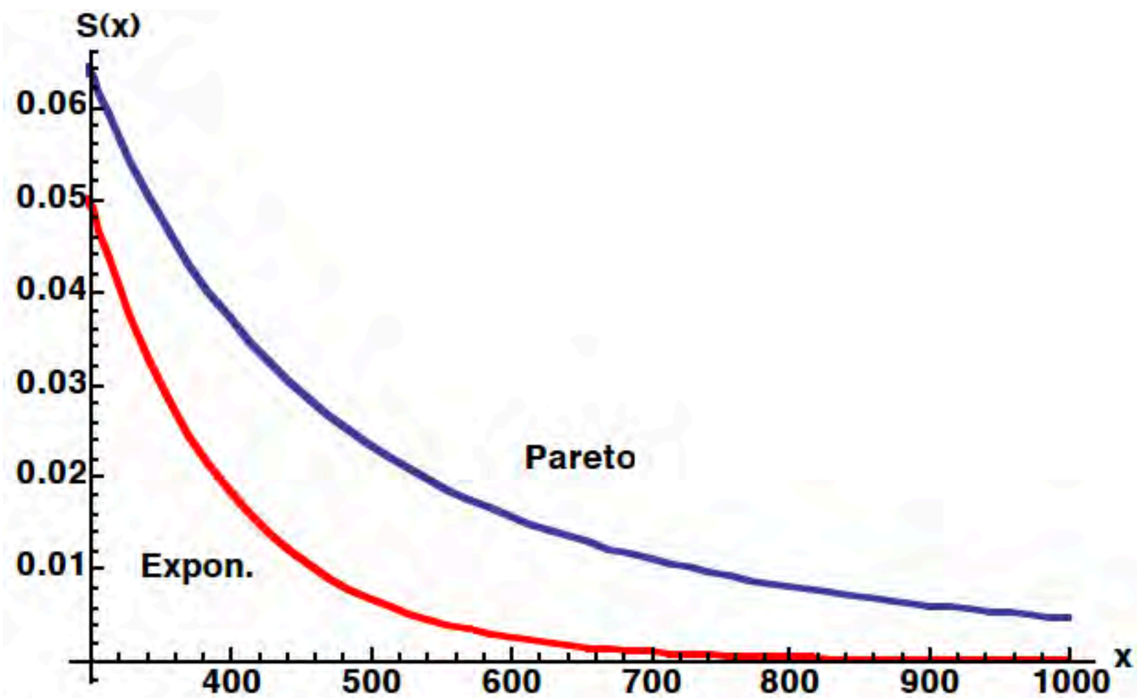
15.18. C. $Y = \ln(1 + (X/\theta))$. $\Leftrightarrow X = \theta(e^Y - 1) = 100(e^Y - 1)$.
$F(x) = 1 - \{100/(100 + x)\}^2$. $F(y) = 1 - \{100/(100 + 100(e^y - 1))\}^2 = 1 - e^{-2y}$.
Thus Y follows an Exponential Distribution with $\theta = 1/2$, and variance $\theta^2 = 1/4$.

Section 16, Tails of Loss Distributions

Actuaries are often interested in the behavior of a size of loss distribution as the size of claim gets very large. The question of interest is how quickly the right-hand tail probability, as quantified in the survival function S(x) = 1 - F(x), goes to zero as x approaches infinity. If the tail probability goes to zero slowly, one describes that as a "heavy-tailed distribution."

For example, for the Pareto distribution $S(x) = \{\theta/(\theta+x)\}^{\alpha}$, which goes to zero as per $x^{-\alpha}$.
If the tail probability goes to zero quickly, then one describes the distribution as "light-tailed".
For the Exponential Distribution, $S(x) = e^{-x/\theta}$, which goes to zero very quickly as $x \to \infty$.
The heavier tailed distribution will have both its density and its survival function go to zero more slowly as x approaches infinity. For example, for a Pareto, $f(x) = \frac{\alpha\theta^{\alpha}}{(\theta+x)^{\alpha+1}}$, which goes to zero more slowly than the density of an Exponential Distribution, $f(x) = e^{-x/\theta} / \theta$.

For example, here is a comparison starting at 300 of the Survival Function of an Exponential Distribution with θ = 100 versus that of a Pareto Distribution with θ = 200, α = 3, and mean of 100:



The Pareto with a heavier righthand tail has its Survival Function go to zero more slowly as x approaches infinity, than the Exponential. The Exponential has less probability in its righthand tail than the Pareto. The Exponential has a lighter righthand tail than the Pareto.

Exercise: Compare S(1000) for the Exponential Distribution with $\theta = 100$ versus that of a Pareto Distribution with $\theta = 200$, $\alpha = 3$, and mean 100.
[Solution: For the Exponential, $S(1000) = e^{-1000/100} = 0.00454\%$.
For the Pareto, $S(1000) = (200/1200)^3 = 0.46296\%$.
Comment: The Pareto Distribution has a much higher probability of a loss of size greater than 1000 than does the Exponential Distribution.]

Exercise: What are the mean and second moment of a Pareto Distribution with parameters $\alpha = 3$ and $\theta = 10$?
[Solution: The mean is: $\theta/(\alpha-1) = 10/2 = 5$.
The second moment is: $\frac{2\theta^2}{(\alpha-1)(\alpha-2)} = 200 / 2 = 100$.]

Exercise: What are the mean and second moment of a LogNormal Distribution with parameters $\mu = 0.9163$ and $\sigma = 1.1774$?
[Solution: The mean is: $\exp(\mu + \sigma^2/2) = \exp(1.6094) = 5$.
The second moment is: $\exp(2\mu + 2\sigma^2) = \exp(4.605) = 100$.]

Thus a Pareto Distribution with parameters $\alpha = 3$ and $\theta = 10$ and a LogNormal Distribution with parameters $\mu = 0.9163$ and $\sigma = 1.1774$ have the same mean and second moment, and therefore the same variance. However, while their first two moments match, the Pareto has a heavier tail. This can be seen by calculating the density functions for some large values of x.

Exercise: What are f(10), f(100), f(1000), and f(10,000) for a Pareto Distribution with parameters $\alpha = 3$ and $\theta = 10$?
[Solution: For a Pareto, $f(x) = \frac{\alpha\theta^\alpha}{(\theta+x)^{\alpha+1}}$. So that $f(10) = 3000/20^4 = 0.01875$,

$f(100) = 2.05 \times 10^{-5}$, $f(1000) = 2.88 \times 10^{-9}$, $f(10{,}000) = 2.99 \times 10^{-13}$.]

Exercise: What are f(10), f(100), f(1000), and f(10,000) for a LogNormal Distribution with parameters $\mu = 0.9163$ and $\sigma = 1.1774$?

[Solution: For a LogNormal $f(x) = \frac{\exp\left[-\frac{(\ln(x) - \mu)^2}{2\sigma^2}\right]}{x\,\sigma\sqrt{2\pi}}$, so that

$f(10) = 0.0169$, $f(100) = 2.50 \times 10^{-5}$, $f(1000) = 8.07 \times 10^{-10}$, $f(10{,}000) = 5.68 \times 10^{-16}$.]

x	Pareto Density	LogNormal Density
	3	0.9163
	10	1.1774
10	1.88E-02	1.69E-02
100	2.05E-05	2.50E-05
1000	2.88E-09	8.07E-10
10,000	2.99E-13	5.68E-16

While at 10 and 100 the two densities are similar, by the time we get to 1000, the LogNormal Density has started to go to zero more quickly. This LogNormal has a lighter tail than this Pareto. In general any LogNormal has a lighter tail than any Pareto Distribution.

For the LogNormal, ln f(x) = $-0.5\,(\{\ln(x)-\mu\}/\sigma)^2 - \ln(x) - \ln(\sigma) - \ln(2\pi)/2$.
For very large x this is approximately: $-0.5\,\ln(x)^2 / \sigma^2$.
For the Pareto, ln f(x) = $\ln(\alpha) + \alpha\ln(\theta) - (\alpha+1)\ln(\theta + x)$.
For very large x this is approximately: $-(\alpha+1)\ln(x)$.

Since the square of ln(x) eventually gets much bigger than ln(x), the log density of the Lognormal (eventually) goes to minus infinity faster than that of the Pareto. In other words, for very large x, the density of the Lognormal goes to zero more quickly than the Pareto. The LogNormal is lighter-tailed than the Pareto.

There are number of methods by which one can distinguish which distribution or empirical data set has the heavier tail. Light-tailed distributions have more moments that exist. For example, the Gamma Distribution has all of its (positive) moments exist. Heavy-tailed distributions do not have higher moments exist. For example, for the Pareto, only those moments for $n < \alpha$ exist.

In general, computing the n^{th} moment involves integrating $x^n f(x)$ with upper limit of infinity. Thus if f(x) goes to zero as x^{-m} as x approaches infinity, then the integrand is x^{n-m} for large x; thus the mean only exist if $n-m < -1$, in other words if $m > n + 1$. The n^{th} moment will only exist if f(x) goes to zero faster than $x^{-(n+1)}$.

For example, the Burr Distribution has $f(x) = \frac{\alpha\gamma\ x^{\gamma-1}}{\theta^\gamma} \left(\frac{1}{1 + (x/\theta)^\gamma}\right)^{\alpha+1}$ go to zero as per

$x^{(\gamma-1)-\gamma(\alpha+1)} = x^{-(\gamma\alpha+1)}$, so the n^{th} moment exists only if $\alpha\gamma > n$.
For example, a Burr Distribution with $\alpha = 2.2$ and $\gamma = 0.8$ has a first moment but fails to have a second moment, since $\alpha\gamma = 1.76 \le 2$.

If it exists, the larger the coefficient of variation, the heavier-tailed the distribution. For example, for the Pareto with $\alpha > 2$, the Coefficient of Variation $= \sqrt{\frac{\alpha}{\alpha - 2}}$, which increases as α approaches 2. Thus as α decreases, the tail of the Pareto gets heavier.

Skewness:

Similarly, when it exists, the larger the skewness, the heavier the tail of distribution. The Normal Distribution is symmetric and thus has a skewness of zero. For the common size of loss distributions, the skewness is usually positive when it exists.

The Gamma, Pareto and LogNormal all have positive skewness. For small τ the Weibull has positive skewness, but has negative skewness for large enough τ.

The Gamma Distribution has skewness of $2 / \sqrt{\alpha}$, which is always positive.
The skewness of the Pareto Distribution does not exist for $\alpha \le 3$.
For $\alpha > 3$, the Pareto skewness is: $2 \frac{\alpha+1}{\alpha-3} \sqrt{\frac{\alpha-2}{\alpha}} > 0$.

For the LogNormal Distribution the skewness = $\frac{\exp(3\sigma^2) - 3\exp(\sigma^2) + 2}{\{\exp(\sigma^2) - 1\}^{1.5}}$.

The denominator is positive, since $\exp(\sigma^2) > 1$ for $\sigma^2 > 0$.
The numerator is positive since it can be written as $y^3 - 3y + 2$, for $y = \exp(\sigma^2) > 1$.
The derivative is $3y^2 - 3 > 0$ for $y > 1$.
At $y = 1$ this denominator is zero, thus for $y > 1$ this denominator is positive.
Thus the skewness of the LogNormal is positive.

For the Weibull Distribution the skewness is:
$\{\Gamma(1+3/\tau) - 3\Gamma(1+1/\tau)\Gamma(1+2/\tau) + 2(\Gamma(1+1/\tau))^3\} / \{\Gamma(1+2/\tau) - (\Gamma(1+1/\tau)^2)\}^{1.5}$.
Note that the skewness depends on the shape parameter τ but not on the scale parameter θ.
Note that for very large tau, the skewness of the Weibull is approximately -1.5 / tau.
For large tau the skewness is negative, but goes to zero as tau goes to infinity.
The Weibull has positive skewness for $\tau < 3.6$ and a negative skewness for $\tau > 3.6$.

Mean Excess Loss (Mean Residual Lives):[51]

Heavy-tailed distributions have mean excess losses (mean residual lives), e(x) that increase to infinity as x approaches infinity. For example, for the Pareto the mean excess loss increases linearly.

Light-tailed distributions have mean excess losses (mean residual lives) that increase slowly or decrease as x approaches infinity. For example, the Exponential Distribution has a constant mean excess loss, $e(x) = \theta$.

Hazard Rate (Force of Mortality):[52]

The hazard rate / force of mortality is defined as:

$$h(x) = f(x) / S(x).$$

If the force of mortality is large, then the chance of being alive at large ages very quickly goes to zero. If the hazard rate (force of mortality) is large, then the density drops off quickly to zero. Thus if the hazard rate is increasing, the tail is light. Conversely, if the hazard rate decreases as x approaches infinity, then the tail is heavy.

The hazard rate for an Exponential Distribution is constant, $h(x) = 1/\theta$.

Relation of the Tail to the Exponential	Hazard Rate	Mean Residual Life
Heavier	Decreasing	Increasing
Lighter	Increasing	Decreasing

$$S(x) = \exp[-\int_t^{\infty} h(t)\ dt\ ].$$

Thus if the hazard rate is increasing, the Survival Function goes to zero more quickly as x approaches infinity.

[51] Mean Excess Loss will be discussed further in a subsequent section.
[52] Hazard rates are discussed further in a subsequent section.

Heavier vs. Lighter Tails:

Heavier or lighter tail is a comparative concept; there are no strict definitions of heavy-tailed and light-tailed:[53]

Heavier Tailed	**Lighter Tailed**
f(x) goes to zero more slowly	**f(x) goes to zero more quickly**
Few Moments exist	**All (positive) moments exist**
Larger Coefficient of Variation	**Smaller Coefficient of Variation**
Higher Skewness	**Lower Skewness**
e(x) Increases to Infinity	**e(x) goes to a constant**
Decreasing Hazard Rate	**Increasing Hazard Rate**

[53] Very heavy tailed distributions may not even have a (finite) coefficient of variation.
Very heavy tailed distributions may not even have a (finite) skewness.
Very light tailed distributions may have a negative skewness.
The faster the mean excess loss increases to infinity the more heavy the tail.
For very light-tailed distributions (such as the Weibull with $\tau > 1$) the mean excess loss may go to zero as x approaches infinity.

Here is a list of loss distributions, arranged in increasing heaviness of the tail:[54] [55] [56]

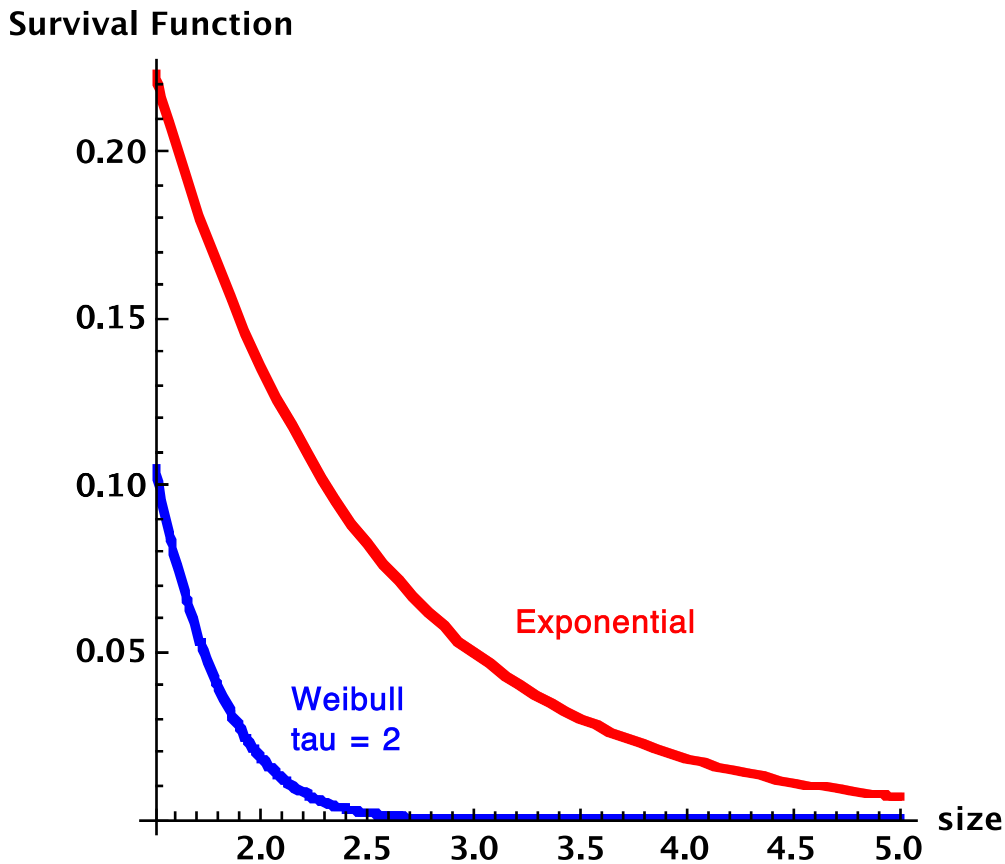
Distribution	Mean Excess Loss (Mean Residual Life)	Do All Positive Moments Exist
Normal	decreases to zero approximately as 1/x	Yes
Weibull for $\tau > 1$	decreases to zero less quickly than 1/x	Yes
Gamma for $\alpha > 1$	decreases to a constant	Yes
Exponential	constant	Yes
Gamma for $\alpha < 1$	increases to a constant	Yes
Inverse Gaussian	increases to a constant	Yes
Weibull for $\tau < 1$	increases to infinity less than linearly	Yes
Trans. Gamma for $\tau < 1$	increases to infinity less than linearly	Yes
LogNormal	increases to infinity just less than linearly	Yes
Inverse Gamma	increases to infinity linearly	No
Pareto	increases to infinity linearly	No
Single Parameter Pareto	increases to infinity linearly	No
Burr	increases to infinity linearly	No
Generalized Pareto	increases to infinity linearly	No
Inverse Trans. Gamma	increases to infinity linearly	No

[54] The Pareto, Single Parameter Pareto, and Inverse Gamma all have tails that are not very different.
The Gamma and Inverse Gaussian have tails that are not very different.
[55] The Gamma Distribution with $\alpha < 1$ is heavier tailed than the Exponential ($\alpha = 1$).
The Gamma Distribution with $\alpha > 1$ is lighter tailed than the Exponential ($\alpha = 1$).
One way to remember which one is heavier than an Exponential, is that as $\alpha \to \infty$, the Gamma Distribution is a sum of many independent identically distributed Exponentials, which approaches a Normal Distribution.
The Normal Distribution is lighter tailed, and therefore so is a Gamma Distribution for $\alpha > 1$.
[56] *While the moments exist for the LogNormal, the Moment Generating Function does not.*

Distributions such as the Pareto are said to have a heavy, thick, or fat (righthand) tail.

Distributions such as the Exponential are said to have a light or thin (righthand) tail.[57]

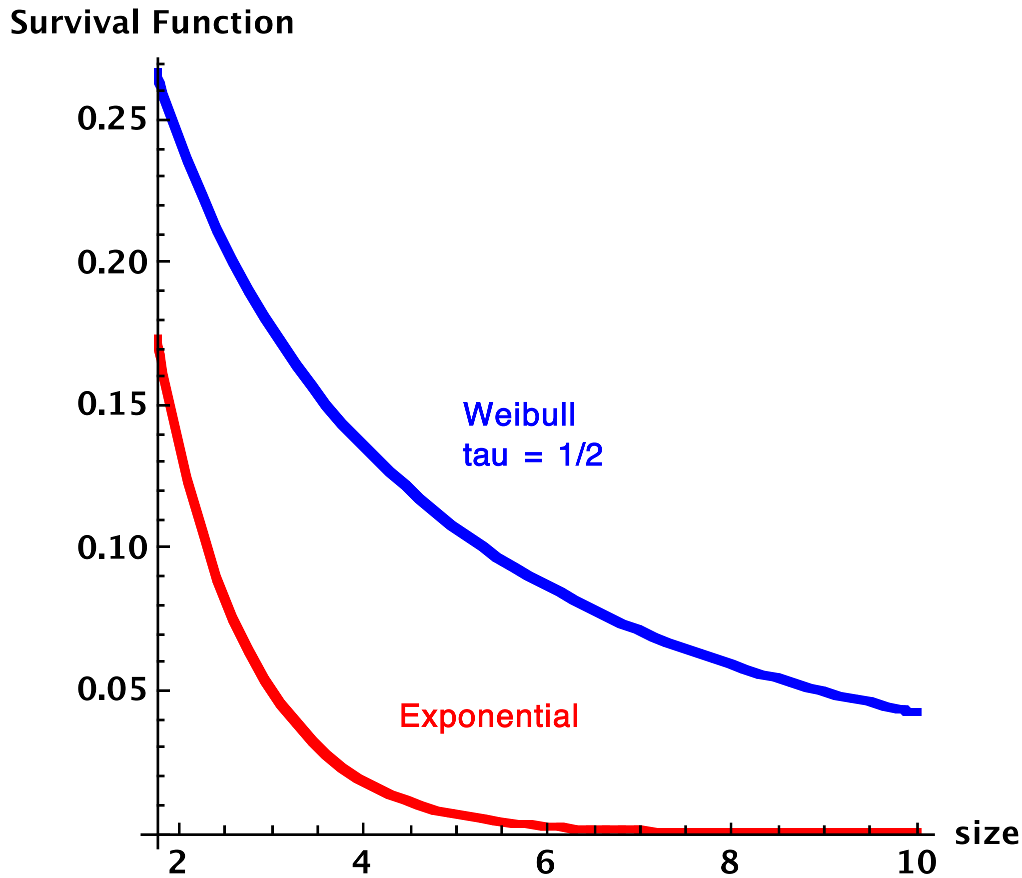
Weibull with $\tau = 2$ and $\theta = 1$, $S(x) = \exp[-x^2]$, as $x \to \infty$, $S(x) \to 0$ more quickly than Exponential.



Weibull with $\tau = 2$ has a lighter righthand tail than an Exponential Distribution.

[57] Distributions used to model human lifetimes, such as Gompertz Law, have lighter tails than the Exponential.

Weibull with $\tau = 1/2$ and $\theta = 1$, $S(x) = \exp[-\sqrt{x}\,]$, as $x \to \infty$,
$S(x) \to 0$ <u>less</u> quickly than Exponential.



Weibull with $\tau = 1/2$ has a <u>heavier</u> righthand tail than an Exponential Distribution.

Comparing Tails:

There is an analytic technique one can use to more precisely compare the tails of distributions. One takes the limit as x approaches infinity of the ratios of the densities.[58]

Exercise: What is the limit as x approaches infinity of the ratio of the density of a Pareto Distribution with parameters α and θ to the density of a Burr Distribution with parameters, α, θ and γ.

[Solution: For the Pareto $f(x) = (\alpha\theta^\alpha)(\theta + x)^{-(\alpha+1)}$.

For the Burr, (using g to distinguish it from the Pareto),

$g(x) = \alpha\gamma(x/\theta)^\gamma\,(1+(x/\theta)^\gamma)^{-(\alpha+1)}/x$.

$\lim_{x \to \infty} f(x) / g(x) = \lim_{x \to \infty} (\alpha\theta^\alpha)(\theta + x)^{-(\alpha+1)} \;/\; \{\alpha\gamma(x/\theta)^\gamma\,(1+(x/\theta)^\gamma)^{-(\alpha+1)}/x\} =$

$\lim_{x \to \infty} \theta^\alpha x^{-(\alpha+1)} / \{(\gamma x^{\gamma-1}/\theta^\gamma)\theta^{\gamma(\alpha+1)}x^{-\gamma(\alpha+1)}\} = \lim_{x \to \infty} \theta^{\alpha-\gamma\alpha} x^{\alpha(\gamma-1)} / \gamma$.

For $\gamma > 1$ the limit is infinity. For $\gamma < 1$ the limit is zero.

For $\gamma = 1$ the limit is one; for $\gamma = 1$, the Burr Distribution is a Pareto.]

Let f(x) and g(x) be the two densities, then if:

$\lim_{x \to \infty} f(x) / g(x) = \infty$, f has a heavier tail than g.

$\lim_{x \to \infty} f(x) / g(x) = 0$, f has a lighter tail than g.

$\lim_{x \to \infty} f(x) / g(x) =$ positive constant, f has a similar tail to g.

Exercise: Compare the tails of Pareto Distribution with parameters α and θ,
and a Burr Distribution with parameters, α, θ and γ.

[Solution: The comparison depends on the γ, the second shape parameter of the Burr.

For $\gamma > 1$, the Pareto has a heavier tail than the Burr.

For $\gamma < 1$, the Pareto has a lighter tail than the Burr.

For $\gamma = 1$, the Burr is equal to the Pareto, thus they have similar, in fact identical tails.]

Note, Loss Models uses the notation $f(x) \sim g(x)$, $x \to \infty$, when $\lim_{x \to \infty} f(x)/g(x) = 1$.

Two distributions have similar tails if $f(x) \sim c\, g(x)$, $x \to \infty$, for some constant $c > 0$.

[58] See Loss Models, Section 3.4.2.

Instead of taking the limit of the ratio of densities, one can equivalently take the limit of the ratios of the survival functions.[59]

Exercise: What is the limit as x approaches infinity of the ratio of the Survival Function of a Pareto Distribution with parameters α and θ to the Survival Function to a Burr Distribution with parameters, α, θ and γ.
[Solution: For the Pareto $S(x) = \theta^{\alpha}(\theta + x)^{-\alpha} = (1 + x/\theta)^{-\alpha}$.
For the Burr, (using T to distinguish it from the Pareto), $T(x) = \{1 + (x/\theta)^{\gamma}\}^{-\alpha}$.
$\lim_{x \to \infty} S(x) / T(x) = \lim_{x \to \infty} \{(1 + (x/\theta)^{\gamma}) / (1 + x/\theta)\}^{\alpha} = \lim_{x \to \infty} \{(x/\theta)^{\gamma-1}\}^{\alpha} = \lim_{x \to \infty} \theta^{\alpha-\gamma\alpha} x^{\alpha(\gamma-1)}$.

For $\gamma > 1$ the limit is infinity. For $\gamma < 1$ the limit is zero. For $\gamma = 1$ the limit is one; for $\gamma = 1$, the Burr Distribution is a Pareto.]

Therefore the comparison of the tails of the Burr and Pareto depends on the value of γ, the second shape parameter of the Burr. For $\gamma > 1$ the Burr has a lighter tail than the Pareto. For $\gamma < 1$ the Burr has a heavier tail than the Pareto. For $\gamma = 1$, the Burr is equal to the Pareto, thus they have similar, in fact identical, tails.

This makes sense, since for $\gamma > 1$, x^{γ} increases more quickly than x. Thus a Burr with $\gamma = 2$ has x^2 in the denominator of its survival function, where the Pareto only has x. Thus the survival function of a Burr with $\gamma = 2$ goes to zero more quickly than the Pareto, indicating it is lighter-tailed than the Pareto. The reverse is true if $\gamma = 1/2$. Then the Burr has $\sqrt{x}$ in the denominator of its survival function, where the Pareto only has x.

These same technique also can be used to compare the tails of distributions from the same family.

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[59] The derivative of the survival function is minus the density. Since as x approaches infinity, S(x) approaches zero, one can apply L'Hospital's Rule. Let the two densities be f and g. Let the two survival functions be S and T. Limit as x approaches infinity of S(x)/T(x) = limit x approaches infinity of S'(x)/T'(x) =
limit x approaches infinity of - f(x)/(- g(x)) = limit x approaches infinity of f(x)/g(x).

Exercise: The first Distribution is a Gamma with parameters α and θ. The second Distribution is a Gamma with parameters a and q. Which distribution has the heavier tail?
[Solution: The density of the first Gamma is: $f_1(x) \sim x^{\alpha-1} \exp(-x/\theta)$.
The density of the second Gamma is: $f_2(x) \sim x^{a-1} \exp(-x/q)$. $f_1(x)/f_2(x) \sim x^{\alpha-a} \exp(x(1/q - 1/\theta))$.
If $1/q - 1/\theta > 0$, then the limit of $f_1(x)/f_2(x)$ as x approaches infinity is ∞.
If $1/q - 1/\theta < 0$, then the limit of $f_1(x)/f_2(x)$ as x approaches infinity is 0.
If $1/q - 1/\theta = 0$, then the limit of $f_1(x)/f_2(x)$ as x approaches infinity is ∞ if $\alpha > a$, and 0 if $a > \alpha$.
Thus we have that: If $\theta > q$, then the first Gamma is heavier-tailed.
If $\theta < q$, then the second Gamma is heavier-tailed.
If $\theta = q$ and $\alpha > a$, then the first Gamma is heavier-tailed.
If $\theta = q$ and $\alpha < a$, then the second Gamma is heavier-tailed.
<u>Comment</u>: Multiplicative constants such as $\Gamma(\alpha)$ or $\theta^{-\alpha}$, which appear in the density, have been ignored since they will not affect whether the limit of the ratio of densities goes to zero or infinity.]

Thus we see that the tails of two Gammas while not very different are not precisely similar.[60] Whichever Gamma has the larger scale parameter is heavier-tailed. If they have the same scale parameter, whichever Gamma has the smaller shape parameter is heavier-tailed.[61]

<u>Inverse Gaussian Distribution vs. Gamma Distribution</u>:

The skewness of the Inverse Gaussian Distribution, $3\sqrt{\mu/\theta}$, is always three times its coefficient of variation, $\sqrt{\mu/\theta}$. In contrast, the Gamma Distribution has it skewness $2/\sqrt{\alpha}$, is always twice times its coefficient of variation, $1/\sqrt{\alpha}$. Thus if a Gamma and Inverse Gaussian have the same mean and variance, then the Inverse Gaussian has a larger skewness; if a Gamma and Inverse Gaussian have the same mean and variance, then the Inverse Gaussian has a heavier tail.

A data set for which a Gamma is a good candidate usually also has an Inverse Gaussian as a good candidate. The fits of the two types of curves differ largely based on the relative magnitude of the skewness of the data set compared to its coefficient of variation. For data sets with less volume, there may be no way statistically to distinguish the fits.

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[60] Using the precise mathematical definitions in <u>Loss Models</u>. Actuaries rarely use this concept to compare the tails of two Gammas. It would be more common to compare a Gamma to let's say a LogNormal. (A LogNormal Distribution has a significantly heavier-tail than a Gamma Distribution.)
[61] If they have the same scale parameters and the same shape parameters, then the two Gammas are identical and have the same tail.

Tails of the Transformed Beta Distribution:

Since many other distributions are special cases of the Transformed Beta Distribution,[62] it is useful to know its tail behavior. The density of the Transformed Beta Distribution is:

$$\frac{\Gamma(\alpha+\tau)}{\Gamma(\tau)\,\Gamma(\alpha)}\,\gamma\theta^{-\gamma\tau}x^{\gamma\tau-1}(1+(x/\theta)^{\gamma})^{-(\alpha+\tau)}.$$

For large x the density acts as $x^{\gamma\tau-1} / x^{\gamma(\alpha+\tau)} = 1/x^{\gamma\alpha+1}$. If we multiply by x^n we get $x^{n-\gamma\alpha-1}$; if we then integrate to infinity, we get a finite answer provided $n - \gamma\alpha - 1 > -1$.
Thus the nth moment exists for $n > \gamma\alpha$.

The larger the product $\gamma\alpha$, the more moments exist and the lighter the (righthand) tail. The α shape parameter is that of a Pareto. The γ shape parameter is the power transform by which the Burr is obtained from the Pareto. Their product, $\gamma\alpha$, determines the (righthand) tail behavior of the Transformed Beta Distribution and its special cases. Provided $\alpha\gamma > 1$, the mean excess loss exists and increases to infinity approximately linearly; for large x, $e(x) \cong x / (\alpha\gamma - 1)$.
This tail behavior carries over to special cases such as: the Burr, Generalized Pareto, Pareto, LogLogistic, ParaLogistic, Inverse Burr, Inverse Pareto, and Inverse ParaLogistic.
All have mean excess losses, when they exist, that increase approximately linearly for large x.[63]

One can examine the behavior of the left hand tail, as x approaches zero, in a similar manner.[64] For small x the density acts as $x^{\gamma\tau-1}$. If we multiply by x^{-n} we get $x^{\gamma\tau-1-n}$; if we then integrate to zero, we get a finite answer provided $\gamma\tau - 1 - n > -1$. Thus the negative n^{th} moment exists for $\gamma\tau > n$. Thus the behavior of the left hand tail is determined by the product of the two shape parameters of the Inverse Burr Distribution.

Thus we see that of the three shape parameters of the Transformed Beta, τ (one more than the power to which x is taken in the Incomplete Beta Function, i.e., the first parameter of the Incomplete Beta Function) affects the left hand tail, α (the shape parameter of the Pareto) affects the righthand tail, and γ (the power transform parameter of the Burr and LogLogistic) affects both tails.

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62 See Figures 5.2 and 5.4 in Loss Models.

63 The mean excess loss of a Pareto, when it exists, is linear in x. $e(x) = (x+\theta)/(\alpha-1)$.

64 Since actuaries are chiefly concerned with the behavior of loss distributions in the righthand tail, as x approaches infinity, assume that unless specified otherwise, "tail behavior" refers to the behavior in the righthand tail as x approaches infinity.

An Example of Distributions fit to Hurricane Data:

Hogg & Klugman in Loss Distributions show the results of fitting different distributions to a set of hurricane data.[65] The hurricane data set is truncated from below at $5 million and consists of 35 storms with total losses adjusted to 1981 levels of more than $5 million.[66] This serves as a good example of the importance of the tails of the distribution to practical applications. Here are the parameters of different distributions fit by Hogg & Klugman via maximum likelihood, as well as their means, coefficients of variation (when they exist) and their skewnesses (when they exist):

Distribution Type	Parameters	Parameters	Parameters	Mean ($ million)	Coefficient of Variation	Skewness
Weibull	θ = 88,588,730	τ = 0.51907		166	2.12	6.11
LogNormal	μ = 17.953	σ = 1.6028		226	3.47	52.26
Pareto	α = 1.1569	θ = 73,674,000		470	N.D.	N.D.
Burr	α = 3.7697	θ = 585,453,983	γ = 0.65994	197	3.75	N.D.
Gen. Pareto	α = 2.8330	θ = 862,660,000	τ = 0.33292	157	2.79	N.D.

It is interesting to compare the tails of the different distributions by comparing the estimated probabilities of a storm greater than $1 billion or $5 billion:

Distribution Type	Probability of storm > 5 million	Probability of storm > 1 billion	Probability of storm > 5 billion	Estimated Annual Frequency of Hurricanes Greater than	
				$1 billion	$5 billion
Weibull	79.86%	2.9637%	0.0300%	4.0591%	0.0410%
LogNormal	94.26%	4.1959%	0.3142%	4.8686%	0.3646%
Pareto	92.68%	4.5069%	0.7475%	5.3185%	0.8821%
Burr	85.28%	3.5529%	0.2122%	4.5567%	0.2722%
Gen. Pareto	43.56%	0.0142%	0.0002%	0.0358%	0.0004%

The lighter-tailed Weibull produces a much lower estimate of the chance of a huge hurricane than a heavier-tailed distribution such as the Pareto. The estimate from the LogNormal, which is heavier-tailed than a Weibull, but lighter-tailed than a Pareto, is somewhere in between.

[65] Loss Distributions was on the syllabus of the old Part 4B exam.
[66] The data is shown in Table 4.1 of Loss Distributions, and Table 14.7 of Loss Models.
See Exercise 14.8 in Loss Models.
In millions of dollars, the trended hurricane sizes are: 6.766, 7.123, 10.562, 14.474, 15.351, 16.983, 18.383, 19.030, 25.304, 29.112, 30.146, 33.727, 40.596, 41.409, 47.905, 49.397, 52.600, 59.917, 63.123, 77.809, 102.942, 103.217, 123.680, 140.136, 192.013, 198.446, 227.338, 329.511, 361.200, 421.680, 513.586, 545.778, 750.389, 863.881, 1638.000.

There were 35 hurricanes greater than 5 million in constant 1981 dollars observed in 32 years. Thus one could estimate the frequency of such hurricanes as 35/32 = 1.09 per year. Then using the curves fit to the data truncated from below one could estimate the frequency of hurricanes greater than size x as: (1.09)S(x) / S(5 million). For example, for the Pareto Distribution the estimated annual frequency of hurricanes greater than 5 billion in 1981 dollars is: (1.09)(.7475%)/(92.68%) = 0.8821%. This is a mean return time of: 1/0.8821% = 113 years. The return times estimated using the other curves are much longer:

Distribution Type	Mean Return Time (Years)	
	of storm > $1 billion	of storm > $5 billion
Weibull	25	2,438
LogNormal	21	274
Pareto	19	113
Burr	22	367
Gen. Pareto	2,796	229,068

It is interesting to note that even the most heavy-tailed of these curves would seem with twenty-twenty hindsight to have underestimated the chance of large hurricanes such as Hurricane Andrew.[67] *The small amount of data does not allow a good estimate of the extreme tail; the observation of just one very large hurricane would have significantly changed the results. Also due to changing or cyclic weather patterns and the increase in homes near the coast, this may just not be an appropriate technique to apply to this particular problem. The preferred technique currently is to simulate possible hurricanes using meteorological data and estimate the likely damage using exposure data on the location and characteristics of insured homes combined with engineering and physics data.*[68]

[67] The losses in Hogg & Klugman are adjusted to 1981 levels. Hurricane Andrew in 8/92 with nearly $16 billion in insured losses, probably exceeded $7 billion dollars in loss in 1981 dollars. It is generally believed that Hurricanes that produce such severe losses have a much shorter average return time than a century.

[68] See for example, "A Formal Approach to Catastrophe Risk Assessment in Management", by Karen M. Clark, PCAS 1986, or "Use of Computer Models to Estimate Loss Costs," by Michael A. Walters and Francois Morin, PCAS 1997.

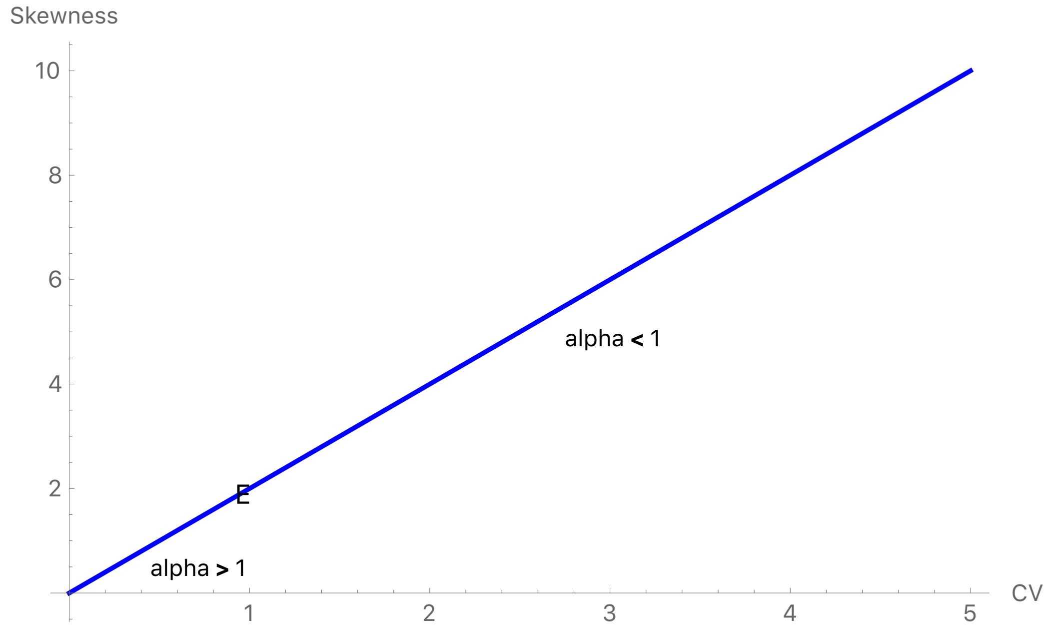
Coefficient of Variation versus Skewness, Two Parameter Distributions:

For the following two parameter distributions: Pareto, LogNormal, Gamma and Weibull, the Coefficient of Variation and Skewness depend on a single shape parameter.
Values are tabulated below:

Shape	Pareto		LogNormal		Gamma		Weibull	
Parameter	C.V.	Skew	C.V.	Skew	C.V.	Skew	C.V.	Skew
0.2	N.A.	N.A.	0.202	0.056	2.236	4.472	15.843	190.1
0.4	N.A.	N.A.	0.417	0.355	1.581	3.162	3.141	11.35
0.6	N.A.	N.A.	0.658	1.207	1.291	2.582	1.758	4.593
0.8	N.A.	N.A.	0.947	3.399	1.118	2.236	1.261	2.815
1	N.A.	N.A.	1.311	9.282	**1.000**	**2.000**	**1.000**	**2.000**
1.2	N.A.	N.A.	1.795	26.840	0.913	1.826	0.837	1.521
1.4	N.A.	N.A.	2.470	87.219	0.845	1.690	0.724	1.198
1.6	N.A.	N.A.	3.455	331	0.791	1.581	0.640	0.962
1.8	N.A.	N.A.	4.953	1503	0.745	1.491	0.575	0.779
2	N.A.	N.A.	7.321	8208	0.707	1.414	0.523	0.631
2.2	3.317	N.A.	11.201	53,948	0.674	1.348	0.480	0.509
2.4	2.449	N.A.	17.786	426,061	0.645	1.291	0.444	0.405
2.6	2.082	N.A.	29.354	4,036,409	0.620	1.240	0.413	0.315
2.8	1.871	N.A.	50.391	4.6E+07	0.598	1.195	0.387	0.237
3	1.732	N.A.	90.012	6.2E+08	0.577	1.155	0.363	0.168
3.2	1.633	25.720	167	1.0E+10	0.559	1.118	0.343	0.106
3.4	1.558	14.117	324	2.0E+11	0.542	1.085	0.325	0.051
3.6	1.500	10.222	652	4.6E+12	0.527	1.054	0.309	0.001
3.8	1.453	8.259	1366	1.3E+14	0.513	1.026	0.294	-0.045
4	1.414	7.071	2981	4.3E+15	0.500	1.000	0.281	-0.087
5	1.291	4.648	268,337	2.7E+24	0.447	0.894	0.229	-0.254
6	1.225	3.810	6.6E+07	1.5E+35	0.408	0.816	0.194	-0.373
7	1.183	3.381	4.4E+10	7.6E+47	0.378	0.756	0.168	-0.463
8	1.155	3.118	7.9E+13	3.5E+62	0.354	0.707	0.148	-0.534
9	1.134	2.940	3.9E+17	1.4E+79	0.333	0.667	0.133	-0.591
10	1.118	2.811	5.2E+21	5.2E+97	0.316	0.632	0.120	-0.638

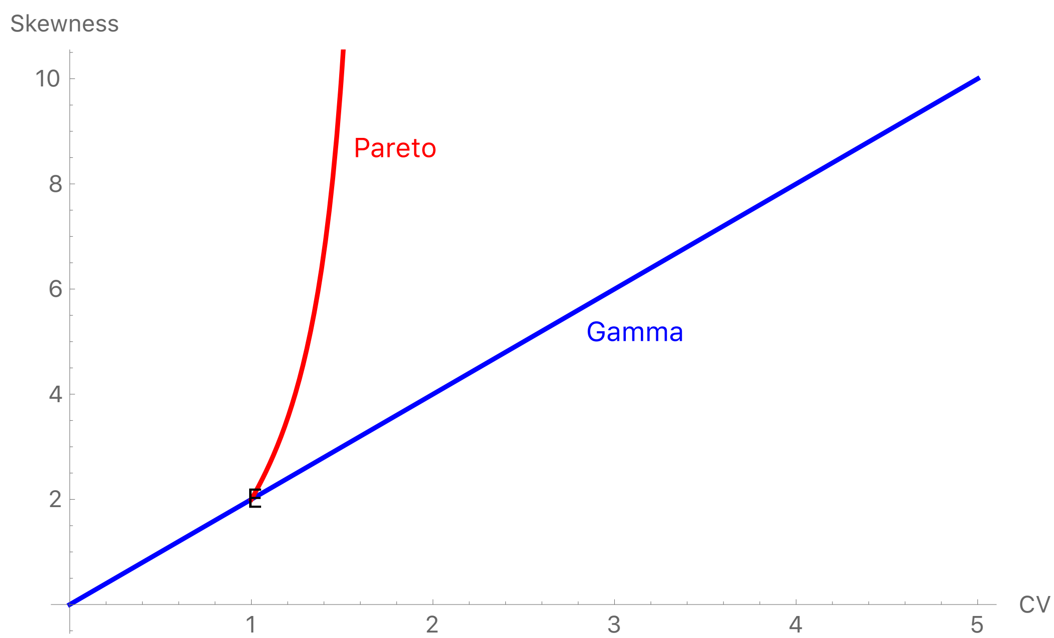
The shape parameters for these distributions are:
Pareto α
LogNormal σ
Gamma α
Weibull τ

As mentioned previously, for the Gamma Distribution the skewness is twice the CV:[69]



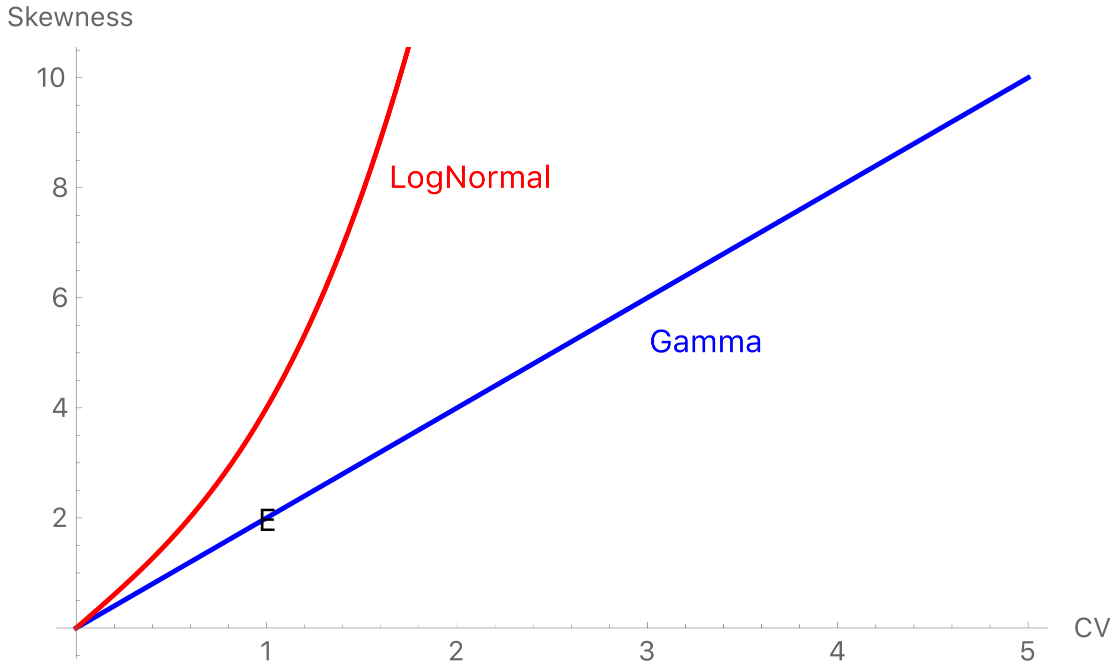
For $\alpha > 1$ the Gamma Distribution is lighter-tailed than an Exponential, and has CV < 1 and skewness < 2. Conversely, for $\alpha < 1$ the Gamma Distribution is heavier-tailed than an Exponential, and has CV > 1 and skewness > 2. The Exponential Distribution ($\alpha = 1$), shown above as E, has CV = 1 and skewness = 2.

For the Pareto Distribution the skewness is more than twice the CV, when they exist.
For the Pareto, the CV > 1 and the skewness > 2:



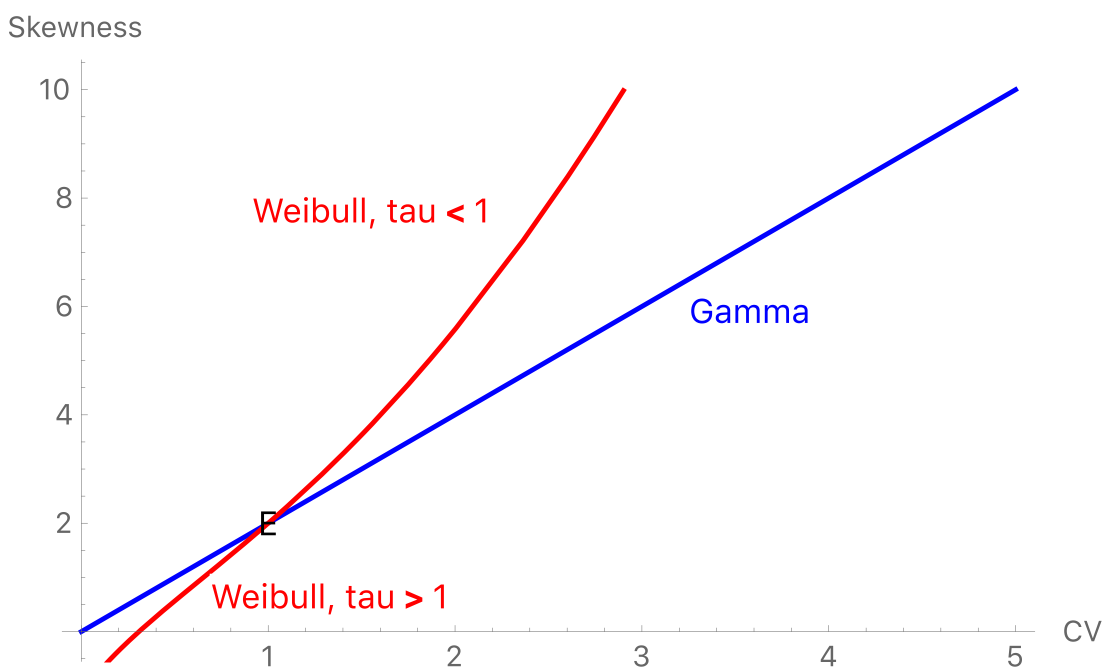
[69] For the Inverse Gaussian, the Skewness is three times the CV.

As a goes to infinity, the Pareto approaches the Exponential which has CV = 1 and skewness = 2. As α approaches 3, the skewness approaches infinity.
Here is a similar graph for the LogNormal Distribution versus the Gamma Distribution:



For the LogNormal, as σ approaches zero, the coefficient of variation and skewness each approach zero. For $\sigma = 1$, CV = 1.311 and the skewness = 9.282.
As σ approaches infinity both the skewness and CV approach infinity.

Finally, here is a similar graph for the Weibull Distribution versus the Gamma Distribution:



For $\tau > 1$ the Weibull Distribution is lighter-tailed than an Exponential, and has CV < 1 and skewness < 2. Conversely, for $\tau < 1$ the Weibull Distribution is heavier-tailed than an Exponential, and has CV > 1 and skewness > 2. The Exponential Distribution ($\tau = 1$), shown above as E, has CV = 1 and skewness = 2.

The CV is positive by definition. The skewness is positive for curves skewed to the right and negative for curves skewed to the left. The Pareto, LogNormal and Gamma all have positive Skewness. The Weibull has positive skewness for tau < 3.60235 and negative skewness for tau > 3.60235.

Existence of Moment Generating Functions:[70]

The moment generating function for a continuous loss distribution is given by:[71]

$$M(t) = \int_0^\infty f(x)\, e^{xt}\, dx = E[e^{xt}].$$

For example for the Gamma Distribution:

$$M(t) = (1 - \theta t)^{-\alpha}, \text{ for } t < 1/\theta.$$

The moments of the function can be obtained as the derivatives of the moment generating function at zero. Thus if the Moment Generating Function exists (within an interval around zero) then so do all the moments. However the converse is not true.

The moment generating function, when it exists, can be written as a power series in t:[72]

$$M(t) = \sum_{n=0}^{n=\infty} E[X^n]\, t^n / n!.$$

In order for the moment generating function to converge (in an interval around zero), the moments $E[X^n]$ may not grow too quickly as n gets large. This is yet another way to distinguish lighter and heavier tailed distributions. Those with Moment Generating Functions are lighter-tailed than those without Moment Generating Functions.

Thus the Weibull for $\tau > 1$, whose m.g.f. exists, is lighter-tailed than the Weibull with $\tau < 1$, whose m.g.f. does not. *The Transformed Gamma has the same behavior as the Weibull; for $\tau > 1$ the Moment Generating Function exists and the distribution is lighter-tailed than $\tau < 1$ for which the Moment Generating Function does not exist. (For $\tau = 1$, one gets a Gamma, for which the Moment Generating Function exists.)*

The LogNormal Distribution has its moments increase rapidly and thus it does not have a Moment Generating Function. The LogNormal is the heaviest-tailed of those distributions which have all their moments.

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[70] See "Mahler's Guide to Aggregate Distributions."
See also Definition 3.8 in Loss Models.
[71] With support from zero to infinity. In general the integral goes over the support of the probability distribution.
[72] This is just the usual Taylor Series, substituting in the moments for the derivatives at zero of the Moment Generating Function.

Problems:

16.1 (2 points) You are given the following information on three (3) size of loss distributions:

Distribution	Coefficient of Variation	Skewness
I	2	3
II	1.22	3.81
III	1	2

Which of these three loss distributions can not be a Gamma Distribution?
A. I B. II C. III D. I, II, III E. None of A,B,C, or D

16.2 (1 points) Which of the following distributions always have positive skewness?
1. Weibull
2. Normal
3. Gamma

A. None of 1, 2, or 3 B. 1 C. 2 D. 3 E. None of A, B, C, or D

16.3 (2 points) Which of the following statements is true?
1. For the Pareto Distribution, the standard deviation (when it exists) is always greater than the mean.
2. For the Pareto Distribution, the skewness (when it exists) is always greater than twice the coefficient of variation.
3. For the LogNormal distribution, f(x) goes to zero more quickly as x approaches infinity than for the Transformed Gamma distribution.

Hint: For the Transformed Gamma distribution, $f(x) = \tau(x/\theta)^{\tau\alpha} \exp[-(x/\theta)^{\tau}] / \{x\ \Gamma(\alpha)\}$.
A. 1, 2 B. 1, 3 C. 2, 3 D. 1, 2, 3 E. None of A, B, C, or D

16.4 (2 points) Rank the tails of the following three distributions, from lightest to heaviest:
1. Weibull with $\tau = 0.5$ and $\theta = 10$.
2. Weibull with $\tau = 1$ and $\theta = 100$.
3. Weibull with $\tau = 2$ and $\theta = 1000$.

A. 1, 2, 3 B. 2,1, 3 C. 1, 3, 2 D. 3, 2, 1 E. None of A, B, C or D

16.5 (3 points) Rank the tails of the following three distributions, from lightest to heaviest:
1. Gamma with $\alpha = 0.7$ and $\theta = 10$.
2. Inverse Gaussian with $\mu = 3$ and $\theta = 4$.
3. Inverse Gaussian with $\mu = 5$ and $\theta = 2$.

A. 1, 2, 3 B. 2, 1, 3 C. 1, 3, 2 D. 3, 2, 1 E. None of A, B, C or D

16.6 (1 point) Rank the tails of the following three distributions, from lightest to heaviest:
1. Exponential
2. Lognormal
3. Single Parameter Pareto

A. 1, 2, 3 B. 2, 1, 3 C. 1, 3, 2 D. 3, 2, 1 E. None of A, B, C or D

16.7 (1 point) The Inverse Exponential Distribution has a righthand tail similar to which of the following distributions?

A. Lognormal B. Pareto $\alpha = 1$ C. Pareto $\alpha = 2$ D. Weibull $\tau < 1$ E. Weibull $\tau > 1$

16.8 (3 points) You are given the following:
- Claim sizes for Risk A follow a Exponential distribution, with mean 400.
- Claim sizes for Risk B follow a Gamma distribution, with parameters $\theta = 200$, $\alpha = 2$.
- r is the ratio of the proportion of Risk B's claims (in number) that exceed d to the proportion of Risk A's claims (in number) that exceed d.

Determine the limit of r as d goes to infinity.

A. 0 B. 1/2 C. 1 D. 2 E. ∞

16.9 (3 points) Compare the righthand tails of the Paralogistic and Inverse Paralogistic distributions.

16.10 (5 points) X follows a Weibull Distribution.
Y^P is the corresponding per payment variable for a deductible, $d > 0$.
Compare the righthand tails of X and Y^P.

16.11 (2 points) $f(x) = \frac{2}{\pi} \frac{1}{1 + x^2}$, $x > 0$.
Compare the righthand tail of the above density to that of an Inverse Pareto.

16.12 (4B, 11/92, Q.2) (1 point) Which of the following are true?
1. The random variable X has a lognormal distribution with parameters μ and σ, if $Y = e^X$ has a normal distribution with mean μ and standard deviation σ.
2. The lognormal and Pareto distributions are positively skewed.
3. The lognormal distribution generally has greater probability in the tail than the Pareto distribution.

A. 1 only B. 2 only C. 1, 3 only D. 2, 3 only E. 1, 2, 3

16.13 (4B, 11/93, Q.21) (1 point) Which of the following statements are true for statistical distributions?
1. Linear combinations of independent normal random variables are also normal.
2. The lognormal distribution is often useful as a model for claim size distribution because it is positively skewed.
3. The Pareto probability density function tapers away to zero much more slowly than the lognormal probability density function.

A. 1 B. 1, 2 C. 1, 3 D. 2, 3 E. 1, 2, 3

16.14 (4B, 11/99, Q.19) (2 points) You are given the following:
- Claim sizes for Risk A follow a Pareto distribution, with parameters $\theta = 10,000$ and $\alpha = 2$.
- Claim sizes for Risk B follow a Burr distribution, $F(x) = 1 - \{1/(1+(x/\theta)^\gamma)\}^\alpha$, with parameters $\theta = 141.42$, $\alpha = 2$, and $\gamma = 2$.
- r is the ratio of the proportion of Risk A's claims (in number) that exceed d to the proportion of Risk B's claims (in number) that exceed d.

Determine the limit of r as d goes to infinity.

A. 0 B. 1 C. 2 D. 4 E. ∞

16.15 (CAS3, 11/03, Q.16) (2.5 points)
Which of the following is/are true, based on the existence of moments test?
I. The Loglogistic Distribution has a heavier tail than the Gamma Distribution.
II. The Paralogistic Distribution has a heavier tail than the Lognormal Distribution.
III. The Inverse Exponential has a heavier tail than the Exponential Distribution.

A. I only
B. I and II only
C. I and III only
D. II and III only
E. I, II, and III

Solutions to Problems:

16.1. E. Distributions I and II can't be a Gamma, for which
the skewness = twice coefficient of variation.

16.2. D. The Normal is symmetric, so it has skewness of zero. The Gamma has skewness of $\alpha^{0.5} > 0$. The Weibull has either a positive or negative skewness, depending on the value of τ.

16.3. A. 1. True. This is the same as saying the CV > 1 for the Pareto. 2. True.
3. False. For the LogNormal, $\ln f(x) = -0.5 (\{\ln(x) - \mu\}/\sigma)^2 - \ln(x) - \ln(\sigma) - \ln(2\pi)/2$.
For very large x this is approximately: $-0.5 \ln(x)^2/\sigma^2$.
For the Transformed Gamma, $\ln f(x) = \ln(\tau) + (\tau\alpha - 1)\ln(x) - \tau\alpha\ln(\theta) - (x/\theta)^\tau - \ln\Gamma(\alpha)$.
For very large x this is approximately: $-x^\tau / \theta^\tau$.
Thus the log density of the LogNormal goes to minus infinity more slowly than that of the Transformed Gamma. Therefore the density function of the LogNormal goes to zero less quickly as x approaches infinity than that of the Transformed Gamma.
The LogNormal has a heavier tail than the Transformed Gamma.

16.4. D. The three survival functions are: $S_1(x) = \exp[-(x/10)^{0.5}]$, $S_2(x) = \exp(-x/100)$, $S_3(x) = \exp[-(x/1000)^2]$. $S_1(x)/S_2(x) = \exp[x/100 - \sqrt{x}/\sqrt{10}]$. The limit as x approaches infinity of $S_1(x)/S_2(x)$ is ∞, since x increases more quickly than $\sqrt{x}$. Thus the first Weibull is heavier-tailed than the second. Similarly, the limit as x approaches infinity of $S_2(x)/S_3(x)$ is ∞, since x increases more quickly than x^2. Thus the second Weibull is heavier-tailed than the third.
Alternately, just calculate the densities or log densities for an extremely large value of x, for example 1 billion = 10^9. (The log densities are more convenient to work with; the ordering of the densities and the log densities are the same.)
For the Weibull, $f(x) = \tau(x/\theta)^\tau \exp(-(x/\theta)^\tau) /x$. $\ln f(x) = \ln(\tau) + \tau\ln(x/\theta) - \ln(x) - (x/\theta)^\tau$.
For the first Weibull, ln f(1 billion) = ln(0.5) + (0.5)ln(100 million) - ln(1 billion) - $\sqrt{100 \text{ million}}$ ≅ -10,000. For the second Weibull, ln f(1 billion) = ln(1) + (1)ln(10 million) - ln(1 billion) - 10 million ≅ -10,000,000.
For the third Weibull, ln f(1 billion) = ln(2) + (2)ln(1 million) - ln(1 billion) - (1 million)^2 ≅ -1,000,000,000,000. Thus f(1 billion) is much larger for the first Weibull than second Weibull, while f(1 billion) is much larger for the second Weibull than third Weibull. Thus the third Weibull has the lightest tail, while the first Weibull has the heaviest tail.
Comment: For the Weibull, the smaller the shape parameter τ, the heavier the tail. The values of the scale parameter θ, have no effect on the heaviness of the tail. However by changing the scale, the third Weibull with $\theta = 1000$ does take longer before its density falls below the others than if it instead had $\theta = 1$. The (right) tail behavior refers to the behavior as x approaches infinity, thus how long it takes the density to get smaller does not affect which has a lighter tail.
While the third Weibull might be lighter-tailed, for some practical applications with a maximum covered loss you may be uninterested in the large values of x at which its density is smaller than the others.

16.5. B. The three densities functions are:
$f_1(x) = 10^{-0.7}\ x^{-0.3} \exp(-x/10)/\ \Gamma(0.7)$, $f_2(x) = \sqrt{4/(2\pi)}\ x^{-1.5} \exp(-4(x/3 - 1)^2/\ (2x)) =$
$\sqrt{2/\pi}\ x^{-1.5} \exp(-2x/9 + 4/3 - 2/x)$. $f_3(x) = \sqrt{1/\pi}\ x^{-1.5} \exp(-x/25 + 2/5 - 1/x)$.
We will take the limit as x approaches infinity of the ratios of these densities, ignoring any annoying multiplicative constants such as $10^{-0.7}/\ \Gamma(0.7)$ or $\sqrt{2/\pi}\ e^{4/3}$.
$f_1(x)/f_2(x) \sim x^{-0.3} \exp(-x/10)\ /\ x^{-1.5} \exp(-2x/9 - 2/x) = x^{1.2} \exp(0.122x + 2/x)$.
The limit as x approaches infinity of $f_1(x)/f_2(x)$ is ∞.
Thus the Gamma is heavier-tailed than the first Inverse Gaussian with $\mu = 3$ and $\theta = 4$.
$f_1(x)/f_3(x) \sim x^{-0.3} \exp(-x/10)\ /\ x^{-1.5} \exp(-x/25 - 1/x) = x^{1.2} \exp(-0.06x + 1/x)$.
The limit as x approaches infinity of $f_1(x)/f_3(x)$ is 0, since exp(-0.06x) goes to zero very quickly.
Thus the Gamma is lighter-tailed than the second Inverse Gaussian with $\mu = 5$ and $\theta = 2$.
Thus the second Inverse Gaussian has the heaviest tail, followed by the Gamma, followed by the first Inverse Gaussian.
Comment: In general the Inverse Gaussian and the Gamma have somewhat similar tails; they both have their mean residual lives go to a constant as x approaches infinity.
Which is heavier-tailed depends on the particular parameters of the distributions.
Let's assume we have a Gamma with shape parameter α and scale parameter β (using beta rather than using theta which is also a parameter of the Inverse Gaussian,) and Inverse Gaussian with parameters μ and θ.
Then the density of the Gamma $f_1(x) \sim x^{\alpha-1} \exp(-x/\beta)$.
Then the density of the Inverse Gaussian $f_2(x) \sim x^{-1.5} \exp[-x\theta/(2\mu^2) - \theta/(2x)]$.
$f_1(x)/f_2(x) \sim x^{\alpha+0.5} \exp[x(\theta/(2\mu^2) - 1/\beta) + \theta/\ 2x]$.
If $\theta/(2\mu^2) > 1/\beta$, then the limit as x approaches infinity of $f_1(x)/f_2(x)$ is ∞, and the Gamma is heavier-tailed than the Inverse Gaussian.
If $\theta/(2\mu^2) < 1/\beta$, then the limit as x approaches infinity of $f_1(x)/f_2(x)$ is 0, and the Gamma is lighter-tailed than the Inverse Gaussian.
If $\theta/\ (2\mu^2) = 1/\beta$, then $f_1(x)/f_2(x) \sim x^{\alpha+0.5} \exp[\theta/(2x)]$, and the limit as x approaches infinity of $f_1(x)/f_2(x)$ is ∞, and the Gamma is heavier-tailed than the Inverse Gaussian.

16.6. A. The Single Parameter Pareto does not have all of its moments, and thus is heavier tailed than the other two. The Lognormal has an increasing mean excess loss, while that for the Exponential is constant. Thus the Lognormal is heavier tailed than the Exponential.
Comment: The Single Parameter Pareto has a tail similar to that of the Pareto.

16.7. B. The Inverse Exponential does not have a mean and neither does the Pareto for $\alpha = 1$. More specifically, the density of the Inverse Exponential is: $\theta e^{-\theta/x}/x^2$ which is approximately θ/x^2 for large x, while the density of the Pareto for $\alpha = 1$ is: $\theta/(x+\theta)^2$ which is also approximately θ/x^2 for large x.
Comment: The Inverse Gamma Distribution has a similar to tail to the Pareto Distribution for the same shape parameter α. The Inverse Exponential is the Inverse Gamma for $\alpha = 1$.

16.8. A. $S_A(d) = e^{-d/400}$. $f_B(x) = xe^{-x/200}/40{,}000$.

$$S_B(d) = \int_d^\infty x\, e^{-x/200}/40{,}000\ dx = (1/40{,}000)(-200xe^{-x/200} - 40{,}000e^{-x/200}\ \Big]_{x=d}^{x=\infty}$$

$= (1 + d/200)e^{-d/200}$.
$r = S_B(d) / S_A(d) = (1 + d/200)/ e^{0.0025d}$. As d goes to infinity the denominator increases faster than the numerator; thus as d goes to infinity, r goes to **zero**.
Comment: Similar to 4B, 11/99, Q.19.

16.9. For the Paralogistic: $f(x) = \dfrac{\alpha^2 x^{\alpha-1}}{\theta^\alpha\{1 + (x/\theta)^\alpha\}^{\alpha+1}}$.

For large x, this is approximately proportional to: $\dfrac{x^{\alpha-1}}{(x^\alpha)^{\alpha+1}} = \dfrac{1}{x^{\alpha^2+1}}$.

For the Inverse Paralogistic: $f(x) = \dfrac{\tau^2 (x/\theta)^{\tau^2}}{x\{1 + (x/\theta)^\tau\}^{\tau+1}}$.

For large x, this is approximately proportional to: $\dfrac{x^{\tau^2}}{x\,(x^\tau)^{\tau+1}} = \dfrac{1}{x^{\tau+1}}$.

Thus for $\alpha^2 + 1 = \tau + 1$, in other words for $\tau = \alpha^2$, the two righthand tails are similar.
For $\tau < \alpha^2$, Inverse Paralogistic has a heavier righthand tail than the Paralogistic.
For $\tau > \alpha^2$, Inverse Paralogistic has a lighter righthand tail than the Paralogistic.

16.10. For the Weibull, $S(x) = \exp[-(x/\theta)^\tau]$.
The survival function of Y^P is: $S(y+d) / S(d) = \exp[-(\{y+d\}/\theta)^\tau] / \exp[-(d/\theta)^\tau]$.
Thus the ratio of the survival function of Y^P at x and that of the Weibull is:
$\exp[-(\{x+d\}/\theta)^\tau + (x/\theta)^\tau + (d/\theta)^\tau] = \exp[\{x^\tau + d^\tau - (x+d)^\tau\} / \theta^\tau]$.
The behavior of this ratio as x approaches infinity depends on that of: $x^\tau + d^\tau - (x+d)^\tau$.
For $\tau = 1$, $x^\tau + d^\tau - (x+d)^\tau = 0$, $\exp[\{x^\tau + d^\tau - (x+d)^\tau\} / \theta^\tau] = 1$, and the two tails are equal.
For example, for $\tau = 2$, $x^\tau + d^\tau - (x+d)^\tau = -2dx$, which approaches minus infinity as x approaches infinity. In general, for $\tau > 1$, $x^\tau + d^\tau - (x+d)^\tau$ approaches minus infinity,
$\exp[\{x^\tau + d^\tau - (x+d)^\tau\} / \theta^\tau]$ approaches zero,
and thus Y^P has a lighter righthand tail than the Weibull.
For example, for $\tau = 1/2$, $x^\tau + d^\tau - (x+d)^\tau = x^{1/2} + d^{1/2} - (x+d)^{1/2} = x^{1/2} + d^{1/2} - x^{1/2} (1 + d/x)^{1/2}$
$\cong x^{1/2} + d^{1/2} - x^{1/2} \{1 + (1/2)(d/x)\} = d^{1/2} + (1/2)(d/x^{1/2})$.
This approaches a constant $d^{1/2}$ as x approaches infinity.
In general, for $\tau < 1$, $x^\tau + d^\tau - (x+d)^\tau$ approaches a constant d^τ,
$\exp[\{x^\tau + d^\tau - (x+d)^\tau\} / \theta^\tau]$ approaches a positive constant,
and thus Y^P has a similar righthand tail to the Weibull.
Comment: One could instead work with the ratio of densities rather than survival functions.
For $\tau = 1$ we have an Exponential, and Y^P follows the same Exponential as X.

16.11. For large x, the given density is approximately proportional to x^{-2}.
For the Inverse Pareto: $f(x) = \frac{\tau\theta x^{\tau-1}}{(x+\theta)^{\tau+1}}$.

For large x, this is approximately proportional to: $\frac{x^{\tau-1}}{x^{\tau+1}} = x^{-2}$.

Thus the two righthand tails are similar.
Comment: The given density is the one-sided standard Cauchy density, not on the syllabus.

16.12. B. 1. False. Ln(X) has a Normal distribution if X has a LogNormal distribution.
2. True. The skewness of the Pareto does not exist for $\alpha \le 3$.
For $\alpha > 3$, the Pareto skewness is: $2\{(\alpha+1)/(\alpha-3)\}\sqrt{(\alpha-2)/\alpha} > 0$.

LogNormal Skewness = $(\exp[3\sigma^2] - 3\exp[\sigma^2] + 2) / (\exp[\sigma^2] - 1)^{1.5}$. The denominator is positive, since $\exp(\sigma^2) > 1$ for $\sigma^2 > 0$. The numerator is positive since it can be written as:
$y^3 - 3y + 2$, for $y = \exp(\sigma^2) > 1$. (The derivative of $y^3 - 3y + 2$ is $3y^2 - 3$, which is positive for $y > 1$. At $y = 1$, $y^3 - 3y + 2$ is zero, thus for $y > 1$ it is positive.)
Since the numerator and denominator are both positive, so is the skewness.
3. False. The Pareto is heavier-tailed than the Lognormal distribution. This can be seen by a comparison of the mean residual lives. That of the lognormal increases less than linearly, while the mean residual life of the Pareto increases linearly. Another way to see this is that all of the moments of the LogNormal distribution exist, while higher moments of the Pareto distribution do not exist.
Comment: The LogNormal and the Pareto distributions are both heavy-tailed and heavy-tailed distributions have positive skewness, (are skewed to the right.) These are statements that practicing actuaries should know.

16.13. E. 1. True. 2. True. 3. True.
Comment: Statement 3 is another way of saying that the Pareto has a heavier tail than the LogNormal.

16.14. E. $S_A(d) = \{10{,}000/(10{,}000+d)\}^2$. $S_B(d) = \{1/(1+(d/141.42)^2)\}^2 = (20{,}000/(20{,}000+d^2))^2$.

$r = S_A(d) / S_B(d) = \left(\frac{20{,}000+d^2}{2(10{,}000+d)}\right)^2$. As d goes to infinity the numerator increases faster than the denominator; thus as d goes to infinity, r goes to **infinity**.

Comment: For $\gamma > 1$, the Burr Distribution has a lighter tail than the Pareto Distribution, while for $\gamma < 1$, the Burr Distribution would have a heavier tail than the Pareto Distribution with the same α.

16.15. E. I. The Loglogistic does not have all its moments, while the Gamma does.
$\Rightarrow$ The Loglogistic Distribution has a heavier tail than the Gamma Distribution.
II. The Paralogistic does not have all its moments, while the Lognormal does.
$\Rightarrow$ The Paralogistic Distribution has a heavier tail than the Lognormal Distribution.
III. The Inverse Exponential does not have all its moments, while the Exponential does.
$\Rightarrow$ The Inverse Exponential Distribution has a heavier tail than the Exponential Distribution.

Section 17, Limited Expected Values

As discussed previously, the Limited Expected Value $E[X \wedge x]$ is the average size of loss with all losses limited to a given maximum size x. Thus the Limited Expected Value, $E[X \wedge x]$, is the mean of the data censored from above at x.

The Limited Expected Value is closely related to other important quantities: the Loss Elimination Ratio, the Mean Excess Loss, and the Excess (Pure Premium) Ratio. *The Limited Expected Value can be used to price Increased Limit Factors. The ratio of losses expected for an increased limit L, compared to a basic limit B, is $E[X \wedge L] / E[X \wedge B]$.*

The Limited Expected Value is generally the sum of two pieces. Each loss of size less than or equal to u contributes its own size, while each loss greater than u contributes just u to the average.

For a discrete distribution:

$$E[X \wedge u] = \sum_{x_i \le u} x_i \text{ Prob}[X = x_i] + u \text{ Prob}[X > u].$$

For a continuous distribution:

$$E[X \wedge u] = \int_0^u t\, f(t)\, dt + u\, S(u).$$

Rather than calculating this integral, make use of Appendix A in the tables attached to the exam, which has formulas for the limited expected value for each distribution.[73]
For example, the formula for the Limited Expected Value of the Pareto is:[74]

$$E[X \wedge x] = \frac{\theta}{\alpha - 1} \left\{1 - \left(\frac{\theta}{\theta + x}\right)^{\alpha - 1}\right\}, \alpha \neq 1.$$

Exercise: For a Pareto with $\alpha = 4$ and $\theta = 1000$, compute $E[X]$, $E[X \wedge 500]$ and $E[X \wedge 5000]$.

[Solution: $E[X] = \frac{\theta}{\alpha - 1} = 333.33$. $E[X \wedge x] = \frac{1000}{4-1} \left\{1 - \left(\frac{1000}{1000 + x}\right)^{4-1}\right\}$.
$E[X \wedge 500] = 234.57$. $E[X \wedge 5000] = 331.79$.]

[73] In some cases the formula for the Limited Expected Value (Limited Expected First Moment) is not given. In those cases, one takes $k = 1$, in the formula for the Limited Expected Moments.
[74] For $\alpha = 1$, $E[X \wedge x] = -\theta \ln(\theta/(\theta+x))$.

Here are the formulas for the limited expected value for some distributions:

Distribution	Limited Expected Value, $E[X \wedge x]$
Exponential	$\theta\ (1 - e^{-x/\theta})$
Pareto	$\frac{\theta}{\alpha - 1} \left\{1 - \left(\frac{\theta}{\theta + x}\right)^{\alpha - 1}\right\}$, $\alpha \neq 1$
LogNormal	$\exp(\mu + \sigma^2/2)\ \Phi\left[\frac{\ln(x) - \mu - \sigma^2}{\sigma}\right] + x\ \{1 - \Phi\left[\frac{\ln(x) - \mu}{\sigma}\right]\}$
Gamma	$\alpha\ \theta \Gamma[\alpha+1\ ;\ x/\theta] + x\ \{1 - \Gamma[\alpha ; x/\theta]\}$
Weibull	$\theta\ \Gamma(1 + 1/\tau)\ \Gamma[1 + 1/\tau\ ;\ (x/\theta)^\tau] + x\ \exp[-(x/\theta)^\tau]$
Single Parameter Pareto	$\frac{\alpha\ \theta}{\alpha - 1} - \frac{\theta^\alpha}{(\alpha - 1)\ x^{\alpha - 1}} = \theta\ \frac{\alpha - \left(\frac{\theta}{x}\right)^{\alpha - 1}}{\alpha - 1}$, $x \geq \theta$

Exercise: For a LogNormal Distribution with μ = 9.28 and σ = 0.916, determine $E[X \wedge 25{,}000]$.
[Solution: $E[X \wedge x] = \exp(\mu + \sigma^2/2)\ \Phi[(\ln x - \mu - \sigma^2)/\sigma] + x\ \{1 - \Phi[(\ln x - \mu)/\sigma]\}$.
$E[X \wedge 25{,}000]$ =
$\exp(9.6995)\Phi[0.01] + 25{,}000\ \{1 - \Phi[0.92]\}$ = (16,310)(0.5040) + (25,000)(1 - 0.8212) = 12,705.]

<u>Relationship to the LER, Excess Ratio, and Mean Excess Loss</u>:

The following relationships hold between the mean, the Limited Expected Value $E[X \wedge x]$, the Excess Ratio R(x), the Mean Excess Loss e(x), and the Loss Elimination Ratio LER(x):

mean = $E[X \wedge$ infinity].

e(x) = { mean - $E[X \wedge x]$ } / S(x).

R(x) = 1 - { $E[X \wedge x]$ / mean } = 1 - LER(x).

R(x) = e(x) S(x) / mean.

LER(x) = $E[X \wedge x]$ / mean.

Layer Average Severity:

The Limited Expected Value can be useful when dealing with layers of loss. For example, suppose we are estimating the expected value (per loss) of the layer of loss greater than \$1 million but less than \$5 million.[75] Note here we are taking the average over all losses, including those that are too small to contribute to the layer. This **Layer Average Severity** is equal to the Expected Value Limited to \$5 million minus the Expected Value Limited to \$1 million.

Layer Average Severity = E[X∧ top of Layer] - E[X∧ bottom of layer].

The Layer Average Severity is the insurer's average payment per loss to an insured, when there is a deductible of size equal to bottom of the layer and a maximum covered loss equal to the top of the layer.
expected payment per <u>loss</u> = average amount paid per <u>loss</u> =
E[X∧ Maximum Covered Loss] - E[X∧ Deductible Amount].

Exercise: Losses follow a Pareto with $\alpha = 4$ and $\theta = 1000$. There is a deductible of 500 and a maximum covered loss of 5000. What is the insurer's average payment per loss?
[Solution: From previous solutions: E[X∧500] = 234.57. E[X∧5000] = 331.79.
The Layer Average Severity = E[X∧5000] - E[X∧500] = 331.79 - 234.59 = 97.20.]

Each small loss, $x \le d$, contributes nothing to a layer from d to u.
Each medium size loss, $d < x \le u$, contributes x - d to a layer from d to u.
Each large loss, $u < x$, contributes u - d to a layer from d to u.

Therefore, Layer Average Severity = $\int_d^u (t - d)\, f(t)\, dt + S(u)\,(u - d)$.

Average Non-zero Payment:

Besides the average amount paid per loss to the insured, one can also calculate the average amount paid per non-zero payment by the insurer. Loss Models refers to this as the expected payment per payment.[76]

With a deductible, there are many instances where the insured suffers a small loss, but the insurer makes no payment. Therefore, if the denominator only includes those situations where the insurer makes a non-zero payment, the average will be bigger. The average payment per payment is greater than or equal to the average payment per loss.

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[75] *This might be useful for pricing a reinsurance contract.*
[76] See Section 8.2 of Loss Models.

Exercise: Losses follow a Pareto with $\alpha = 4$ and $\theta = 1000$.
There is a deductible of 500, and a maximum covered loss of 5000.
What is the average payment per non-zero payment by the insurer?
[Solution: From the previous solution the average payment per loss to the insured is 97.20.
However, the insurer only makes a payment S(500) = 19.75% of the time the insured has a loss. Thus the average per non-zero payment by the insurer is: 97.20 / 0.1975 = 492.08.]

expected payment per <u>payment</u> = $\dfrac{E[X \wedge \text{Maximum Covered Loss}] - E[X \wedge \text{Deductible}]}{S(\text{Deductible})}$

<u>Coinsurance</u>:

Sometimes, the insurer will only pay a percentage of the amount it would otherwise pay.[77]
As discussed previously, this is referred to as a coinsurance clause. For example with a 90% coinsurance factor, after the application of any maximum covered loss and/or deductible, the insurer would only pay 90% of what it would pay in the absence of the coinsurance clause.

Exercise: Losses follow a Pareto with $\alpha = 4$ and $\theta = 1000$. There is a deductible of 500, a maximum covered loss of 5000, and a coinsurance factor of 80%. What is the insurer's average payment per loss to the insured?
[Solution: From a previous solution in the absence of the coinsurance factor, the average payment is 97.20.
With the coinsurance clause each payment is multiplied by 0.8, so the average is:
(0.8)(97.20) = 77.76.]

In general each payment is multiplied by the coinsurance factor, thus so is the average. This is just a special case of multiplying a variable by a constant. The n^{th} moment is multiplied by the constant to the n^{th} power. The variance is therefore multiplied by the square of the coinsurance factor.

Exercise: Losses follow a Pareto with $\alpha = 4$ and $\theta = 1000$.
There is a deductible of 500, a maximum covered loss of 5000, and a coinsurance factor of 80%. What is the insurer's average payment per non-zero payment by the insurer?
[Solution: From a previous solution the average payment per loss to the insured is 77.76.
However, the insurer only makes a payment S(500) = 19.75% of the time the insured has a loss.
Thus the average per non-zero payment by the insurer is: 77.76 / 0.1975 = 393.66.]

[77] For example, *coinsurance clauses are sometimes used in Health Insurance, Homeowners Insurance, or Reinsurance.*

Formulas for Average Payments:[78]

Given **Deductible Amount d, Maximum Covered Loss u, and coinsurance factor c, then the average payment per (non-zero) payment by the insurer is:**

$$\mathbf{c}\ \frac{E[X \wedge u] - E[X \wedge d]}{S(d)} = c\ e(d).$$

Given **Deductible Amount d, Maximum Covered Loss u, and coinsurance factor c, then the insurer's average payment per loss to the insured is: c (E[X∧u] - E[X∧d])**.

The insurer's average payment per loss to the insured is the Layer of Loss between the Deductible Amount d to the Maximum Covered Loss u, E[X∧u] - E[X∧d], all multiplied by the coinsurance factor. The average per non-zero payment by the insurer is the insurer's average payment per loss to the insured divided by the ratio of the number of non-zero payments by the insurer to the number of losses by the insured, S(d).

$E[(X-d)_+]$:

The expected losses excess of d are: $\mathbf{E[(X-d)_+] = E[X] - E[X \wedge d]}$.[79]

Exercise: For a LogNormal Distribution with $\mu = 9.28$ and $\sigma = 0.916$, determine $E[(X - 25,000)_+]$.
[Solution: From a previous exercise: $E[X \wedge 25,000] = 12,705$.
$E[X] = \exp(9.28 + 0.916^2/2) = 16,310$. $E[(X - 25,000)_+] = 16,310 - 12,705 = 3605$.]

In general, for a LogNormal Distribution:

$$E[(X - x)_+] = E[X] - E[X \wedge x] =$$

$$\exp(\mu + \sigma^2/2) - \{\exp(\mu + \sigma^2/2)\ \Phi\left[\frac{\ln(x) - \mu - \sigma^2}{\sigma}\right] + x\ \{1 - \Phi\left[\frac{\ln(x) - \mu}{\sigma}\right]\}\} =$$

$$\exp(\mu + \sigma^2/2)\ \{1 - \Phi\left[\frac{\ln(x) - \mu - \sigma^2}{\sigma}\right]\} - x\ \{1 - \Phi\left[\frac{\ln(x) - \mu}{\sigma}\right]\} =$$

$$\exp(\mu + \sigma^2/2)\ \Phi[d_1] - x\ \Phi[d_2], \text{ where } d_1 = \frac{\mu + \sigma^2 - \ln(x)}{\sigma} \text{ and } d_2 = \frac{\mu - \ln(x)}{\sigma} = d_1 - \sigma.$$ [80]

[78] See Theorem 8.7 in Loss Models.
More general formulas that include the effects of inflation will be discussed in a subsequent section.
[79] This is a special case of the formula for the average payment per loss, with no maximum covered loss (u = ∞) and no coinsurance (c = 1)
[80] This is closely related to the Black-Scholes formula for pricing options, not on the syllabus of this exam.

Limited Expected Value as an Integral of the Survival Function:

The Limited Expected Value can be written as an integral of the Survival Function,
S(x) = 1 - F(x).

$$E[X \wedge x] = \int_0^x t\, f(t)\, dt \;+ x\, S(x).$$

Using integration by parts and the fact that the integral of f(x) is -S(x):[81]

$$E[X \wedge x] = \left\{- S(x)x + \int_0^x S(t)\, dt \right\} + x\, S(x) = \int_0^x S(t)\, dt\,.$$

Thus the Limited Expected Value can be written as an integral of the Survival Function from 0 to the limit, for a distribution with support starting at zero.[82]

$$E[X \wedge x] = \int_0^x S(t)\, dt\,.$$

Since the mean is E[X∧∞], it follows that the **mean can be written as an integral of the Survival Function from 0 to the infinity, for a distribution with support starting at zero**.[83]

$$E[X] = \int_0^\infty S(t)\, dt\,.$$

The losses in the layer from a to b is given as a difference of Limited Expected Values:

$$E[X \wedge b] - E[X \wedge a] = \int_a^b S(t) dt\,.$$

Thus the Losses in a Layer can be written as an integral of the Survival Function from the bottom of the Layer to the top of the Layer.

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[81] Note that the derivative of S(x) is dS(x) /dx = d(1-F(x)) / dx = - f(x). Thus an indefinite integral of f(x) is -S(x) = F(x) - 1. (There is always an arbitrary constant in an indefinite integral.)
[82] Thus this formula does not apply to the Single Parameter Pareto Distribution. For the Single Parameter Pareto Distribution with support starting at θ, E[X∧x] = θ + integral from θ to x of S(t). More generally, E[X∧x] is the sum of the integral from -∞ to 0 of -F(t) and the integral from 0 to x of S(t).
[83] Do not apply this formula to a Single Parameter Pareto Distribution. For a continuous distribution with support on (a, b), the mean is: a + the integral from a to b of S(x). For the Single Parameter Pareto Distribution with support (θ, ∞), E[X] = θ + integral from θ to ∞ of S(x).

Expected Amount by Which Aggregate Claims are Less than a Given Value:

The amount by which x is less than b is defined as $(b - X)_+$: 0 if x > b and b - x if x ≤ b.

For example, $(10 - 2)_+ = 8$, while $(10 - 15)_+ = 0$.

x	10 - X	$(10 - X)_+$	$X \wedge 10$	$(10 - X)_+ + (X \wedge 10)$
2	8	8	2	10
7	3	3	7	10
15	-5	0	10	10

So we see that $(10 - x)_+ + (x \wedge 10) = 10$, regardless of x.

In general, $(b - X)_+ + (X \wedge b) = b. \Rightarrow E[(b - X)_+] + E(X \wedge b) = b. \Rightarrow E[(b - X)_+] = b - E[X \wedge b]$.

More generally, the expected amount by which losses are less than b is:

$$E[(b - X)_+] = \int_0^b (b - x)\, f(x)\, dx = b \int_0^b f(x)\, dx - \int_0^b x\, f(x)\, dx = bF(b) - \{E[X \wedge b] - bS(b)\} = b - E[X \wedge b].$$

Therefore, the expected amount by which losses are less than b is:
$\mathbf{E[(b - X)_+] = b - E[X \wedge b].}$

This can also be seen via a Lee Diagram, as discussed in a subsequent section.

The expected amount by which aggregate losses are less than a given amount is sometimes called the "savings." [84]

For example, assume policyholder dividends are 1/3 of the amount by which that policyholder's aggregate annual claims are less than 1000. Then let L be aggregate annual claims. Then

$$\text{Policyholder Dividend} = \begin{cases} (1000 - L)/3 & L < 1000 \\ 0 & L \geq 1000 \end{cases}$$

Then the expected policyholder dividend is one third times the average amount by which aggregate claims are less than 1000.
Therefore, the expected dividend is: $(1000 - E[L \wedge 1000])/3$.

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[84] *Insurance Savings as used in Retrospective Rating is discussed for example in Gillam and Snader, "Fundamentals of Individual Risk Rating."*

Exercise:
The aggregate annual claims for a policyholder follow the following discrete distribution:
Prob[X = 200] = 30%, Prob[X = 500] = 40%, Prob[X = 2000] = 20%, Prob[X = 5000] = 10%.
Policyholder dividends are 1/4 of the amount by which that policyholder's aggregate annual claims are less than 1000, (no dividend is paid if annual claims exceed 1000.)
[Solution: $E[X \wedge 1000] = (0.3)(200) + (0.4)(500) + (0.3)(1000) = 560$.
Therefore, the expected amount by which aggregate annual claims are less than 1000 is:
$1000 - E[X \wedge 1000] = 1000 - 560 = 440$. Expected policyholder dividend is: 440/4 = 110.
Alternately, if aggregate claims are 200, then the dividend is: (1000 - 200)/4 = 200.
If aggregate claims are 500, then the dividend is: (1000 - 500)/4 = 125.
If aggregate claims are 2000 or 5000, then no dividend is paid.
Expected dividend (including those cases where no dividend is paid) is:
(0.3)(200) + (0.4)(125) = 110.]

Use the formula, $E[(b - X)_+] = b - E[X \wedge b]$, if the distribution is continuous rather than discrete.

Exercise: Assume aggregate annual claims for a policyholder are LogNormally distributed, with $\mu = 4$ and $\sigma = 2.5$.[85] Policyholder dividends are 1/3 of the amount by which that policyholder's aggregate annual claims are less than 1000. No dividend is paid if annual claims exceed 1000.
What are the expected policyholder dividends?
[Solution: For the LogNormal distribution,
$E[X \wedge x] = \exp(\mu + \sigma^2/2)\,\Phi[(\ln x - \mu - \sigma^2)/\sigma] + x\,\{1 - \Phi[(\ln x - \mu)/\sigma]\}$.
$E[X \wedge 1000] = \exp(7.125)\Phi(-1.337) + (1000)\{1 - \Phi(1.163)\} =$
$(1242.6)(0.0906) + (1000)(1 - 0.8776) = 235$.
Therefore, the expected dividend is $(1000 - E[X \wedge 1000])/3 = 255$.]

Sometimes, the dividend or bonus is stated in terms of the loss ratio, which is losses divided by premiums. In this case, the same technique can be used to determine the average dividend or bonus.

Exercise: An insurance agent will receive a bonus if his loss ratio is less than 75%.
The agent will receive a percentage of earned premium equal to 1/5 of the difference between 75% and his loss ratio. The agent receives no bonus if his loss ratio is greater than 75%.
His earned premium is 10 million. His incurred losses are distributed according to a Pareto distribution with $\alpha = 2.5$ and $\theta = 12$ million. Calculate the expected value of his bonus.
[Solution: A loss ratio of 75% corresponds to (0.75)(10 million) = \$7.5 million in losses.
If his losses are L, his loss ratio is L/10 million. If L < 7.5 million, his bonus is:
(1/5)(0.75 - L/10 million)(10 million) = (1/5)(7.5 million - L).
Therefore, his bonus is 1/5 the amount by which his losses are less than \$7.5 million.
For the Pareto distribution, $E[X \wedge x] = \{\theta/(\alpha-1)\}\{1 - (\theta/(\theta+x)^{\alpha-1}\}$.
Therefore, $E[X \wedge 7.5 \text{ million}] = (12 \text{ million}/1.5)\{1 - (12/(12+7.5)^{1.5}\} = 4.138$ million.
Therefore, the expected bonus is: (1/5){7.5 million - 4.138 million} = 672 thousand.]

[85] Note that we are applying the mathematical concept of a limited expected value to the distribution of aggregate losses in the same manner as was done to a distribution of sizes of loss.

$E[(X-d)_+]$ versus $E[(d - X)_+]$:

$E[(X-d)_+] = E[X] - E[X \wedge d]$, is the expected losses excess of d.

$E[(d-X)_+] = d - E[X \wedge d]$, is the expected amount by which losses are less than d.

Therefore, $E[(X-d)_+] - E[(d-X)_+] = E[X] - d = E[X - d]$.

In fact, $(X-d)_+ - (d-X)_+ = (x-d, \text{ if } x \geq d) - (d - x, \text{ if } x < d) = X - d$.

Exercise: For a Poisson distribution, determine $E[(N-1)_+]$.
[Solution: $E[N \wedge 1] = 0\ f(0) + 1\ \text{Prob}[N \geq 1] = \text{Prob}[N \geq 1]$.
$E[(N-1)_+] = E[N] - E[N \wedge 1] = \lambda - \text{Prob}[N \geq 1] = \lambda + e^{-\lambda} - 1$.
Alternately, $E[(N-1)_+] = E[(1-N)_+] + E[N] - 1 = \text{Prob}[N = 0] + \lambda - 1 = e^{-\lambda} + \lambda - 1$.]

Exercise: In baseball a team bats in an inning until it makes 3 outs. In the fifth inning of today's game, each batter for the Bad News Bears has a 45% chance of walking and a 55% chance of striking out, independent of any other batter. What is the expected number of runs the Bad News Bears will score in the fifth inning?
(If there are three men on base, a walk forces in a run. Assume no wild pitches, passed balls, etc. Assume nobody steals any bases, is picked off base, etc.)
[Solution: Treat a walk as a failure for the defense, and striking out as a success for the defense. An inning ends when there are three successes. The number of walks (failures) is Negative Binomial with r = 3 and β = (chance of failure)/(chance of success) = 0.45/0.55 = 9/11.
$f(0) = 1/(1 + \beta)^r = (11/20)^3 = 0.1664$. $f(1) = r\beta/(1 + \beta)^{r+1} = (3)(9/11)(11/20)^4 = 0.2246$.
$f(2) = \{(r)(r+1)/2\}\beta^2/(1 + \beta)^{r+2} = (3)(4/2)(9/11)^2(11/20)^5 = 0.2021$.
$E[N \wedge 3] = 0f(0) + 1f(0) + 2f(2) + 3\{1 - f(0) - f(1) - f(2)\} = 0.2246 + (2)(0.2021) + (3)(0.4069) = 1.8495$. If there are 3 or fewer walks in the inning, they score no runs. With a total of 4 walks they score 1 run, with a total 5 walks they score 2 runs, etc. $\Rightarrow$
Number of runs scored = $(N - 3)_+$.
Expected number of runs scored = $E[(N - 3)_+] = E[N] - E[N \wedge 3] = (3)(9/11) - 1.8495 = 0.605$.]

Average Size of Losses in an Interval:

As discussed previously, the Limited Expected Value is generally the sum of two pieces.
Each loss of size less than x contributes its own size, while each loss greater than or equal to x contributes just x to the average:

$$E[X \wedge x] = \int_0^x y\, f(y)\, dy + x\, S(x).$$

This formula can be rewritten to put the integral in terms of the limited expected value $E[X \wedge x]$ and the survival function S(x), both of which are given in the Appendix A of Loss Models:

$$\int_0^x y\, f(y)\, dy = E[X \wedge x] - x\, S(x).$$

This integral represents the dollars of loss on losses of size 0 to x.
Dividing by the probability of such claims, F(x), would give the average size of such losses.
Dividing instead by the mean would give the percentage of losses represented by those losses.

The dollars of loss represented by the losses in an interval from a to b is just the difference of two integrals of the type we have been discussing:

$$\int_0^b y\, f(y)\, dy - \int_0^a y\, f(y)\, dy = E[X \wedge b] - bS(b) - \{E[X \wedge a] - aS(a)\}.$$

Dividing by F(b) - F(a) would give the average size of loss for losses in this interval.

Average Size of Losses in the Interval [a, b] = $\dfrac{\{E[X \wedge b] - b\, S(b)\} - \{E[X \wedge a] - a\, S(a)\}}{F(b) - F(a)}$.

Exercise: For a LogNormal Distribution, with parameters $\mu = 8$ and $\sigma = 3$, what is the average size of those losses with sizes between \$1 million and \$5 million?
[Solution: For the LogNormal: F(5 million) = $\Phi[\{\ln(5 \text{ million})-\mu\} / \sigma] = \Phi[\{\ln(5 \text{ million})-8\}/3]$ = $\Phi[(15.425-8)/3] = \Phi[2.475] = 0.9933$.
F(1 million) = $\Phi[\{\ln(1 \text{ million})-8\}/3] = \Phi[1.939] = 0.9737$.
$E[X \wedge 5 \text{ million}] = \exp(\mu + \sigma^2/2)\Phi[(\ln(5 \text{ mil}) - \mu - \sigma^2)/\sigma] + (5 \text{ mil})\{1 - \Phi[(\ln(5 \text{ mil}) - \mu)/\sigma]\}$ =
$\exp(\mu + \sigma^2/2)\Phi[(\ln(5 \text{ mil}) - \mu - \sigma^2)/\sigma] + (5 \text{ mil}) \{1 - \Phi[(\ln(5 \text{ mil}) - \mu)/\sigma]\}$ =
(268,337)Φ[-0.525] + (5,000,000)(1 - Φ[2.475]) =
(268,337)(0.2998) + (5,000,000)(0.0067) = 113,679.
$E[X \wedge 1 \text{ million}] = \exp(\mu + \sigma^2/2)\Phi[(\ln(1 \text{ mil}) - \mu - \sigma^2)/\sigma] + (1 \text{ mil})\{1 - \Phi[(\ln(1 \text{ mil}) - \mu)/\sigma]\}$ =
(268,337)Φ[-1.061] + (1,000,000)(1- Φ[1.939]) =
(268,337)(0.1444) + (1,000,000)(0.0263) = 65,048.
Thus, the average size of loss for those losses of size between \$1 million and \$5 million is:
$(\{E[X \wedge 5m] - (5m) S(5m)\} - \{E[X \wedge 1m] - (1m)S(1m)\}) / \{F(5m) - F(1m)\}$ =
({113,679 - (5,000,000)(0.0067)} - {65,048 - (1,000,000)(0.0263)}) / (0.9933 - 0.9737) =
41,700/0.0196 = \$2.13 million.
<u>Comment</u>: Note that the average size of loss is not at the midpoint of the interval, which is \$3 million. In the case of the LogNormal, $E[X \wedge x] - xS(x) = \exp(\mu+\sigma^2/2)\Phi[(\ln(x)-\mu-\sigma^2)/\sigma]$.
Thus, the mean loss size for the interval a to b is:
$\exp(\mu + \sigma^2/2)\{\Phi[(\ln b - \mu - \sigma^2)/\sigma] - \Phi[(\ln a - \mu - \sigma^2)/\sigma]\} / \{\Phi[(\ln b - \mu)/\sigma] - \Phi[(\ln a - \mu)/\sigma]\}$,
which would have saved some computation in this case.]

Dividing instead by the mean would give the percentage of dollars of total losses represented by those claims.

Proportional of Total Losses from Losses in the Interval [a, b] is:

$$\frac{\{E[X \wedge b] - b\,S(b)\} - \{E[X \wedge a] - a\,S(a)\}}{E[X]}.$$

Exercise: For a LogNormal Distribution, with parameters $\mu = 8$ and $\sigma = 3$, what percentage of the total losses are from those losses with sizes between \$1 million and \$5 million?
[Solution: $E[X] = \exp(\mu + \sigma^2/2) = e^{12.5} = 268,337$.
From a previous solution, $(\{E[X \wedge 5m] - (5m) S(5m)\} - \{E[X \wedge 1m] - (1m)S(1m)\}) = 41,700$.
41,700/268,337 = 15.5%.
<u>Comment</u>: In the case of the LogNormal, the percentage of losses from losses of size a to b =
$\exp(\mu + \sigma^2/2) \{\Phi[(\ln b - \mu - \sigma^2)/\sigma] - \Phi[(\ln a - \mu - \sigma^2)/\sigma]\} / \exp(\mu + \sigma^2/2)$ =
$\Phi[(\ln b - \mu - \sigma^2)/\sigma] - \Phi[(\ln a - \mu - \sigma^2)/\sigma]$.]

Questions about the losses in an interval have to be distinguished from those about layers of loss. For example, the losses in the layer from \$100,000 and \$1 million are part of the dollars from losses of size greater than \$100,000. Each loss of size between \$100,000 and \$1 million contributes its size minus \$100,000 to this layer, while those of size greater than \$1 million contribute the width of the layer, \$900,000, to this layer.[86]

Payments Subject to a Minimum:[87]

Assume a disabled worker is paid his weekly wage, subject to a minimum payment of 300.[88]
Let X be a workers weekly wage. Then, while he is unable to work, he is paid Max[X, 300].

Min[X, 300] + Max[X, 300] = X + 300.
Therefore, E[Max[X, 300]] = 300 + E[X] - E[Min[X, 300]] = 300 + E[X] - E[X $\wedge$ 300].

Let Y = amount the worker is paid = Max[X, 300].
Then Y - 300 = 0 if $X \le 300$, and Y - 300 = X - 300 if $X > 300$.
Therefore, E[Y - 300] = E[$(X - 300)_+$] = E[X] - E[X $\wedge$ 300].

$\Rightarrow$ E[Y] = 300 + E[X] - E[X $\wedge$ 300], matching the previous result.

Exercise: Weekly wages are distributed as follows:
200 @ 20%, 300 @30%, 400 @30%, 500 @ 10%, 1000 @10%.
Determine the average weekly payment to a worker who is disabled.
[Solution: E[X] = (20%)(200) + (30%)(300) + (30%)(400) + (10%)(500) + (10%)(1000) = 400.
E[X $\wedge$ 300] = (20%)(200) + (30%)(300) + (30%)(300) + (10%)(300) + (10%)(300) = 280.
300 + E[X] - E[X $\wedge$ 300] = 300 + 400 - 280 = 420.
Alternately, one can list all of the possibilities:

Wage	Payment	Probability
200	300	20%
300	300	30%
400	400	30%
500	500	10%
1000	1000	10%

(20%)(300) + (30%)(300) + (30%)(400) + (10%)(500) + (10%)(1000) = 420.]

Another way to look at this is that the average payment is:
mean wage + (the average amount by which the wage is less than 300) =
E[X] + (300 - E[X $\wedge$ 300]) = 300 + E[X] - E[X $\wedge$ 300], matching the previous result.

In general, **E[Max[X, a]] = a + E[X] - E[X $\wedge$ a]**.

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[86] See the earlier section on Layers of Loss.
[87] See for example, SOA M, 11/06, Q. 20.
[88] This is a very simplified version of benefits under Workers Compensation.

Payments Subject to both a Minimum and a Maximum:[89]

Assume a disabled worker is paid his weekly wage, subject to a minimum payment of 300, and a maximum payment of 700.[90] Let X be a workers weekly wage. Then, while he is unable to work, he is paid Min[Max[X, 300], 700].

Let Y = amount the worker is paid = Min[Max[X, 300], 700].
Then Y - 300 = 0 if $X \le 300$, Y - 300 = X - 300 if $300 < X < 700$, and Y - 300 = 400 if $X \ge 700$.
Therefore, E[Y - 300] = the layer from 300 to 700 = $E[X \wedge 700] - E[X \wedge 300]$.
$\Rightarrow E[Y] = 300 + E[X \wedge 700] - E[X \wedge 300]$.

Exercise: Weekly wages are distributed as follows:
200 @ 20%, 300 @30%, 400 @30%, 500 @ 10%, 1000 @10%.
Determine the average weekly payment to a worker who is disabled.
[Solution: $E[X \wedge 300]$ = (20%)(200) + (80%)(300) = 280.
$E[X \wedge 700]$ = (20%)(200) + (30%)(300) + (30%)(400) + (10%)(500) + (10%)(700) = 370.
300 + $E[X \wedge 700] - E[X \wedge 300]$ = 300 + 370 - 280 = 390.
Alternately, one can list all of the possibilities:

Wage	Payment	Probability
200	300	20%
300	300	30%
400	400	30%
500	500	10%
1000	700	10%

(20%)(300) + (30%)(300) + (30%)(400) + (10%)(500) + (10%)(700) = 390.]

Another to way to arrive at the same result is that the average payment is:
mean wage + (average amount by which the wage is less than 300) - (layer above 700) =
$E[X] + (300 - E[X \wedge 300]) - (E[X] - E[X \wedge 700]) = 300 + E[X \wedge 700] - E[X \wedge 300]$,
matching the previous result.

In general, $\mathbf{E[Min[Max[X , a] , b]] = a + E[X \wedge b] - E[X \wedge a]}$.

We note that if $b = \infty$, in other words the payments are not subject to a maximum, this reduces to the result previously discussed for that case, $E[Max[X, a]] = a + E[X] - E[X \wedge a]$.

If instead a = 0, in other words the payment is not subject to a minimum, this reduces to $E[Min[X , b]] = E[X \wedge b]$, which is the definition of the limited expected value.

[89] This mathematics is a simplified version of the premium calculation under a Retrospectively Rated Policy. See "Mahler's Guide to P&C Ratemaking and Reserving."
[90] This is a simplified version of benefits under Workers Compensation.

Normal Distribution:

For the Standard Normal:

$$\int_{-\infty}^{x} t\, f(t)\, dt = \int_{-\infty}^{x} t \exp[-t^2/2] / \sqrt{2\pi}\ dt = -\exp[-t^2/2] / \sqrt{2\pi} \Big]_{t=-\infty}^{t=x} = -\exp[-x^2/2] / \sqrt{2\pi} = -\phi(x).$$

For the nonstandard Normal:

$$\int_{-\infty}^{x} t\, f(t)\, dt = \int_{-\infty}^{x} t\, \phi[(t-\mu)/\sigma] / \sigma\ dt = \int_{-\infty}^{(x-\mu)/\sigma} (\sigma y + \mu)\, \phi[y]\ dy =$$

$$\sigma \int_{-\infty}^{(x-\mu)/\sigma} y\, \phi[y]\ dy + \mu \int_{-\infty}^{(x-\mu)/\sigma} \phi[y]\ dy = -\sigma\, \phi[(x-m)/s] + \mu\, \Phi[(x-m)/s].$$

Thus, $E[X \wedge x] = \int_{-\infty}^{x} t\, f(t)\, dt + x\, S(x) = -\sigma\, \phi[(x-\mu)/\sigma] + \mu\, \Phi[(x-\mu)/\sigma] + x\, (1 - \Phi[(x-\mu)/\sigma])$

$= x - \sigma\, \phi[(x-\mu)/\sigma] - (x - \mu)\, \Phi[(x-\mu)/\sigma]$.

Exercise: For a Normal Distribution with $\mu = 33$ and $\sigma = 10$, determine $E[X \wedge 38]$.
[Solution: $38 - (10)\, \phi[(38 - 33) / 10] - (38 - 33)\, \Phi[(38 - 33) / 10] = 38 - 10\, \phi[0.5] - 5\, \Phi[0.5] =$
$38 - (10) \exp[-0.5^2/2] / \sqrt{2\pi} - (5)(0.6915) = 31.02$.]

Problems:

17.1 (1 point) You are given the following:
- The size of loss distribution is given by
  $f(x) = 2e^{-2x}$, $x > 0$
- Under a basic limits policy, individual losses are capped at 1.
- The expected annual claim frequency is 13.

What are the expected annual total loss payments on a basic limits policy?
A. less than 5.0
B. at least 5.0 but less than 5.5
C. at least 5.5 but less than 6.0
D. at least 6.0 but less than 6.5
E. at least 6.5

Use the following information for the next 7 questions.
Assume the unlimited losses follow a LogNormal Distribution with parameters $\mu = 7$ and $\sigma = 3$.
Assume an average of 200 losses per year.

17.2 (1 point) What is the total cost expected per year?
A. less than $19 million
B. at least $19 million but less than $20 million
C. at least $20 million but less than $21 million
D. at least $21 million but less than $22 million
E. at least $22 million

17.3 (2 points) If the insurer pays no more than $1 million per loss, what is the insurer's total cost expected per year?
A. less than $7 million
B. at least $7 million but less than $8 million
C. at least $8 million but less than $9 million
D. at least $9 million but less than $10 million
E. at least $10 million

17.4 (2 points) If the insurer pays no more than $5 million per loss, what is the insurer's total cost expected per year?
A. less than $7 million
B. at least $7 million but less than $8 million
C. at least $8 million but less than $9 million
D. at least $9 million but less than $10 million
E. at least $10 million

17.5 (1 point) What are the dollars in the layer from \$1 million to \$5 million expected per year?
A. less than \$3.5 million
B. at least \$3.5 million but less than \$3.7 million
C. at least \$3.7 million but less than \$3.9 million
D. at least \$3.9 million but less than \$4.1 million
E. at least \$4.1 million

17.6 (1 point) What are the total dollars excess of \$5 million per loss expected per year?
A. less than \$7 million
B. at least \$7 million but less than \$8 million
C. at least \$8 million but less than \$9 million
D. at least \$9 million but less than \$10 million
E. at least \$10 million

17.7 (2 points) What is the average size of loss for those losses between \$1 million and \$5 million in size?
A. less than \$1.4 million
B. at least \$1.4 million but less than \$1.7 million
C. at least \$1.7 million but less than \$2.0 million
D. at least \$2.0 million but less than \$2.3 million
E. at least \$2.3 million

17.8 (1 point) What is the expected total cost per year of those losses between \$1 million and \$5 million in size?
A. less than \$3.5 million
B. at least \$3.5 million but less than \$3.7 million
C. at least \$3.7 million but less than \$3.9 million
D. at least \$3.9 million but less than \$4.1 million
E. at least \$4.1 million

17.9 (2 points) A Pareto Distribution with parameters $\alpha = 2.5$ and $\theta = \$15,000$ appears to be a good fit to liability claims. What is the expected average size of loss for a policy issued with a \$250,000 limit of liability?
A. less than 9200
B. at least 9200 but less than 9400
C. at least 9400 but less than 9600
D. at least 9600 but less than 9800
E. at least 9800

Use the following information for the next 4 questions:

- The weekly wages for workers in a state follow a Pareto Distribution with $\alpha = 4$ and $\theta = 1800$.
- Injured workers are paid weekly benefits equal to 2/3 of their pre-injury average weekly wage, but subject to a maximum benefit of the state average weekly wage and a minimum benefit of 1/4 of the state average weekly wage.
- Injured workers have the same wage distribution as all workers.
- The duration of payments is independent of the worker's wage.

17.10 (1 point) What is the state average weekly wage?
A. less than $500
B. at least $500 but less than $530
C. at least $530 but less than $560
D. at least $560 but less than $590
E. at least $590

17.11 (2 points) For a Pareto Distribution with parameters $\alpha = 4$ and $\theta = 1800$,
what is $E[X \wedge 900]$?
A. less than $400
B. at least $400 but less than $430
C. at least $430 but less than $460
D. at least $460 but less than $490
E. at least $490

17.12 (2 points) For a Pareto Distribution with parameters $\alpha = 4$ and $\theta = 1800$,
what is $E[X \wedge 225]$?
A. less than $100
B. at least $100 but less than $130
C. at least $130 but less than $160
D. at least $160 but less than $190
E. at least $190

17.13 (3 points) What is the average weekly benefit received by injured workers?
A. less than $300
B. at least $300 but less than $320
C. at least $320 but less than $340
D. at least $340 but less than $360
E. at least $360
Hint: Use the solutions to the previous three questions.

Use the following information for the next 15 questions:
Losses follow an Exponential Distribution with θ = 10,000.

17.14 (1 point) What is the average loss?
A. less than 8500
B. at least 8500 but less than 9000
C. at least 9000 but less than 9500
D. at least 9500 but less than 10,000
E. at least 10,000

17.15 (1 point) Assuming a 25,000 policy limit, what is the average payment by the insurer?
A. less than 9000
B. at least 9000 but less than 9100
C. at least 9100 but less than 9200
D. at least 9200 but less than 9300
E. at least 9300

17.16 (1 point) Assuming a 1000 deductible (with no maximum covered loss), what is the average payment per loss?
A. less than 9000
B. at least 9000 but less than 9100
C. at least 9100 but less than 9200
D. at least 9200 but less than 9300
E. at least 9300

17.17 (1 point) Assuming a 1000 deductible (with no maximum covered loss), what is the average payment per non-zero payment by the insurer?
A. less than 8500
B. at least 8500 but less than 9000
C. at least 9000 but less than 9500
D. at least 9500 but less than 10,000
E. at least 10,000

17.18 (1 point) Assuming a 1000 deductible and a 25,000 maximum covered loss, what is the average payment per loss?
A. less than 8500
B. at least 8500 but less than 9000
C. at least 9000 but less than 9500
D. at least 9500 but less than 10,000
E. at least 10,000

17.19 (1 point) Assuming a 1000 deductible and a 25,000 maximum covered loss, what is the average payment per (non-zero) payment by the insurer?
A. less than 9000
B. at least 9000 but less than 9100
C. at least 9100 but less than 9200
D. at least 9200 but less than 9300
E. at least 9300

17.20 (1 point) Assuming a 75% coinsurance factor (with no deductible or maximum covered loss), what is the average payment by the insurer?
A. less than 6700
B. at least 6700 but less than 6800
C. at least 6800 but less than 6900
D. at least 6900 but less than 7000
E. at least 7000

17.21 (1 point) Assuming a 75% coinsurance factor and a 1000 deductible (with no maximum covered loss), what is the average payment per loss?
A. less than 6700
B. at least 6700 but less than 6800
C. at least 6800 but less than 6900
D. at least 6900 but less than 7000
E. at least 7000

17.22 (1 point) Assuming a 75% coinsurance factor, a 1000 deductible and a 25,000 maximum covered loss, what is the average payment per non-zero payment by the insurer?
A. less than 6700
B. at least 6700 but less than 6800
C. at least 6800 but less than 6900
D. at least 6900 but less than 7000
E. at least 7000

17.23 (2 points) What is the average size of the losses in the interval from 1000 to 25,000? Assume no deductible, no maximum covered loss, and no coinsurance factor.
A. less than 7500
B. at least 7500 but less than 8000
C. at least 8000 but less than 8500
D. at least 8500 but less than 9000
E. at least 9000

17.24 (2 points) What is the proportion of total dollars of loss from the losses in the interval from 1000 to 25,000? Assume no deductible, no maximum covered loss, and no coinsurance factor.
A. less than 74%
B. at least 74% but less than 76%
C. at least 76% but less than 78%
D. at least 78% but less than 80%
E. at least 80%

17.25 (3 points) Assuming a 1000 deductible, what is the average size of the insurer's payments for those payments greater than 500 and at most 4000?
A. less than 2100
B. at least 2100 but less than 2130
C. at least 2130 but less than 2160
D. at least 2160 but less than 2190
E. at least 2190

17.26 (3 points) Assuming a 75% coinsurance factor, and a 1000 deductible, what is the average size of the insurer's payments for those payments greater than 500 and at most 4000?
A. less than 2100
B. at least 2100 but less than 2130
C. at least 2130 but less than 2160
D. at least 2160 but less than 2190
E. at least 2190

17.27 (4 points) Assuming a 75% coinsurance factor, a 1000 deductible and a 25,000 maximum covered loss, what is the average size of the insurer's payments for those payments greater than 15,000 and at most 19,000?
A. less than 17,400
B. at least 17,400 but less than 17,500
C. at least 17,500 but less than 17,600
D. at least 17,600 but less than 17,700
E. at least 17,700

17.28 (1 point) Assuming a 75% coinsurance factor, a 1000 deductible and a 25,000 maximum covered loss, what is the mean of the insurer's payments per loss?
A. less than 4000
B. at least 4000 but less than 5000
C. at least 5000 but less than 6000
D. at least 6000 but less than 7000
E. at least 7000

17.29 (3 points) You are given the following information about a policyholder:
- His loss ratio is calculated as incurred losses divided by earned premium.
- He will receive a policyholder dividend as a percentage of earned premium equal to 1/4 of the difference between 60% and his loss ratio.
- He receives no policyholder dividend if his loss ratio is greater than 60%.
- His earned premium is 40,000.
- His incurred losses are distributed via a LogNormal Distribution, with $\mu = 6$ and $\sigma = 3$.

Calculate the expected value of his policyholder dividend.
(A) 4800 (B) 5000 (C) 5200 (D) 5400 (E) 5600

Use the following information for the next two questions:
- In the state of Minnehaha, each town is responsible for its snow removal.
- However, a state fund shares the cost if a town has a lot of snow during a winter.
- In exchange, a town is required to pay into this state fund when it has a winter with a small amount of snow.
- Let x be the number of inches of snow a town has during a winter.
- If $x < 20$, then the town pays the state fund $c(20 - x)$, where c varies town.
- If $x > 50$, then the state fund pays the town $c(x - 50)$.
- c = 1000 for the town of Frostbite Falls.

17.30 (3 points) The number of inches of snow the town of Frostbite Falls has per winter is equally likely to be: 8, 10, 16, 21, 35, 57, 70, or 90.
What is the expected net amount the state fund pays Frostbite Falls (expected amount state fund pays town minus expected amount town pays the state fund) per winter?
A. 3000 B. 3500 C. 4000 D. 4500 E. 5000

17.31 (5 points) The number of inches of snow the town of Frostbite Falls has per winter is LogNormal, with $\mu = 2.4$ and $\sigma = 1.5$.
What is the expected net amount the state fund pays Frostbite Falls (expected amount state fund pays town minus expected amount town pays the state fund) per winter?
A. 7000 B. 7500 C. 8000 D. 8500 E. 9000

17.32 (2 points) N follows a Poisson Distribution, with $\lambda = 2.5$. Determine $E[(N - 3)_+]$.
A. 0.2 B. 0.3 C. 0.4 D. 0.5 E. 0.6

17.33 (2 points) The lifetime of batteries is Exponential with mean 6. Batteries are sold for $100 each. If a battery lasts less than 2 years, the manufacturer will pay the purchaser the pro rata share of the purchase price. For example if the battery lasts only 1.5 years, the manufacturer will pay the purchaser (100)(2 - 1.5)/2 = 25.
What is the expected amount paid by the manufacturer per battery sold?
(A) 11 (B) 13 (C) 15 (D) 17 (E) 19

17.34 (4 points) XYZ Insurance Company writes insurance in a state with a catastrophe fund for hurricanes. For any hurricane on which XYZ has more than $30 million in losses in this state, the Catastrophe Fund will pay XYZ 75% of its hurricane losses above $30 million, subject to a maximum payment from the fund of $90 million.
The amount XYZ pays in this state on a hurricane that hits this state is distributed via a LogNormal Distribution, with $\mu = 15$ and $\sigma = 2$. What is expected value of the amount XYZ will receive from the Catastrophe Fund due to the next hurricane to hit this state?
(A) 4 million (B) 5 million (C) 6 million (D) 7 million (E) 8 million

Use the following information for the next two questions:

- Losses follow a Pareto Distribution, with parameters $\alpha = 5$ and $\theta = 40,000$.
- Three losses are expected each year.
- For each loss less than or equal to 5,000, the insurer makes no payment.

17.35 (2 points) You are given the following:
For each loss greater than 5,000, the insurer pays the amount of the loss up to the maximum covered loss of 25,000, less a 5000 deductible.
(Thus for a loss of 7000 the insurer pays 2000; for a loss of 80,000 the insurer pays 20,000.)
Determine the insurer's expected annual payments.
A. Less than 7,500
B. At least 7,500, but less than 12,500
C. At least 12,500, but less than 17,500
D. At least 17,500, but less than 22,500
E. At least 22,500

17.36 (2 points) For each loss greater than 5,000, the insurer pays the entire amount of the loss up to the maximum covered loss of 25,000.
Determine the insurer's expected annual payments.
A. Less than 7,500
B. At least 7,500, but less than 12,500
C. At least 12,500, but less than 17,500
D. At least 17,500, but less than 22,500
E. At least 22,500

17.37 (2 points) Losses follow an Exponential Distribution with $\theta = 20{,}000$.
Calculate the percent of expected losses within the layer 5,000 to 50,000.
A. Less than 50%
B. At least 50%, but less than 55%
C. At least 55%, but less than 60%
D. At least 60%, but less than 65%
E. At least 65%

17.38 (4 points) Losses follow a LogNormal Distribution with $\mu = 9.4$ and $\sigma = 1$.
Calculate the percent of expected losses within the layer 5,000 to 50,000.
A. Less than 50%
B. At least 50%, but less than 55%
C. At least 55%, but less than 60%
D. At least 60%, but less than 65%
E. At least 65%

17.39 (3 points) Losses follow a Pareto Distribution with $\alpha = 3$ and $\theta = 40{,}000$.
Calculate the percent of expected losses within the layer 5,000 to 50,000.
A. Less than 50%
B. At least 50%, but less than 55%
C. At least 55%, but less than 60%
D. At least 60%, but less than 65%
E. At least 65%

17.40 (3 points) N follows a Geometric Distribution, with $\beta = 2.5$. Determine $E[(N - 3)_+]$.
A. 0.9 B. 1.0 C. 1.1 D. 1.2 E. 1.3

17.41 (3 points) Losses follow a Pareto Distribution with $\alpha = 3$ and $\theta = 12{,}000$.
Policy A has a deductible of 3000. Policy B has a maximum covered loss of u.
The average payment per loss under Policy A is equal to that under Policy B. Determine u.
A. 4000 B. 5000 C. 6000 D. 7000 E. 8000

17.42 (1 point) X is 5 with probability 80% and 25 with probability 20%.
If $E[(b - X)_+] = 8$, determine b.
A. 10 B. 15 C. 20 D. 25 E. 30

17.43 (2 points) X is Exponential with $\theta = 2$. Y is equal to $1 - X$ if $X < 1$, and Y is 0 if $X \geq 1$.
What is the expected value of Y?
A. 0.15 B. 0.17 C. 0.19 D. 0.21 E. 0.23

17.44 (3 points) Let R be the weekly wage for a worker compared to the statewide average.
R follows a LogNormal Distribution with $\sigma = 0.4$. Determine the percentage of overall wages earned by workers whose weekly wage is less than twice the statewide average.
A. 88% B. 90% C. 92% D. 94% E. 96%

17.45 (2 points) You observe the following 35 losses: 6, 7, 11, 14, 15, 17, 18, 19, 25, 29, 30, 34, 40, 41, 48, 49, 53, 60, 63, 78, 85, 103, 124, 140, 192, 198, 227, 330, 361, 421, 514, 546, 750, 864, 1638. What is the (empirical) Limited Expected Value at 50?
A. less than 38
B. at least 38 but less than 39
C. at least 39 but less than 40
D. at least 40 but less than 41
E. at least 41

17.46 (2 points) Alex's pay is based on the annual profit made by his employer.
Alex is paid 2% of the profit, subject to a minimum payment of 100.
The annual profits for Alex's company, X, follow a distribution F(x).
Which of the following represents Alex's expected payment?
A. $100F(100) + E[X]/50 - E[X \wedge 100]$
B. $100F(5000) + E[X]/50 - E[X \wedge 5000]/50$
C. $100 + 0.02(E[X] - E[X \wedge 5000])$
D. $0.02(E[X \wedge 5000] - E[X \wedge 100]) + 100S(5000)$
E. None of A, B, C, or D

17.47 (2 points) In the previous question, assume $F(x) = 1 - \{20,000/(20,000 + x)\}^3$.
Determine Alex's expected payment.
A. 200 B. 230 C. 260 D. 290 E. 320

17.48 (2 points) The size of losses follows a Gamma distribution with parameters $\alpha = 3$, $\theta = 100$.
What is the limited expected value at 500, $E[X \wedge 500]$?
Hint: Use Theorem A.1 in Appendix A of Loss Models:

$$\Gamma(n; x) = 1 - \sum_{j=0}^{j=n-1} x^j e^{-x}/j! \text{ , for n a positive integer.}$$

A. less than 275
B. at least 275 but less than 280
C. at least 280 but less than 285
D. at least 285 but less than 290
E. at least 290

17.49 (2 points) Donald Adams owns the Get Smart Insurance Agency.
Let L be the annual losses from the insurance policies that Don's agency writes for the Control Insurance Company. L follows a Single Parameter Pareto distribution with $\alpha = 3$ and $\theta = 100{,}000$.
Don gets a bonus from the Control Insurance Company calculated as (170,000 - L)/4 if this quantity is positive and 0 otherwise. Calculate Don's expected bonus.
A. Less than 10,000
B. At least 10,000, but less than 12,000
C. At least 12,000, but less than 14,000
D. At least 14,000, but less than 16,000
E. At least 16,000

17.50 (1 point) In the previous question, calculate the expected value of Don's bonus conditional on his bonus being positive.

17.51 (1 point) X follows the density f(x), with support from 0 to infinity.

$$\int_0^{1000} f(x)\,dx = 0.87175. \quad \int_0^{1000} x\,f(x)\,dx = 350.61.$$

Determine $E[X \wedge 1000]$.
A. Less than 480
B. At least 480, but less than 490
C. At least 490, but less than 500
D. At least 500, but less than 510
E. At least 510

17.52 (3 points) The size of loss is modeled by a two parameter Pareto distribution with $\theta = 5000$ and $\alpha = 3$. An insurance has the following provisions:
(i) It pays 75% of the first 2000 of any loss.
(ii) It pays 90% of any portion of a loss that is greater than 10,000.
Calculate the average payment per loss.
A Less than 1050
B. At least 1050, but less than 1100
C. At least 1100, but less than 1150
D. At least 1150, but less than 1200
E. At least 1200

17.53 (3 points) The mean number of minutes used per month by owners of cell phones varies between owners via a Single Parameter Pareto Distribution with $\alpha = 1.5$ and $\theta = 20$.
The Telly Savalas Phone Company is planning to sell a new unlimited calling plan.
Only those whose current average usage is greater than the overall average will sign up for the plan.
In addition, those who sign up will use on average 50% more minutes than currently.
What is the expected number of minutes used per month under the new plan?
A. 150 B. 180 C. 210 D. 240 E. 270

17.54 (2 points) Define the first moment distribution, G(x), as the percentage of total loss dollars that come from those losses of size less than x.
If the size of loss distribution follows a LogNormal Distribution, with parameters μ and σ, determine the form of the first moment distribution.

17.55 (3 points) Define the quartiles as the 25th, 50th, and 75th percentiles.
Define the trimmed mean as the average of those values between the first (lower) quartile and the third (upper) quartile.
Determine the trimmed mean for an Exponential Distribution.

17.56 (2 points) For a Pareto Distribution with $\alpha = 1$, derive the formula for the Limited Expected Value that is shown in Appendix A of Loss Models, attached to the exam.

17.57 (4 points) The value of a Property Claims Service (PCS) index is determined by the catastrophe losses for the insurance industry in a certain region of the country over a certain period of time. Each $100 million of catastrophe losses corresponds to one point on the index.
A 100/150 call spread would pay: $(200)\{(S - 150)_+ - (S - 100)_+\}$,

where S is the value of the PCS index at expiration and $X_+ = \begin{cases} 0 \text{ if } x < 0 \\ x \text{ if } x \geq 0 \end{cases}$.

You assume that the catastrophe losses in a certain region follow a LogNormal Distribution with parameters $\mu = 20$ and $\sigma = 2$.
What is the expected payment on a 100/150 call spread on the PCS Index for this region?
A. 200 B. 300 C. 400 D. 500 E. 600

17.58 (2 points) For the Normal Distribution with mean μ and standard deviation σ:
$E[X \wedge x] = x - (x-\mu)\,\Phi[(x-\mu)/\sigma] - \sigma\,\phi[(x-\mu)/\sigma]$.
For $\mu = 62$ and $\sigma = 20$, calculate $E[X \wedge 75]$.

17.59 (3 points) Define the quantile Q_α to be such that $F[Q_\alpha] = \alpha$.
For α between 0 and 1/2, compute the Windsorized mean by:
1. Replace all values below Q_α by Q_α.
2. Replace all values above $Q_{1-\alpha}$ by $Q_{1-\alpha}$.
3. Take the average.
Determine the algebraic form of the Windsorized mean for an Exponential Distribution.

17.60 (5 points) As a state insurance department actuary in the state of Pennisota, you have to oversee the operations of the State Medical Reinsurance Fund. The Fund covers the layer from \$250,000 to \$1 million (\$750,000 in excess of \$250,000) for each Medical Malpractice claim incurred in Pennisota.
The severity distribution for Medical Malpractice Insurance in the state of Pennisota is Pareto with $\alpha = 3$ and $\theta = 400{,}000$.
The State is considering two possible legislative bills for Medical Malpractice in Pennisota.
Legislative Bill 1 would increase the size of all claims by 5%.
Legislative Bill 2 would only increase the size of claims with values originally less than \$500,000 by 10%, subject to the constraint that claims that are originally less than \$500,000 would become no greater than \$500,000. Under Legislative Bill 2 the value of all claims in excess of \$500,000 would not change.
a. (2 points) Estimate for Bill 1 its percentage impact on the losses paid by the State Medical Reinsurance Fund.
b. (3 points) Estimate for Bill 2 its percentage impact on the losses paid by the State Medical Reinsurance Fund.

17.61 (4 points) Define π_p as the p^{th} percentile.
Define the trimmed mean as the average of those values between π_{1-p} and π_p.
For p = 95%, determine the algebraic form of the trimmed mean for a Pareto Distribution.

<u>**17.62**</u> (3 points) Losses follow a Pareto Distribution with $\alpha = 2.6$ and $\theta = 5000$.
An insurance policy has a deductible of 2000 applied after a maximum covered loss of 3000.
A new insurance policy will instead apply a deductible of 1000 prior to a limit of 1000.
What is the percentage increase in expected losses paid under the new policy?
A. 30% D. 35% C. 40% D. 45% E. 50%

17.63 (2 points) A general contractor will be paid a bonus if a skyscraper is completed in less than 450 days. (The skyscraper will have to pass final inspection.)
Let S be the number of days taken to complete the skyscraper.
The bonus is 30,000 times the amount by which S is less than 450.
No bonus is paid if $S \geq 450$.
$E[S] = 500$.
$E[(S-450)_+] = 90$.
Determine the expected bonus paid.
A. 0.8 million B. 0.9 million C. 1.0 million D. 1.1 million E. 1.2 million

17.64 (3 points) The distribution of annual loss ratios for Permanent Assurance Company is LogNormal with $\mu = -0.65$ and $\sigma = 0.6$.
Conditional on the loss ratio being greater than 0.7, compute the average loss ratio.
A. 95% D. 100% C. 105% D. 110% E. 115%

17.65 (8 points) You assume that the distribution of the wealth in a country follows a Single Parameter Pareto Distribution.
(a) If α = 1.5, determine the average wealth of those whose wealth exceeds 10 million > θ.
(b) For α > 1 and y > θ, determine the average wealth of those whose wealth exceeds y.
(c) Let Π_p and Π_q be two percentiles of the distribution, with q > p. Determine Π_q / Π_p.
(d) What part of the total wealth of the top 10% of the wealth distribution is owned by the top 1%?

17.66 (2 points) An insurer has excess-of-loss reinsurance on auto insurance. You are given:
● The individual losses have a Pareto distribution with

$$F(x) = 1 - \left(\frac{4000}{4000 + x}\right)^3, \ x > 0.$$

● Reinsurance will pay the excess of each loss over 3000.
If the reinsurance instead paid the excess of each loss over 5000, what would be the percent increase in the expected payment per loss by the insurer?
A. 15% D. 17% C. 19% D. 21% E. 23%

17.67 (3 points) For a LogNormal Distribution with μ = 8 and σ = 1.2, determine $E[(X - 10{,}000)_+]$.
A. 1500 B. 2000 C. 2500 D. 3000 E. 3500

17.68 (4, 5/88, Q.61) (3 points) Losses for a given line of insurance are distributed according to the probability density function f(x) = 0.015 - 0.0001x, 0 < x < 100.
An insurer has issued policies each with a deductible of 10 for this line.
On these policies, what is the average expected payment by the insurer per non-zero payment by the insurer?
A. Less than 30
B. At least 30, but less than 35
C. At least 35, but less than 40
D. At least 40, but less than 45
E. 45 or more

17.69 (4, 5/90, Q.53) (2 points) Loss Models defines two functions:
1. the limited expected value function, $E[X \wedge x]$ and
2. the Mean Excess Loss function, e(x)
If F(x) = Pr{X ≤ x} and the expected value of X is denoted by E[X], then which of the following equations expresses the relationship between $E[X \wedge x]$ and e(x)?
A. $E[X \wedge x] = E[X] - e(x) / \{1 - F(x)\}$
B. $E[X \wedge x] = E[X] - e(x)$
C. $E[X \wedge x] = E[X] - e(x)(1 - F(x))$
D. $E[X \wedge x] = E[X](1 - F(x)) - e(x)$
E. None of the above

17.70 (4B, 11/93, Q.16) (1 point)
Which of the following statements are true regarding loss distribution models?
1. For small samples, method of moments estimators have smaller variances than maximum likelihood estimators.
2. The limited expected value function evaluated at any point d ≥ 0 equals

$$E[X \wedge d] = \int_0^d x\, f_X(x)\, dx + d\{1 - F_X(d)\}$$, where $f_X(x)$ and $F_X(x)$ are the probability density

and distribution functions, respectively, of the loss random variable X.
3. A consideration in model selection is agreement between the empirical and fitted limited expected value functions.

A. 2 B. 1, 2 C. 1, 3 D. 2, 3 E. 1, 2, 3

17.71 (4B, 5/95, Q.22) (2 points) You are given the following:
- Losses follow a Pareto distribution, with parameters $\theta = 1000$ and $\alpha = 2$.
- 10 losses are expected each year.
- The number of losses and the individual loss amounts are independent.
- For each loss that occurs, the insurer's payment is equal to the **entire** amount of the loss if the loss is greater than 100. The insurer makes no payment if the loss is less than or equal to 100.

Determine the insurer's expected annual payments.
A. Less than 8,000
B. At least 8,000, but less than 9,000
C. At least 9,000, but less than 9,500
D. At least 9,500, but less than 9,900
E. At least 9,900

17.72 (4B, 11/98, Q.28) (2 points) You are given the following:
- Losses follow a lognormal distribution, with parameters $\mu = 10$ and $\sigma = 1$.
- One loss is expected each year.
- For each loss less than or equal to 50,000, the insurer makes no payment.
- For each loss greater than 50,000, the insurer pays the **entire** amount of the loss up to the maximum covered loss of 100,000.

Determine the insurer's expected annual payments.
A. Less than 7,500
B. At least 7,500, but less than 12,500
C. At least 12,500, but less than 17,500
D. At least 17,500, but less than 22,500
E. At least 22,500

17.73 (3, 5/00, Q.25) (2.5 points) An insurance agent will receive a bonus if his loss ratio is less than 70%. You are given:
(i) His loss ratio is calculated as incurred losses divided by earned premium on his block of business.
(ii) The agent will receive a percentage of earned premium equal to 1/3 of the difference between 70% and his loss ratio.
(iii) The agent receives no bonus if his loss ratio is greater than 70%.
(iv) His earned premium is 500,000.
(v) His incurred losses are distributed according to the Pareto distribution:

$F(x) = 1 - \{600{,}000 / (x + 600{,}000)\}^3$, $x > 0$.

Calculate the expected value of his bonus.
(A) 16,700 (B) 31,500 (C) 48,300 (D) 50,000 (E) 56,600

17.74 (3, 11/00, Q.27 & 2009 Sample Q.116) (2.5 points) Total hospital claims for a health plan were previously modeled by a two-parameter Pareto distribution with $\alpha = 2$ and $\theta = 500$.
The health plan begins to provide financial incentives to physicians by paying a bonus of 50% of the amount by which total hospital claims are less than 500.
No bonus is paid if total claims exceed 500.
Total hospital claims for the health plan are now modeled by a new Pareto distribution with $\alpha = 2$ and $\theta = K$. The expected claims plus the expected bonus under the revised model equals expected claims under the previous model.
Calculate K.
(A) 250 (B) 300 (C) 350 (D) 400 (E) 450

17.75 (3, 11/02, Q.37 & 2009 Sample Q.96) (2.5 points) Insurance agent Hunt N. Quotum will receive no annual bonus if the ratio of incurred losses to earned premiums for his book of business is 60% or more for the year. If the ratio is less than 60%, Hunt's bonus will be a percentage of his earned premium equal to 15% of the difference between his ratio and 60%. Hunt's annual earned premium is 800,000. Incurred losses are distributed according to the Pareto distribution, with $\theta = 500{,}000$ and $\alpha = 2$. Calculate the expected value of Hunt's bonus.
(A) 13,000 (B) 17,000 (C) 24,000 (D) 29,000 (E) 35,000

17.76 (1 point) In the previous question, (3, 11/02, Q. 37), calculate the expected value of Hunt's bonus, given that Hunt receives a (positive) bonus.
(A) 46,000 (B) 48,000 (C) 50,000 (D) 52,000 (E) 54,000

17.77 (CAS3, 11/03, Q.21) (2.5 points)
The cumulative loss distribution for a risk is $F(x) = 1 - 10^6 / (x + 10^3)^2$.
Calculate the percent of expected losses within the layer 1,000 to 10,000.
A. 10% B. 12% C. 17% D. 34% E. 41%

17.78 (SOA3, 11/03, Q.3 & 2009 Sample Q.84) (2.5 points) A health plan implements an incentive to physicians to control hospitalization under which the physicians will be paid a bonus B equal to c times the amount by which total hospital claims are under 400 ($0 \le c \le 1$). The effect the incentive plan will have on underlying hospital claims is modeled by assuming that the new total hospital claims will follow a two-parameter Pareto distribution with $\alpha = 2$ and $\theta = 300$.
E(B) = 100. Calculate c.
(A) 0.44 (B) 0.48 (C) 0.52 (D) 0.56 (E) 0.60

17.79 (SOA3, 11/04, Q.7 & 2009 Sample Q.123) (2.5 points) Annual prescription drug costs are modeled by a two-parameter Pareto distribution with $\theta = 2000$ and $\alpha = 2$.
A prescription drug plan pays annual drug costs for an insured member subject to the following provisions:
(i) The insured pays 100% of costs up to the ordinary annual deductible of 250.
(ii) The insured then pays 25% of the costs between 250 and 2250.
(iii) The insured pays 100% of the costs above 2250 until the insured has paid 3600 in total.
(iv) The insured then pays 5% of the remaining costs.
Determine the expected annual plan payment.
(A) 1120 (B) 1140 (C) 1160 (D) 1180 (E) 1200

17.80 (CAS3, 11/05, Q.22) (2.5 points) An insurance agent gets a bonus based on the underlying losses, L, from his book of business.
L follows a Pareto distribution with parameters $\alpha = 3$ and $\theta = 600{,}000$.
His bonus, B, is calculated as (650,000 - L)/3 if this quantity is positive and 0 otherwise.
Calculate his expected bonus.
A Less than 100,000
B. At least 100,000, but less than 120,000
C. At least 120,000, but less than 140,000
D. At least 140,000, but less than 160,000
E. At least 160,000

17.81 (SOA M, 11/05, Q.14) (2.5 points) You are given:
(i) T is the future lifetime random variable.
(ii) $h(t) = \mu$, $t \ge 0$, where h(t) is the hazard rate.
(iii) Var[T] = 100.
Calculate $E[T \wedge 10]$.
(A) 2.6 (B) 5.4 (C) 6.3 (D) 9.5 (E) 10.0

17.82 (CAS3, 5/06, Q.37) (2.5 points) Between 9 am and 3 pm Big National Bank employs 2 tellers to service customer transactions. The time it takes Teller X to complete each transaction follows an exponential distribution with a mean of 10 minutes. Transaction times for Teller Y follow an exponential distribution with a mean of 15 minutes. Both Teller X and Teller Y are continuously busy while the bank is open.
On average every third customer transaction is a deposit and the amount of the deposit follows a Pareto distribution with parameter $\alpha = 3$ and $\theta = \$5000$.
Each transaction that involves a deposit of at least $7500 is handled by the branch manager.
Calculate the expected total deposits made through the tellers each day.
A. Less than $31,000
B. At least $31,000, but less than $32,500
C. At least $32,500, but less than $35,000
D. At least $35,000, but less than $37,500
E. At least $37,500

17.83 (SOA M, 11/06, Q.20) (2.5 points)
For a special investment product, you are given:
(i) All deposits are credited with 75% of the annual equity index return, subject to a minimum guaranteed crediting rate of 3%.
(ii) The annual equity index return is normally distributed with a mean of 8% and a standard deviation of 16%.
(iii) For a random variable X which has a normal distribution with mean m and standard deviation σ, you are given the following limited expected values:

	E[X ∧ 3%]	
	μ = 6%	μ = 8%
σ = 12%	-0.43%	0.31%
σ = 16%	-1.99%	-1.19%
	E[X ∧ 4%]	
	μ = 6%	μ = 8%
σ = 12%	0.15%	0.95%
σ = 16%	-1.43%	-0.58%

Calculate the expected annual crediting rate.
(A) 8.9% (B) 9.4% (C) 10.7% (D) 11.0% (E) 11.6%

17.84 (SOA M, 11/06, Q.31 & 2009 Sample Q.286) (2.5 points)
Michael is a professional stuntman who performs dangerous motorcycle jumps at extreme sports events around the world.
The annual cost of repairs to his motorcycle is modeled by a two parameter Pareto distribution with $\theta = 5000$ and $\alpha = 2$.
An insurance reimburses Michael's motorcycle repair costs subject to the following provisions:
(i) Michael pays an annual ordinary deductible of 1000 each year.
(ii) Michael pays 20% of repair costs between 1000 and 6000 each year.
(iii) Michael pays 100% of the annual repair costs above 6000 until Michael has paid 10,000 in out-of-pocket repair costs each year.
(iv) Michael pays 10% of the remaining repair costs each year.
Calculate the expected annual insurance reimbursement.
(A) 2300 (B) 2500 (C) 2700 (D) 2900 (E) 3100

Solutions to Problems:

17.1. C. The distribution is an Exponential Distribution with $\theta = 1/2$.
For the Exponential Distribution $E[X \wedge x] = \theta\ (1 - e^{-x/\theta})$.
The average size of the capped losses is: $E[X \wedge 1] = (1/2)(1 - e^{-2}) = 0.432$.
Thus the expected annual total loss payments on a basic limits policy are: (13)(0.432) = **5.62**.
Alternately, one can use the relation between the mean excess loss and the Limited Expected Value: $e(x) = \{\text{mean} - E[X \wedge x]\} / \{1 - F(x)\}$, therefore $E[X \wedge x] = \text{mean} - e(x)\{1 - F(x)\}$.
For the Exponential Distribution, the mean excess loss is a constant = θ = mean.
Therefore $E[X \wedge x] = \text{mean} - e(x)\{1 - F(x)\} = \theta - \theta(e^{-x/\theta})$. Proceed as before.

17.2. B. mean = $\exp(\mu + \sigma^2/2) = 98{,}716$. Therefore, with 200 claims expected per year, the expected total cost per year is: (200)(98716) = $**19.74 million.**

17.3. A. $E[X \wedge x] = \exp(\mu + \sigma^2/2)\Phi[(\ln x - \mu - \sigma^2)/\sigma] + x\ \{1 - \Phi[(\ln x - \mu)/\sigma]\}$.
$E[X \wedge 1 \text{ mill.}] = \exp(7 + 9/2)\Phi[(\ln(1{,}000{,}000) - 7 - 9)/3] + (1{,}000{,}000)\{1 - \Phi[(\ln(1{,}000{,}000) - 7)/3]\}$
$= (98{,}716)\Phi[-0.73] + (1{,}000{,}000)(1-\Phi[2.27]) = (98{,}716)(1 - 0.7673) + (1{,}000{,}000)(1 - 0.9884)$
$= 22{,}971 + 11{,}600 = 34{,}571$.
With a limit of $1 million per claim and 200 claims expected per year, the expected total cost per year is: 200 $E[X \wedge 1 \text{ million}]$ = (200)(34,571) = **$6.91 million**.

17.4. E. $E[X \wedge 5 \text{ million}] = \exp(7 + 9/2)\Phi[(\ln(5{,}000{,}000) - 7 - 9)/3] +$
$(5{,}000{,}000)\ \{1 - \Phi[(\ln(5{,}000{,}000) - 7)/3]\} = (98{,}716)\Phi[-0.19] + (5{,}000{,}000)(1 - \Phi[2.81])$
$= (98{,}716)(1 - 0.5753) + (5{,}000{,}000)(1 - 0.9975) = 41{,}925 + 12{,}500 = 54{,}425$.
200 $E[X \wedge 5 \text{ million}]$ = **$10.88 million.**

17.5. D. The dollars in the layer from $1 million to $5 million is the difference between the dollars limited to $5 million and the dollars limited to $1 million. Using the answers to the two previous questions: $10.88 million - $6.91 million = **$3.97 million.**
Comment: In terms of the limited expected values and the expected number of losses N, the dollars in the layer from $1 million to $5 million equals: $N\{E[X \wedge 5 \text{ million}] - E[X \wedge 1 \text{ million}]\}$.
In this case N = 200.

17.6. C. The dollars excess of $5 million per loss is the difference between the total cost and the cost limited to $5 million per loss. Using the answers to two prior questions:
$19.74 million - $10.88 million = **$8.86 million.**
Comment: The dollars excess of $5 million per losses equals:
$N\{E[X \wedge \infty] - E[X \wedge 5 \text{ million}]\} = N\{\text{mean} - E[X \wedge 5 \text{ million}]\}$. In this case N = 200 losses.

17.7. D. First calculate the dollars of loss on these losses per total number of losses:
$\{E[X \wedge 5 \text{ million}] - 5 \text{ million } S(5 \text{ million})\} - \{E[X \wedge 1 \text{ million}] - 1 \text{ million } S(1 \text{ million})\}$ =
{54,425 - (5 million)(1 - 0.9975)} - {34,517 - (1 million)(1 - 0.9884)} = 41,925 - 22,917 = $19,008.
Then divide by the probability of a loss being of this size:
$F(5 \text{ million}) - F(1 \text{ million}) = \Phi[(\ln(5{,}000{,}000) - 7)/3] - \Phi[(\ln(1{,}000{,}000) - 7)/3]$ =
$\Phi[2.81] - \Phi[2.27]$ = (0.9975 - 0.9884) = 0.0091. $19,008 / 0.0091 = **$2.09 million.**

17.8. C. Either one can calculate the expected number of losses of this size per year as (200){F(5 million) - F(1 million)} = (200){0.9975 - 0.9884} = 1.8 and multiply by the average size calculated in the previous question. (1.8)($2.09 million) = **$3.8 million.** Alternately, one can multiply the expected number of losses per year times the dollars on these losses per loss calculated in a previous question: (200)($19,008) = $3.8 million.

17.9. E. For the Pareto Distribution, $E[X \wedge x] = \{\theta/(\alpha-1)\}\{1-(\theta/(\theta+x))^{\alpha-1}\}$.
$E[X \wedge 250{,}000] = \{15{,}000/1.5\}\{1 - (15{,}000/(15{,}000+250{,}000))^{2.5-1}\}$ = 10,000(0.9865) = **9865.**

17.10. E. The mean of a Pareto is: $\theta/(\alpha-1)$ = 1800/3 = **600**.

17.11. B. For the Pareto Distribution, $E[X \wedge x] = \{\theta/(\alpha-1)\}\{1 - (\theta/(\theta+x))^{\alpha-1}\}$.
$E[X \wedge 900] = \{1800/3\}\{1 - (1800/(1800+900))^3\}$ = **422.22.**

17.12. D. For the Pareto Distribution, $E[X \wedge x] = \{\theta/(\alpha-1)\}\{1 - (\theta/(\theta+x))^{\alpha-1}\}$.
$E[X \wedge 225] = \{1800/3\}\{1 - (1800/(1800+225))^3\}$ = **178.60.**

17.13. **B.** The average weekly wage is \$600, from a previous solution. Thus the maximum benefit is \$600, while the minimum benefit is \$600/4 = \$150. These correspond to pre-injury wages of \$600/(2/3) = \$900 and \$150/(2/3) = \$225 respectively. (If a workers pre-injury wage is more than \$900 his benefit is only \$600. If his pre-injury wage is less than \$225, his benefit is still \$150.)
Let x be the worker's pre-injury wage, then the worker's benefits are:
\$150 if x ≤ \$225, 2x/3 if x ≥ \$225 and x ≤ \$900, \$600 if x ≥ \$900.
Thus the average benefit is made up of three terms (low, medium, and high wages):

$$150\ F(225) + (2/3) \int_{225}^{900} x\ f(x)\ dx\ + 600\ S(900).$$

$$\int_{225}^{900} x\ f(x)\ dx = \int_{0}^{900} x\ f(x)\ dx - \int_{0}^{225} x\ f(x)\ dx = E[X \wedge 900] - 900S(900) - \{E[X \wedge 225] - 225S(225)\}.$$

Thus the average benefit is:
150F(225) + 150S(225) + 600S(900) - 600S(900) + (2/3)(E[X∧900] - E[X∧225]) =
150 + (2/3)(E[X∧900] -E[X∧225]) = 150 + (2/3)(422.22 - 178.60) = **312.41**.
Alternately, the benefits can be described as:
150 + (2/3)(layer of wages between 900 and 225) = 150 + (2/3)(E[X∧900] -E[X∧225]).
<u>Comment</u>: Extremely unlikely to be asked on the exam. *Relates to the calculation of Law Amendment Factors used in Workers' Compensation Ratemaking. Geometrically oriented students may benefit by reviewing the subsection on payments subject to both a minimum and a maximum in the subsequent section on Lee Diagrams.*

17.14. E. E[X] = θ = **10,000**.

17.15. C. E[X∧25,000] = 10,000 $(1-e^{-25,000/10,000})$ = **9179**.

17.16. B. E[X∧x] = θ $(1 - e^{-x/\theta})$. E[X∧1000] = 10,000 $(1 - e^{-1000/10,000})$ = 952.
E[X] - E[X∧1000] = 10,000 - 952 = **9048**.

17.17. E. {E[X] - E[X∧1000]}/S(1000) = (10,000 - 952)/0.9048 = **10,000**.
Alternately, the average size of the data truncated and shifted from below is the mean excess loss. For the Exponential e(x) = θ = **10,000**.
<u>Comment</u>: For the Exponential, {E[X] - E[X∧x]}/S(x) = θ. Thus for the Exponential Distribution, in the absence of any maximum covered loss, the average size of the insurer's payment per non-zero payment by the insurer does not depend on the deductible amount; the mean excess loss is constant for the Exponential Distribution.

17.18. A. E[X∧25,000] = 10,000 $(1 - e^{-25,000/10,000})$ = 9179**.**
E[X∧25,000] - E[X∧1000] = 9179 - 952 = **8,227**.

17.19. B. {E[X∧25,000] - E[X∧1000]} / S(1000) = (9179 - 952)/0.9048 = **9093**.

17.20. E. Each payment is 75% of the insured's loss, so the average is:
(0.75)E[X] = (0.75)(10,000) = **7500.**

17.21. B. Each payment is 75% of the what it would have been without any coinsurance, so the average is: $(0.75)(E[X] - E[X \wedge 1000]) = (0.75)(10,000 - 952) =$ **6786.**

17.22. C. Each payment is 75% of the what it would have been without any coinsurance, so the average is $(0.75)(E[X \wedge 25,000] - E[X \wedge 1000])/ S(1000) = (0.75)(9179 - 952)/0.9048 =$ **6819.**

17.23. D. Average Size of Losses in the Interval [1000, 25,000] =
$\{E[X \wedge 25,000] - 25,000S(25,000) - (E[X \wedge 1000] - 1000S(1000))\}/ \{F(25,000) - F(1000)\} =$
$\{9179 - 25,000(0.08208) - (952 - (1000)(0.9048))\} / (0.9179 - 0.0952) = 7080/0.8227 =$ **8606.**

17.24. A. $\{E[X \wedge 25,000] - 25,000S(25,000) - (E[X \wedge 1000] - 1000S(1000))\} / E[X] =$
$\{9179 - 25,000(0.08208) - (952 - (1000)(0.9048))\} / 10,000 = 7080/10,000 =$ **70.8%.**

17.25. C. The payments from 500 to 4000 correspond to losses of size between 1500 and 5000. These losses have average size:
$\{E[X \wedge 5000] - 5000S(5000) - (E[X \wedge 1500] - 1500S(1500))\} / \{F(5000) - F(1500)\} =$
$\{3935 - 5000(0.6065) - (1393 - (1500)(0.8607))\} / (0.3935 - 0.1393) = 3148.$
The average size of the payments is 1000 less: 3148 - 1000 = **2148.**

17.26. B. The payments from 500 to 4000 correspond to losses of size between
1000 + (500/0.75) = 1667 and 1000+(4000/0.75) = 6333. These losses have average size:
$\{E[X \wedge 6333] - 6333S(6333) - (E[X \wedge 1667] - 1667S(1667))\} / \{F(6333) - F(1667)\} =$
$\{4692 - 6333(0.5308) - (1535 - (1667)(0.8465))\} / (0.4692 - 0.1535) = 3820.$
The average size of the payments is 1000 less and then multiplied by 0.75:
(3820 - 1000)(0.75) = **2115.**

17.27. B. The most the insurer will pay is: (0.75)(25,000 - 1000) = 18,000.
For any loss of size greater than or equal to 25,000 the insurer pays 18,000.
Let X be the size of loss.
Then the payment is 18,000 if X ≥ 25,000, and (0.75)(X - 1000) if 25,000 > X > 1000.
A payment of 15,000 corresponds to a loss of: (15,000/0.75) + 1000 = 21,000.
Thus the dollars of payments greater than 15,000 and at most 19,000 is the payments on losses of size greater than 21,000, which we split into two pieces:

$$\int_{21,000}^{25,000} 0.75(x - 1000)\, f(x)\, dx + 18,000\, S(25,000) =$$

$$0.75 \int_{21,000}^{25,000} x\, f(x)\, dx - 750\{F(25,000)-F(21,000)\} + 18,000S(25,000) =$$

$0.75\,\{E[X \wedge 25,000] - 25,000S(25,000) - (E[X \wedge 21,000] - 21,000S(21,000))\}$
$\quad + 750\{S(25,000) - S(21,000)\} + 18,000S(25,000) =$
$0.75E[X \wedge 25,000] - 0.75E[X \wedge 21,000] + 15,750S(21,000) - 18,750S(25,000) - 750S(21,000)$
$\quad + 750S(25,000) + 18,000S(25,000) =$
$0.75E[X \wedge 25,000] - 0.75E[X \wedge 21,000] + 15,000S(21,000) =$
0.75(9179.2) - (0.75)(8775.4) + 15,000(0.12246) = 2139.8.
In order to get the average size we need to divide the payments by the percentage of the number of losses represented by losses greater than 21,000, S(21,000) = 0.12246:
2139.8 / 0.12246 = **17,473.**
<u>Comment</u>: Long and difficult. In this case it may be easier to calculate the integral of xf(x)dx, rather than put it in terms of the Limited Expected Values and Survival Functions.

17.28. D. $0.75\{E[X \wedge 25,000] - E[X \wedge 1000]\}$ = (0.75)(9179 - 952) = **6170.**

17.29. B. A loss ratio of 60% corresponds to (0.6)(40000) = $24,000 in losses.
If his losses are x, and x < 24,000, then he gets a dividend of (1/4)(24,000 - x).
The expected dividend is:

$$(1/4) \int_0^{24,000} (24,000 - x)\, f(x)\, dx = (1/4)\{24,000\, F(24,000) - (E[X \wedge 24,000] - 24,000S(24,000))\}$$

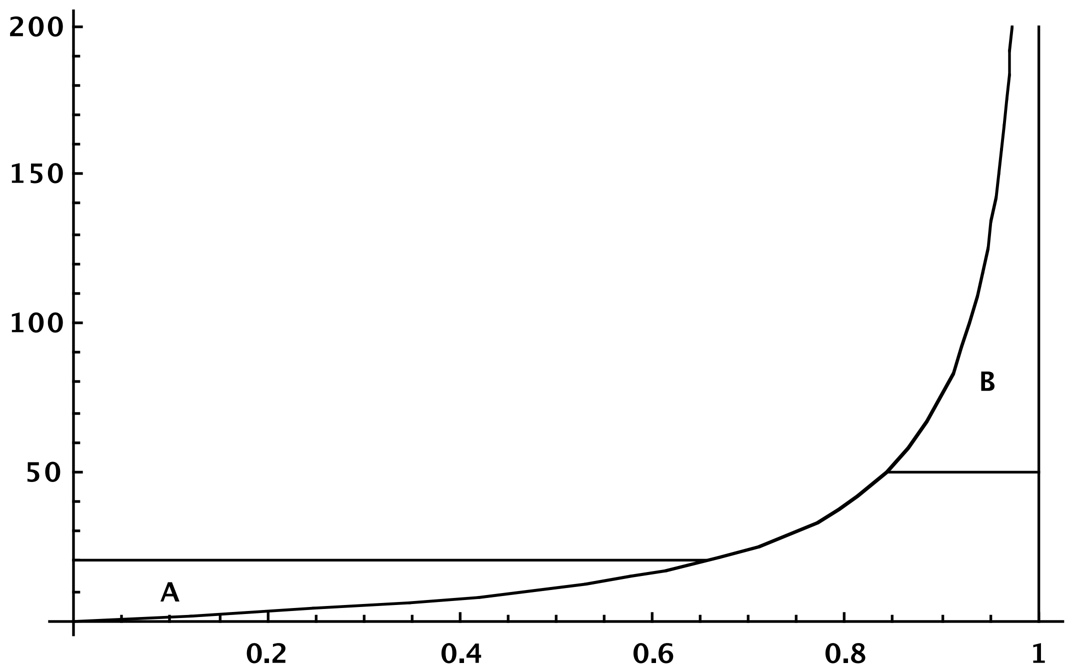
$= (1/4)\{24,000 - (E[X \wedge 24,000]\}$. For a LogNormal Distribution,
$E[X \wedge x] = \exp(\mu + \sigma^2/2)\, \Phi[(\ln x - \mu - \sigma^2)/\sigma] + x\,\{1 - \Phi[(\ln x - \mu)/\sigma]\}$. Therefore,
$E[X \wedge 24,000] = \exp(6 + 9/2)\Phi[(\ln 24,000 - 6 - 9)/3] + 24,000\,\{1 - \Phi[(\ln 24,000 - 6)/3]\} =$
$(36,316)\Phi[-1.64] + (24,000)\{1 - \Phi[1.36]\} = (36,316)(1 - 0.9495) + (24,000)(1 - 0.9131) = 3920.$
Therefore, his expected dividend is: (1/4)(24,000 - 3920) = $**5020**.

17.30. E. If x < 20, then Frostbite Falls pays the state fund 1000(20 - x).
The expected amount by which x is less than 20, (the “savings” at 20), is: 20 - E[X ∧ 20].
E[X ∧ 20] = (8 + 10 + 16 + 100)/8 = 16.75.
Therefore, the expected amount paid by the town to the state fund per winter is:
(1000)(20 - E[X ∧ 20]) = 3250.
If x > 50, then the state fund pays Frostbite Falls 1000(x - 50). The expected amount by which x is more than 50, (the inches of snow excess of 50), is: E[X] - E[X ∧ 50].
E[X] = (8 + 10 + 16 + 21 + 35 + 57 + 70 + 90)/8 = 38.375.
E[X ∧ 50] = (8 + 10 + 16 + 21 + 35 + 150)/8 = 30.
Therefore, the expected amount paid by the state fund to the town per winter is:
(1000)(E[X] - E[X ∧ 50]) = (1000)(38.375 - 30) = 8375.
Expected amount state fund pays town minus expected amount town pays the state fund is:
8375 - 3250 = **5125**.
Alternately, one can list what happens in each possible situation:

Snow	Paid by State
8	-12,000
10	-10,000
16	-4,000
21	0
35	0
57	7,000
70	20,000
90	40,000
Average	**5,125**

Comment: (12 + 10 + 4)/8 = 3.250 = 20 - E[X ∧ 20]. (7 + 20 + 40)/8 = 8.375 = E[X] - E[X ∧ 50].
A very simplified example of retrospective rating.
See Introduction to Ratemaking and Loss Reserving for Property and Casualty Insurance.

17.31. A. If x < 20, then Frostbite Falls pays the state fund 1000(20 - x).
The expected amount by which x is less than 20, (the "savings" at 20), is: 20 - E[X ∧ 20].
$E[X \wedge 20] = \exp(\mu + \sigma^2/2)\ \Phi[(\ln 20 - \mu - \sigma^2)/\sigma] + 20\{1 - \Phi[(\ln 20 - \mu)/\sigma]\} =$
$(33.954)\Phi[-1.10] + (20)\{1 - \Phi[0.40]) = (33.954)(0.1357) + (20)(1 - 0.6554) = 11.50.$
Therefore, the expected amount paid by the town to the state fund per winter is:
(1000)(20 - E[X ∧ 20]) = 8500.
If x > 50, then the state fund pays Frostbite Falls 1000(x - 50). The expected amount by which x is more than 50, (the inches of snow excess of 50), is: E[X] - E[X ∧ 50].
$E[X] = \exp(\mu + \sigma^2/2) = 33.954.$
$E[X \wedge 50] = \exp(\mu + \sigma^2/2)\ \Phi[(\ln 50 - \mu - \sigma^2)/\sigma] + 50\{1 - \Phi[(\ln 50 - \mu)/\sigma]\} =$
$(33.954)\Phi[-0.49] + (20)\{1 - \Phi[1.01]) = (33.954)(0.3121) + (50)(1 - 0.8438) = 18.41.$
Therefore, the expected amount paid by the state fund to the town per winter is:
(1000)(E[X] - E[X ∧ 50]) = (1000)(33.954 - 18.41) = 15,544.
Expected amount state fund pays town minus expected amount town pays the state fund is:
15,544 - 8500 = **7,044**.
<u>Comment</u>: In the following Lee Diagram, other than the constant c = 1000, the expected amount paid by the town to the state (when there is little snow) corresponds to Area A, below a horizontal line at 20 and above the curve. Other than the constant c = 1000, the expected amount paid by the state to the town (when there is a lot of snow) corresponds to Area B, above a horizontal line at 50 and below the curve.



17.32. C. $E[(N-3)_+] = E[N] - E[N \wedge 3] = \lambda$ - (Prob[N = 1] + 2Prob[N = 2] + 3Prob[N ≥ 3]) =
$\lambda - \lambda e^{-\lambda} - \lambda^2 e^{-\lambda} - (3)(1 - e^{-\lambda} - \lambda e^{-\lambda} - \lambda^2 e^{-\lambda}/2) = \lambda + 3e^{-\lambda} + 2\lambda e^{-\lambda} + \lambda^2 e^{-\lambda}/2 - 3 =$
$2.5 + 3e^{-2.5} + 2(2.5)\lambda e^{-2.5} + (2.5^2)e^{-2.5}/2 - 3 =$ **0.413**.
Alternately, $E[(N-3)_+] = E[(3-N)_+] + E[N] - 3$ = 3Prob[N = 0] + 2Prob[N = 1] + Prob[N = 2] + λ - 3 =
$\lambda + 3e^{-\lambda} + 2\lambda e^{-\lambda} + \lambda^2 e^{-\lambda}/2 - 3 =$ **0.413**.

17.33. C. The expected amount by which lifetimes are less than 2 is:
$2 - E[X \wedge 2] = 2 - (6)(1 - e^{-2/6}) = 0.2992$.
The expected amount paid per battery is: (100)(0.2992/2) = **14.96**.

17.34. B. For the LogNormal Distribution,
$E[X \wedge x] = \exp(\mu + \sigma^2/2)\ \Phi[(\ln x - \mu - \sigma^2)/\sigma] + x\ \{1 - \Phi[(\ln x - \mu)/\sigma]\}$.
$E[X \wedge$ 30 million] =
$\exp(15 + 2^2/2)\Phi[(\ln 30{,}000{,}000 - 15 - 2^2)/2] + 30{,}000{,}000\ \{1 - \Phi[(\ln 30{,}000{,}000 - 15)/2]\} =$
$24{,}154{,}953\ \Phi[-0.89] + 30{,}000{,}000\ \{1 - \Phi[1.11]\} =$
(24,154,953)(0.1867) + (30,000,000)(1 - 0.8665) = 8.51 million.
$E[X \wedge$ 150 million] =
$\exp(15 + 2^2/2)\ \Phi[(\ln 15{,}0000{,}000 - 15 - 2^2)/2] + 150{,}000{,}000\ \{1 - \Phi[(\ln 150{,}000{,}000 - 15)/2]\} =$
$24{,}154{,}953\ \Phi[-0.09] + 150{,}000{,}000\ \{1 - \Phi[1.91]\} =$
(24,154,953)(0.4641) + (150,000,000)(1 - 0.9719) = 15.43 million.
The maximum payment of \$90 million correspond to a loss by XYZ of: 30 + 90/0.75 =
150 million. Therefore the average payment to XYZ per hurricane is:
0.75 ($E[X \wedge$ 150 million] - $E[X \wedge$ 30 million]) = (0.75)(15.43 - 8.51) = **5.2 million**.
<u>Comment</u>: The portion of hurricanes on which XYZ receives non-zero payments is:
S(30 million) = $1 - \Phi[(\ln 30{,}000{,}000 - 15)/2] = 1 - \Phi[1.11] = 0.01335$.
Therefore, the average payment per nonzero payment is: (0.75)(15.43 - 8.51) / 0.1335 =
38.9 million. *A very simplified version of the Florida Hurricane Catastrophe Fund.*

17.35. C. Per loss, the insurer would pay the layer from 5,000 to 25,000, which is:
$E[X \wedge 25{,}000] - E[X \wedge 5{,}000]$. For the Pareto: $E[X \wedge x] = \{\theta/(\alpha-1)\}\{1-(\theta/(\theta+x))^{\alpha-1}\} =$
$10{,}000\ \{1 - (40{,}000/(40{,}000+x))^4\}$. $E[X \wedge 25{,}000] = 10{,}000\ \{1 - (40/65)^4\} = 8566$.
$E[X \wedge 5{,}000] = 10{,}000\ \{1 - (40/45)^4\} = 3757$. $E[X \wedge 25{,}000] - E[X \wedge 5{,}000] = 8566 - 3757 = 4809$.
Three losses expected per year, thus the insurer’s expected payment is: (3)(4809) = **14,427**.

17.36. E. Without the feature that the insurer pays the entire loss (up to 25,000) for each loss greater than 5,000, the insurer would pay the layer from 5,000 to 25,000, which is:
$E[X \wedge 25{,}000] - E[X \wedge 5{,}000]$. As calculated in the solution to the previous question,
$E[X \wedge 25{,}000] - E[X \wedge 5{,}000] = 8566 - 3757 = 4809$. However, that extra provision adds 5,000 per large loss, or $5{,}000S(5000) = 5{,}000\{\theta/(\theta+5000)\}^\alpha = 5000\ (40/45)^5 = 2775$. Thus per loss the insurer pays: $5{,}000(S(5000) + E[X \wedge 25{,}000] - E[X \wedge 5{,}000] = 2775 + 4809 = 7584$.
There are three losses expected per year, thus the insurer’s expected payment is:
(3)(7584) = **22,752**.

17.37. E. $E[X \wedge x] = \theta (1 - e^{-x/\theta})$. $E[X \wedge 50{,}000] = 20{,}000 (1 - e^{-50{,}000/20{,}000}) = 18{,}358$.
$E[X \wedge 5000] = 20{,}000 (1 - e^{-5000/20{,}000}) = 4424$.
$(E[X \wedge 50{,}000] - E[X \wedge 5000]) / E[X] = (18{,}358 - 4424) / 20{,}000 =$ **69.7%**.
Alternately, the expected losses within the layer 5,000 to 50,000 is:

$$\int_{5000}^{50{,}000} S(x)\, dx = \int_{5000}^{50{,}000} e^{-x/20{,}000}\, dx = 13{,}934.$$

The percent of expected losses within the layer 5,000 to 50,000 is: 13,934/20,000 = **69.7%**.
Alternately, for the Exponential Distribution, $LER(x) = 1 - e^{-x/\theta}$.
$LER(50{,}000) - LER(5000) = e^{-5000/20{,}000} - e^{-50{,}000/20{,}000} = e^{-0.25} - e^{-2.5} =$ **69.7%**.

17.38. D. $E[X] = \exp(\mu + \sigma^2/2) = e^{9.9} = 19{,}930$.
$E[X \wedge 5000] = \exp(\mu + \sigma^2/2)\, \Phi[(\ln 5000 - \mu - \sigma^2)/\sigma] + 5000\{1 - \Phi[(\ln 5000 - \mu)/\sigma]\} =$
$(19{,}930)\Phi[-1.88] + (5000)\{1 - \Phi[-0.88]) = (19{,}930)(0.0301) + (5000)(0.8106) = 4653$.
$E[X \wedge 50{,}000] = \exp(\mu + \sigma^2/2)\, \Phi[(\ln 50{,}000 - \mu - \sigma^2)/\sigma] + 50{,}000\{1 - \Phi[(\ln 50{,}000 - \mu)/\sigma]\} =$
$(19{,}930)\Phi[0.42] + (50{,}000)(1 - \Phi[1.42]) = (19{,}930)(0.6628) + (50{,}000)(1 - 0.9222) = 17{,}100$.
The percent of expected losses within the layer 5,000 to 50,000 is:
$(E[X \wedge 50{,}000] - E[X \wedge 5000])/E[X] = (17{,}100 - 4653)/19{,}930 =$ **62.5%**.

17.39. C. $E[X \wedge x] = \{\theta/(\alpha-1)\}\{1 - (\theta/(\theta + x))^{\alpha-1}\} = 20{,}000\{1 - (40{,}000/(40{,}000 + x))^2\}$.
$E[X \wedge 5000] = 4198$. $E[X \wedge 50{,}000] = 16{,}049$. $E[X] = \theta/(\alpha-1) = 20{,}000$.
The percent of expected losses within the layer 5,000 to 50,000 is:
$(E[X \wedge 50{,}000] - E[X \wedge 5000])/E[X] = (16{,}049 - 4198)/20{,}000 =$ **59.3%**.

17.40. A. $E[(N - 3)_+] = E[N] - E[N \wedge 3] = \beta - (\text{Prob}[N = 1] + 2\text{Prob}[N = 2] + 3\text{Prob}[N \geq 3]) =$
$\beta - \beta/(1+\beta)^2 - 2\beta^2/(1+\beta)^3 - 3\beta^3/(1+\beta)^3 = \{\beta(1+\beta)^3 - \beta(1+\beta) - 2\beta^2 - 3\beta^3\}/(1+\beta)^3 = \beta^4/(1+\beta)^3$
$= 2.5^4/3.5^3 =$ **0.911**.
Alternately, $E[(N-3)_+] = E[(3-N)_+] + E[N] - 3 = 3\text{Prob}[N = 0] + 2\text{Prob}[N = 1] + \text{Prob}[N = 2] + \beta - 3 =$
$\beta + 3/(1+\beta) + 2\beta/(1+\beta)^2 + \beta^2/(1+\beta)^3 - 3 = 2.5 + 3/3.5 + (2)(2.5)/3.5^2 + 2.5^2/3.5^3 - 3 =$ **0.911**.
Alternately, the Geometric shares the memoryless property of the Exponential $\Rightarrow$
$E[(N-3)_+] / \text{Prob}[N \geq 3] = E[N] = \beta$. $\Rightarrow E[(N-3)_+] = \beta\, \text{Prob}[N \geq 3] = \beta\, \beta^3/(1+\beta)^3 = \beta^4/(1+\beta)^3 =$ **0.911**.
<u>Comment</u>: For integral j, for the Geometric, $E[(N - j)_+] = \beta^{j+1}/(1+\beta)^j$.

17.41. E. For Policy A the average payment per loss is: $E[X] - E[X \wedge 3000] =$
$\theta/(\alpha-1) - \{\theta/(\alpha-1)\}\{1 - (\theta/(\theta+3000))^{\alpha-1}\} = 6000(12/15)^2 = 6000(0.64)$.
For Policy B the average payment per loss is: $E[X \wedge u] = \{\theta/(\alpha-1)\}\{1 - (\theta/(\theta+u))^{\alpha-1}\} =$
$6000\{1 - (12{,}000/(12{,}000+u))^2\}$. Setting this equal to 6000(0.64):
$6000(0.64) = 6000\{1 - (12{,}000/(12{,}000+u))^2\} \Rightarrow (12{,}000/(12{,}000+u))^2 = 0.36. \Rightarrow u =$ **8000**.

17.42. B. $E[(25 - X)_+] = (25 - 5)(80\%) + (0)(20\%) = 16 > 8. \Rightarrow$ b must be less than 25.
Therefore, $E[(b - X)_+] = (0.8)(b - 5) = 8. \Rightarrow b =$ **15**.

17.43. D. $E[Y] = E[(1 - X)_+] = 1 - E[X \wedge 1] = 1 - \theta(1 - e^{-1/\theta}) = 1 - 2(1 - e^{-1/2}) =$ **0.213**.

Alternately, $E[Y] = \int_0^1 (1 - x)\, e^{-x/2}/2 \, dx = -e^{-x/2} + xe^{-x/2} + 2e^{-x/2} \Big]_{x=0}^{x=1} = 2e^{-1/2} - 1 =$ **0.213**.

17.44. D. Since by definition E[R] = 1, the LogNormal Distribution has mean of 1.
$\exp[\mu + \sigma^2/2] = 1. \Rightarrow \mu = -\sigma^2/2 = -0.08.$
Percentage of overall wages earned by workers with R < 2 is:
$\{E[X \wedge 2] - 2S(2)\} / E[X] = \Phi[(\ln 2 - \mu - \sigma^2)/\sigma] = \Phi[(\ln 2 + 0.08 - 0.4^2)/0.4] = \Phi[1.53] =$ **93.7%**.
Comment: *Such wage tables are used to price the impact of changes in the laws governing Workers Compensation Benefits.*

17.45. B. Each loss below 50 is counted as its size, while each of the 19 losses greater than or equal to 50 counts as 50. $E[X \wedge 50] =$
{6 + 7+ 11+ 14 + 15+17+ 18 + 19 + 25+ 29 + 30 + 34 + 40 + 41 + 48 + 49 + (19)(50)} / 35 =
(403 + 950)/35 = **38.66**.

17.46. C. 100/2% = 5000. If the profit is less then 5000, then Alex gets 100.
Thus we want: E[Max[0.02 X, 100]] = 0.02 E[max[X, 5000]].

$$E[\text{Max}[X, 5000]] = 5000F(5000) + \int_{5000}^{\infty} x f(x) dx = 5000F(5000) + \int_0^{\infty} x f(x) dx - \int_0^{5000} x f(x) dx$$

$= 5000F(5000) + E[X] - \{E[X \wedge 5000] - 5000S(5000)\} = 5000 + E[X] - E[X \wedge 5000].$
0.02 E[Max[X, 5000] = **100 + 0.02(E[X] - E[X ∧ 5000])**.
Alternately, let Y = max[X, 5000]. Y - 5000 = 0 if X ≤ 5000, and Y - 5000 = X - 5000 if X > 5000.
Therefore, $E[Y - 5000] = E[(X - 5000)_+] = E[X] - E[X \wedge 5000].$
$\Rightarrow E[Y] = 5000 + E[X] - E[X \wedge 5000].$
Expected value of Alex's pay is: 0.02E[Y] = **100 + 0.02(E[X] - E[X ∧ 5000])**.
Comment: Similar to SOA M, 11/06, Q.20.

17.47. B. F is a Pareto Distribution with $\alpha = 3$ and $\theta = 20{,}000$.
E[X] = 20,000/(3 - 1) = 10,000.
$E[X \wedge 5000] = (10{,}000)(1 - \{20{,}000/(20{,}000 + 5000)\}^2) = 3600.$
Alex's expected payment is: $100 + 0.02(E[X] - E[X \wedge 5000]) = 100 + (0.02)(10{,}000 - 3600) =$ **228**.

17.48. C. $\Gamma[3 ; 5] = 1 - e^{-5}(1 + 5 + 5^2/2) = 0.875.$ $\Gamma[4 ; 5] = 1 - e^{-5}(1 + 5 + 5^2/2 + 5^3/6) = 0.735.$
For the Gamma Distribution, $E[X \wedge 500] = (\alpha\theta)\Gamma[\alpha+1 ; 500/\theta] + 500 \{1 - \Gamma[\alpha ; 500/\theta]\} =$
$300\Gamma[4 ; 5] + 500\{1 - \Gamma[3 ; 5]\} = (300)(0.735) + (500)(1 - 0.875) =$ **283**.

17.49. A. $E[X \wedge x] = \alpha\theta/(\alpha - 1) - \theta^3/\{(\alpha - 1)x^{\alpha-1}\}$.

$E[L \wedge 170{,}000] = (3)(100{,}000)/(3 - 1) - 100{,}000^3/\{(3 - 1)\ 170{,}000^{3-1}\} = 132{,}699$.

$E[(170{,}000 - L)_+] = 170{,}000 - E[L \wedge 170{,}000] = 170{,}000 - 132{,}699 = 37{,}301$.

$E[\text{Bonus}] = E[(170{,}000 - L)_+/4] = 37{,}301/4 =$ **9325**.

Comment: Similar to CAS3, 11/05, Q.22.

17.50. His bonus is positive when L < 170,000.

$$F(170{,}000) = 1 - \left(\frac{100{,}000}{170{,}000}\right)^3 = 0.79646.$$

E[Bonus | Bonus > 0] = E[Bonus] / Prob[Bonus > 0] = E[Bonus] / F(170,000) =
9325 / 0.79646 = **11,708**.

17.51. A. $E[X \wedge 1000] = \int_0^{1000} x\, f(x)\, dx + 1000\ S(1000) = 350.61 + (1000)(1 - 0.87175) =$ **478.86**.

Comment: Based on a LogNormal Distribution with $\mu = 6.0$ and $\sigma = 0.8$.

17.52. D. For this Pareto Distribution, $E[X \wedge x] = (5000/2)\ \{1 - 5000^2/(5000 + x)^2\}$.

$E[X \wedge 2000] = 1224$. $E[X \wedge 10{,}000] = 2222$. $E[X] = \theta/(\alpha-1) = 2500$.

The average payment per loss is:

$(75\%)\ E[X \wedge 2000] + (90\%)(E[X] - E[X \wedge 10{,}000]) = (75\%)(1224) + (90\%)(2500 - 2222) =$ **1168**.

17.53. E. The mean of the Single Parameter Pareto is: $\alpha\theta/(\alpha - 1) = (1.5)(20)/(1.5 - 1) = 60$.
Thus we want the average size of loss for those losses of size greater than 60.

$E[X \wedge x] = \alpha\theta/(\alpha - 1) - \theta^\alpha / \{x^{\alpha-1}(\alpha - 1)\}$.

$E[X \wedge 60] = (1.5)(20)/(1.5 - 1) - 20^{1.5} / \{60^{1.5-1}(1.5 - 1)\} = 36.906$.

Average size of loss for those losses of size greater than 60 is:

$\{E[X] - (E[X \wedge 60] - 60S(60))\}/S(60) = (60 - 36.906)/\{(20/60)^{1.5}\} + 60 = 180$.

Taking into account the 50% increase: (1.5)(180) = **270**.

Alternately, the average size of those losses of size greater than 60 is:

$$\int_{60}^{\infty} x\, f(x)\, dx\ /\ S(60) = \frac{\int_{60}^{\infty} x\ 1.5\ 20^{1.5}\ x^{-2.5}\ dx}{(20/60)^{1.5}} = (1.5)\ (60^{1.5}) \int_{60}^{\infty} x^{-1.5}\ dx$$

$= (1.5)(60^{1.5})\ (2)(60^{-0.5}) = 180$. Taking into account the 50% increase: (1.5)(180) = **270**.

17.54. The contribution of the small losses, those losses of size less than x is: $E[X \wedge x] - x\,S(x)$.

The percentage of loss dollars from those losses of size less than x is: $\frac{E[X \wedge x] - x\,S(x)}{E[X]}$.

For the LogNormal Distribution, $E[X \wedge x] - x\,S(x) = \exp[\mu + \sigma^2/2]\,\Phi[\frac{\ln[x] - \mu - \sigma^2}{\sigma}]$.

Thus for the LogNormal Distribution, $G(x) = \frac{E[X \wedge x] - x\,S(x)}{E[X]} = \Phi[\frac{\ln[x] - \mu - \sigma^2}{\sigma}]$.

G(x) is also LogNormal with parameters: $\mu + \sigma^2$, and σ.

17.55. $0.25 = 1 - \exp[-Q_{0.25} / \theta]$. $\Rightarrow Q_{0.25} = \theta \ln[4/3]$.

$0.75 = 1 - \exp[-Q_{0.75} / \theta]$. $\Rightarrow Q_{0.75} = \theta \ln[4]$.

$$\int_{\theta\ln[4/3]}^{\theta\ln[4]} x \exp[-x/\theta]\,dx = -x\exp[-x/\theta] - \theta\exp[-x/\theta] \Big]_{x=\theta\ln[4/3]}^{x=\theta\ln(4)} = \theta \ln[4/3]\,3/4 + \theta\,3/4 - \theta\ln[4]/4 - \theta/4$$

$= \theta\{1/2 + \ln[4]/2 - \ln[3]\,3/4\}$.

One half of the total probability is between the first and third quartile.

Trimmed Mean = $\theta\{1/2 + \ln[4]/2 - \ln[3]\,3/4\} / (1/2) = \theta\{1 + \ln[4] - \ln[3]\,3/2\}$ = **$0.7384\,\theta$**.

Alternately, $E[X \wedge x] = \theta\,(1 - e^{-x/\theta})$.

$E[X \wedge Q_{0.25}] = 0.25\,\theta$. $E[X \wedge Q_{0.75}] = 0.75\,\theta$.

The average size of those losses of size between $Q_{0.25}$ and $Q_{0.75}$ is:

$$\frac{\{E[X \wedge Q_{0.75}] - Q_{0.75}\,S(Q_{0.75})\} - \{E[X \wedge Q_{0.25}] - Q_{0.25}\,S(Q_{0.25})\}}{F(Q_{0.75}) - F(Q_{0.25})} =$$

$$\frac{\{0.75\theta - (\ln[4]\theta)(0.25)\} - \{0.25\theta - (\ln[4/3]\theta)(0.75)\}}{0.75 - 0.25} = \frac{\theta\,\{0.5 + 0.5\ln[4] - 0.75\ln[3]\}}{1/2} =$$

$\theta\{1 + \ln[4] - \ln[3]\,3/2\}$ = **$0.7384\,\theta$**.

<u>Comment</u>: Here the trimmed mean excludes 25% probability in each tail.
One could instead for example exclude 10% probability in each tail.
The trimmed mean could be applied to a small set of data in order to estimate the mean of the distribution from which the data was drawn. For a symmetric distribution such as a Normal Distribution, the trimmed mean would be an unbiased estimator of the mean. If instead you assumed the data was from a skewed distribution such as an Exponential, then the trimmed mean would be a biased estimator of the mean. If the data was drawn from an Exponential, then the trimmed mean divided by 0.7384 would be an unbiased estimator of the mean.
The trimmed mean would be a robust estimator; it would not be significantly affected by unusual values in the sample. In contrast, the sample mean can be significantly affected by one unusually large value in the sample.

17.56. For $\alpha = 1$, $S(x) = \frac{\theta}{\theta + x}$.

$$E[X \wedge x] = \int_0^x S(t)\, dt = \int_0^x \frac{\theta}{\theta + t}\, dt = \theta \ln(\theta + t)\Big]_{t=0}^{t=x} = \theta \ln(\theta+x) - \theta \ln(\theta) = -\theta \ln\left[\frac{\theta}{\theta + x}\right].$$

<u>Comment</u>: The mean only exists if $\alpha > 1$. However, since the values entering its computation are limited, the limited expected value exists as long as $\alpha > 0$.

17.57. D. $(S - 150)_+ - (S - 100)_+$ is the amount in the layer from 100 to 150 on the index.
This is 1/(100 million) times the layer from 10 billion to 15 billion on catastrophe losses.
(100 million times 150 is 15 billion.)
Thus the payment on the spread is 1/500,000 times the layer from 10 billion to 15 billion on catastrophe losses.

For the LogNormal, $E[X \wedge x] = \exp(\mu+\sigma^2/2)\ \Phi\left[\frac{\ln(x) - \mu - \sigma^2}{\sigma}\right] + x\{1 - \Phi\left[\frac{\ln(x) - \mu}{\sigma}\right]\}$.

$E[X \wedge 10 \text{ billion}] =$

$\exp[20 + 2^2/2]\ \Phi[\frac{\ln[10 \text{ billion}] - 20 - 2^2}{2}] + (10 \text{ billion})\ \{1 - \Phi[\frac{\ln[10 \text{ billion}] - 20}{2}]\} =$

(3.5849 billion) Φ[-0.49] + (10 billion) {1 - Φ[1.51]} =
(3.5849 billion) (0.3121) + (10 billion) {1 - 0.9345} = 1.774 billion.

$E[X \wedge 15 \text{ billion}] =$

$\exp[20 + 2^2/2]\ \Phi[\frac{\ln[15 \text{ billion}] - 20 - 2^2}{2}] + (15 \text{ billion})\ \{1 - \Phi[\frac{\ln[15 \text{ billion}] - 20}{2}]\} =$

(3.5849 billion) Φ[-0.28] + (15 billion) {1 - Φ[1.72]} =
(3.5849 billion) (0.3897) + (15 billion) {1 - 0.9573} = 2.038 billion.

$\frac{E[X \wedge 15 \text{ billion}] - E[X \wedge 10 \text{ billion}]}{500{,}000}$ = (2.038 billion - 1.774 billion) / 500,000 = **528**.

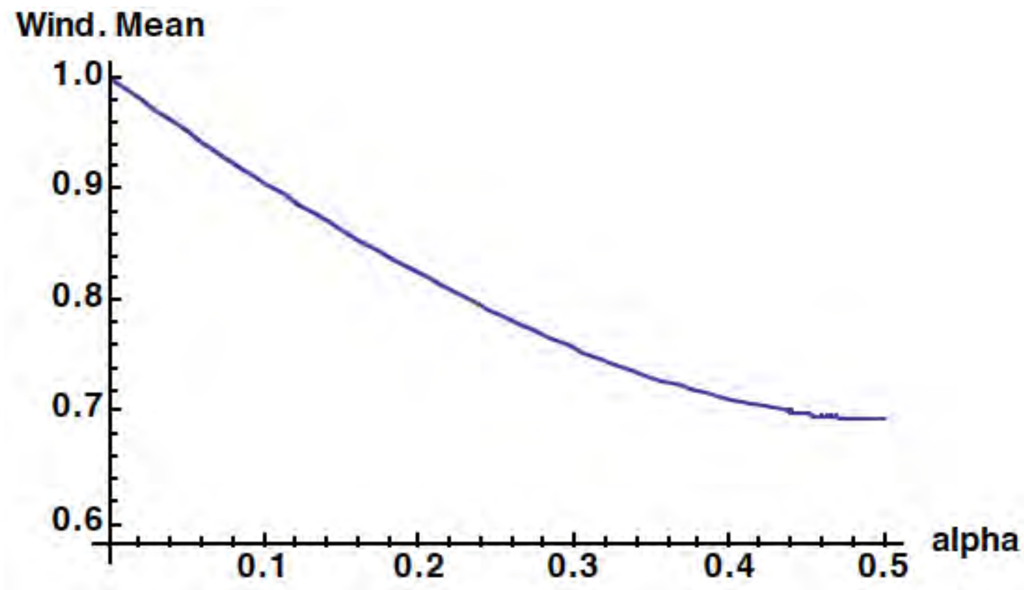
<u>Comment</u>: Not intended as a realistic model of catastrophe losses.
Catastrophe losses would be from hurricanes, earthquakes, etc.
An insurer could hedge its catastrophe risk by buying a lot of these or similar call spreads. An insurer who owned many of these call spreads, would be paid money in the event of a lot of catastrophe losses in this region for the insurance industry. This should offset to some extent the insurer’s own losses due to these catastrophes, in a manner somewhat similar to reinsurance.
528 is the amount expected to be paid by someone who sold one of these calls (in other words owned a put.) The probability of paying anything is low, but this person who sold a call could pay up to a maximum of: (200)(50) = 10,000.

17.58. $E[X \wedge 75] = 75 - (75 - 62)\,\Phi[(75\text{-}62)/20] - (20)\,\phi[(75\text{-}62)/20] = 75 - 13\,\Phi[0.65] - 20\,\phi[0.65]$
$= 75 - (13)(0.7422) - (20)\exp[-0.65^2/2] / \sqrt{2\pi}$ = **58.89**.

17.59. The small values each contribute Q_α. Their total contribution is αQ_α.
The large values each contribute $Q_{1-\alpha}$. Their total contribution is $\alpha Q_{1-\alpha}$.
The medium values each contribute their value x.

Their total contribution is: $\int_{Q_\alpha}^{Q_{1-\alpha}} x\, f(x)\, dx$ =

$E[X \wedge Q_{1-\alpha}] - Q_{1-\alpha}\, S(Q_{1-\alpha}) - \{E[X \wedge Q_\alpha] - Q_\alpha\, S(Q_\alpha)\}$ =
$E[X \wedge Q_{1-a}] - \alpha Q_{1-\alpha} - \{E[X \wedge Q_\alpha] - (1\text{-}\alpha)Q_\alpha\}$.
Thus adding up the three contributions, the Windsorized mean is:
$\alpha Q_\alpha + \alpha Q_{1-\alpha} + E[X \wedge Q_{1-\alpha}] - \alpha Q_{1-\alpha} - \{E[X \wedge Q_\alpha] - (1\text{-}\alpha)Q_\alpha\}$ =
$E[X \wedge Q_{1-\alpha}] - E[X \wedge Q_\alpha] + Q_\alpha$.
For the Exponential, $Q_\alpha = -\theta \ln(1 - \alpha)$. $Q_{1-\alpha} = -\theta \ln(\alpha)$.
$E[X \wedge x] = \theta\,(1 - e^{-x/\theta})$. $E[X \wedge Q_\alpha] = \theta\alpha$. $E[X \wedge Q_{1-\alpha}] = \theta(1\text{-}\alpha)$.
Thus the Windsorized mean is: $\theta(1\text{-}\alpha) - \theta\alpha - \theta \ln(1 - \alpha)$ = $\boldsymbol{\theta\,\{1 - 2\alpha - \ln(1\text{-}\alpha)\}}$.
Comment: The trimmed mean excludes probability in each tail.
In contrast, the Windsorized mean substitutes for extreme values the corresponding quantile.
The Windsorized mean could be applied to a small set of data in order to estimate the mean of the distribution from which the data was drawn.
For example if α = 10%, then all values below the 10th percentile are replaced by the 10th percentile, and all values above the 90th percentile are replaced by the 90th percentile, prior to taking an average. For a symmetric distribution such as a Normal Distribution, the Windsorized mean would be an unbiased estimator of the mean. If instead you assumed the data was from a skewed distribution such as an Exponential, then the Windsorized mean would be a biased estimator of the mean.
The Windsorized mean would be a robust estimator; it would not be significantly affected by unusual values in the sample. In contrast, the sample mean can be significantly affected by one unusually large value in the sample.
For the Exponential, here is a graph of the Windsorized mean divided by the mean, in other words the Windsorized mean for θ = 1, as a function of alpha:



As alpha increases, we are substituting for more of the values in the tails.

17.60. a) For the Pareto Distribution:

$E[X \wedge x] = \frac{\theta}{\alpha - 1} \left\{1 - \left(\frac{\theta}{\theta + x}\right)^{\alpha - 1}\right\} = (200{,}000) \{1 - \left(\frac{400{,}000}{400{,}000 + x}\right)^2\}.$

The current cost is proportional to: E[X∧1 million] - E[X∧250,000] =
183,673 - 124,260 = 59,413.
The cost under Bill #1 is proportional to: 1.05 (E[X∧1 million/1.05] - E[X∧250,000/1.05]) =
(1.05)(182,504 - 121,408) = 64,151.
Impact of Bill #1 is: 64,151/59,413 - 1 = **8.0%**.
b) Split the reinsurer's coverage into two layers:
250K to 500K and 500K to 1000K.
The second layer is unaffected by Bill #2. The first layer is affected by Bill #2 in the exact same way as if there had been 10% inflation to all losses.
(If 454,454 < X ≤ 500,000, then after 10% inflation the contribution to the first layer is as if the loss became $500,000. If X > $500,000, then after 10% inflation the contribution to the first layer is as if the loss stayed the same.)
Thus the cost under Bill # 2 is:
1.1 E[X∧454,454] - 1.1 E[X∧227,273] + E[X∧1 million] - E[X∧500,000] =
(1.1)(156,179) - (1.1)(118,673) + 183,673 - 160,494 = 64,436.
Impact of Bill #2 is: 64,436/59,413 - 1 = **8.5%**.
Alternately, if X is the original cost and Y is the cost under Bill #2, then:
If X ≤ 500,000 / 1.1 = 454,454, then Y = 1.1 X.
If 454,454 < X ≤ 500,000, then Y = 500,000.
If 500,000 < X, then Y = X.
Let f, F, and S refer to the original Pareto, then the cost under Bill # 2 is:

$$\int_{250,000/1.1}^{500,000/1.1} (1.1x - 250,000)\ f(x)\ dx\ + (500,000 - 250,000)\{F(500,000) - F(454,454)\} +$$

$$\int_{500,000}^{1,000,000} (x - 250,000)\ f(x)\ dx\ + 750,000\ S(1,000,000) =$$

$$1.1 \int_{227,273}^{454,545} x\ \ f(x)\ d\ - 250,000\{F(454,545) - F(227,273)\}$$

$$+ 250,000\{F(500,000) - F(454,545)\} +$$

$$\int_{500,000}^{1,000,000} x\ \ f(x)\ dx\ - 250,000\{F(1,000,000) - F(500,000)\} + 750,000\ S(1,000,000) =$$

1.1{E[X∧454,545] - 454,545S(454,545) - E[X∧227,273] + 227,273S(227,273)}
+ (250,000){S(454,545) - S(227,273)} - (250,000){S(500,000) - S(454,545)}
E[X∧1 million] - 1,000,000S(1,000,000) - E[X∧500,000] + 500,000S(500,000)
+ 250,000{S(1,000,000) - S(500,000)} + 750,000 S(1,000,000) =
1.1 E[X∧454,545] - 1.1 E[X∧227,273] + E[X∧1 million] - E[X∧500,000] =
(1.1)(156,179) - (1.1)(118,673) + 183,673 - 160,494 = 64,436.
Impact of Bill #2 is: 64,436/59,413 - 1 = **8.5%**.
<u>Comment</u>: Based on CAS9, 11/99, Q.40.

17.61. Using the formula for VaR for the Pareto, $\pi_p = \theta\{(1-p)^{-1/\alpha} - 1\}$.

$\pi_{0.05} = \theta\{0.95^{-1/\alpha} - 1\}$. $\pi_{0.95} = \theta\{0.05^{-1/\alpha} - 1\}$.

$E[X \wedge x] = \frac{\theta}{\alpha - 1}\left\{1 - \left(\frac{\theta}{\theta + x}\right)^{\alpha - 1}\right\}$, $\alpha \neq 1$.

$E[X \wedge \pi_{0.05}] = \frac{\theta}{\alpha - 1}\left\{1 - \left(\frac{1}{0.95^{-1/\alpha}}\right)^{\alpha - 1}\right\} = \frac{\theta}{\alpha - 1}\left\{1 - 0.95^{1-1/\alpha}\right\}$.

$E[X \wedge \pi_{0.95}] = \frac{\theta}{\alpha - 1}\left\{1 - \left(\frac{1}{0.05^{-1/\alpha}}\right)^{\alpha - 1}\right\} = \frac{\theta}{\alpha - 1}\left\{1 - 0.05^{1-1/\alpha}\right\}$.

The trimmed mean, the average size of those losses of size between $\pi_{0.05}$ and $\pi_{0.95}$ is:

$$\frac{\{E[X \wedge \pi_{0.95}] - \pi_{0.95}\, S(\pi_{0.95})\} - \{E[X \wedge \pi_{0.05}] - \pi_{0.05}\, S(\pi_{0.05})\}}{F(\pi_{0.95}) - F(\pi_{0.05})} =$$

$\{\frac{\theta}{\alpha - 1}\ (0.95^{1-1/\alpha} - 0.05^{1-1/\alpha}) + 0.95\ \pi_{0.05} - 0.05\ \pi_{0.95}\} / 0.9 =$

$\theta\{\frac{1}{\alpha - 1}\ (0.95^{1-1/\alpha} - 0.05^{1-1/a}) + 0.95^{1-1/\alpha} - 0.05^{1-1/\alpha} - 0.9\} / 0.9 =$

$\theta\{\frac{\alpha\ (0.95^{1-1/\alpha} - 0.05^{1-1/\alpha})}{(0.9)\ (\alpha-1)} - 1\}$, $\alpha \neq 1$.

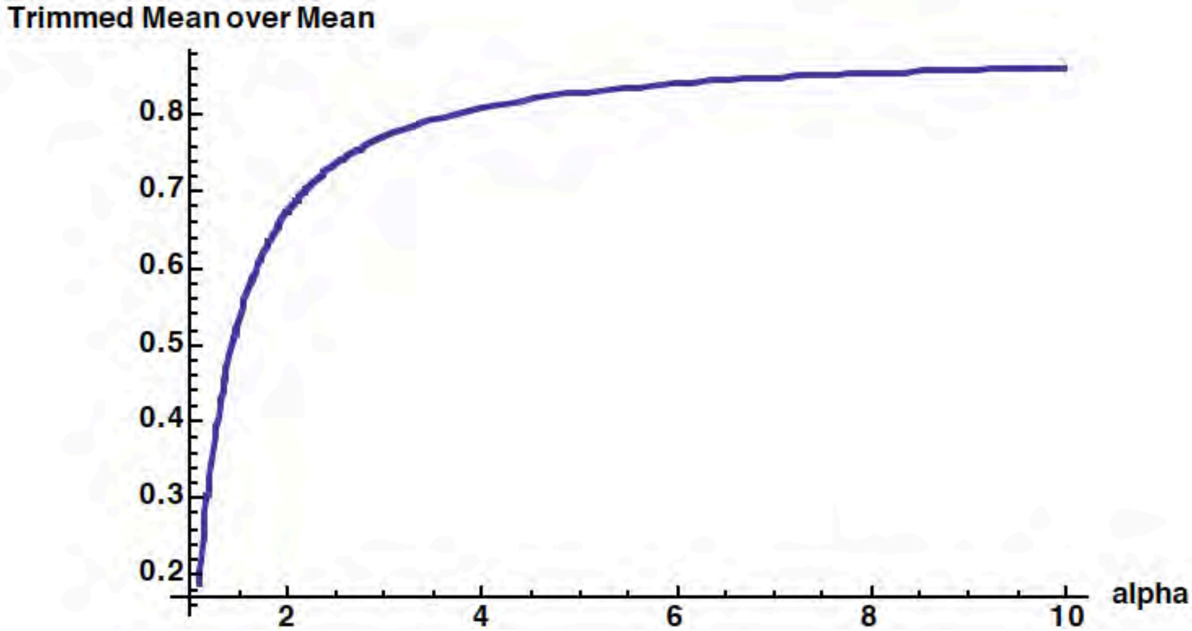
For $\alpha = 1$, $E[X \wedge x] = -\theta \ln[\frac{\theta}{\theta + x}]$.

$\pi_{0.05} = \theta\{0.95^{-1} - 1\} = \theta/19$. $\pi_{0.95} = \theta\{0.05^{-1} - 1\} = 19\theta$.

$E[X \wedge \pi_{0.05}] = \theta \ln(20/19)$. $E[X \wedge \pi_{0.95}] = \theta \ln(20)$.

Therefore, the trimmed mean is: $\theta\{\ln(20) - \ln(20/19) + 0.95/19 - (0.05)(19)\} / 0.9 = 2.2716\ \theta$.

<u>Comment</u>: Here the trimmed mean excludes 5% probability in each tail.
One could instead for example exclude 10% probability in each tail.
Even though we have excluded an equal probability in each tail, for the positively skewed Pareto Distribution, the trimmed mean is less than the mean.
As α approaches 1, the mean approaches infinity,
while the trimmed mean approaches 2.2716 θ.
Here is a graph of the ratio of the trimmed mean to the mean:



For example, for $\alpha = 3$, the trimmed mean is 0.3844366θ , while the mean is $\theta/2$;
for $\alpha = 3$ the ratio of the trimmed mean to the mean is 0.768872.

17.62. D. For the Pareto Distribution,

$$E[X \wedge x] = \frac{\theta}{\alpha - 1} \left\{1 - \left(\frac{\theta}{\theta + x}\right)^{\alpha - 1}\right\} = 3125 \{1 - \left(\frac{5000}{5000 + x}\right)^{1.6}\}.$$

The average payment per loss for the first policy is:
$E[X \wedge 3000] - E[X \wedge 2000] = 1651.81 - 1300.91 = 350.90$.
The second policy pays nothing for $X < 1000$, pays 1000 for $X > 2000$,
and pays $X - 1000$ for $1000 \le X \le 2000$. This is equivalent to $d = 1000$ and $u = 2000$.
The average payment per loss for the second policy is:
$E[X \wedge 2000] - E[X \wedge 1000] = 1300.91 - 790.68 = 510.23$.
The percentage increase in expected losses is: 510.23/350.90 = **45.4%**.

17.63. E. $90 = E[(S-450)_+] = E[S] - E[S \wedge 450]. \Rightarrow E[S \wedge 450] = 500 - 90 = 410$.
Therefore, the expected amount by which S is less than 450 is: $450 - E[S \wedge 450] = 40$.
Therefore, the expected bonus is (40)(30,000) = **1,200,000**.

17.64. D. $F(0.7) = \Phi[\frac{\ln(0.7) + 0.65}{0.6}] = \Phi[\frac{\ln(0.7) + 0.65}{0.6}] = \Phi[0.49] = 0.6879.$

$E[X] = \exp[-0.65 + 0.6^2/2] = 0.6250.$

$E[X \wedge x] = \exp(\mu + \sigma^2/2)\, \Phi\left[\frac{\ln(x) - \mu - \sigma^2}{\sigma}\right] + x\, \{1 - \Phi\left[\frac{\ln(x) - \mu}{\sigma}\right]\}.$

$E[X \wedge 0.7] = \exp[-0.65 + 0.6^2/2]\, \Phi[\frac{\ln(0.7) + 0.65 - 0.6^2}{0.6}] + (0.7)\, \{1 - \Phi[\frac{\ln(0.7) + 0.65}{0.6}]\}.$

$= (0.6250)\, \Phi[-0.11] + (0.7)(1 - \Phi[0.49]) = (0.6250)(1 - 0.4562) + (0.7)(1 - 0.6879) = 0.5036.$

Thus, conditional on the loss ratio being greater than 0.7, the average loss ratio is:

$\frac{E[X] - \{E[X \wedge 0.7] + 0.7\, S(0.7)\}}{1 - F(0.7)} = \frac{0.6250 - \{0.5036 + (0.7)(1 - 0.6879)\}}{1 - 0.6879} = \mathbf{1.089}.$

17.65. (a) $S(10) = (\theta/10)^{1.5}$.

wealth of such individuals is: $\int_{10}^{\infty} x\, f(x)\, dx = \int_{10}^{\infty} x\, (1.5)(\theta^{1.5})/x^{2.5}\, dx = (1.5/0.5)\, \theta^{1.5} / 10^{0.5}$.

$\Rightarrow$ average wealth of such individuals is: $\{(1.5/0.5)\, \theta^{1.5} / 10^{0.5}\} / (\theta/10)^{1.5} = (3)(10) = 30$ million.

(b) $S(y) = (\theta/y)^{\alpha}$. wealth of such individuals is: $\int_{y}^{\infty} x\, f(x)\, dx = \int_{y}^{\infty} x\, \alpha\, \theta^{\alpha}/x^{\alpha+1}\, dx = \frac{\alpha}{\alpha - 1}\, \theta^{\alpha} / y^{\alpha-1}$.

$\Rightarrow$ average wealth of such individuals is: $\{\frac{\alpha}{\alpha - 1}\, \theta^{\alpha} / y^{\alpha-1}\} / (\theta/y)^{\alpha} = \frac{\alpha}{\alpha - 1}\, y$.

Alternately, for the Single Parameter Pareto Distribution, $e(x) = x / (\alpha-1)$.

Thus the average size of the data left truncated at y is: $y + y / (\alpha-1) = \frac{\alpha}{\alpha - 1}\, y$.

(c) $\Pi_p = \text{VaR}_p(X) = \theta\, (1-p)^{-1/\alpha}$. $\Rightarrow \Pi_q / \Pi_p = \left(\frac{1 - p}{1 - q}\right)^{1/\alpha}$.

(d) From part (c), $\Pi_{99\%} / \Pi_{90\%} = \left(\frac{1 - 0.9}{1 - 0.99}\right)^{1/\alpha} = 10^{1/\alpha}$. From part (b), the average wealth of each group is proportional to its lower endpoint, the corresponding percentile.

Thus the ratio of the average wealth of the top 1% to the top 10% is $10^{1/\alpha}$.

However, there are only 1/10 as many individuals in the top 1% as the top 10%.

Thus the ratio of the total wealth of the top 1% to the top 10% is: $10^{1/\alpha} / 10 = 0.1^{(\alpha-1)/\alpha}$.

Alternately, $\text{ES}_p(X) = \frac{\alpha}{\alpha - 1}\, \theta(1-p)^{-1/\alpha}$. $\Rightarrow \text{ES}_{99\%}(X) / \text{ES}_{90\%}(X) = (0.01/0.1)^{-1/\alpha} = 10^{1/\alpha}$.

Thus the ratio of the average wealth of the top 1% to the top 10% is $10^{1/\alpha}$. Proceed as before.

Comment: As alpha approaches one, in other words as the distribution of wealth becomes more unequal between individuals, $\frac{\alpha}{\alpha - 1}$ approaches infinity.

For example, for $\alpha = 1.5$, the ratio of the total wealth of the top 1% to the top 10% is:
$0.1^{(\alpha-1)/\alpha} = 0.1^{1/3} = 46.4\%$.

For $\alpha = 1.5$, in the same way, the top 0.1% owns 46.4% of the total wealth of the top 1%.

If instead $\alpha = 2.5$, then the ratio of the total wealth of the top 1% to the top 10% is:
$0.1^{(\alpha-1)/\alpha} = 0.1^{0.6} = 25.1\%$.

17.66. C. $E[X \wedge x] = \frac{\theta}{\alpha - 1} \left\{1 - \left(\frac{\theta}{\theta + x}\right)^{\alpha - 1}\right\}$ = (4000/2) {1 - $\left(\frac{4000}{4000 + x}\right)^2$}.

$E[X \wedge 3000]$ = (2000) {1 - $(4/7)^2$} = 1346.94.
$E[X \wedge 5000]$ = (2000) {1 - $(4/9)^2$} = 1604.94.
1604.94/1346.94 - 1 = **19.16%**.
Comment: The reinsurer's expected payment per ground-up loss goes from 2000 - 1346.94 = 653.06, to 2000 - 1604.94 = 395.06.

17.67. B. E[X] = $\exp(8 + 1.2^2/2)$ = 6124.

$$E[X \wedge 10{,}000] = \exp(8 + 1.2^2/2)\, \Phi\left[\frac{\ln(10{,}000) - 8 - 1.2^2}{1.2}\right] + 10{,}000 \left\{1 - \Phi\left[\frac{\ln(10{,}000) - 8}{1.2}\right]\right\} =$$

6124 $\Phi[-0.19]$ + 10,000 {1 - $\Phi[1.00]$} = (6124)(0.4247) + (10,000)(1 - 0.8413) = 4188.
$E[(X - 10{,}000)_+]$ = 6124 - 4188 = **1936**.

17.68. C. For a loss of size x, the insurer pays 0 if x < 10, and x - 10 if 100 ≥ x ≥ 10. (There are no losses greater than 100.) The average payment, excluding from the average small losses on which the insurer makes no payment is:

$$\frac{\int_{10}^{100} (x-10) f(x)\, dx}{\int_{10}^{100} f(x)\, dx} = \frac{\int_{10}^{100} (x-10)(0.015 - 0.0001x)\, dx}{\int_{10}^{100} (0.015 - 0.0001x)\, dx} = 32.4 / 0.855 = \mathbf{37.9}.$$

Alternately, $S(10) = \int_{10}^{100} f(x)\, dx = \int_{10}^{100} (0.015 - 0.0001x)\, dx = 0.855$.

$$E[X] = \int_0^{100} x\, f(x)\, dx = \int_0^{100} x\,(0.015 - 0.0001x)\, dx = 41.67.$$

$$E[X \wedge 10] = \int_0^{10} x\, f(x)\, dx + 10S(10) = \int_0^{10} x\,(0.015 - 0.0001x)\, dx + (10)(0.855)$$

= 0.72 + 8.55 = 9.27.
Average payment per payment is: $(E[X] - E[X \wedge 10])/S(10)$ = (41.67 - 9.27)/0.855 = **37.9**.

17.69. C. e(x) = (losses excess of x) / (claims excess of x) = $(E[X] - E[X \wedge x]) / S(x)$.
Therefore, $E[X \wedge x] = E[X] - e(x)\{1 - F(x)\}$.

17.70. D. 1. False (not true), For small samples, either of the two methods may have smaller variance. For large samples, the method of maximum likelihood has the smallest variance.
2. True. This is the definition of the Limited Expected Value. 3. True.

17.71. E. Expected amount paid per loss = $\int_{100}^{\infty} x\, f(x)\, dx = \int_{0}^{\infty} x\, f(x)\, dx - \int_{0}^{10} x\, f(x)\, dx =$

Mean - {E[X∧100] - 100S(100)}. S(100) = $\{\theta/(\theta+100)\}^2 = (1000/1100)^2 = 0.8264$.
E[X∧100] = $\{\theta/(\alpha-1)\}\{1 - (\theta/(\theta+100))^{\alpha-1}\} = \{1000/(2-1)\}\{1 - (1000/1100)^{2-1}\} = 90.90$.
Mean = $\theta/(\alpha-1)$ = 1000. Therefore, Expected amount paid per loss =
1000 - {90.90 - 82.64} = 991.74. Expect 10 losses per year, so the average cost per year is: (10)(991.5) = **$9915**.
Alternately, the expected cost per year of 10 losses is:

$$10 \int_{100}^{\infty} x\, f(x)\, dx = (10)(2)(1000^2) \int_{100}^{\infty} x\, (1000+x)^{-3}\, dx =$$

$$10^7 \{-x\, (1000+x)^{-2}\} \Big]_{x=100}^{x=\infty} + 10^7 \int_{100}^{\infty} (1000+x)^{-2}\, dx = 10^7 \{100/1100^2 + 1/1100\} = \mathbf{\$9917}.$$

Alternately, the average severity per loss > $100 is: 100 + e(100) = 100 + $(\theta+100)/(\alpha - 1)$
= 1100 + 100 = $1200. Expected number of losses > $100 = 10S(100) = 8.2645.
Expected annual payment = ($1200) (8.2645) = **$9917**.
Comment: This is the franchise deductible.

17.72. C. For the LogNormal: $E[X \wedge x] = \exp(\mu + \sigma^2/2)\ \Phi[(\ln x - \mu - \sigma^2)/\sigma] + x\ \{1 - \Phi[(\ln x - \mu)/\sigma]\}$.
$E[X \wedge 100{,}000] = \exp(10 + 1^2/2)\ \Phi[(\ln 100{,}000 - 10 - 1^2)/1] + 100{,}000\{1 - \Phi[(\ln 100{,}000 - 10)/1]\}$
$= e^{10.5}\Phi(0.51) + 100{,}000\{1 - \Phi(1.51)\} = 36{,}316(0.6950) + 100{,}000(1 - 0.9345) = 31{,}790$.
$E[X \wedge x] - xS(x) = \exp(\mu + \sigma^2/2)\ \Phi[(\ln x - \mu - \sigma^2)/\sigma]$. $E[X \wedge 50{,}000] - 50{,}000S(50{,}000) =$
$e^{10.5}\Phi[(\ln 50{,}000 - 10 - 1^2)/1] = 36{,}316\ \Phi(-0.18) = 36{,}316(0.4286) = 15{,}565$.
Without the feature that the insurer pays the entire loss (up to $100,000) for each loss greater than $50,000, the insurer would pay the layer from 50,000 to 100,000, which is
$E[X \wedge 100{,}000] - E[X \wedge 50{,}000]$. That extra provision adds 50,000 per large loss, or
50,000 S(50000). Thus the insurer pays: $50{,}000S(50000) + E[X \wedge 100{,}000] - E[X \wedge 50{,}000] =$
$E[X \wedge 100{,}000] - \{E[X \wedge 50{,}000] - 50{,}000S(50{,}000)\} = 31{,}790 - 15{,}565 =$ **16,225**.
Alternately, the insurer pays for all dollars of loss in the layer less than $100,000, except it pays nothing for losses of size less than $50,000. The former is: $E[X \wedge 100{,}000]$;
the latter is: $E[X \wedge 50{,}000] - 50{,}000S(50{,}000)$. Thus the insurer pays:
$E[X \wedge 100{,}000] - \{E[X \wedge 50{,}000] - 50{,}000S(50{,}000)\}$. Proceed as above.
Alternately, the insurer pays all dollars for losses greater than $50,000, except it pays nothing in the layer above $100,000. The former is:
$E[X] - \{E[X \wedge 50{,}000] - 50{,}000(S(50{,}000)\}$; the latter is: $E[X] - E[X \wedge 100{,}000]$.
Thus subtracting the two values the insurer pays:
$E[X \wedge 100{,}000] - \{ E[X \wedge 50{,}000] - 50{,}000(S(50{,}000) \}$. Proceed as above.
Alternately, the insurer pays all dollars for losses greater than $50,000 and less than $100,000, and pays $100,000 per loss greater than $100,000. The former is:
$\{E[X \wedge 100{,}000] - 100{,}000S(100{,}000)\} - \{E[X \wedge 50{,}000] - 50{,}000S(50{,}000)\}$; the latter is:
100,000S(100,000). Thus adding the two contributions the insurer pays:
$E[X \wedge 100{,}000] - \{ E[X \wedge 50{,}000] - 50{,}000S(50{,}000)\}$. Proceed as above.

17.73. E. A loss ratio of 70% corresponds to (0.7)(500000) = $350,000 in losses.
If the losses are x, and x < 350,000, then the agent gets a bonus of (1/3)(350,000 - x).
On the other hand, if x ≥ 350,000, then the bonus is zero.
Therefore the expected bonus is:

$$(1/3)\int_0^{350,000} (350,000 - x) f(x)\, dx = (1/3)(350,000)\int_0^{350,000} f(x)\, dx - (1/3)\int_0^{350,000} x f(x)\, dx =$$

(1/3)(350,000)F(350,000) - (1/3){E[X∧350,000] - 350,000S(350,000)} =
(1/3){350,000 - (E[X∧350,000] }.
The distribution of losses is Pareto with $\alpha = 3$ and $\theta = 600,000$. Therefore,
$E[X\wedge 350,000] = \{\theta/(\alpha-1)\}\{1 - (\theta/(\theta + 350,000))^{\alpha-1}\} = (600000/2)(1 - (600/950)^2) = 180,332$.
Therefore, the expected bonus is: (1/3) (350,000 - 180,332) = **56,556**.
Alternately, the expected amount by which losses are less than b is: b - E[L∧ b].
Therefore, expected bonus = (1/3)(expected amount by which losses are less than 350,000) =
$E[(350,000 - L)_+]/3$ = (1/3)(350,000 - E[L∧ 350,000]). Proceed as before.

Alternately, his losses must be less than 350,000 to receive a bonus.
$S(350,000) = (600/(350 + 600))^3 = 0.25193$ = probability that he receives no bonus.
The mean of the "small" losses (< 350,000) is: {E[L∧ 350,000] - 350,000S(350,000)}/F(350,000)
= (180,332 - (350,000)(0.25193))/(1 - 0.25193) = 123,192.
123,192 / 500,000 = 24.638%, is the expected loss ratio when he gets a bonus.
Therefore, the expected bonus when he gets a bonus is: 500,000(70% - 24.638%)/3 = 75,603.
His expected overall bonus is: (1 - 0.25193)(75,603) + (0.25193)(0) = **56,556**.
Comment: Note that since if x ≥ 350,000 the bonus is zero, we only integrate from zero to 350,000. Therefore, it is not the case that E[Bonus] = (1/3)(350,000 - E[X]).

17.74. C. Let total dollars of claims be A. Let B = the Bonus.
Then B = (500-A)/2 if A < 500 and 0 if A ≥ 500. Let y = A if A < 500 and 500 if A ≥ 500.
Then E[y] = E[A∧ 500]. 2B + y = 500, regardless of A. Therefore 2E[B] + E[y] = 500.
Therefore E[B] = (500 - E[A∧ 500])/2 = 250 - E[A∧ 500]/2.
For the Pareto Distribution, $E[X] = \theta/(\alpha-1)$, and $E[X\wedge x] = \{\theta/(\alpha-1)\}\{1 - (\theta/(x+\theta))^{\alpha-1}\}$.
For the revised model, E[A∧ 500] = K{1- (K/(500+K))} = 500K / (500 + K).
Thus for the revised model, E[B] = 250 - 250K/(500 + K) = 125,000/(500 + K).
Expected aggregate claims under the revised model are: K/(2-1) = K.
Expected aggregate claims under the previous model are: 500/(2-1) = 500.
So we are given that: K + 125,000/(500 + K) = 500.
$500K + K^2 + 125,000 = 250,000 + 500K$. $K^2 = 125,000$. K = **353.**
Comment: The expected amount by which claims are less than 500 is:
$E[(500 - A)_+] = 500 - E[A\wedge 500]$.

17.75. E. A loss ratio of 60% corresponds to: (60%)(800,000) = 480,000 in losses.
For the Pareto distribution, $E[X \wedge x] = \{\theta/(\alpha-1)\}\{1 - (\theta/(\theta+x))^{\alpha-1}\}$.
$E[X \wedge 480{,}000] = \{(500{,}000)/(2-1)\}(1 - (500{,}000/(480{,}000 + 500{,}000))^{2-1}) = 244{,}898$.
If losses are less than 480,000 a bonus is paid.
Bonus = (15%)(amount by which losses < 480,000).
Expected bonus = $(15\%)E[(480{,}000 - L)_+]$ =
$(15\%)\{480{,}000 - E[L \wedge 480{,}000]\}$ = (15%)(480,000 - 244,898) = **35,265**.

17.76. B. From the previous solution, his expected bonus is: 35,265.
He gets a bonus when the aggregate losses are less than 480,000.
The probability of this is: $F(480{,}000) = 1 - \{500/(500 + 480)\}^2 = 0.73969$.
Expected value of Hunt's bonus, given that Hunt receives a (positive bonus) is:
35,265/0.73969 = **47,675**.
Comment: This question asks about an analog to the expected payment per (non-zero) payment, while the exam question asks about an analog to the expected payment per loss. In this question we only consider situations where the bonus is positive, while the exam question includes those situations where the bonus is zero.

17.77. E. The expected losses within the layer 1,000 to 10,000 is:

$$\int_{1000}^{10{,}000} S(x)\,dx = \int_{1000}^{10{,}000} 10^6/(x + 10^3)^2\,dx = -10^6/(x + 10^3)\Big]_{x=1000}^{x=10{,}000} = 10^6(1/2000 - 1/11000) =$$

1000(1/2 - 1/11).

$$E[X] = \int_0^\infty S(x)\,dx = \int_0^\infty 10^6/(x + 10^3)^2\,dx = -10^6/(x + 10^3)\Big]_{x=0}^{x=\infty} = 1000.$$

Therefore the percent of expected losses within the layer 1,000 to 10,000 is:
1000(1/2 - 1/11)/1000 = 1/2 - 1/11 = **40.9%**.
Alternately, this is a Pareto Distribution with $\alpha = 2$ and $\theta = 1000$.
$E[X \wedge x] = \{\theta/(\alpha-1)\}\{1 - (\theta/(\theta + x))^{\alpha-1}\} = 1000\{1 - 1000/(1000 + x)\} = 1000x/(1000 + x)$.
$E[X \wedge 1000] = 500$. $E[X \wedge 10{,}000] = 909$. $E[X] = \theta/(\alpha-1) = 1000$.
The percent of expected losses within the layer 1,000 to 10,000 is:
$(E[X \wedge 10{,}000] - E[X \wedge 1000]) / E[X]$ = (909 - 500)/1000 = **40.9%**.

17.78. A. $E[X \wedge x] = \{\theta/(\alpha-1)\}\{1 - (\theta/(\theta+x))^{\alpha-1}\}$. $E[X \wedge 400] = 300(1 - 3/7) = 171.43$.
$100 = E[B] = c(400 - E[X \wedge 400]) = c(400 - 171.43) = c228.57$. ⇒ c = 100/228.57 = **0.4375**.

17.79. C. At 5100 in loss, the insured pays: 250 + (25%)(2250 - 250) + (5100 - 2250) = 3600.
⇒ For annual losses > 5100, the insured pays 5% of the amount > 5100.
⇒ The insurer pays: 75% of the layer from 250 to 2250, 0% of the layer 2250 to 5100, and 95% of the layer from 5100 to ∞.
$E[X \wedge x] = \{\theta/(\alpha-1)\}\{1 - (\theta/(\theta+x))^{\alpha-1}\}$ = (2000){1 - 2000/(2000 + x)} = 2000x/(2000 + x).
$E[X \wedge 250]$ = (2000)(250) / (2000 + 250) = 222.
$E[X \wedge 2250]$ = (2000)(2250) / (2000 + 2250) = 1059.
$E[X \wedge 5100]$ = (2000)(5100) / (2000 + 5100) = 1437.
$E[X] = \theta/(\alpha-1)$ = 2000 / (2 - 1) = 2000.
The expected annual plan payment:
(75%)($E[X \wedge 2250] - E[X \wedge 250]$) + (95%)($E[X] - E[X \wedge 5100]$) =
(75%)(1059 - 222) + (95%)(2000 - 1437) = **1163**.
Comment: *Provisions are similar to those in the 2006 Medicare Prescription Drug Program.*
Here is a detailed breakdown of the layers of loss:

Layer	Expected Losses in Layer	Insured Share	Insurer Share
5100 to ∞	563	5%	95%
2250 to 5100	378	100%	0%
250 to 2250	837	25%	75%
0 to 250	222	100%	0%
Total	2000		

$E[X] - E[X \wedge 5100]$ = 2000 - 1437 = 563. $E[X \wedge 5100] - E[X \wedge 2250]$ = 1437 - 1059 = 378.
$E[X \wedge 2250] - E[X \wedge 250]$ = 1059 - 222 = 837. $E[X \wedge 250]$ = 222.
For example, for an annual loss of 1000, insured pays: 250 + (25%)(1000 - 250) = 437.5, and insurer pays: (75%)(1000 - 250) = 562.5.
For an annual loss of 4000, insured pays: 250 + (25%)(2250 - 250) + (4000 - 2250) = 2500, and insurer pays: (75%)(2250 - 250) = 1500.
For an annual loss of 8000, insured pays: 250 + (25%)(2250 - 250) + (5100 - 2250) + (5%)(8000 - 5100) = 3745, and insurer pays: (75%)(2250 - 250) + (95%)(8000 - 5100) = 4255.

17.80. C. For this Pareto,
$E[L \wedge 650{,}000] = \{600{,}000/(3 - 1)\}\{1 - (600{,}000/(650{,}000 + 600{,}000))^2\}$ = 230,880.
$E[(650{,}000 - L)_+] = 650{,}000 - E[L \wedge 650{,}000]$ = 650,000 - 230,880 = 419,120.
E[Bonus] = $E[(650{,}000 - L)_+/3]$ = 419,120/3 = **139,707**.

17.81. C. A constant force of mortality is an Exponential Distribution.
Variance = θ^2 = 100. ⇒ θ = 10.
E[T ∧ 10] = $\theta(1 - e^{-10/\theta})$ = (10)(1 - e^{-1}) = **6.32**.

17.82. B. Teller X completes on average 6 transactions per hour, while Teller Y completes on average 4 transactions per hour. (6)(6 + 4) = 60 transactions by tellers expected in total.
1/3 of all transactions are deposits, and therefore we expect 20 deposits.
Expected number of deposits handled by tellers: 20 F(7500).
Average size of those deposit of size less than 7500 is:
{E[X ∧ 7500] - 7500S(7500)} / F(7500).
Expected total deposits made through the tellers each day:
20{E[X ∧ 7500] - 7500S(7500)7500]} = 20{(5000/2)(1 - $(5/12.5)^2$) - 7500$(5/12.5)^3$} =
(20){2100 - (7500)(0.064)} = **32,400**.
Comment: While the above is the intended solution of the CAS, it is not what I would have done to solve this poorly worded exam question.
Let y be total number of deposits expected per day.
Then we expect S(7500)y deposits to be handled by the manager, and F(7500)y deposits to be handled by the tellers. Expect 60 - F(7500)y non-deposits to be handled by the tellers.
1/3 of all transactions are deposits, presumably including those handled by the manager.
{60 + S(7500)y}/3 = y. ⇒ y = 60/{3 - S(7500)}.
Expected number of deposits handled by tellers: F(7500)y = F(7500) 60/{3 - S(7500)}.
Multiply by the average size of those deposit of size less than 7500:
(F(7500) 60/{3 - S(7500)}) {E[X ∧ 7500] - 7500S(7500)}/F(7500) =
60{E[X ∧ 7500] - 7500S(7500)}/(3 - S(7500))
= (60){2100 - (7500)(0.064)}/(3 - 0.064) = 33,106.
Resulting in a different answer than the intended solution.

17.83. B. 3%/75% = 4%. If the index return is less then 4%, then the depositor gets 3%.
Thus we want: E[Max[0.75 X, 3%]] = 75% E[max[X, 4%]].

$$E[\max[X, 4\%]] = 4F(4) + \int_4^\infty x f(x)\,dx = 4F(4) + \int_0^\infty x f(x)\,dx - \int_0^4 x f(x)\,dx$$

$= 4F(4) + E[X] - \{E[X \wedge 4] - 4S(4)\} = 4 + E[X] - E[X \wedge 4] = 4 + 8 - (-0.58) = 12.58$.
75% E[max[X, 4%]] = (75%)(12.58%) = **9.43%**.
Alternately, let Y = max[X, 4]. Then Y - 4 = 0 if $X \le 4$, and Y - 4 = X - 4 if X > 4.
Therefore, $E[Y - 4] = E[(X - 4)_+] = E[X] - E[X \wedge 4]$.

$\Rightarrow E[Y] = 4 + E[X] - E[X \wedge 4] = 4 + 8 - (-0.58) = 12.58$. (75%)(12.58%) = **9.43%**.
<u>Comment</u>: In general, Min[X, 4] + Max[X, 4] = X + 4.
Therefore, $E[\text{Max}[X, 4]] = E[X] + 4 - E[X \wedge 4]$.

17.84. C. Let X be such that Michael just has paid 10,000 in out-of-pocket repair costs:
10000 = 1000 + (20%)(6000 - 1000) + (X - 6000). $\Rightarrow$ X = 14,000.
Thus the insurance pays 80% of the layer from 1000 to 6000, plus 90% of the layer above 14,000.
For this Pareto Distribution, $E[X \wedge x] = 5000\{1 - 5000/(5000 + x)\} = 5000x/(5000 + x)$.
$E[X \wedge 1000] = 833$. $E[X \wedge 6000] = 2727$. $E[X \wedge 14{,}000] = 3684$. $E[X] = \theta/(\alpha-1) = 5000$.
Expected annual payment by the insurer is:
$80\%(E[X \wedge 6000] - E[X \wedge 1000]) + 90\%(E[X] - E[X \wedge 14{,}000]) =$
80%(2727 - 833) + 90%(5000 - 3684) = **2700**.
<u>Comment</u>: Similar to SOA3, 11/04, Q.7.
Here is a detailed breakdown of the layers of loss:

Layer	Expected Losses in Layer	Michael's Share	Insurer Share
14,000 to ∞	5000 - 3684 = 1316	10%	90%
6000 to 14,000	3684 - 2727 = 957	100%	0%
1000 to 6000	2727 - 833 = 1894	20%	80%
0 to 1000	833	100%	0%
Total	5000		

Section 18, Limited Higher Moments

One can get limited higher moments in a manner parallel to the limited expected value.
Just as the limited expected value at u, $E[X \wedge u]$, is the first moment of data limited to u, the limited second moment, $E[(X \wedge u)^2]$, is the second moment of the data limited to u.
First limit the losses, then square, then take the expected value.

Exercise: Prob[X = 2] = 70%, and Prob[X = 9] = 30%. Determine $E[X \wedge 5]$ and $E[(X \wedge 5)^2]$.
[Solution: $E[X \wedge 5] = (70\%)(2) + (30\%)(5) = 2.9$. $E[(X \wedge 5)^2] = (70\%)(2^2) + (30\%)(5^2) = 10.3$.
Comment: $\text{Var}[X \wedge 5] = 10.3 - 2.9^2 = 1.89$.]

As with the limited expected value, one can write the limited second moment as a contribution of small losses plus a contribution of large losses:

$$E[(X \wedge u)^2] = \int_0^u t^2 f(t)\, dt + S(u)\, u^2.$$

The losses of size larger than u, each contribute u^2, while the losses of size u or less, each contribute their size squared. $E[(X \wedge u)^2]$ can be computed by integration in the same manner as the moments and Limited Expected Values. As shown in Appendix A attached to the exam, here are the formulas for the limited higher moments for some distributions:[91]

Distribution	$E[(X \wedge x)^n]$
Exponential	$n!\, \theta^n\, \Gamma(n+1\,;\, x/\theta) + x^n e^{-x/\theta}$
Pareto	$\{n!\, \theta^n\, \Gamma(\alpha-n) / \Gamma(\alpha)\}\, \beta[n+1, \alpha-n\,;\, x/(\theta+x)] + x^n(\theta/(\theta+x))^\alpha$
Gamma	$\{\theta^n\, \Gamma(\alpha+n)\, \Gamma(\alpha+n; x/\theta)\, /\, \Gamma(\alpha)\} + x^n\{1 - \Gamma(\alpha; x/\theta)\}$
LogNormal	$\exp[n\mu + 0.5 n^2\sigma^2]\ \Phi\left[\frac{\ln(x) - \mu - n\sigma^2}{\sigma}\right] + x^n\left\{1 - \Phi\left[\frac{\ln(x) - \mu}{\sigma}\right]\right\}$
Weibull	$\theta^n\, \Gamma(1 + n/\tau)\, \Gamma(1 + n/\tau\,;\, (x/\theta)^\tau) + x^n \exp[-(x/\theta)^\tau]$
Single Parameter Pareto	$\frac{\alpha\, \theta^n}{\alpha - n} - \frac{n\, \theta^\alpha}{(\alpha - n)\, x^{\alpha-n}}$, $x \geq \theta$

[91] The formula for the limited moments of the Pareto involving Incomplete Beta Functions, reduces to the formula shown subsequently for n=2. However, it requires integration by parts and a lot of algebraic manipulation.

One obtains the Limited Expected Value by setting n = 1, while one obtains the limited second moment for n = 2.[92]

Distribution	$E[(X \wedge x)^2]$
Exponential	$2\theta^2 - 2\theta^2 e^{-x/\theta} - 2\theta x e^{-x/\theta}$
Single Parameter Pareto	$\frac{\alpha\,\theta^2}{\alpha - 2} - \frac{2\,\theta^\alpha}{(\alpha - 2)\,x^{\alpha-2}}$, $x \ge \theta$
Pareto	$\frac{2\theta^2}{(\alpha-1)\,(\alpha-2)}\,\{1 - (1 + x/\theta)^{1-\alpha}(1 + (\alpha-1)x/\theta)\}$
LogNormal	$\exp[2\mu + 2\sigma^2]\,\Phi\left[\frac{\ln(x) - \mu - 2\sigma^2}{\sigma}\right] + x^2\,\{1 - \Phi\left[\frac{\ln(x) - \mu}{\sigma}\right]\}$

Exercise: For a LogNormal Distribution with $\mu = 7$ and $\sigma = 0.5$, what is the $E[(X \wedge 1000)^2]$?
[Solution: $E[(X \wedge 1000)^2] = e^{14.5}\,\Phi[\{\ln(1000) - 7.5\} / 0.5] + 1000^2\{1 - \Phi[\{\ln(1000) - 7\} / 0.5]\}$
$= 1{,}982{,}759\,\Phi[-1.184] + 1{,}000{,}000\{1 - \Phi[-0.184]\}$
$= 1{,}982{,}759\,(0.1182) + 1{,}000{,}000(1 - 0.4270) = 807{,}362$.]

Generally, $E[(X \wedge u)^2]$ is less than $E[X^2]$. For low censorship points u or more skewed distributions the difference can be quite substantial. For example, in the above exercise,
$E[X^2] = \exp[2\mu + 2\sigma^2] = e^{14.5} = 1{,}982{,}759$,
while $E[(X \wedge 1000)^2] = 807{,}362$.

[92] The limited second moments of a Exponential and Pareto are not shown in Loss Models in these forms, but as shown below these formulas are correct.

Second Limited Moment in Terms of Other Quantities of Interest:

It is sometimes useful to write $E[(X \wedge u)^2]$ in terms of the Survival Function, the Excess Ratio R(x) or the Loss Elimination Ratio LER(x) as follows. Using integration by parts and the fact that the integral of f(x) is -S(x):

$$E[(X \wedge u)^2] = \int_0^u t^2 f(t)\, dt + S(u)\, u^2 = -S(t)\, t^2 \Big]_{t=0}^{t=u} + \int_0^u S(t)\, 2t\, dt + S(u)u^2 = \int_0^u S(t)\, 2t\, dt.$$

In particular, for u = ∞, one can write the second moment as twice the integral of the survival function times x:[93]

$$E[X^2] = 2 \int_0^\infty S(t)\, t\, dt.$$

More generally, $E[(X \wedge u)^n] = \int_0^u n\, t^{n-1} S(t)\, dt$.[94]

For u = ∞, $E[X^n] = \int_0^\infty n\, t^{n-1} S(t)\, dt$.

Using integration by parts and the fact that the integral of S(t) is LER(x) μ :

$$E[(X \wedge u)^2] = 2 \int_0^\infty S(t)\, t\, dt = 2x\mu\, LER(x) \Big]_{t=0}^{t=u} - 2\mu \int_0^u LER(t)\, dt.$$

$$E[(X \wedge u)^2] = 2\mu \{LER(u)u - \int_0^u LER(t)\, dt\} = 2\mu\{\int_0^u R(t)\, dt - R(u)u\}.$$

[93] Recall that the mean can be written as an integral of the survival function. One can proceed in the same manner to get higher moments in terms of integrals of the survival function times a power of x.

[94] The form shown here is true for distributions with support x > 0. More generally, the nth limited moment is the sum of an integral from -∞ to 0 of $-n\, t^{n-1} F(t)$ and an integral from 0 to L of $n\, t^{n-1} S(t)$.
See Equation 3.9 in Loss Models.

So for example for the Pareto distribution: $\mu = \theta / (\alpha-1)$, $R(x) = \{\theta/(\theta+x)\}^{\alpha-1}$.

$$\int_0^u R(t)\, dt = -\theta^{\alpha-1}/\{(\alpha-2)(\theta+t)^{\alpha-2}\}\Big]_{t=0}^{t=u} = \{\theta^{\alpha-1}/(\alpha-2)\}\{\theta^{2-\alpha} - (\theta+u)^{2-\alpha}\}.$$

$E[(X \wedge u)^2] = \{2\theta^{\alpha}/(\alpha-1)(\alpha-2)\}\{\theta^{2-\alpha} - (\theta+u)^{2-\alpha} - (\alpha-2)u(\theta+u)^{1-\alpha}\}$.

$E[(X \wedge u)^2] = \{2\theta^2/(\alpha-1)(\alpha-2)\}\{1 - (1 + u/\theta)^{2-\alpha} - (\alpha-2)(u/\theta)(1 + u/\theta)^{1-\alpha}\}$.

$E[(X \wedge u)^2] = E[X^2]\,\{1 - (1 + u/\theta)^{1-\alpha}[1 + (\alpha-1)u/\theta]\}$.

Letting u go to infinity, it follows that: $E[X^2] = 2\, E[X] \int_0^\infty R(x)\, dx$.

$$\Rightarrow \int_0^\infty R(x)\, dx = \frac{E[X^2]}{2\, E[X]}.$$

Now the excess ratio R(x) is: $\dfrac{\text{expected excess losses}}{\text{mean}} = \dfrac{\int_x^\infty S(y)\, dy}{E[X]}$.

Therefore, $\displaystyle\int_{x=0}^\infty \int_{y=x}^\infty S(y)\, dy\, dx = E[X] \int_0^\infty R(x)\, dx = E[X] \frac{E[X^2]}{2\, E[X]} = E[X^2]/2.$

I will use the above result to show that the variance is equal to $\displaystyle\int_{x=0}^\infty \left(\int_{y=x}^\infty S(y)\, dy\right)^2 \frac{f(x)}{S(x)^2}\, dx$.

Using integration by parts, let $u = \left(\displaystyle\int_{y=x}^\infty S(y)\, dy\right)^2$ and $dv = \dfrac{f(x)}{S(x)^2}\, dx$.

$du = 2 \displaystyle\int_{y=x}^\infty S(y)\, dy\ (-S(x))$. $\qquad v = 1/S(x)$.

Therefore, $$\int_{x=0}^\infty \left(\int_{y=x}^\infty S(y)\, dy\right)^2 \frac{f(x)}{S(x)^2}\, dx = \left(\int_{y=x}^\infty S(y)\, dy\right)^2 \frac{1}{S(x)}\Bigg]_{x=0}^{x=\infty} + 2 \int_{x=0}^\infty \int_{y=x}^\infty S(y)\, dy\, dx.$$

Now as was shown previously, $E[X^2] = 2 \int_0^{\infty} t\, S(t)\, dt$.

Therefore, if there is a finite second moment, $\int_0^{\infty} t\, S(t)\, dt$ must be finite.

If in the extreme righthand tail S(t) ~ $1/t^2$, then this integral would be infinite.
Therefore, in the extreme right hand tail, S(t) must go down faster than $1/t^2$.

If in the extreme righthand tail S(t) ~ $1/t^{2+\varepsilon}$, then $\int_{y=x}^{\infty} S(y)\, dy \,/\, \sqrt{S(x)} \sim (1/x^{1+\varepsilon})\, x^{1+\varepsilon/2} = 1/x^{\varepsilon/2}$.

Therefore, if in the extreme right hand tail, S(t) goes down faster than $1/t^2$,

then $\int_{y=x}^{\infty} S(y)\, dy \,/\, \sqrt{S(x)}$ approaches zero as x approaches infinity.

Therefore, as x approaches infinity, $\left(\int_{y=x}^{\infty} S(y)\, dy\right)^2 \frac{1}{S(x)}$ approaches zero.

Thus, $\int_{x=0}^{\infty} \left(\int_{y=x}^{\infty} S(y)\, dy\right)^2 \frac{f(x)}{S(x)^2}\, dx = -E[X]^2 / S(0) + 2\, E[X^2]/2 = E[X]^2 - E[X]^2 = \text{Var}[X]$.

<u>Pareto Distribution</u>:

As discussed previously, $\int_0^\infty R(x)\ dx = \frac{E[X^2]}{2\ E[X]}$.[95]

For example, for the Pareto Distribution, $R(x) = \{\theta/(\theta+x)\}^{\alpha-1}$.

$$\int_0^\infty R(x)\ dx = \theta^{\alpha-1} \int_0^\infty (\theta + x)^{\alpha-1}\ dx = \theta^{\alpha-1}\ \theta^{\alpha-2}/(\alpha - 2) = \theta / (\alpha-2) = \frac{2\ \theta^2 / \{(\alpha-1)(\alpha-2)\}}{2\ \theta / (\alpha-1)} = \frac{E[X^2]}{2\ E[X]}.$$

As discussed previously, for the Pareto Distribution,
$E[(X \wedge x)^2] = E[X^2]\ \{1 - (1 + x/\theta)^{1-\alpha}(1 + (\alpha-1)x/\theta)\}$.[96]

Exercise: For a Pareto with $\alpha = 4$ and $\theta = 1000$, compute $E[X^2]$, $E[(X \wedge 500)^2]$ and $E[(X \wedge 5000)^2]$.

[Solution: $E[X^2] = \frac{2\theta^2}{(\alpha-1)\ (\alpha-2)} = 333{,}333$,

$E[(X \wedge 500)^2] = E[X^2]\ \{1 - (1 + 500/\theta)^{1-\alpha}(1 + (\alpha-1)500/\theta)\} = 86{,}420$, and

$E[(X \wedge 5000)^2] = E[X^2]\ \{1 - (1 + 5000/\theta)^{1-\alpha}(1 + (\alpha-1)5000/\theta)\} = 308{,}642$.]

The limited higher moments can also be used to calculate the variance, coefficient of variation, and skewness of losses subject to a maximum covered loss.

Exercise: For a Pareto Distribution with $\alpha = 4$ and $\theta = 1000$, and for a maximum covered loss of 5000, compute the variance and coefficient of variation (per single loss.)
[Solution: From a previous solutions $E[X \wedge 5000] = 331.79$ and $E[(X \wedge 5000)^2] = 308{,}642$.
Thus the variance is: $308{,}642 - 331.79^2 = 198{,}557$. Thus the CV is: $\sqrt{198{,}557} / 331.79 = 1.34$.]

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95 Where R(x) is the excess ratio, and the distribution has support starting at zero.
96 While this formula was derived above, it is <u>not</u> in the Appendix attached to the exam.

Second Moment of a Layer of Loss:

Also, one can use the Limited Second Moment to calculate the second moment of a layer of loss.[97]

The second moment of the layer from d to u is:[98]

$$E[(X \wedge u)^2] - E[(X \wedge d)^2] - 2d \{E[X \wedge u] - E[X \wedge d]\}.$$

Exercise: For a Pareto with $\alpha = 4$ and $\theta = 1000$, compute the second moment of the layer from 500 to 5000.
[Solution: From the solutions to previous exercises, $E[X \wedge 500] = 234.57$, $E[X \wedge 5000] = 331.79$, $E[(X \wedge 500)^2] = 86{,}420$, and $E[(X \wedge 5000)^2] = 308{,}642$.
The second moment of the layer from 500 to 5000 is:
$E[(X \wedge 5000)^2] - E[(X \wedge 500)^2] - 2(500)\{E[X \wedge 5000] - E[X \wedge 500]\} =$
$308{,}642 - 86{,}420 - (1000)(331.79 - 234.57) = 125{,}002$.
Comment: Note this is the second moment per loss, including those losses that do not penetrate the layer, in the same way that $E[X \wedge 5000] - E[X \wedge 500]$ is the first moment of the layer per loss.]

With no maximum covered loss, in other words with $u = \infty$ in the above formula, then:
$E[(X-d)_+^2] = E[X^2] - E[(X \wedge d)^2] - 2d \{E[X] - E[X \wedge d]\}$.

Exercise: For a Pareto with $\alpha = 4$ and $\theta = 1000$, compute the second moment of $(X - 500)_+$.
[Solution: $E[X^2] - E[(X \wedge 500)^2] - (2)(500) \{E[X] - E[X \wedge 500]\} =$
$333{,}333 - 86{,}420 - (1000)(333.33 - 234.57) = 148{,}153$.
Comment: Recall that $(X - 500)_+$ includes the zero payments.]

Exercise: For a Pareto with $\alpha = 4$ and $\theta = 1000$, compute the second moment of $(X - 5000)_+$.
[Solution: $E[X^2] - E[(X \wedge 5000)^2] - (2)(5000) \{E[X] - E[X \wedge 5000]\} =$
$333{,}333 - 308{,}642 - (10{,}000)(333.33 - 331.79) = 9291$.]

With instead no deductible, in other words with $d = 0$ in the above formula, then the second moment of the layer from 0 to u is $E[(X \wedge u)^2]$, as it should be.

[97] Recall that the expected value of a Layer of Loss is the difference of the Limited Expected Value at the top of the layer and the Limited Expected Value at the bottom of the layer. For second and higher moments the relationships are more complicated.
[98] See Theorem 8.8 in Loss Models. Here we are referring to the second moment of the per loss variable; similar to the average payment per loss, we are including those times a small loss contributes zero to the layer.

<u>Derivation of the Second Moment of a Layer of Loss</u>:

For the layer from d to u, the medium size losses contribute x - d, while the large losses contribute the width of the interval u - d.

Therefore, the second moment of the layer from d to u is:

$$\int_d^u (x - d)^2 f(x)\ dx + (u - d)^2 S(u) = \int_d^u (x^2 - 2dx + d^2) f(x)\ dx + u^2 S(u) - 2duS(u) + d^2 S(u) =$$

$$\int_0^u x^2 f(x)\ dx + u^2 S(u) - \int_0^d x^2 f(x)\ dx - d^2 S(d) + d^2 S(d) - 2d \int_d^u x f(x)\ dx$$

$$+ d^2\{F(u) - F(d)\} - 2duS(u) + d^2 S(u) =$$

$E[(X \wedge u)^2] - E[(X \wedge d)^2] - 2d\{E[X \wedge u] - uS(u) - (E[X \wedge d] - dS(d))\}$
$+ d^2\{S(d) + F(u) - F(d)\} - 2duS(u) + d^2 S(u) =$
$E[(X \wedge u)^2] - E[(X \wedge d)^2] - 2d\{E[X \wedge u] - (E[X \wedge d]\}$
$+ d\{2uS(u) - 2dS(d) + dS(d) + dF(u) - dF(d) -2uS(u) + dS(u)\} =$
$E[(X \wedge u)^2] - E[(X \wedge d)^2] - 2d\{E[X \wedge u] - E[X \wedge d]\} + d\{d(F(u) + S(u)) - d(F(d) + S(d)\} =$
$E[(X \wedge u)^2] - E[(X \wedge d)^2] - 2d\{E[X \wedge u] - E[X \wedge d]\} + d(d - d) =$
$E[(X \wedge u)^2] - E[(X \wedge d)^2] - 2d\{E[X \wedge u] - E[X \wedge d]\}$.

<u>Variance of a Layer of Loss</u>:

Given the first and second moments layer of loss, one can compute the variance and the coefficient of variation of a layer of loss.

Exercise: For a Pareto with $\alpha = 4$ and $\theta = 1000$, compute the variance of the losses in the layer from 500 to 5000.
[Solution: From the previous exercise, the second moment is 125,002 and the mean is $E[X \wedge 5000] - E[X \wedge 500] = 331.79 - 234.57 = 97.22$.
The variance of the layer from 500 to 5000 is: $125{,}002 - 97.22^2 = 115{,}550$.]

Exercise: For a Pareto with $\alpha = 4$ and $\theta = 1000$, compute the coefficient of variation of the losses in the layer from 500 to 5000.
[Solution: From the previous exercise, the variance of the layer from 500 to 5000 is: $125{,}002 - 97.22^2 = 115{,}550$ and the mean is 97.22. Thus the CV = $115{,}550^{0.5} / 97.22 = 3.5$.]

Using the formulas for the first and seconds moments of a layer of loss,
the variance of the layer of losses from d to u is:

$$\mathbf{E[(X \wedge u)^2] - E[(X \wedge d)^2] - 2d\,\{E[X \wedge u] - E[X \wedge d]\} - \{E[X \wedge u] - E[X \wedge d]\}^2}.$$

Since the average payment per loss under a maximum covered loss of u and a deductible of d is the layer from d to u, this provides a formula for the variance of the average payment per loss under a maximum covered loss of u and a deductible of d.

Exercise: Assume losses are given by a LogNormal Distribution with $\mu = 8$ and $\sigma = 0.7$.
An insured has a deductible of 1000, and a maximum covered loss of 10,000.
What is the expected amount paid per loss?
[Solution: For the LogNormal Distribution:
$E[X \wedge x] = \exp(\mu + \sigma^2/2)\Phi[(\ln x - \mu - \sigma^2)/\sigma] + x\{1 - \Phi[(\ln x - \mu)/\sigma]\}$.
$E[X \wedge 1000] = e^{8.245}\ \Phi[\{\ln(1000) - 8 - 0.49\} / 0.7] + 1000\{1 - \Phi[\{\ln(1000) - 8\} / 0.7]\ \}$
$= 3808.54\ \Phi[-2.260] + 1000\{1 - \Phi[-1.560]\}$
$= 3808.54\ (0.0119) + 1000(1 - 0.0594) = 986$.
$E[X \wedge 10{,}000] = e^{8.245}\ \Phi[\{\ln(10{,}000) - 8 - 0.49\} / 0.7] + 10{,}000\{1 - \Phi[\{\ln(10{,}000) - 8\} / 0.7]\ \}$
$= 3808.54\ \Phi[1.029] + 10{,}000\{1 - \Phi[1.729]\} = 3808.54(0.8483) + 10{,}000(1 - 0.9581) = 3650$.
$E[X \wedge 10{,}000] - E[X \wedge 1000] = 3650 - 986 = 2664$.]

Exercise: Assume losses are given by a LogNormal Distribution with $\mu = 8$ and $\sigma = 0.7$.
An insured has a deductible of 1000, and a maximum covered loss of 10,000.
What is the variance of the amount paid per loss?
[Solution: For the LogNormal Distribution:
$E[(X \wedge x)^2] = \exp[2\mu + 2\sigma^2]\Phi[\{\ln(x) - (\mu + 2\sigma^2)\}/\sigma] + x^2\{1 - \Phi[\{\ln(x) - \mu\}/\sigma]\}$.
$E[(X \wedge 1000)^2] = e^{16.98}\ \Phi[\{\ln(1000) - 8.98\} / 0.7] + 1000^2\{1 - \Phi[\{\ln(1000) - 8\} / 0.7]\ \}$
$= 23{,}676{,}652\ \Phi[-2.960] + 1{,}000{,}000\{1 - \Phi[-1.560]\}$
$= 23{,}676{,}652\ (0.0015) + 1{,}000{,}000(1 - 0.0594) = 976{,}115$.
$E[(X \wedge 10{,}000)^2] = e^{16.98}\ \Phi[\{\ln(10{,}000) - 8.98\} / 0.7] + 10{,}000^2\{1 - \Phi[\{\ln(10{,}000) - 8\} / 0.7]\ \}$
$= 23{,}676{,}652\ \Phi[0.329] + 100{,}000{,}000\{1 - \Phi[1.729]\}$
$= 23{,}676{,}652\ (0.6289) + 100{,}000{,}000(1 - 0.9581) = 19{,}080{,}246$.
$E[(X \wedge u)^2] - E[(X \wedge d)^2] - 2d\{E[X \wedge u] - (E[X \wedge d]\} - \{E[X \wedge u] - (E[X \wedge d]\}^2$
$= 19{,}080{,}246 - 976{,}115 - (2000)(2664) - 2664^2 = 5.68$ million.
<u>Comment</u>: It would take too long to perform all of these computations for a single exam question!]

If one has a coinsurance factor of c, then each payment is multiplied by c, therefore the variance is multiplied by c^2.

<u>Exercise</u>: Assume losses are given by a LogNormal Distribution with $\mu = 8$ and $\sigma = 0.7$.
An insured has a deductible of 1000, a maximum covered loss of 10,000, and a coinsurance factor of 80%. What is the variance of the amount paid per loss?
[Solution: $(0.8^2)(5.68$ million$) = 3.64$ million.]

Variance of Non-Zero Payments:

Exercise: For a Pareto with $\alpha = 4$ and $\theta = 1000$, compute the average non-zero payment given a deductible of 500 and a maximum covered loss of 5000.
[Solution: $(E[X \wedge 5000] - E[X \wedge 500]) / S(500) = (331.79 - 234.57) / 0.1975 = 492.2$.]

One can compute the second moment of the non-zero payments in a manner similar to the second moment of the payments per loss. Given a deductible of d and a maximum covered loss of u, the 2nd moment of the non-zero payments is:[99]

$$\int_d^u (x - d)^2 \{f(x)/S(d)\}\, dx + (u - d)^2 S(u)/S(d) = (\text{2nd moment of the payments per loss}) / S(d) =$$

$\{E[(X \wedge u)^2] - E[(X \wedge d)^2] - 2d(E[X \wedge u] - E[X \wedge d])\} / S(d)$.
So just as with the first moment, the second moment of the non-zero payments has S(d) in the denominator. If one has a coinsurance factor of c, then the second moment is multiplied by c^2.

Exercise: For a Pareto with $\alpha = 4$ and $\theta = 1000$, compute the second moment of the non-zero payments given a deductible of 500 and a maximum covered loss of 5000.
[Solution: $\{E[(X \wedge 5000)^2] - E[(X \wedge 500)^2] - 2(500)\{E[X \wedge 5000] - E[X \wedge 500]\}\}/S(500) =$
$\{308{,}642 - 86{,}420 - (1000)(331.79 - 234.57)\} / (1000/1500)^4 = 125{,}002/0.1975309 = 632{,}823$.]

Thus given a deductible of d and a maximum covered loss of u the variance of the non-zero payments is: $\{E[(X \wedge u)^2] - E[(X \wedge d)^2] - 2d(E[X \wedge u] - E[X \wedge d])\} / S(d) - \{(E[X \wedge u] - E[X \wedge d]) / S(d)\}^2$.

Exercise: For a Pareto with $\alpha = 4$ and $\theta = 1000$, compute the variance of the non-zero payments given a deductible of 500 and a maximum covered loss of 5000.
[Solution: From the solutions to previous exercises, variance $= 632{,}823 - 492.2^2 = 390{,}562$.]

If one has a coinsurance factor of c, then each payment is multiplied by c, therefore the variance is multiplied by c^2.

Exercise: For a Pareto with $\alpha = 4$ and $\theta = 1000$, compute the variance of the non-zero payments given a deductible of 500, a maximum covered loss of 5000, and a coinsurance factor of 85%.
[Solution: $(0.85^2)(390{,}562) = 282{,}181$.]

[99] Note that the density is f(x)/S(d) from d to u, and has a point mass of S(u)/S(d) at u.

<u>Exponential Distribution</u>:

As shown in Appendix A of <u>Loss Models</u> attached to the exam, for the Exponential Distribution:
$E[(X \wedge x)^n] = n!\ \theta^n\ \Gamma(n+1\ ;\ x/\theta) + x^n\ e^{-x/\theta}$.

However, the Incomplete Gamma for positive integer alpha is:[100]

$$\Gamma(\alpha\ ;\ x) = 1 - \sum_{i=0}^{\alpha-1} \frac{x^i\ e^{x}}{i!}.$$

Thus, $\Gamma(3\ ;\ x/\theta) = 1 - e^{-x/\theta} - (x/\theta)e^{-x/\theta} - (x/\theta)^2 e^{-x/\theta}/2$.

Therefore, for the Exponential, $E[(X \wedge x)^2] = 2!\ \theta^2\ \Gamma(2+1\ ;\ x/\theta) + x^2\ e^{-x/\theta} =$
$(2)(\theta^2)\{1 - e^{-x/\theta} - (x/\theta)e^{-x/\theta} - (x/\theta)^2 e^{-x/\theta}/2\} + x^2\ e^{-x/\theta} = 2\theta^2 - 2\theta^2 e^{-x/\theta} - 2\theta x e^{-x/\theta}$.

Exercise: For an Exponential Distribution with $\theta = 10$, what is the $E[(X \wedge 30)^2]$?
[Solution: $E[(X \wedge x)^2] = 2\theta^2 - 2\theta^2 e^{-x/\theta} - 2\theta x e^{-x/\theta}$. $E[(X \wedge 30)^2] = 200 - 200e^{-3} - 600e^{-3} = 160.2$.]

Exercise:
What is the variance of the payment per loss, Y^L, for an Exponential with deductible d?
[Solution: The first moment is: $E[X] - E[X \wedge d] = \theta\ e^{-d/\theta}$.
Second Moment is: $E[X^2] - E[(X \wedge d)^2] - 2\ d\ \{E[X] - E[X \wedge d]\} =$
$2\theta^2 - \{2\theta^2 - 2\theta^2 e^{-d/\theta} - 2\theta d e^{-dx/\theta}\} - (2d)(\theta - \theta(1 - e^{-d/\theta}))\} = 2\theta^2 e^{-d/\theta}$.
Thus $Var[Y^L] = 2\theta^2 e^{-d/\theta} - (\theta\ e^{-d/\theta})^2 = \theta^2\ (2e^{-d/\theta} - e^{-2d/\theta})$.
Alternately, the payment per payment variable, Y^P, is also Exponential with mean θ.
$E[Y^L] = E[Y^P]\ S(d) = \theta\ e^{-d/\theta}$.
Second Moment of Y^L is: (Second moment of Y^P) $S(d) = 2\theta^2 e^{-d/\theta}$. Proceed as before.]

Exercise: Losses follow an Exponential Distribution with $\theta = 200$.
For a deductible of 100, what is the variance of the payment per loss variable, Y^L?
[Solution: $\theta^2\ (2e^{-d/\theta} - e^{-2d/\theta}) = 200^2\ (2\ e^{-0.5} - e^{-1}) = 33{,}807$.]

[100] See Theorem A.1 in <u>Loss Models</u>.

Problems:

Use the following information for the next 7 questions.

Assume the unlimited losses follow a LogNormal Distribution with parameters μ = 10 and σ = 1.5. Assume an average of 10,000 losses per year. For simplicity, ignore any variation in costs due to variations in the number of losses per year.

18.1 (2 points) What is the coefficient of variation of the total cost expected per year?
A. less than 0.014
B. at least 0.014 but less than 0.018
C. at least 0.018 but less than 0.022
D. at least 0.022 but less than 0.026
E. at least 0.026

18.2 (3 points) If the insurer pays no more than $250,000 per loss,
what is the coefficient of variation of the insurer's total cost expected per year?
A. less than 0.014
B. at least 0.014 but less than 0.018
C. at least 0.018 but less than 0.022
D. at least 0.022 but less than 0.026
E. at least 0.026

18.3 (3 points) If the insurer pays no more than $1,000,000 per loss,
what is the coefficient of variation of the insurer's total cost expected per year?
A. less than 0.014
B. at least 0.014 but less than 0.018
C. at least 0.018 but less than 0.022
D. at least 0.022 but less than 0.026
E. at least 0.026

18.4 (1 point) If the insurer pays the layer from $250,000 to $1 million per loss,
what are the insurer's total costs expected per year?
A. less than $135 million
B. at least $135 million but less than $140 million
C. at least $140 million but less than $145 million
D. at least $145 million but less than $150 million
E. at least $150 million

18.5 (3 points) If the insurer pays the layer from $250,000 to $1 million per loss,
what is the coefficient of variation of the insurer's total cost expected per year?
A. less than 0.03
B. at least 0.03 but less than 0.05
C. at least 0.05 but less than 0.07
D. at least 0.07 but less than 0.09
E. at least 0.09

18.6 (3 points) What is the coefficient of skewness of the total cost expected per year?
A. less than 0.15
B. at least 0.15 but less than 0.20
C. at least 0.20 but less than 0.25
D. at least 0.25 but less than 0.30
E. at least 0.30

18.7 (3 points) If the insurer pays no more than $250,000 per loss, what is the coefficient of skewness of the insurer's total cost expected per year?
A. less than 0.015
B. at least 0.015 but less than 0.020
C. at least 0.020 but less than 0.025
D. at least 0.025 but less than 0.030
E. at least 0.030

Use the following information for the next 7 questions
Losses follow an Exponential Distribution with θ = 10,000.

18.8 (1 point) What is the variance of losses?
A. less than 105 million
B. at least 105 million but less than 110 million
C. at least 110 million but less than 115 million
D. at least 115 million but less than 120 million
E. at least 120 million

18.9 (2 points) Assuming a 25,000 policy limit, what is the variance of payments by the insurer?
A. less than 35 million
B. at least 40 million but less than 45 million
C. at least 45 million but less than 50 million
D. at least 50 million but less than 55 million
E. at least 55 million

18.10 (3 points) Assuming a 1000 deductible (with no maximum covered loss),
what is the variance of the payments per loss?
A. less than 95 million
B. at least 95 million but less than 100 million
C. at least 100 million but less than 105 million
D. at least 105 million but less than 110 million
E. at least 110 million

18.11 (2 points) Assuming a 1000 deductible (with no maximum covered loss),
what is the variance of non-zero payments by the insurer?
A. less than 95 million
B. at least 95 million but less than 100 million
C. at least 100 million but less than 105 million
D. at least 105 million but less than 110 million
E. at least 110 million

18.12 (3 points) Assuming a 1000 deductible and a 25,000 maximum covered loss, what is the variance of the payments per loss?
A. less than 55 million
B. at least 55 million but less than 56 million
C. at least 56 million but less than 57 million
D. at least 57 million but less than 58 million
E. at least 58 million

18.13 (3 points) Assuming a 1000 deductible and a 25,000 maximum covered loss, what is the variance of the non-zero payments by the insurer?
A. less than 53 million
B. at least 53 million but less than 54 million
C. at least 54 million but less than 55 million
D. at least 55 million but less than 56 million
E. at least 56 million

18.14 (2 points) Assuming a 75% coinsurance factor, a 1000 deductible and a 25,000 maximum covered loss, what is the variance of the insurer's payments per loss?
A. less than 15 million
B. at least 15 million but less than 20 million
C. at least 20 million but less than 25 million
D. at least 25 million but less than 30 million
E. at least 30 million

18.15 (2 points) Let X be the result of rolling a fair six-sided die, with the numbers 1 through 6 on its faces. Calculate Var($X \wedge 4$).
(A) 1.1 (B) 1.2 (C) 1.3 (D) 1.4 (E) 1.5

18.16 (3 points) The size of loss distribution has the following characteristics:
(i) E[X] = 245.
(ii) $E[X \wedge 100] = 85$.
(iii) S(100) = 0.65.
(iv) $E[X^2 \mid X > 100] = 250{,}000$.
There is an ordinary deductible of 100 per loss.
Calculate the second moment of the payment per loss.
(A) 116,000 (B) 118,000 (C)120,000 (D) 122,000 (E) 124,000

Use the following information for the next four questions:

- Losses follow a LogNormal Distribution with parameters $\mu = 9.7$ and $\sigma = 0.8$.
- The insured has a deductible of 10,000, maximum covered loss of 50,000, and a coinsurance factor of 90%.

18.17 (3 points) What is the average payment per loss?
A. less than 7,000
B. at least 7,000 but less than 8,000
C. at least 8,000 but less than 9,000
D. at least 9,000 but less than 10,000
E. at least 10,000

18.18 (2 points) What is $E[(X \wedge 50{,}000)^2]$?
A. less than 600 million
B. at least 600 million but less than 700 million
C. at least 700 million but less than 800 million
D. at least 800 million but less than 900 million
E. at least 900 million

18.19 (2 points) What is $E[(X \wedge 10{,}000)^2]$?
A. less than 80 million
B. at least 80 million but less than 90 million
C. at least 90 million but less than 100 million
D. at least 100 million but less than 110 million
E. at least 110 million

18.20 (2 points) What is the variance of the payment per loss?
A. less than 110 million
B. at least 110 million but less than 120 million
C. at least 120 million but less than 130 million
D. at least 130 million but less than 140 million
E. at least 140 million

18.21 (3 points) You are given:

Claim Size	Number of Claims
0-25	30
25-50	32
50-100	20
100-200	8

Assume a uniform distribution of claim sizes within each interval.
Estimate $E[(X \wedge 80)^2]$.
(A) 2300 (B) 2400 (C) 2500 (D) 2600 (E) 2700

Use the following information for the next 5 questions:

- Losses are uniform from 0 to 40.

18.22 (1 point) What is $E[X \wedge 10]$?
A. 8.5 B. 8.75 C. 9.0 D. 9.25 E. 9.5

18.23 (1 point) What is $E[X \wedge 25]$?
A. 15 B. 16 C. 17 D. 18 E. 19

18.24 (2 points) What is $E[(X \wedge 10)^2]$?
A. 79 B. 80 C. 81 D. 82 E. 83

18.25 (2 points) What is $E[(X \wedge 25)^2]$?
A. 360 B. 365 C. 370 D. 375 E. 380

18.26 (2 points) What is the variance of the layer of loss from 10 to 25?
A. 37 B. 39 C. 41 D. 43 E. 45

Use the following information for the next 7 questions.

Assume the following discrete size of loss distribution:

10	50%
50	30%
100	20%

18.27 (2 points) What is the coefficient of variation of this size of loss distribution?
A. less than 0.6
B. at least 0.6 but less than 0.8
C. at least 0.8 but less than 1.0
D. at least 1.0 but less than 1.2
E. at least 1.2

18.28 (3 points) What is the coefficient of skewness of this size of loss distribution?
A. less than 0
B. at least 0 but less than 0.2
C. at least 0.2 but less than 0.4
D. at least 0.4 but less than 0.6
E. at least 0.6

18.29 (2 points) If the insurer pays no more than 25 per loss,
what is the coefficient of variation of the distribution of the size of payments?
A. less than 0.6
B. at least 0.6 but less than 0.8
C. at least 0.8 but less than 1.0
D. at least 1.0 but less than 1.2
E. at least 1.2

<u>18.30</u> (2 points) If the insurer pays no more than 60 per loss,
what is the coefficient of variation of the distribution of the size of payments?
A. less than 0.6
B. at least 0.6 but less than 0.8
C. at least 0.8 but less than 1.0
D. at least 1.0 but less than 1.2
E. at least 1.2

18.31 (3 points) If the insurer pays no more than 60 per loss,
what is the coefficient of skewness of the distribution of the size of payments?
A. less than 0
B. at least 0 but less than 0.2
C. at least 0.2 but less than 0.4
D. at least 0.4 but less than 0.6
E. at least 0.6

18.32 (1 point) If the insurer pays the layer from 30 to 70 per loss,
what is the insurer's expected payment per loss?
A. 10 B. 12 C. 14 D. 16 E. 18

18.33 (2 points) If the insurer pays the layer from 30 to 70 per loss,
what is the coefficient of variation of the insurer's payments per loss?
A. less than 0.6
B. at least 0.6 but less than 0.8
C. at least 0.8 but less than 1.0
D. at least 1.0 but less than 1.2
E. at least 1.2

18.34 (2 points) X follows the density f(x), with support from 0 to infinity.

$$\int_0^{500} f(x)\,dx = 0.685. \quad \int_0^{500} x\,f(x)\,dx = 217. \quad \int_0^{500} x^2\,f(x)\,dx = 76{,}616.$$

Determine the variance of the limited loss variable with u = 500, $X \wedge 500$.
A. 14,000 B. 15,000 C. 16,000 D. 17,000 E. 18,000

18.35 (4 points) The size of loss follows a Single Parameter Pareto Distribution
with $\alpha = 3$ and $\theta = 200$.
A policy has a deductible of 250, a maximum covered loss of 1000,
and a coinsurance of 80%.
Determine the variance of Y^P, the per payment variable.
A. 12,000 B. 13,000 C. 14,000 D. 15,000 E. 16,000

<u>18.36</u> (3 points)
The loss severity random variable X follows an exponential distribution with mean θ.
Determine the coefficient of variation of the excess loss variable Y = max(X - d, 0).

18.37 (21 points) Let U be the losses in the layer from a to b.
Let V be the losses in the layer from c to d. $a < b \le c < d$.
(i) (3 points) Determine an expression for the covariance of U and V in terms of Limited Expected Values.
(ii) (2 points) For an Exponential Distribution with mean of 8, determine the covariance of the losses in the layer from 0 to 10 and the losses in the layer from 10 to infinity.
(iii) (3 points) For an Exponential Distribution with mean of 8, determine the variance of the losses in the layer from 0 to 10.
Hint: For the Exponential, $E[(X \wedge x)^2] = 2\theta^2 - 2\theta^2 e^{-x/\theta} - 2\theta x e^{-x/\theta}$.
(iv) (3 points) For an Exponential Distribution with mean of 8, determine the variance of the losses in the layer from 10 to infinity.
(v) (1 point) For an Exponential Distribution with mean of 8, determine the correlation of the losses in the layer from 0 to 10 and the losses in the layer from 10 to infinity.
(vi) (2 points) For a Pareto Distribution with $\alpha = 3$ and $\theta = 16$, determine the covariance of the losses in the layer from 0 to 10 and the losses in the layer from 10 to infinity.
(vii) (3 points) For a Pareto Distribution with $\alpha = 3$ and $\theta = 16$, determine the variance of the losses in the layer from 0 to 10.
Hint: For the Pareto, $E[(X \wedge x)^2] = \frac{2\,\theta^2}{(\alpha-1)\,(\alpha-2)}\,\{1 - (1 + x/\theta)^{1-\alpha}\,(1 + (\alpha-1)x/\theta)\}$.
(viii) (3 points) For a Pareto Distribution with $\alpha = 3$ and $\theta = 16$, determine the variance of the losses in the layer from 10 to infinity.
(ix) (1 point) For a Pareto Distribution with $\alpha = 3$ and $\theta = 16$, determine the correlation of the losses in the layer from 0 to 10 and the losses in the layer from 10 to infinity.

18.38 (14 points) Let X be the price of a stock at time 1.
X is distributed via a LogNormal Distribution with $\mu = 4$ and $\sigma = 0.3$.
Let Y be the payoff on a one-year 70-strike European Call on this stock.
$Y = 0$ if $X \le 70$, and $Y = X - 70$ if $X > 70$.
(i) (1 point) What is the mean of X?
(ii) (2 points) What is the variance of X?
(iii) (3 points) What is the mean of Y?
(iv) (4 points) What is the variance of Y?
(v) (3 points) What is the covariance of X and Y?
(vi) (1 point) What is the correlation of X and Y?

Use the following information for the next 3 questions:

Limit	Limited Expected Value	Limited Second Moment
100,000	55,556	4444 million
250,000	80,247	12,346 million
500,000	91,837	20,408 million
1,000,000	97,222	27,778 million
Infinite	100,000	40,000 million

18.39 (2 points)
Determine the coefficient of variation of the layer of loss from 100,000 to 500,000.
(A) Less than 3
(B) At least 3, but less than 4
(C) At least 4, but less than 5
(D) At least 5, but less than 6
(E) At least 6

18.40 (2 points)
Determine the coefficient of variation of the layer of loss from 250,000 to 1 million.
(A) Less than 3
(B) At least 3, but less than 4
(C) At least 4, but less than 5
(D) At least 5, but less than 6
(E) At least 6

18.41 (2 points) Determine the coefficient of variation of the layer of loss excess of 250,000.
(A) Less than 3
(B) At least 3, but less than 4
(C) At least 4, but less than 5
(D) At least 5, but less than 6
(E) At least 6

Use the following information for the next 2 questions:
- Losses are uniform from 2 to 20.
- There is a deductible of 5.

18.42 (1 point) Determine the variance of Y^P, the per-payment variable.
A. 17 B. 18 C. 19 D. 20 E. 21

18.43 (3 points) Determine the variance of Y^L, the per-loss variable.
A. 23 B. 26 C. 29 D. 32 E. 35

18.44 (3 points) The loss severity random variable X follows the Pareto distribution with $\alpha = 5$ and $\theta = 400$.
Determine the coefficient of variation of the excess loss variable Y = max(X - 300, 0).
(A) 6.5 (B) 7.0 (C) 7.5 (D) 8.0 (E) 8.5

18.45 (4 points) Severity follows a LogNormal Distribution with $\mu = 7$ and $\sigma = 0.6$.
There is a 1000 deductible.
Determine the variance of the payment per loss variable.
(A) 300,000 (B) 400,000 (C) 500,000 (D) 600,000 (E) 700,000

18.46 (3 points) You are given the following information:

Limit	Limited Expected Value	Limited Second Moment	Survival Function
1000	976.66	963,617	0.91970
5000	2828.18	9,830,381	0.12465
Infinite	3000	12,000,000	

There is a 1000 deductible and a 5000 maximum covered loss.
Determine the standard deviation of the per payment variable, Y^P.
A. 1050 B. 1100 C. 1150 D. 1200 E. 1250

18.47 (3, 11/00, Q.21 & 2009 Sample Q.115) (2.5 points)
A claim severity distribution is exponential with mean 1000.
An insurance company will pay the amount of each claim in excess of a deductible of 100.
Calculate the variance of the amount paid by the insurance company for one claim, including the possibility that the amount paid is 0.
(A) 810,000 (B) 860,000 (C) 900,000 (D) 990,000 (E) 1,000,000

18.48 (1, 5/01, Q.35) (1.9 points) The warranty on a machine specifies that it will be replaced at failure or age 4, whichever occurs first.
The machine's age at failure, X, has density function 1/5 for $0 < x < 5$.
Let Y be the age of the machine at the time of replacement.
Determine the variance of Y.
(A) 1.3 (B) 1.4 (C) 1.7 (D) 2.1 (E) 7.5

18.49 (4, 11/03, Q.37 & 2009 Sample Q.28) (2.5 points) You are given:

Claim Size (X)	Number of Claims
(0, 25]	25
(25, 50]	28
(50, 100]	15
(100, 200]	6

Assume a uniform distribution of claim sizes within each interval.
Estimate $E(X^2) - E[(X \wedge 150)^2]$.
(A) Less than 200
(B) At least 200, but less than 300
(C) At least 300, but less than 400
(D) At least 400, but less than 500
(E) At least 500

18.50 (SOA3, 11/03, Q.28) (2.5 points) For (x):
(i) K is the curtate future lifetime random variable.
(ii) $q_{x+k} = 0.1(k + 1)$, k = 0, 1, 2,..., 9
Calculate Var(K $\wedge$ 3).
(A) 1.1 (B) 1.2 (C) 1.3 (D) 1.4 (E) 1.5

18.51 (4, 5/07, Q.13) (2.5 points)
The loss severity random variable X follows the exponential distribution with mean 10,000.
Determine the coefficient of variation of the excess loss variable Y = max(X - 30,000, 0).
(A) 1.0 (B) 3.0 (C) 6.3 (D) 9.0 (E) 39.2

Solutions to Problems:

18.1. E. For the sum of N independent losses, both the Variance and the mean are N times that for a single loss. The standard deviation is multiplied by $\sqrt{N}$.

Thus the CV, which is the ratio of the standard deviation to the mean, is divided by $\sqrt{N}$.

Per loss, mean = $\exp(\mu + \sigma^2/2) = e^{11.125}$, and second moment is $\exp(2\mu + 2\sigma^2) = e^{24.5}$.
Therefore for a single loss, CV = $\sqrt{E[X^2]/E[X]^2 - 1} = \sqrt{e^{24.5} / e^{22.25} - 1} = \sqrt{e^{2.25} - 1} = 2.91$.

For 10,000 losses we divide by $\sqrt{10{,}000}$ = 100, thus the CV for the total losses is **0.0291**.

18.2. A. $E[X \wedge x] = \exp(\mu + \sigma^2/2)\ \Phi[(\ln x - \mu - \sigma^2)/\sigma] + x\ \{1 - \Phi[(\ln x - \mu)/\sigma]\}$.
$E[X \wedge 250{,}000] = \exp(11.125)\Phi[(\ln(250{,}000) - 10 - 2.25)/1.5] +$
$(250{,}000)\{1 - \Phi[(\ln(250{,}000) - 10)/1.5]\} = (67{,}846)\Phi[0.12] + (250{,}000)(1 - \Phi[1.62])$
$= (67{,}846)(0.5478) + (250{,}000)(1 - 0.9474) = 50.3$ thousand.
$E[(X \wedge L)^2] = \exp[2\mu + 2\sigma^2]\Phi[\{\ln(L) - (\mu + 2\sigma^2)\} / \sigma] + L^2\{1 - \Phi[\{\ln(L) - \mu\} / \sigma]\}$.
$E[(X \wedge 250{,}000)^2] = \exp(24.5)\Phi[-1.38] + 6.25 \times 10^{10}\{1 - \Phi[1.62]\} =$
$(4.367 \times 10^{10})(0.0838) + (6.25 \times 10^{10})(1 - 0.9474) = 6.95 \times 10^9$.
Therefore for a single loss Coefficient of Variation =
$\sqrt{E[(X \wedge 250{,}000)^2]/E[X \wedge 250{,}000]^2 - 1} = \sqrt{6.95 \times 10^9 / 2.53 \times 10^9 - 1} = 1.32$.

For 10,000 losses we divide by $\sqrt{10{,}000}$ = 100, thus the CV is **0.0132**.

18.3. C. $E[X \wedge x] = \exp(\mu + \sigma^2/2)\ \Phi[(\ln x - \mu - \sigma^2)/\sigma] + x\ \{1 - \Phi[(\ln x - \mu)/\sigma]\}$.
$E[X \wedge 1{,}000{,}000] =$
$\exp(11.125)\Phi[(\ln(1{,}000{,}000) - 10 - 2.25)/1.5] + (1{,}000{,}000)\ \{1 - \Phi[(\ln(1{,}000{,}000) - 10)/1.5]\}$
$= (67{,}846)\Phi[1.04] + (1{,}000{,}000)(1 - \Phi[2.54]) = (67{,}846)(0.8508) + (1{,}000{,}000)(1 - 0.9945)$
$= 63.2$ thousand.
$E[(X \wedge L)^2] = \exp[2\mu + 2\sigma^2]\ \Phi[\{\ln(L) - (\mu + 2\sigma^2)\}/\sigma] + L^2\{1 - \Phi[\{\ln(L) - \mu\}/\sigma]\}$.
$E[(X \wedge 1{,}000{,}000)^2] = \exp(24.5)\Phi[-0.46] + (10^{12})\{1 - \Phi[2.54]\} =$
$(4.367 \times 10^{10})(0.3228) + (10^{12})\{1 - 0.9945\} = 1.960 \times 10^{10}$.
Therefore Coefficient of Variation = $\sqrt{E[(X \wedge 1 \text{ million})^2]/E[X \wedge 1 \text{ million}]^2 - 1}$ =

$\sqrt{1.960 \times 10^{10} / 3.99 \times 10^9 - 1} = 1.98$.

For 10,000 losses we divide by $\sqrt{10{,}000}$ = 100, thus the CV is **0.0198**.

Comment: The Coefficient of Variation of the Limited Losses is less than that of the unlimited losses. The CV of the losses limited to 250,000 is lower than that of the losses limited to $1 million.

18.4. A. Since the insurer expects 10,000 losses per year, the expected dollars in the layer from 250,000 to $1 million are:
$10{,}000\{E[X \wedge 1 \text{ million}] - E[X \wedge 250{,}000]\} = 10{,}000(63.2 \text{ thousand} - 50.3 \text{ thousand}) =$
129 million.

18.5. C. The mean for the layer is: E[X∧ 1 million] - E[X∧ 250,000] =
63.2 thousand - 50.3 thousand = 13.1 thousand. The second moment for the layer is:
$E[(X\wedge 1 \text{ million})^2] - E[(X\wedge 250{,}000)^2] - 2(250{,}000)(E[X\wedge 1 \text{ million}] - E[X\wedge 250{,}000]) =$
$1.960 \times 10^{10} - 6.95 \times 10^9 - 6.55 \times 10^9 = 6.10 \times 10^9$.
Therefore for a single loss, Coefficient of Variation = $\sqrt{6.10 \times 10^9 / 1.72 \times 10^8 - 1} = 5.9$.
For 10,000 losses we divide by $\sqrt{10{,}000} = 100$, thus the CV is **0.059**.
Comment: The CV of a layer depends on how high the layer is, the width of the layer, as well as the loss distribution. A higher layer usually has a larger CV.

18.6. E. For the sum of N independent losses, both the Variance and the Third Central Moment are N times that for a single loss. (For the sum of independent random variables, second and third central moments each add.) Thus the skewness, which is the ratio of the Third Central Moment to the variance to the 3/2 power, is divided by $\sqrt{N}$.
Per loss, mean **=** $\exp(\mu + \sigma^2/2) = e^{11.125}$, second moment is: $\exp(2\mu + 2\sigma^2) = e^{24.5}$, and third moment is: $\exp(3\mu + 4.5\sigma^2) = e^{40.125}$. Therefore, the variance is: $e^{24.5} - e^{22.25} = 3.907 \times 10^{10}$.
The third central moment is: $e^{40.125} - 3e^{24.5}e^{11.125} + 2e^{33.375} = 2.620 \times 10^{17}$.
Thus for one loss the skewness is: $2.585 \times 10^{17} / (3.907 \times 10^{10})^{1.5} = 33.5$. For 10,000 losses we divide by $\sqrt{10{,}000} = 100$, thus the skewness for the total losses is **0.335**.

18.7. B. $E[X\wedge 250{,}000] = \exp(11.125)\ \Phi[(\ln(250{,}000) - 10 - 2.25)/1.5] +$
$(250{,}000)\ \{1 - \Phi[(\ln(250{,}000) - 10)/1.5]\} = 50.3$ thousand.
$E[(X\wedge 250{,}000)^2] = \exp(24.5)\ \Phi[-1.38] + 6.25 \times 10^{10}\{1 - \Phi[1.62]\} = 6.95 \times 10^9$.
Thus the variance of a limited loss is $6.95 \times 10^9 - 2.53 \times 10^9 = 4.42 \times 10^9$.
$E[(X\wedge L)^3] = \exp[3\mu + 4.5\ \sigma^2]\ \Phi[\{\ln(L) - (\mu + 3\sigma^2)\}/\sigma] + L^3\{1-\Phi[\{\ln(L) - \mu\}/\sigma]\}$.
$E[(X\wedge 250{,}000)^3] = e^{40.125}\Phi[-2.88] + (1.5625 \times 10^{16})\ \{1-\Phi[1.62]\} = 1.355 \times 10^{15}$.
The third central moment is:
$1.355 \times 10^{15} - 3(50.3 \text{ thousand})(6.95 \times 10^9) + 2(50.3 \text{ thousand})^3 = 5.6 \times 10^{14}$.
Thus for one loss the skewness is: $5.6 \times 10^{14} / (4.42 \times 10^9)^{1.5} = 1.9$.
For 10,000 losses we divide by $\sqrt{10{,}000} = 100$; thus the skewness for the total losses is **0.019**.
Comment: The skewness of the limited losses is much smaller than that of the unlimited losses. Rare very large losses have a large impact on the skewness of the unlimited losses.

18.8. A. θ^2 = **100 million**.

18.9. E. The second moment is $E[(X \wedge x)^2] = 2\theta^2 \Gamma(3; x/\theta) + x^2 e^{-x/\theta}$.

$E[(X \wedge 25{,}000)^2] = 200{,}000{,}000\ \Gamma(3; 2.5) + 625{,}000{,}000 e^{-2.5}$ =

(200 million) $\{1 - e^{-2.5}(1 + 2.5 + 2.5^2/2)\}$ + 51.30 million = 142.54 million.

variance = 142.54 million - 9179^2 = **58.28 million**.

Comment: $\Gamma(\alpha; x) = 1 - \sum_{i=0}^{\alpha-1} \frac{x^i e^{x}}{i!}$.

18.10. B. The second moment is: $E[X^2] - E[(X \wedge 1000)^2] - (2)(1000)\{E[X] - (E[X \wedge 1000]\}$.

$E[X] - E[X \wedge 1000]$ = 10,000 - 952 = 9048. $E[X^2] = 2\theta^2$ = 200 million.

$E[(X \wedge 1000)^2] = 200{,}000{,}000\ \Gamma(3; 0.1) + 1{,}000{,}000 e^{-0.1}$ =

200 million$\{\{1 - e^{-0.1}(1+0.1+0.1^2/2)\} + 0.005e^{-2.5}\}$ = 0.936 million.

The second moment is: 200 million - 0.936 million - (2000)(9048) = 180.97 million.

Variance = 180.97 million - 9048^2 = **99.1 million**.

Alternately, the payment per payment variable, Y^P, is also Exponential with mean 10,000.

$E[Y^L] = E[Y^P]\ S(d) = 10{,}000\ e^{-1000/10{,}000}$ = 9048.

Second Moment of Y^L is: (Second moment of Y^P) $S(d) = 2\theta^2 e^{-d/\theta} = (200{,}000{,}000)e^{-1000/10{,}000}$

= 180.97 million. Variance = 180.97 million - 9048^2 = **99.1 million**.

Comment: In this situation, $\text{Var}[Y^L] = 2\theta^2 e^{-d/\theta} - (\theta\ e^{-d/\theta})^2 = \theta^2\ (2e^{-d/\theta} - e^{-2d/\theta})$.

18.11. C. The second moment is: $\int_{1000}^{\infty} \frac{f(x)}{S(1000)}\ (x-1000)^2\ dx$

= (2nd moment of payment per loss) / S(1000) = 180.97 million / $e^{-0.1}$ = 180.97 million / 0.9048
= 200.00 million.

Variance = 200.00 million - $10{,}000^2$ = **100 million**.

Comment: Due to the memoryless property of the Exponential, the variance is the same as in the absence of a deductible.

18.12. D. $E[X \wedge 25{,}000]$ = 9179. $E[X \wedge 1000]$ = 952.

$E[(X \wedge 25{,}000)^2]$ = 142.54 million. $E[(X \wedge 1000)^2]$ = 0.936 million.

second moment of the layer of loss =

$E[(X \wedge 25{,}000)^2] - E[(X \wedge 1000)^2] - (2)(1000)\{E[X \wedge 25{,}000] - (E[X \wedge 1000]\}$ =

142.54 million - 0.936 million - (2000)(9179 - 952) = 125.15 million.

Variance = 125.15 million - $(9179 - 952)^2$ = **57.46 million**.

18.13. D. The second moment is: $\int_{1000}^{25,000} \frac{f(x)}{S(1000)} (x-1000)^2 \, dx + 24,000^2 \, S(25,000)/S(1000)$

= (2nd moment of payment per loss) / S(1000)
= 125.15 million / $e^{-0.1}$ = 125.15 million / 0.9048 = 138.32 million.
The mean is: 9093. Variance = 138.32 million - 9093^2 = **55.63 million**.

18.14. E. The second moment is:
$0.75^2\{E[(X \wedge 25,000)^2] - E[(X \wedge 1000)^2] - (2)(1000)\{E[X \wedge 25,000] - (E[X \wedge 1000]\}\}$
= 0.5625{ 142.54 million - 0.936 million - (2000)(9179 - 952)} = 70.40 million.
Variance = 70.40 million - 6170^2 = **32.33 million**.

18.15. C. $E[X \wedge 4] = (1/6)(1) + (1/6)(2) + (1/6)(3) + (3/6)(4) = 3.$
$E[(X \wedge 4)^2] = (1/6)(1^2) + (1/6)(2^2) + (1/6)(3^2) + (3/6)(4^2) = 10.33.$
$Var(X \wedge 4) = 10.33 - 3^2 =$ **1.33**.

18.16. E. $E[X^2 \mid X > 100] = \int_{100}^{\infty} x^2 f(x) \, dx / S(100). \Rightarrow \int_{100}^{\infty} x^2 f(x) \, dx = S(100) E[X^2 \mid X > 100]$

= (0.65)(250,000) = 162,500.

$\int_{100}^{\infty} x \, f(x) \, dx = \int_{0}^{\infty} x \, f(x) \, dx - \int_{0}^{100} x \, f(x) \, dx = E[X] - \{E[X \wedge 100] - 100S(100)\}$

= 245 - {85 - (100)(0.65)} = 225.

$\int_{100}^{\infty} f(x) \, dx = S(100) = 0.65.$

With a deductible of 100 per loss, the second moment of the payment per loss is:

$\int_{100}^{\infty} (x - 100)^2 f(x) \, dx = \int_{100}^{\infty} x^2 f(x) \, dx - 200 \int_{100}^{\infty} x \, f(x) \, dx + 10,000 \int_{100}^{\infty} f(x) \, dx$

= 162,500 - (200)(225) + (10,000)(0.65) = **124,000**.
Comment: Similar to SOA M, 5/05, Q.17, in "Mahler's Guide to Aggregate Distributions."
The variance of the payment per loss is: $124,000 - (245 - 85)^2 = 98,400$.
With a deductible of d, the second moment of the payment per loss is:
$E[X^2 \mid X > d] \, S(d) - 2d(E[X] - \{E[X \wedge d] - dS(d)\}) + d^2 S(d) =$
$E[X^2 \mid X > d] \, S(d) - 2d \, E[X] + 2d \, E[X \wedge d] - d^2 S(d).$

18.17. E. $E[X \wedge x] = \exp(\mu+\sigma^2/2)\ \Phi[(\ln x - \mu - \sigma^2)/\sigma] + x\ \{1 - \Phi[(\ln x - \mu)/\sigma]\}$.
$E[X \wedge 50{,}000] = \exp(10.02)\Phi[(\ln(50{,}000) - 9.7 - 0.64)/0.8] + (50{,}000)\ \{1 - \Phi[(\ln(50{,}000) - 9.7)/0.8]\}$
$= (22{,}471)\Phi[0.60] + (50{,}000)\{1 - \Phi[1.40]\} = (22{,}471)(0.7257) + (50{,}000)\ (1 - 0.9192) = 20{,}347$.
$E[X \wedge 10{,}000] = \exp(10.02)\Phi[(\ln(10{,}000) - 9.7 - 0.64\ )/0.8] + (10{,}000)\{1 - \Phi[(\ln(10{,}000) - 9.7)/0.8]\}$
$= (22{,}471)\Phi[-1.41] + (10{,}000)\{1 - \Phi[-0.61]\} = (22{,}471)(0.0793) + (10{,}000)\ (0.7291) = 9073$.
The average payment per loss is:
$(0.9)(E[X \wedge 50{,}000] - E[X \wedge 10{,}000]) = (0.9)(20{,}347 - 9073) =$ **10,147**.

18.18. B. $E[(X \wedge x)^2] = \exp[2\mu + 2\sigma^2]\ \Phi[\{\ln(x) - (\mu + 2\sigma^2)\} / \sigma] + x^2\{1 - \Phi[\{\ln(x) - \mu\} / \sigma]\ \}$.
$E[(X \wedge 50{,}000)^2] = \exp[20.68]\ \Phi[\{\ln(50{,}000) - 10.98\}/0.8] + 50{,}000^2\{1 - \Phi[\{\ln(50{,}000) - 9.7\}/0.8]\ \} =$
$957{,}656{,}776\Phi[-0.20] + 2{,}500{,}000{,}000\{1 - \Phi[1.40]\} =$
$(957{,}656{,}776)(0.4207) + (2{,}500{,}000{,}000)\{1 - 0.9192\} =$ **604.9 million**.

18.19. B. $E[(X \wedge 10{,}000)^2] =$
$\exp[20.68]\Phi[\{\ln(10{,}000) - 10.98\} / 0.8] + 10{,}000^2\{1 - \Phi[\{\ln(10{,}000) - 9.7\} / 0.8]\ \} =$
$957{,}656{,}776\Phi[-2.21] + 100{,}000{,}000\{1 - \Phi[-0.61]\} =$
$(957{,}656{,}776)(0.0136) + (100{,}000{,}000)(0.7291) =$ **85.9 million**.

18.20. D. $c^2\{E[(X \wedge L)^2] - E[(X \wedge d)^2] - 2d\{E[X \wedge L] - (E[X \wedge d]\} - \{E[X \wedge L] - (E[X \wedge d]\}^2\}$
$= 0.9^2\{E[(X \wedge 50{,}000)^2] - E[(X \wedge 10{,}000)^2] - 2(10{,}000)\{E[X \wedge 50{,}000] - (E[X \wedge 10{,}000]\}$
$- \{E[X \wedge 50{,}000] - (E[X \wedge 10{,}000]\}^2\} =$
$0.81\{604.8 \text{ million} - 85.9 \text{ million} - 20{,}000(20{,}345 - 9073) - (20{,}345 - 9073)^2\} =$
$(0.81)\{166.4 \text{ million}) =$ **134.8 million**.

18.21. A. For a uniform distribution on (a, b), $E[X^2] = (b^3 - a^3)/\{3(b - a)\}$.
For those intervals above 80, $E[(X \wedge 80)^2] = 80^2 = 6400$.
We need to divide the interval from 50 to 100 into two pieces.
For losses uniform on 50 to 100, 2/5 are expected to be greater than 80, so we assign (2.5)(20) = 8 to the interval 80 to 100 and the remaining 12 to the interval 50 to 80.

Bottom of Interval	Top of Interval	Number of Claims	Expected 2nd Moment Limited to 80
0	25	30	208
25	50	32	1458
50	80	12	4300
80	100	8	6400
100	200	8	6400
Average			**2299**

18.22. B., 18.23. C., 18.24. E., 18.25. B. , 18.26. C.

$E[X \wedge 10] = \int_0^{10} x/40 \ dx + (3/4)(10) = $ **8.75**.

$E[X \wedge 25] = \int_0^{25} x/40 \ dx + (15/40)(25) = $ **17.1875**.

$E[(X \wedge 10)^2] = \int_0^{10} x^2/40 \ dx + (3/4)(10^2) = $ **83.333**.

$E[(X \wedge 25)^2] = \int_0^{25} x^2/40 \ dx \, x + (15/40)(25^2) = $ **364.583**.

Layer Average Severity is: $E[X \wedge 25] - E[X \wedge 10] = 17.1875 - 8.75 = 8.4375$.
2nd moment of the layer = $E[(X \wedge 25)^2] - E[(X \wedge 10)^2] - (2)(10)(E[X \wedge 25] - E[X \wedge 10]) =$
$364.583 - 83.333 - (2)(10)(8.4375) = 112.5$. Variance of the layer = $112.5 - 8.4375^2 =$ **41.31**.
Alternately, the contributions to the layer from each small loss is 0, from each medium loss is x - 10, and each large loss is 15. Thus the second moment of the layer is:

$\int_{10}^{25} (x-10)^2/40 \ dx + (15/40)(15^2) = 28.125 + 84.375 = 112.5$. Proceed as before.

18.27. C. Mean = (50%)(10) + (30%)(50) + (20%)(100) = 40.
Second Moment = $(50\%)(10^2) + (30\%)(50^2) + (20\%)(100^2) = 2800$.
Variance = $2800 - 40^2 = 1200$. coefficient of variation = $\sqrt{1200} / 40 =$ **0.866**.

18.28. E.
Third Central Moment = $(50\%)(10 - 40)^3 + (30\%)(50 - 40)^3 + (20\%)(100 - 40)^3 = 30,000$.
Skewness = Third Central Moment / Variance$^{1.5}$ = $30,000 / 1200^{1.5}$ = **0.722**.

18.29. A. Mean = (50%)(10) + (30%)(25) + (20%)(25) = 17.5.
Second Moment = $(50\%)(10^2) + (30\%)(25^2) + (20\%)(25^2) = 362.5$.
Variance = $362.5 - 17.5^2 = 56.25$.
coefficient of variation = $\sqrt{56.25} / 17.5 =$ **0.429**.

18.30. B. Mean = (50%)(10) + (30%)(50) + (20%)(60) = 32.
Second Moment = $(50\%)(10^2) + (30\%)(50^2) + (20\%)(60^2) = 1520$.
Variance = $1520 - 32^2 = 496$.
coefficient of variation = $\sqrt{496} / 32 =$ **0.696**.

18.31. B. Third Central Moment = $(50\%)(10 - 32)^3 + (30\%)(50 - 32)^3 + (20\%)(60 - 32)^3 = 816$.
Skewness = Third Central Moment / Variance$^{1.5}$ = $816 / 496^{1.5}$ = **0.074**.

18.32. C. (50%)(0) + (30%)(20) + (20%)(40) = **14**.

18.33. D. Second Moment = $(50\%)(0^2) + (30\%)(20^2) + (20\%)(40^2) = 440$.
Variance = $440 - 14^2 = 244$. coefficient of variation = $\sqrt{244}$ / 14 = **1.116**.

18.34. B. $E[X \wedge 500] = \int_0^{500} x\, f(x)\, dx + 500\, S(500) = 217 + (500)(1 - 0.685) = 374.5$.

$E[(X \wedge 500)^2] = \int_0^{500} x^2\, f(x)\, dx + 500^2\, S(500) = 76{,}616 + (500^2)(1 - 0.685) = 155{,}366$.

$\text{Var}[(X \wedge 500)^2] = E[(X \wedge 500)^2] - E[X \wedge 500]^2 = 155{,}366 - 374.5^2 =$ **15,116**.
Comment: *Based on a Gamma Distribution with α = 4.3 and θ = 100.*

18.35. C. From the Tables attached to the exam, for the Single Parameter Pareto, for $x \ge \theta$:
$E[X \wedge x] = \frac{\alpha\,\theta}{\alpha - 1} - \frac{\theta^\alpha}{(\alpha - 1)\, x^{\alpha-1}}$. $E[(X \wedge x)^2] = \frac{\alpha\,\theta^2}{\alpha - 2} - \frac{2\,\theta^\alpha}{(\alpha - 2)\, x^{\alpha-2}}$.

Thus $E[X \wedge 250] = (3)(200) / 2 - 200^3 / \{(2)\,(250^2)\} = 236$.
$E[X \wedge 1000] = (3)(200) / 2 - 200^3 / \{(2)\,(1000^2)\} = 296$. $S(250) = (200/250)^3 = 0.512$.
Thus the mean payment per payment is: (80%) (296 - 236) / 0.512 = 93.75.
$E[(X \wedge 250)^2] = (3)(200^2) / 1 - (2)(200^3) / 250 = 56{,}000$.
$E[(X \wedge 1000)^2] = (3)(200^2) / 1 - (2)(200^3) / 1000 = 104{,}000$.
Thus the second moment of the non-zero payments is:
$(80\%)^2 \{104{,}000 - 56{,}000 - (2)(250)(296 - 236)\} / 0.512 = 22{,}500$.
Thus the variance of the non-zero payments is: $22{,}500 - 93.75^2$ = **13,711**.

18.36. The probability that a loss exceeds d is: $e^{-d/\theta}$.
The non-zero payments excess of a deductible d is the same as the original Exponential.
Zero payments would contribute nothing to the aggregate amount paid.
One can think of $Y = (X - d)_+$ as the aggregate that results from a Bernoulli frequency with $q = e^{-d/\theta}$, and an Exponential severity with mean θ.
This has a variance of: (mean of Bernoulli)(var. of Expon.) + (mean of Expon.)^2(var. of Bernoulli)
$= (q)(\theta^2) + (\theta^2)(q)(1 - q) = (\theta^2)(q)(2 - q)$.
Mean aggregate is: $q\theta$.
Coefficient of variation is: $\sqrt{(\theta^2)(q)(2 - q)}\ / (q\theta) = \sqrt{2/q - 1} = \mathbf{\sqrt{2\, e^{d/\theta} - 1}}$.

Alternately, the payment per payment variable, Y^P, is also Exponential with mean θ.
$E[Y^L] = E[Y^P]\ S(d) = \theta\ e^{-d/\theta}$.
Second Moment of Y^L is: (Second moment of Y^P) $S(d) = 2\theta^2\ e^{-d/\theta}$.
$1 + CV^2$ = (Second moment) / (First Moment)$^2 = 2\,e^{d/\theta}$. $\Rightarrow CV = \mathbf{\sqrt{2\, e^{d/\theta} - 1}}$.
Comment: Similar to 4, 5/07, Q.13.

18.37. (i) $E[U] = E[X \wedge b] - E[X \wedge a]$. $E[V] = E[X \wedge d] - E[X \wedge c]$.
For x < c, V = 0. For c < x < d, U = b - a, and V = x - c. For d < x, U = b - a, and V = d - c.

Therefore, $E[UV] = \int_c^d (b-a)(x-c)f(x)\,dx + (b-a)(d-c)\ S(d) = (b - a)\{\int_c^d (x-c)f(x)\,dx + (d-c)\ S(d)\}$

$= (b - a)$(expected losses in the layer from c to d) $= (b - a)\{E[X \wedge d] - E[X \wedge c]\}$.
$Cov[U, V] = E[UV] - E[U]E[V] = \mathbf{\{b - a - E[X \wedge b] + E[X \wedge a]\}\{E[X \wedge d] - E[X \wedge c]\}}$.
Cov[U, V] = (width of the lower interval minus the layer average severity of the lower interval) (layer average severity of the upper interval).
$Cov[U, V] = \{E[(b - X)_+] - E[(a - X)_+]\}\{E[X \wedge d] - E[X \wedge c]\}$.

(ii) $E[X \wedge 10] = 8(1 - e^{-10/8}) = 5.708$.
Using the result from part (i) with a = 0, b = c = 10, and d = ∞:
Covariance $= \{10 - E[X \wedge 10]\}\{E[X] - E[X \wedge 10]\} = (10 - 5.708)(8 - 5.708) = \mathbf{9.84}$.
(iii) Mean of the layer from 0 to 10 is: $E[X \wedge 10] = 5.708$.
$E[(X \wedge 10)^2] = 2(8^2) - 2(8^2)\ e^{-10/8} - 2(8)(10)\ e^{-10/8} = 45.487$.
Second Moment of the layer from 0 to 10 is:
$E[(X \wedge 10)^2] = 2(8^2) - 2(8^2)\ e^{-10/8} - 2(8)(10)\ e^{-10/8} = 45.487$.
Variance of the layer from 0 to 10 is: $45.487 - 5.708^2 = \mathbf{12.906}$.
(iv) Mean of the layer from 10 to ∞ is: $E[X] - E[X \wedge 10] = 8 - 5.708 = 2.292$.
$E[X^2] = 2\theta^2 = 2(8^2) = 128$.
Second Moment of the layer from 10 to ∞ is:
$E[X^2] - E[(X \wedge 10)^2] - (2)(10)(E[X] - E[X \wedge 10]) = 128 - 45.487 - (20)(2.292) = 36.673$.
Variance of the layer from 10 to ∞ is: $36.673 - 2.292^2 = \mathbf{31.420}$.
(v) Correlation of the layer from 0 to 10 and the layer from 10 to infinity is:
$9.84 / \sqrt{(12.906)(31.420)} = \mathbf{0.489}$.

(vi) $E[X \wedge 10] = \{\theta/(\alpha-1)\}\{1 - (\theta/(\theta+x))^{\alpha-1}\} = (16/2)\{1 - (16/26)^2\} = 4.970$.
Using the result from part (i) with a = 0, b = c = 10, and d = ∞:
Covariance = $\{10 - E[X \wedge 10]\}\{E[X] - E[X \wedge 10]\} = (10 - 4.970)(8 - 4.970) =$ **15.24**.
(vii) Mean of the layer from 0 to 10 is: $E[X \wedge 10] = 4.970$.
Second Moment of the layer from 0 to 10 is:
$E[(X \wedge 10)^2] = 2(16^2)\{1 - (1 + 10/16)^{-2}(1 + (2)(10)/16)\} / \{(2)(1)\} = 37.870$.
Variance of the layer from 0 to 10 is: $37.870 - 4.970^2 =$ **13.169**.
(viii) Mean of the layer from 10 to ∞ is: $E[X] - E[X \wedge 10] = 8 - 4.970 = 3.030$.
$E[X^2] = 2(16^2) / \{(2)(1)\} = 256$.
Second Moment of the layer from 10 to ∞ is:
$E[X^2] - E[(X \wedge 10)^2] - (2)(10)(E[X] - E[X \wedge 10]) = 256 - 37.870 - (20)(3.030) = 157.530$.
Variance of the layer from 10 to ∞ is: $157.530 - 3.030^2 =$ **148.35**.
(ix) Correlation of the layer from 0 to 10 and the layer from 10 to infinity is:
$15.24 / \sqrt{(13.169)(148.35)} =$ **0.345**.

Comment: Since a larger loss contributes more to both layers than a smaller loss, the losses in the layers are positively correlated.
Equation 40 in "On the Theory of Increased Limits and Excess of Loss Pricing,"
by Robert S. Miccolis. PCAS 1977 is a special case of the result derived in part (a).
While the layers are positively correlated, this correlation diminishes as the distance between the layers increases.
See "The Mathematics of Excess Losses," by Leigh J. Halliwell, Variance Volume 6 Issue 1.

18.38. (i) $E[X] = \exp[4 + 0.3^2/2] =$ **57.11**.
(ii) $E[X^2] = \exp[(2)(4) + (2)(0.3^2)] = 3568.85$.
$VAR[X] = 3568.85 - 57.11^2 =$ **307.3**.
(iii) $E[Y] = 0 \text{ Prob}[X < 70] + E[X - 70 \mid X > 70] \text{ Prob}[X > 70] = E[X - 70 \mid X > 70] \text{ Prob}[X > 70] =$
$e(70)\, S(70) = E[X] - E[X \wedge 70]$.

$E[X \wedge 70] = \exp[4 + 0.3^2/2]\, \Phi[(\ln(70) - 4 - 0.3^2)/0.3] + 70\, \{1 - \Phi[(\ln(70) - 4)/0.3] =$

$57.11\, \Phi[0.53] + (70)\{1 - \Phi[0.83]\} = (57.11)(0.7019) + (70)(1 - 0.7967) = 54.32$.
Thus $E[Y] = E[X] - E[X \wedge 70] = 57.11 - 54.32 =$ **2.79**.

(iv) $E[Y^2] = 0 \text{ Prob}[X < 70] + E[(X - 70)^2 \mid X > 70] \text{ Prob}[X > 70] =$
(second moment of the layer from 70 to infinity) Prob[X > 70] =
$E[X^2] - E[(X \wedge 70)^2] - (2)(70)\{E[X] - E[X \wedge 70]\}$.

For LogNormal Distribution the second limited moment is:

$$E[(X \wedge x)^2] = \exp[2\mu + 2\sigma^2]\, \Phi\left[\frac{\ln(x) - \mu - 2\sigma^2}{\sigma}\right] + x^2\, \{1 - \Phi\left[\frac{\ln(x) - \mu}{\sigma}\right]\}.$$

$E[(X \wedge 70)^2] = \exp[(2)(4) + (2)(0.3^2)]\, \Phi[(\ln(70) - 4 - (2)(0.3^2))/0.3] + 70^2\, \{1 - \Phi[(\ln(70) - 4)/0.3]\}$

$= 3568.85\, F[0.23] + 4900\, \{1 - \Phi[0.83]\} = (3568.85)(0.5910) + (4900)(1 - 0.7967) = 3105.36$.

Thus, $E[Y^2] = E[X^2] - E[(X \wedge 70)^2] - (2)(70)\{E[X] - E[X \wedge 70]\} =$

$3568.85 - 3105.36 - (140)(57.11 - 54.32) = 72.89$.
$VAR[Y] = E[Y^2] - E[Y]^2 = 72.89 - 2.79^2 =$ **65.11**.
(v) $E[XY] = E[X (X - 70) \mid X > 70] \text{ Prob}[X > 70] =$
$E[(70)(X - 70) - (X - 70)(X - 70) \mid X > 70] \text{ Prob}[X > 70] =$
$\{E[(70)(X - 70) \mid X > 70] - E[(X - 70)(X - 70) \mid X > 70]\} \text{ Prob}[X > 70] =$
$E[(70)(X - 70) \mid X > 70] \text{ Prob}[X > 70] - E[(X - 70)^2 \mid X > 70] \text{ Prob}[X > 70] =$
$70\, E[X - 70 \mid X > 70] \text{ Prob}[X > 70] - E[Y^2 \mid X > 70] \text{ Prob}[X > 70] =$
$70\, E[Y] + E[Y^2] = (70)\,(2.79) + 72.89 = 268.19$.
$Cov[X, Y] = E[XY] - E[X]\, E[Y] = 268.19 - (57.11)(2.79) =$ **108.85**.

(vi) $Corr[X, Y] = \dfrac{Cov[X, Y]}{\sqrt{VAR[X]\, VAR[Y]}} = \dfrac{108.85}{\sqrt{(307.3)\,(65.11)}} =$ **0.77**.

18.39. A. The first moment of the layer is: 91,837 - 55,556 = 36,281.
The second moment of the layer is:
20,408 million - 4444 million - (2)(100,000)(36,281) = 8707.8 million.
Variance of the layer is: 8707.8 million - $36{,}281^2$ = 7392 million.
Standard Deviation of the layer is: 85,977.
CV of the layer is: 85,977/36,281 = **2.37**.
Comment: Based on a Pareto Distribution with $\alpha = 3$ and $\theta = 200{,}000$.

18.40. C. The first moment of the layer is: 97,222 - 80,247 = 16,975.
The second moment of the layer is:
27,778 million - 12,346 million - (2)(250,000)(16,975) = 6944.5 million.
$1 + CV^2 = 6944.5 \text{ million} / 16{,}975^2 = 24.100. \Rightarrow CV =$ **4.81**.

18.41. E. The mean is: $E[X] - E[X \wedge 250{,}000] = 100{,}000 - 80{,}247 = 19{,}753$.
Second moment of the payment per loss variable is:
$E[X^2] - E[(X \wedge 250{,}000)^2] - (2)(250{,}000)\ (E[X] - E[X \wedge 250{,}000]) =$
40,000 million - 12,346 million - (500,000)(19,753) = 17,778 million.
$1 + CV^2$ = 17,778 million / $19{,}753^2 = 45.563$. ⇒ CV = **6.68**.

18.42. C. The non-zero payments are uniform from 0 to 15, with variance:
$(15 - 0)^2 / 12 =$ **18.75**.

18.43. A. The non-zero payments are uniform from 0 to 15,
with mean: 7.5, variance: $(15 - 0)^2 / 12 = 18.75$,
and second moment: $18.75 + 7.5^2 = 75$.
The probability of a non-zero payment is: 15/18 = 5/6.
Thus Y^L is a two-point mixture of a uniform distribution from 0 to 15 and a distribution that is always zero, with weights 5/6 and 1/6.
The mean of the mixture is: (5/6)(7.5) + (1/6)(0) = 6.25.
The second moment of the mixture is: $(5/6)(75) + (1/6)(0^2) = 62.5$.
The variance of this mixture is: $62.5 - 6.25^2 =$ **23.44**.
Alternately, Y^L can be thought of as a compound distribution, with Bernoulli frequency with mean 5/6 and Uniform severity from 0 to 15.
The variance of this compound distribution is:
(Mean Freq.)(Var. Sev.) + $(\text{Mean Sev.})^2$(Var. Freq.) =
$(5/6)(18.75) + (7.5)^2\ \{(5/6)(1/6)\} =$ **23.44**.

18.44. A. The probability that a loss exceeds 300 is: $(4/7)^5 = 0.06093$.
The losses truncated and shifted from below at 300 is also a Pareto Distribution but with
$\alpha = 5$ and $\theta = 400 + 300 = 700$.
One can think of $Y = (X - 300)_+$ as the aggregate that results from a Bernoulli frequency with
$q = 0.06093$, and a Pareto severity with $\alpha = 5$ and $\theta = 700$.
This Pareto has mean: 700/4 = 175, second moment: $(2)(700^2) / \{(4)(3)\} = 81{,}667$,
and variance: $81{,}667 - 175^2 = 51{,}042$.
This has a variance of: (mean of Bernoulli)(var. of Pareto) + $(\text{mean of Pareto})^2$(var. of Bernoulli)
$= (0.06093)(51{,}042) + (175^2)(0.06093)(1 - 0.06093) = 4862$.
Mean aggregate is: (0.06093)(175) = 10.663.
Coefficient of variation is: $\sqrt{4862}$ / 10.663 = **6.54**.
Alternately, this is mathematically equivalent to a two point mixture, with 0.06093 weight to a Pareto with $\alpha = 5$ and $\theta = 700$ (the non-zero payments) and (1 - 0.06093) weight to a distribution that is always zero. The mean is: (0.06093)(175) + (1 - 0.06093)(0) = 10.663.
The second moment is the weighted average of the two second moments:
(0.06093)(81,667) + (1 - 0.06093)(0) = 4976.
Therefore, $1 + CV^2 = 4976/10.663^2 = 43.76$. ⇒ CV = **6.54**.
Comment: Similar to 4, 5/07, Q.13, which involves an Exponential rather than a Pareto.

18.45. D. $E[X] = \exp[7 + 0.6^2/2] = 1313$.
$E[X^2] = \exp[(2)(7) + (2)(0.6^2)] = 2,470,670$.
$E[X \wedge x] = \exp(\mu + \sigma^2/2)\ \Phi[(\ln x - \mu - \sigma^2)/\sigma] + x\{1 - \Phi[(\ln x - \mu)/\sigma]\}$.
$E[X \wedge 1000] = 1313\ \Phi[\{\ln(1000) - 7 - 0.6^2\} / 0.6] + 1000\{1 - \Phi[\{\ln(1000) - 7\} / 0.6]\}$
$= 1313\ \Phi[-0.75] + (1000)(1 - \Phi[-0.15]) = (1313)(0.2266) + (1000)(1 - 0.4404) = 857$.
$E[(X \wedge x)^2] = \exp[2\mu + 2\sigma^2]\Phi[\{\ln(x) - (\mu + 2\sigma^2)\} / \sigma] + x^2\{1 - \Phi[\{\ln(x) - \mu\} / \sigma]\}$.
$E[(X \wedge 1000)^2] = 2,470,670\ \Phi[\{\ln(1000) - 7.72\} / 0.6] + 1000^2\{1 - \Phi[\{\ln(1000) - 7\} / 0.6]\}$
$= 2,470,670\ \Phi[-1.35] + 1,000,000\ \{1 - \Phi[-0.15]\} = (2,470,670)(0.0885) + (1,000,000)(0.5596)$
$= 778,254$.
Mean of the payment per loss variable is: $E[X] - E[X \wedge 1000] = 1313 - 857 = 456$.
Second moment of the payment per loss variable is:
$E[X^2] - E[(X \wedge 1000)^2] - (2)(1000)\ (E[X] - E[X \wedge 1000])$
$= 2,470,670 - 778,254 - (2)(1000)(456) = 780,416$.
Variance of the payment per loss variable is: $780,416 - 456^2 =$ **572,480**.

18.46. E. The first moment of the per payment variable is:
$(E[X \wedge 5000] - E[X \wedge 1000]) / S(1000) = (2828.18 - 976.66) / 0.91970 = 2013$.
The second moment of the per payment variable is:
$\{E[(X \wedge 5000)^2] - E[(X \wedge 1000)^2] - (2)(1000)(E[X \wedge 5000] - E[X \wedge 1000])\} / S(1000) =$
$\{9.830,381 - 963,617 - (2)(1000)(2828.18 - 976.66)\} / 0.91970 = 5,614,574$.
Variance of the per payment variable is: $5,614,574 - 2013^2 = 1,562,405$.
$\sqrt{1,562,405}$ **= 1250**.

18.47. D. An Exponential distribution truncated and shifted from below is the same Exponential Distribution, due to the memoryless property of the Exponential. Thus the nonzero payments are Exponential with mean 1000. The probability of a nonzero payment is the probability that a loss is greater than the deductible of 100; $S(100) = e^{-100/1000} = 0.90484$.

Thus the payments of the insurer can be thought of as a compound distribution, with Bernoulli frequency with mean 0.90484 and Exponential severity with mean 1000. The variance of this compound distribution is:

(Mean Freq.)(Var. Sev.) + (Mean Sev.)^2(Var. Freq.) =

$(0.90484)(1000^2) + (1000)^2 \{(0.90484)(0.09516)\}$ = **990,945**.

Equivalently, the payments of the insurer in this case are a two point mixture of an Exponential with mean 1000 and a distribution that is always zero, with weights 0.90484 and 0.09516. This has a first moment of: (1000)(0.90484) + (0.09516)(0) = 904.84, and a second moment of: $\{(2)(1000^2)\}(0.90484) + (0.09516)(0^2)$ = 1,809,680.

Thus the variance is: $1,809,680 - 904.84^2$ = **990,945**.

Alternately, the payment per payment variable, Y^P, is also Exponential with mean 1000.

$E[Y^L] = E[Y^P]\ S(d) = 1000\ e^{-100/1000} = 904.8$.

Second Moment of Y^L is: (Second moment of Y^P) $S(d) = 2\theta^2 e^{-d/\theta} = (2,000,000)e^{-100/1000}$

= 1,809,675. Variance = $1,809,675 - 904.8^2$ = **991,011**.

Alternately, for the Exponential Distribution, $E[X] = \theta = 1000$, and $E[X^2] = 2\theta^2$ = 2 million.

For the Exponential Distribution, $E[X \wedge x] = \theta\ (1 - e^{-x/\theta})$.

$E[X \wedge 100] = 1000(1 - e^{-100/1000}) = 95.16$.

For the Exponential, $E[(X \wedge x)^n] = n!\ \theta^n\ \Gamma(n+1; x/\theta) + x^n\ e^{-x/\theta}$.

$E[(X \wedge 100)^2] = (2)1000^2\Gamma(3; 100/1000) + 100^2 e^{-100/1000}$.

According to Theorem A.1 in Loss Models, for integral α, the incomplete Gamma function $\Gamma(\alpha; y)$ is 1 minus the first a densities of a Poisson Distribution with mean y.

$\Gamma(3; y) = 1 - e^{-y}(1 + y + y^2/2)$. $\Gamma(3; 0.1) = 1 - e^{-0.1}(1 + 0.1 + 0.1^2/2) = 0.0001546$.

Therefore, $E[(X \wedge 100)^2] = (2\text{ million})(0.0001546) + 10,000e^{-0.1} = 9357$.

The first moment of the layer from 100 to ∞ is: $E[X] - E[X \wedge 100] = 1000 - 95.16 = 904.84$.

The second moment of the layer from 100 to ∞ is:

$E[X^2] - E[(X \wedge 100)^2] - (2)(100)(E[X] - E[X \wedge 100]) =$

2,000,000 - 9357 - (200)(904.84) = 1,809,675.

Therefore, the variance of the layer from 100 to ∞ is: $1,809,675 - 904.84^2$ = **990,940**.

Alternately, one can work directly with the integrals, using integration by parts.

The first moment of the layer from 100 to ∞ is:

$$\int_{100}^{\infty} (x - 100)e^{-x/1000}/1000\ dx = \int_{100}^{\infty} x e^{-x/1000}/1000\ dx - (1/10)\int_{100}^{\infty} e^{-x/1000}\ dx =$$

$$-xe^{-x/1000} - 1000e^{-x/1000} \Big]_{x=100}^{x=\infty} - 100e^{-0.1} = 100e^{-0.1} + 1000e^{-0.1} - 100e^{-0.1} = 904.84.$$

The second moment of the layer from 100 to ∞ is:

$$\int_{100}^{\infty} (x - 100)^2 e^{-x/1000}/1000 \; dx$$

$$= \int_{100}^{\infty} x^2 e^{-x/1000}/1000 \; dx \; - \int_{100}^{\infty} x \, e^{-x/1000}/5 \; dx \; + 10 \int_{100}^{\infty} e^{-x/1000} \; dx \; =$$

$$-x^2 \, e^{-x/1000} - 2000x e^{-x/1000} - 2{,}000{,}000 e^{-x/1000} \Big]_{x=100}^{x=\infty} + 200x e^{-x/1000} + 200{,}000 e^{-x/1000} \Big]_{x=100}^{x=\infty} +$$

$10{,}000e^{-0.1} = e^{-0.1}\{10{,}000 + 200{,}000 + 2{,}000{,}000 - 20{,}000 - 200{,}000 + 10{,}000\} =$
$2{,}000{,}000e^{-0.1} = 1{,}809{,}675$.

Therefore, the variance of the layer from 100 to ∞ is: $1{,}809{,}675 - 904.84^2 =$ **990,940**.

Comment:
Very long and difficult, unless one uses the memoryless property of the Exponential Distribution.

18.48. C. E[Y] = (2)(4/5) + 4(1/5) = 2.4.

$$E[Y^2] = \int_0^4 x^2/5 \; dx \; + 4^2(1/5) = 64/15 + 16/5 = 7.4667.$$

Var[Y] = $7.4667 - 2.4^2 =$ **1.7067**.

18.49. C. For X ≤ 150, X = X ∧ 150.

So the only contribution to $E[X^2] - E[(X \wedge 150)^2]$ comes from any losses of size > 150.

Losses uniform on (100, 200] ⇒ expect 3 claims greater than 150, out of a total of 74.

⇒ Prob[X>150] = 3/74.

Uniform on (150, 200], $E[X^2]$ = variance + mean^2 = $50^2/12 + 175^2 = 30{,}833$.

On (150, 200], each loss is at least 150, and therefore $E[(X \wedge 150)^2] = 150^2 = 22{,}500$.

$E[X^2] - E[(X \wedge 150)^2]$ = Prob[X>150] $E[X^2 - (X \wedge 150)^2 \mid X$ uniform on (150, 200]] =
(3/74) (30,833 - 22,500) = **338**.

18.50. A. $q_x = 0.1$. $q_{x+1} = 0.2$. $q_{x+2} = 0.3$.
Prob[K = 0] = Prob[Die 1st Year] = $q_x = 0.1$.
Prob[K = 1] = P[Alive @ x+1] P[Die 2nd Year | Alive @ x+1] = $(1 - q_x)\ q_{x+1}$ = (0.9)(0.2) = 0.18.
Prob[K = 2] = P[Alive @ x+2] P[Die 3rd Year | Alive @ x+2] =
$(1 - q_x)\ (1 - q_{x+1})\ q_{x+2}$ = (0.9)(0.8)(0.3) = 0.216.
Prob[K ≥ 3] = 1 - (0.1 + 0.18 + 0.0216) = 0.504.

K	Prob	$K \wedge 3$	$(K \wedge 3)^2$
0	0.1	0	0
1	0.18	1	1
2	0.216	2	4
≥ 3	0.504	3	9
Avg.		2.124	5.580

(0.1)(0) + (0.18)(1) + (0.216)(2) + (0.504)(3) = 2.124.
(0.1)(0) + (0.18)(1) + (0.216)(4) + (0.504)(9) = 5.580.
$\text{Var}(K \wedge 3) = E[(K \wedge 3)^2] - E[K \wedge 3]^2 = 5.580 - 2.124^2 =$ **1.069**.

18.51. C. The probability that a loss exceeds 30,000 is: $e^{-30,000/10,000} = 0.049787$.
The losses truncated and shifted from below at 30,000 is the same as the original Exponential.
One can think of $Y = (X - 30,000)_+$ as the aggregate that results from a Bernoulli frequency with $q = 0.049787$, and an Exponential severity with mean 10,000.
This has a variance of: (mean of Bernoulli)(var. of Expon.) + (mean of Expon.)^2(var. of Bernoulli)
$= (0.049787)(10,000^2) + (10,000^2)(0.049787)(1 - 0.049787) = 9,710,096$.
Mean aggregate is: (0.049787)(10,000) = 497.87.
Coefficient of variation is: $\sqrt{9,710,096}$ / 497.87 = **6.26**.

Alternately, this is mathematically equivalent to a two point mixture, with 0.049787 weight to an Exponential with mean 10,000 (the non-zero payments) and (1 - 0.049787) weight to a distribution that is always zero.
The mean is: (0.049787)(10,000) + (1 - 0.049787)(0) = 497.87.
The second moment is the weighted average of the two second moments:
$(0.049787)(2)(10,000^2) + (1 - 0.049787)(0) = 9,957,414$.
Therefore, $1 + CV^2 = 9,957,414/497.87^2 = 40.17$. ⇒ CV = **6.26**.
Alternately, $E[Y^2] = E[(X - 30,000)_+^2 \mid X > 30,000]\ \text{Prob}[X > 30,000]$
= (Second moment of an Exponential Distribution with θ = 10,000) $e^{-10,000/30,000}$
$= (2)(10,000^2)(0.049787) = 9,957,414$. Proceed as before.

Alternately, $E[Y] = \int_{30,000}^{\infty} (x - 30,000)\ \exp[-x/10,000]/10,000\ dx =$

$\int_0^{\infty} y\ \exp[-(y+30,000)/10,000]/10,000\ dy = e^{-3} \int_0^{\infty} y\ \exp[-y/10,000]/10,000\ dy$

$= e^{-3}\ (10,000) = 497.87$.

$E[Y^2] = \int_{30000}^{\infty} (x - 30,000)^2 \exp[-x/10,000]/10,000\ dx$

$= \int_0^{\infty} y^2 \exp[-(y+30,000)/10,000]/10,000\ dy = e^{-3} \int_0^{\infty} y^2 \exp[-y/10,000]/10,000\ dy$

$= e^{-3}\ (2)(10,000^2) = 9,957,414$. Proceed as before.

Section 19, Mean Excess Loss

As discussed previously, the Mean Excess Loss or Mean Residual Life (complete expectation of life), e(x), is defined as the mean value of those losses greater than size x, where each loss is reduced by x. Thus one only includes those losses greater than size x, and only that part of each such loss greater than x.

$$e(x) = E[X - x \mid X > x] = \frac{\int_x^\infty (t-x)\, f(t)\, dt}{S(x)} = \frac{\int_x^\infty t\, f(t)\, dt}{S(x)} - x.$$

The Mean Excess Loss at d, **e(d) = average payment per payment with a deductible d**.
The Mean Excess Loss is the mean of the loss distribution left truncated and shifted at x:
e(x) = (average size of those losses greater in size than x) - x.

Therefore, the average size of those losses greater in size than x = e(x) + x.

On the exam, usually the easiest way to compute the Mean Excess Loss for a distribution is to use the formulas for the Limited Expected Value in Appendix A of Loss Models, and the identity:

$$\mathbf{e(x) = \frac{E[X] - E[X \wedge x]}{S(x)}}.$$

Therefore, e(0) = mean, provided the distribution has support, x > 0.[101]

Exercise: $E[X \wedge \$1 \text{ million}]$ = \$234,109. E[X] = \$342,222. S(\$1 million) = 0.06119.
Determine e(\$1 million).
[Solution: e(\$1 million) = (342,222 - 234,109) / 0.06119 = \$1.767 million.]

[101] Thus e(0) = mean, with the notable exception of the Single Parameter Pareto.

Formulas for the Mean Excess Loss for Various Distributions:

Distribution	Mean Excess Loss, e(x)
Exponential	θ
Pareto	$\frac{\theta + x}{\alpha - 1}$, $\alpha > 1$
LogNormal	$\exp(\mu + \sigma^2/2) \frac{1 - \Phi\left[\frac{\ln(x) - \mu - n\sigma^2}{\sigma}\right]}{1 - \Phi\left[\frac{\ln(x) - \mu}{\sigma}\right]} - x$
Gamma	$\alpha\theta \frac{1 - \Gamma(\alpha+1 ; x/\theta)}{1 - \Gamma(\alpha ; x/\theta)} - x$
Weibull	$\theta\Gamma(1 + 1/\tau)\{1 - \Gamma(1 + 1/\tau ; (x/\theta)^\tau)\} \exp[(x/\theta)^\tau] - x$
Single Parameter Pareto	$\frac{x}{\alpha - 1}$
Inverse Gaussian	$\mu \frac{\Phi\left[\left(1 - \frac{x}{\mu}\right)\sqrt{\frac{\theta}{x}}\right] + e^{2\theta/\mu}\, \Phi\left[-\left(\frac{x}{\mu} + 1\right)\sqrt{\frac{\theta}{x}}\right]}{\Phi\left[\left(1 - \frac{x}{\mu}\right)\sqrt{\frac{\theta}{x}}\right] - e^{2\theta/\mu}\, \Phi\left[-\left(\frac{x}{\mu} + 1\right)\sqrt{\frac{\theta}{x}}\right]} - x$
Burr	$\{\theta\Gamma(\alpha - 1/\gamma)\Gamma(1+1/\gamma)/\Gamma(\alpha)\}\{\beta[\alpha - 1/\gamma, 1+1/\gamma; 1/(1+(x/\theta)^\gamma)]\}(1+(x/\theta)^\gamma)^\alpha$, $\alpha\gamma > 1$
Trans. Gamma	$\theta\{\Gamma(\alpha+(1/\tau)) / \Gamma(\alpha)\}\{1 - \Gamma(\alpha+(1/\tau) ; (x/\theta)^\tau)\} / \{1 - \Gamma[\alpha ; (x/\theta)^\tau]\} - x$
Gen. Pareto	$\{\theta\tau / (\alpha-1)\}\beta[\alpha-1, \tau+1; \theta/(\theta+x)] / \beta[\alpha, \tau; \theta/(\theta+x)]\}$, $\alpha > 1$
Normal	$\sigma^2\phi[(x-\mu)/\sigma] / \{1 - \Phi[(x-\mu)/\sigma]\} + \mu - x$

It should be noted that for heavier-tailed distributions, just as with the mean, the Mean Excess Loss only exists for certain values of the parameters. Otherwise it is infinite.

For example, for the Pareto for $\alpha \le 1$, the mean excess loss is infinite or does not exist.

The Exponential is the only continuous distribution with a constant Mean Excess Loss.
If F(x) represents the distribution of the ages of death, then e(x) is the (remaining) life expectancy of a person of age x. A constant Mean Excess Loss is independent of age and is equivalent to a force of mortality that is independent of age.

Exercise: For a Pareto with $\alpha = 4$ and $\theta = 1000$, determine e(800).
[Solution: $E[X] = \theta/(\alpha-1) = 333.3333$. $E[X \wedge 800] = \{\theta/(\alpha-1)\}\{1 - (\theta/(\theta + 800))^{\alpha-1}\} = 276.1774$.
$S(800) = \{\theta/(\theta+ 800)\}^{\alpha} = (1/1.8)^4 = 0.09526$.
$e(800) = (333.3333 - 276.1774)/(0.09526) = 600$.
Alternately, $e(x) = (\theta + x) / (\alpha - 1) = (1000 + 800)/(4 - 1) = 600$.]

Recall that e(d) = average payment per payment with a deductible d. For an Exponential Distribution, since the mean excess loss is constant, the average payment per payment remains constant when a deductible is introduced or when an existing deductible changes size.

In contrast, for the Pareto Distribution, $e(d) = (\theta + d) / (\alpha - 1)$. Therefore, for a Pareto Distribution, the average payment per payment increases when a deductible is introduced; when an existing deductible is increased so is the average payment per payment.

Mean Excess Loss in terms of the Survival Function:

The Mean Excess Loss can be written in terms of the survival function, S(x) = 1 - F(x).
By definition, e(x) is the ratio of loss dollars excess of x divided by S(x).

$$e(x) = \int_x^{\infty} (t-x)\, f(t)dt / S(x) = \{\int_x^{\infty} t\, f(t)dt - S(x)x\} / S(x).$$

Using integration by parts and the fact that the integral of f(x) is -S(x):[102]

$$e(x) = \{S(x)x + \int_x^{\infty} S(t)\, dt - S(x)x\} / S(x).$$

$$e(x) = \int_x^{\infty} S(t)\, dt \,/ S(x).$$

So the Mean Excess Loss at x is the integral of the survival function from x to infinity divided by the survival function at x.[103]

For example, for the Pareto Distribution, $S(x) = \theta^{\alpha}(\theta+x)^{-\alpha}$.
Therefore, $e(x) = \{\theta^{\alpha}(\theta+x)^{1-\alpha}/(\alpha-1)\} / \{\theta^{\alpha}(\theta+x)^{-\alpha}\} = (\theta+x) / (\alpha - 1)$.
This matches the formula given above for the Mean Excess Loss of the Pareto Distribution.

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[102] Note that the derivative of S(x) is d(1-F(x))/dx = - f(x). Remember there is an arbitrary constant for indefinite integrals. Thus the indefinite integral of f(x) is either -F(x) or S(x) = 1-F(x).
[103] The Mean Excess Loss as defined here is the same as the complete expectation of life as defined in Life Contingencies. ${}_sp_x = S(x+s) / S(x)$, and therefore: $\mathring{e}_x = \int_0^{\infty} {}_sp_x \, ds$.

Behavior of e(x) in the Righthand Tail:

The behavior of the Mean Excess Loss as the loss size approaches infinity for some distributions:[104] [105]

Distribution	Behavior of e(x) as $x \to \infty$	For Extremely Large x
Exponential	**constant**	$e(x) = \theta$
Single Par. Pareto	increases linearly	$e(x) = x / (\alpha-1)$
Pareto	**increases linearly**	$e(x) = (\theta+x)/(\alpha-1)$
LogNormal	increases to infinity less than linearly	$e(x) \approx x\,\sigma^2 / \ln(x)$
Gamma, $\alpha>1$	decreases towards a horizontal asymptote	$e(x) \to \theta$
Gamma, $\alpha<1$	increases towards a horizontal asymptote[3]	$e(x) \to \theta$
Inverse Gaussian	increases to a constant	$e(x) \to 2\mu^2/\theta$
Weibull, $\tau>1$	decreases to zero	$e(x) \cong \theta^\tau x^{1-\tau} / \tau$
Weibull, $\tau<1$	increases to infinity less than linearly	$e(x) \cong \theta^\tau x^{1-\tau} / \tau$
Trans. Gamma, $\tau>1$	decreases to zero	$e(x) \cong \theta^\tau x^{1-\tau} / \tau$
Trans. Gamma, $\tau<1$	increases to infinity less than linearly	$e(x) \cong \theta^\tau x^{1-\tau} / \tau$
Burr	increases to infinity approximately linearly	$e(x) \cong x/(\alpha\gamma-1)$
Gen. Pareto	increases to infinity approximately linearly	$e(x) \cong x/(\alpha-1)$
Inv. Trans. Gamma	increases to infinity approximately linearly	$e(x) \cong x/(\alpha\tau-1)$
Normal	decreases to zero approximately as 1/x	$e(x) \cong \sigma^2 / (x-\mu)$

Recall that the mean and thus the Mean Excess Loss fails to exist for a Pareto with $\alpha \le 1$.[106]

Also the Gamma with $\alpha = 1$ and the Weibull with $\tau = 1$ are Exponential distributions, and thus have constant Mean Excess Loss.[107]

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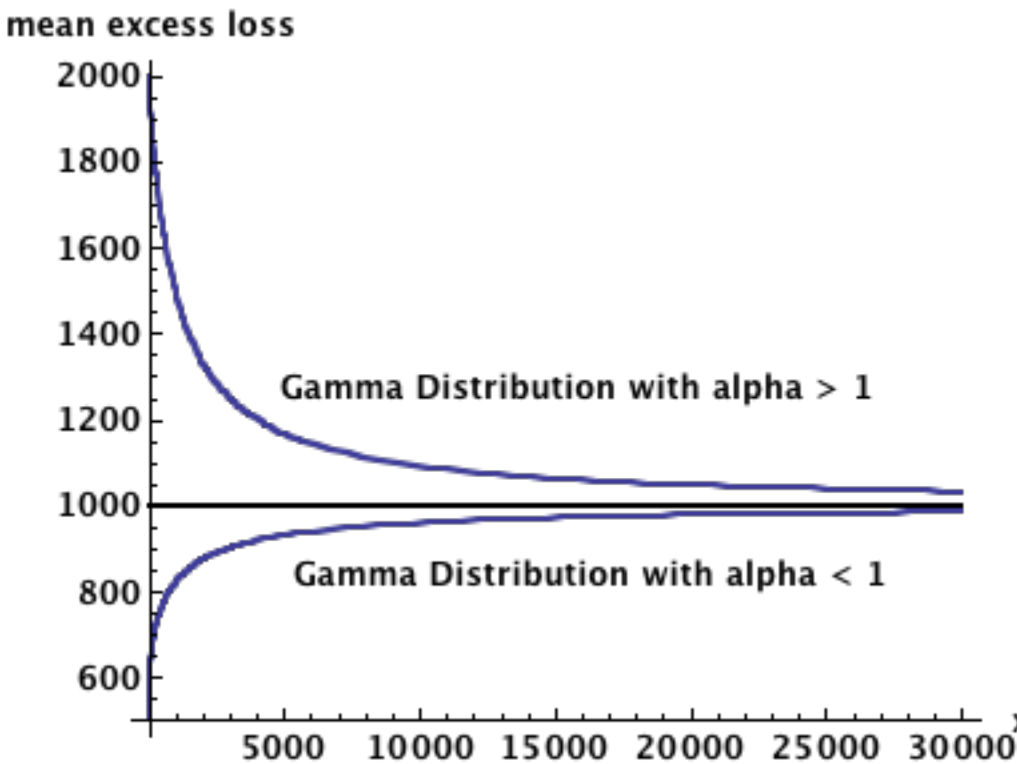
[104] For the Gamma Distribution for large x, $e(x) \cong \theta - (\alpha-1)/x$.
[105] For the LogNormal, e(x) approaches its asymptotic behavior very slowly.
Thus the formula derived subsequently $e(x) \cong x / \{(\ln(x) - \mu) / \sigma^2 - 1\}$, will provide a somewhat better approximation than the formula $e(x) \cong x\,\sigma^2 / \ln(x)$, until one reaches truly immense loss sizes.
[106] The mean and thus the Mean Excess Loss fails to exist for: Inverse Transformed Gamma with $\alpha\tau \le 1$, Generalized Pareto with $\alpha \le 1$, and Burr with $\alpha\gamma \le 1$.
[107] The Transformed Gamma with $\tau = 1$ is a Gamma distribution, and thus in this case has the behavior of the Mean Excess Loss depend on whether alpha is greater than, less than, or equal to one.

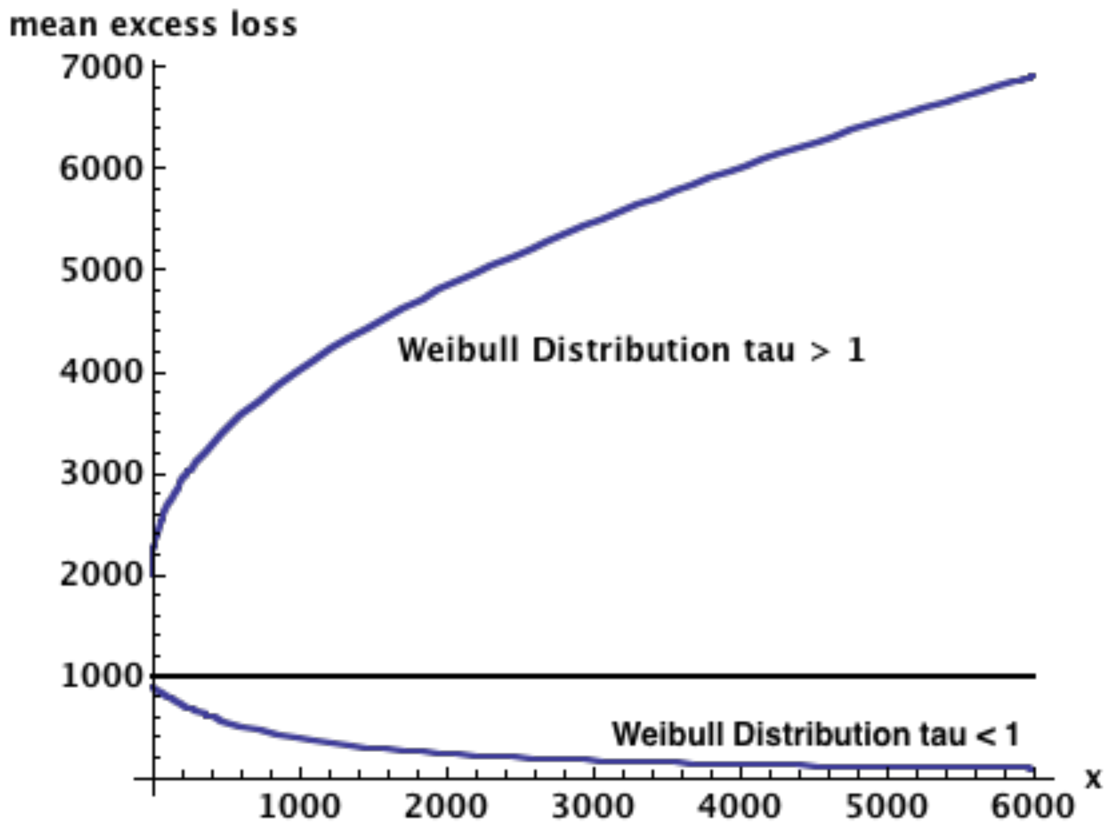
Those curves with heavier tails have the Mean Excess Loss increase with x. Comparing the Mean Excess Loss provides useful information on the fit of the curves to the data. Small differences in the tail of the distributions that may not have been evident in the graphs of the Distribution Function, are made evident by graphing the Mean Excess Loss.[108]

Below is shown for various Gamma distributions the behavior of the Mean Excess Loss as the loss size increases. For $\alpha = 1$, the Exponential Distribution has a constant mean excess loss equal to θ, in this case 1000. For $\alpha > 1$, the mean excess loss decreases to θ.[109]
For a Gamma Distribution with $\alpha < 1$, the mean excess loss increases to θ.[110]
The tail of a Gamma Distribution is similar to that of an Exponential Distribution with the same θ.



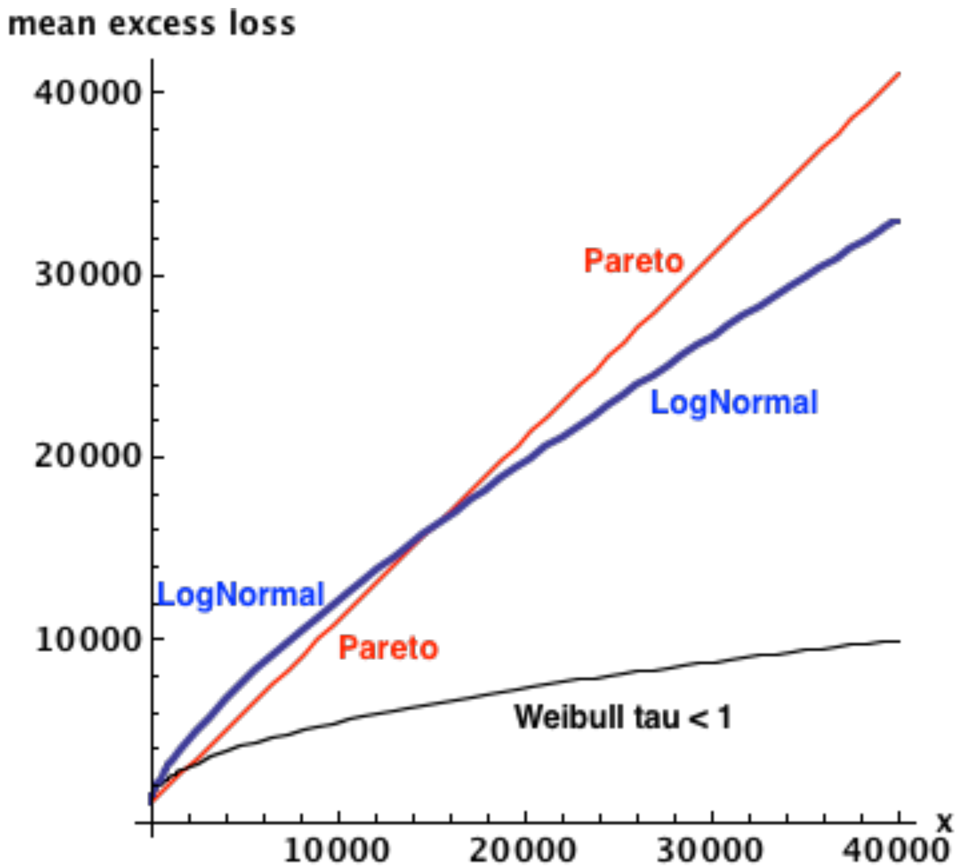
[108] I have found the Mean Excess Loss particularly useful at distinguishing between the tails of the different distributions when using them to estimate Excess Ratios.
[109] The graph shows alpha = 2.
[110] The graph shows alpha = 1/2.

For the Weibull with $\tau = 1$, the Exponential Distribution has a constant mean excess loss equal to θ, in this 1000. For $\tau > 1$, the mean excess loss decreases to 0.[111]
For $\tau < 1$, the mean excess loss increases to infinity less than linearly.[112]



[111] The graph shows tau = 2.
[112] The graph shows tau = 1/2.

The Pareto and LogNormal Distribution each have heavy tails. However, the Pareto Distribution has its mean excess loss increase linearly, while that of the LogNormal increases slightly less than linearly. Thus the Pareto has a heavier (righthand) tail than the LogNormal, which in turn has a heavier tail than the Weibull.[113]



All three distributions have mean residual lives that increase to infinity. Note that it takes a while for the mean residual life of the Pareto to become larger than that of the LogNormal.[114]

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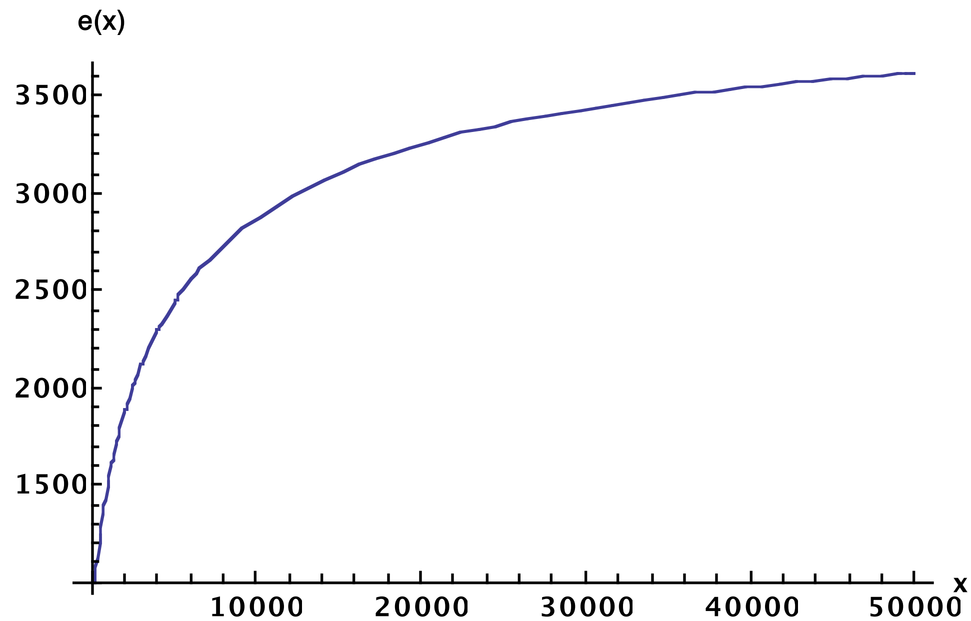
[113] The mean excess losses are graphed for a Weibull Distribution with $\theta = 500$ and $\tau = 1/2$, a LogNormal Distribution with $\mu = 5.5$ and $\sigma = 1.678$, and a Pareto Distribution with $\alpha = 2$ and $\theta = 1000$.
All three distributions have a mean of 1000.

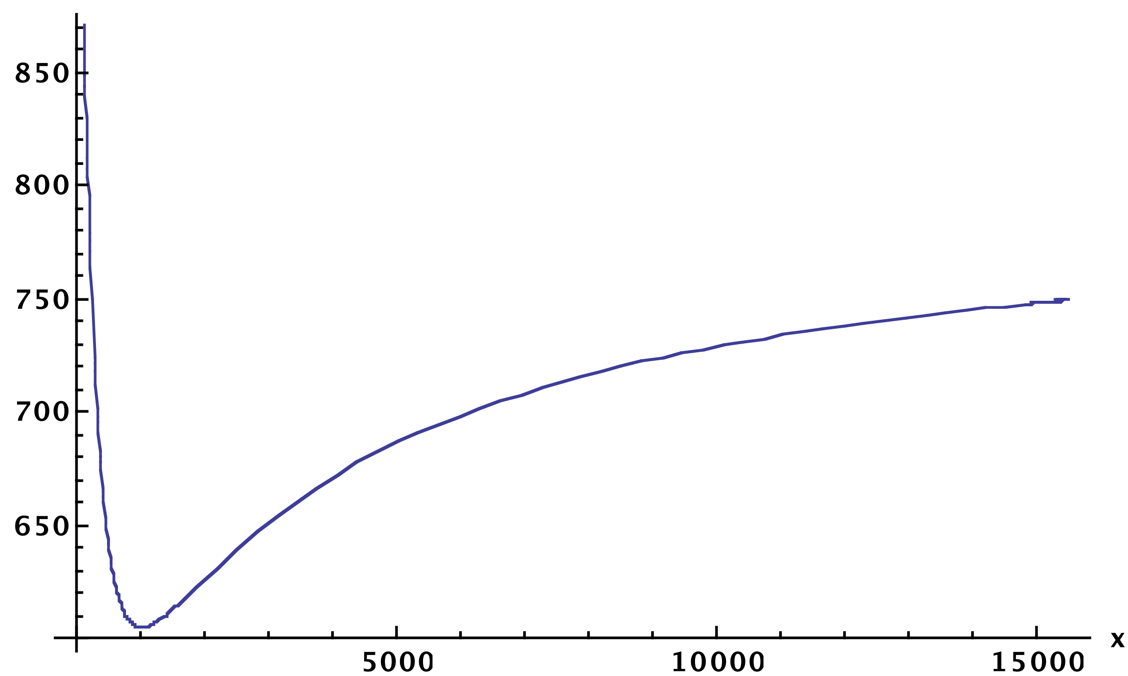
[114] In this case, the Pareto has a larger mean residual life for loss sizes 15 times the mean and greater.

The mean residual life of an Inverse Gaussian increases to a constant; e(x)→$2\mu^2/\theta$.
Thus the Inverse Gaussian has a tail that is somewhat similar to a Gamma Distribution.

Here is e(x) for an Inverse Gaussian with μ = 1000 and θ = 500:



Here is the e(x) for an Inverse Gaussian with μ = 1000 and θ = 2500:



In this case, e(x) initially decreases and then increases towards: (2)(1000^2)/2500 = 800.

Determining the Tail Behavior of the Mean Excess Loss:

The fact that the Mean Excess Loss is the integral of S(t) from x to infinity divided by S(x), is the basis for a method of determining the behavior of e(x) as x approaches infinity. One applies L'Hospital's Rule twice.

$$\lim_{x \to \infty} e(x) = \lim_{x \to \infty} \int_x^{\infty} S(t)\,dt \Big/ S(x) = \lim_{x \to \infty} S(x) / f(x) = \lim_{x \to \infty} -f(x) / f'(x).$$

For example for the Gamma distribution:

$f(x) = \theta^{-\alpha} x^{\alpha-1} e^{-x/\theta} / \Gamma(\alpha)$. $f'(x) = (\alpha-2)\theta^{-\alpha} x^{\alpha-2} e^{-x/\theta} / \Gamma(\alpha) - \theta^{-(\alpha+1)} x^{\alpha-1} e^{-x/\theta} / \Gamma(\alpha)$.

$$\lim_{x \to \infty} e(x) = \lim_{x \to \infty} -f(x) / f'(x) = \lim_{x \to \infty} 1 / \{1/\theta - (\alpha - 1)/ x\} = \theta.$$

When the Mean Excess Loss increases to infinity, it may be useful to look at the limit of x/e(x). Again one applies L'Hospital's Rule twice.

$$\lim_{x \to \infty} x/e(x) = \lim_{x \to \infty} xS(x) \Big/ \int_x^{\infty} S(t)\,dt = \lim_{x \to \infty} (-f(x)x + S(x))/ (-S(x)) =$$

$$\lim_{x \to \infty} (-f(x) - xf'(x) - f(x))/ f(x) = \lim_{x \to \infty} \{-xf'(x) / f(x)\} - 2.$$

For example for the LogNormal distribution:

$f(x) = \xi / x$, where $\xi = \exp[-0.5 (\{\ln(x) - \mu\}/\sigma)^2] / \{\sigma\sqrt{2\pi}\}$.

$f'(x) = -\xi / x^2 - \{(\ln(x) - \mu)/ (x\sigma^2)\} (\xi / x)$.

$$\lim_{x \to \infty} x/e(x) = \lim_{x \to \infty} -xf'(x) / f(x) - 2 = \lim_{x \to \infty} \{1 + (\ln(x) - \mu)/ \sigma^2\} - 2 \cong \ln(x)/\sigma^2.$$

Thus for the LogNormal distribution the Mean Excess Loss increases to infinity, but a little less quickly than linearly: $e(x) \approx x / \{(\ln(x) - \mu)/\sigma^2 - 1\} \cong x\,\sigma^2 / \ln(x)$.

As another example, for the Burr distribution:

$f(x) = \alpha\gamma x^{\gamma-1}\theta^{-\gamma}(1 + (x/\theta)^{\gamma})^{-(\alpha+1)}$. $f'(x) = (\gamma-1)f(x)/x - (\alpha+1)\gamma\, x^{\gamma-1} f(x) / (1 + (x/\theta)^{\gamma})$.

$$\lim_{x \to \infty} x/e(x) = \lim_{x \to \infty} -xf'(x)/ f(x) - 2 = \lim_{x \to \infty} -(\gamma-1) + (\alpha+1)\gamma\, x^{\gamma}\theta^{-\gamma}/(1+(x/\theta)^{\gamma}) - 2$$

$$= -\gamma + 1 + \alpha\gamma + \gamma - 2 = \alpha\gamma - 1.$$

The Mean Excess Loss for the Burr Distribution increases to infinity approximately linearly: $e(x) \approx x / (\alpha\gamma - 1)$, provided $\alpha\gamma > 1$.

Moments, CV, and the Mean Excess Loss:

When the relevant moments are finite and the distribution has support x > 0, then one can compute the moments of the distribution in terms of the mean excess loss, e(x).[115]
We have E[X] = e(0).[116] We will now show how to write the second moment in terms of an integral of the mean excess loss and the survival function.
As shown in a previous section:

$$E[X^2] = 2 \int_{t=0}^{\infty} S(t)\, t \, dt$$

Note that the integral of S(t)/E[X] from x to infinity is the excess ratio, R(x), and thus R'(x) = -S(x)/E[X]. Using this fact and integration by parts:

$$E[X^2]/E[X] = 2 \int_0^{\infty} t\, S(t)/E[X] \, dt = -2t\, R(t) \Big]_{t=0}^{t=\infty} + 2 \int_0^{\infty} R(t) \, dt .$$

For a finite second moment, t R(t) goes to zero as x goes to infinity, therefore:[117]

$$E[X^2] = 2E[X] \int_0^{\infty} R(t) \, dt = 2 \int_0^{\infty} S(t)e(t) \, dt .$$

Exercise: For a Pareto Distribution, what is the integral from zero to infinity of S(x)e(x)?
[Solution: $S(x) = (1+x/\theta)^{-\alpha}$. $e(x) = (x+\theta)/(\alpha-1) = (1+ x/\theta)\, \theta/(\alpha-1)$.
$S(x)e(x) = (1+x/\theta)^{-(\alpha-1)}\, \theta/(\alpha-1)$.

$$\int_0^{\infty} S(t)\, e(t) \, dt = \theta/(\alpha-1) \int_0^{\infty} (1+t/\theta)^{-(\alpha-1)} \, dt = \{\theta/(\alpha-1)\}\, \theta(\alpha-2)\, (1+t/\theta)^{-(\alpha-2)} \Big]_{t=0}^{t=\infty}$$

$= \theta^2 / \{(\alpha-1)(\alpha-2)\}$.
Comment: The integral is one half of the second moment for a Pareto, consistent with the above result.]

Assume that the first two moments are finite and the distribution has support x > 0, and e(x) > e(0) = E[X] for all x. Then:

$$E[X^2] = 2 \int_0^{\infty} S(t)\, e(t) \, dt > 2 \int_0^{\infty} S(t)\, E[X] \, dt = 2E[X] \int_0^{\infty} S(t) \, dt\, t = 2E[X]E[X].$$

---

[115] Since e(x) determines the distribution, it follows that e(x) determines the moments if they exist.
[116] The numerator of e(0) is the losses excess of zero, i.e. all the losses.
The denominator of e(0) is the number of losses larger than 0, i.e., the total number of losses.
The support of the distribution has been assumed to be x > 0.
[117] I have used the fact that E[X]R(t) = S(t)e(t). Both are the losses excess of t.

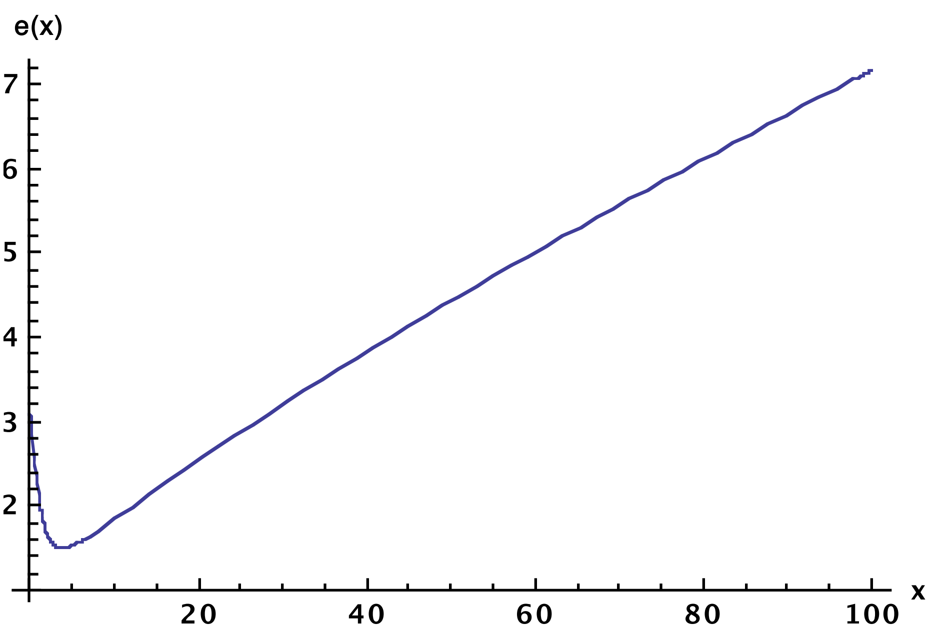
$\Rightarrow E[X^2] > 2E[X^2]$. $\Rightarrow E[X^2] / E[X]^2 > 2$. $\Rightarrow CV^2 = E[X^2] / E[X]^2 - 1 > 1$. $\Rightarrow CV > 1$.

When the first two moments are finite and the distribution has support x > 0, then
if e(x) > e(0) = E[X] for all x, then the coefficient of variation is greater than one.[118]
Note that e(x) > e(0), if e(x) is monotonically increasing.

Examples where this result applies are the Gamma $\alpha < 1$, Weibull $\tau < 1$, Transformed Gamma with $\alpha < 1$ and $\tau < 1$, and the Pareto.[119] In each case CV > 1. Note that each of these distributions is heavier-tailed than an Exponential, which has CV = 1.

While in the tail e(x) for the LogNormal approaches infinity, it is not necessarily true for the LogNormal that e(x) > e(0) for all x. The mean excess loss of the LogNormal can decrease before it finally increases as per x/ln(x) in the tail.

For example, here is a graph of the Mean Excess Loss for a LogNormal with $\mu = 1$ and $\sigma = 0.5$:



In any case, the CV of the LogNormal is: $\sqrt{\exp(\sigma^2) - 1}$. Thus for the LogNormal,

CV < 1 for $\sigma < \sqrt{\ln(2)} \cong 0.82$, while CV > 1 for $\sigma > \sqrt{\ln(2)} \cong 0.82$.

When the first two moments are finite and the distribution has support x > 0, then
if e(x) < e(0) = E[X] for all x, then CV < 1. Note that e(x) < e(0), if e(x) is monotonically decreasing.

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[118] See Section 3.4.5 in Loss Models.
[119] For $\alpha > 2$, so that the CV of the Pareto exists.

Examples where this result applies are the Gamma $\alpha > 1$, Weibull $\tau > 1$, and the Transformed Gamma with $\alpha > 1$ and $\tau > 1$. In each case CV < 1. Note that each of these distributions is lighter-tailed than an Exponential, which has CV = 1.

One can get similar results to those above for higher moments.

Exercise: Assuming the relevant moments are finite and the distribution has support x > 0, express the integral from zero to infinity of $S(x)x^n$, in terms of moments.
[Solution: One applies integration by parts and the fact that dS(x)/dx = - f(x):

$$\int_0^\infty S(t)\, t^n\, dt = S(t)\, t^{n+1} / (n+1) \Big]_{t=0}^{t=\infty} + \int_0^\infty f(t)\, t^{n+1} / (n+1)\, dt = E[X^{n+1}]/(n+1).$$

Where I've used the fact, that if the n+1st moment is finite, then $S(x)x^{n+1}$ must go to zero as x approaches infinity.
<u>Comment</u>: For n = 0 one gets the result that the mean is the integral of the survival function from zero to infinity. For n = 1 one gets the result used above, that the integral of xS(x) from zero to infinity is half of the second moment.]

Exercise: Assuming the relevant moments are finite and the distribution has support x > 0, express the integral from zero to infinity of $S(x)e(x)x^n$, in terms of moments.
[Solution: S(x)e(x) = R(x)E[X].
Then one applies integration by parts, differentiating R(t) and integrating t^n.
Since the integral of S(t)/E[X] from x to infinity is R(x), the derivative of R(x) is -S(x)/E[X].

$$\int_0^\infty S(t)\, e(t)\, t^n\, dt = E[X] \int_0^\infty R(t)\, t^n\, dt =$$

$$E[X]\ \{R(t)\, t^{n+1} / (n+1)\} \Big]_{t=0}^{t=\infty} + E[X] \int_0^\infty (S(t) / E[X])\, t^{n+1} / (n+1)\, dt$$

$$= (1/(n+1)) \int_0^\infty S(t)\, t^{n+1}\, dt = \{1/(n+1)\}\, E[X^{n+2}]/(n+2) = E[X^{n+2}] / \{(n+1)(n+2)\}.]$$

Where I've used the result of the previous exercise and the fact, that if the n+2nd moment is finite, then $R(x)x^{n+1}$ must go to zero as x approaches infinity.]

Thus we can express moments, when they exist, either as integrals of S(t)e(t) times powers of t, integrals of R(t) times powers of t, or as S(t) times powers of t.

Assuming the relevant moments are finite and the distribution has support x > 0, then if e(x) > e(0) = E[X] for all x, we have for n ≥ 1:

$$E[X^{n+1}] / \{n(n+1)\} = \int_0^\infty S(t)\, e(t)\, t^{n-1}\, dt \ > \ E[X] \int_0^\infty S(t)\, t^{n-1}\, dt \ = E[X]E[X^n]/n.$$

Thus if e(x) > e(0) for all x, $E[X^{n+1}] > (n+1)E[X]E[X^n]$, n ≥ 1.
For n = 1 we get a previous result: $E[X^2] > 2E^2[X]$.
For n = 2 we get: $E[X^3] > 3E[X]E[X^2]$.

Conversely, if e(x) < e(0) for all x, $E[X^{n+1}] < (n+1)E[X]E[X^n]$, n ≥ 1.

Equilibrium Distribution:

Given that X follows a distribution with survival function S_X, for x > 0, then Loss Models defines the density of the corresponding "equilibrium distribution" as:[120]

$$g(y) = S_X(y) / E[X], \quad y > 0.$$

Exercise: Demonstrate that the above is actually a probability density function.
[Solution: $S_X(y) / E[X] \geq 0$.

$$\int_0^\infty \{S_X(y) / E[X]\} \, dy = \frac{1}{E[X]} \int_0^\infty S_X(y) \, dy = E[X]/E[X] = 1. \,]$$

Exercise: If severity is Exponential with mean θ = 10, what is the density of the corresponding equilibrium distribution?
[Solution: $g(y) = S_X(y) / E[X] = \exp(-y/10)/ 10$.]

In general, if the severity is Exponential, then the corresponding equilibrium distribution is also Exponential with the same mean.

Exercise: If severity is Pareto, with α = 5 and θ = 1000,
what is the corresponding equilibrium distribution?
[Solution: $g(y) = S_X(y) / E[X] = (1 + y/1000)^{-5} / 250$.
This is the density of another Pareto Distribution, but with α = 4 and θ = 1000.]

The distribution function of the corresponding equilibrium distribution is the loss elimination ratio of the severity distribution:

$$G(y) = \int_0^y \{S_X(y) / E[X]\} \, dy = \frac{1}{E[X]} \int_0^y S_X(y) \, dy = E[X \wedge y]/ E[X] = LER(y).$$

Therefore the survival function of the corresponding equilibrium distribution is the excess ratio of the severity distribution.

For example, if severity is Pareto, the excess ratio, $R(x) = \{\theta/(\theta+x)\}^{\alpha-1}$, which is the survival function for a Pareto with the same scale parameter and a shape parameter one less. Thus if severity is Pareto, (with α > 1), then the distribution of the corresponding equilibrium distribution is also Pareto, with the same scale parameter and shape parameter of α - 1.

[120] See Equation 3.11 in Loss Models.

The mean of the corresponding equilibrium distribution is:[121]

$$\int_0^\infty y \{S_X(y) / E[X]\}\ dy = \frac{1}{E[X]} \int_0^\infty y\ S_X(y)\ dy = \frac{E[X^2]}{2\ E[X]}.$$

The second moment of the corresponding equilibrium distribution is:

$$\int_0^\infty y^2 \{S_X(y) / E[X]\}\ dy = \frac{1}{E[X]} \int_0^\infty y^2\ S_X(y)\ dy = \frac{E[X^3]}{3\ E[X]}.$$

Exercise: If severity is Pareto, with $\alpha = 5$ and $\theta = 1000$, what are the mean and variance of the corresponding equilibrium distribution?
[Solution: The mean of the Pareto is: 1000/4 = 250. The second moment of the Pareto is: $2(1000^2) / \{(5-1)(5-2)\} = 166{,}667$. The third moment of the Pareto is: $6(1000^3) / \{(5-1)(5-2)(5-3)\} = 250$ million. The mean of the corresponding equilibrium distribution is: $E[X^2]/ \{2E[X]\} = 166{,}667 / 500 = 333.33$.
The second moment of the corresponding equilibrium distribution is: $E[X^3] / \{3E[X]\} =$ 250 million/ 750 = 333,333. Thus the variance of the corresponding equilibrium distribution is: $333{,}333 - 333.33^2 = 222{,}222$. Alternately, the corresponding equilibrium distribution is a Pareto Distribution, but with $\alpha = 4$ and $\theta = 1000$. This has mean: 1000/3 = 333.33, second moment: $2(1000^2)/\{(3)(2)\} = 333{,}333$, and variance: $333{,}333 - 333.33^2 = 222{,}222$.]

The hazard rate of the corresponding equilibrium distribution is:

$$\frac{\text{density of the corresponding equilibrium distribution}}{\text{survival function the corresponding equilibrium distribution}} = \frac{E[X^3]}{3\ E[X]} = \frac{S(x)}{E[X]\ R(x)} =$$

$$\frac{S(x)}{\text{expected losses excess of x}} = 1 / e(x).$$

The hazard rate of the corresponding equilibrium distribution is the inverse of the mean excess loss.

Curtate Expectation of Life:

The mean excess loss is mathematically equivalent to what is called the complete expectation of life, $\overset{\circ}{e}_x \cong e(x)$.

Exercise: Five individuals live 53.2, 66.3, 70.8, 81.0, and 83.5 years.
What is the observed $e(60) = \overset{\circ}{e}_{60}$ for this group?
[Solution: (6.3 + 10.8 + 21 + 23.5)/4 = 15.4.]

[121] See Section 3.4.5 in Loss Models.

If instead we ignore any fractions of a year lived, then we get what is called the curtate expectation of life, e_x. Loss Models does not cover the curtate expectation of life

Exercise: Five individuals live 53.2, 66.3, 70.8, 81.0, and 83.5 years.
What is the observed e_{60} for this group?
[Solution: (6 + 10 + 21 + 23)/4 = 15.0.]

Since we are ignoring any fractions of a year lived, $e_x \le \overset{\circ}{e}_x$.
On average we are ignoring about 1/2 year of life, therefore, $e_x \cong \overset{\circ}{e}_x - 1/2$.

Just as we can write $e(x) = \overset{\circ}{e}_x$ in terms of an integral of the Survival Function:

$$\overset{\circ}{e}_x = \int_x^\infty S(t)dt \ / \ S(x)$$

one can write the curtate expectation of life in terms of a summation of Survival Functions:

$$e_x = \sum_{t = x+1}^{\infty} S(t) \ / \ S(x). \qquad e_0 = \sum_{t=1}^{\infty} S(t).$$

Exercise: Determine e_x for an Exponential Distribution with mean θ.

[Solution: $e_x = \sum_{t=x+1}^{\infty} S(t) \ / \ S(x) = \sum_{t=1}^{\infty} e^{-(x+t)/\theta} / e^{-x/\theta} = \sum_{t=1}^{\infty} e^{-t/\theta} = e^{-1/\theta} / (1 - e^{-1/\theta}) = 1/(e^{1/\theta} - 1)$.

Comment: $e_x = 1/(e^{1/\theta} - 1) \cong 1/\{1/\theta + 1/(2\theta^2)\} = \theta/\{1 + 1/(2\theta)\} \cong \theta\{1 - 1/(2\theta)\} = \theta - 1/2$.]

For example, for an Exponential Distribution with mean $\theta = 10$, $e_x = 1/(e^{0.1} - 1) = 9.508$.
This compares to $\overset{\circ}{e}_x = \theta = 10$.

Exercise: Determine e_0 for a Pareto Distribution with $\theta = 1$ and $\alpha = 2$.
[Solution: $e_0 = S(1) + S(2) + S(3) + ... = (1/2)^2 + (1/3)^2 + (1/4)^2 + (1/5)^2 + ... = \pi^2/6 - 1 = 0.645$.
Comment: $e(0) = E[X] = \theta/(\alpha - 1) = 1$.]

Problems:

19.1 (2 points) Assume you have a Pareto distribution with $\alpha = 5$ and $\theta = \$1000$.
What is the Mean Excess Loss at \$2000?
A. less than \$500
B. at least \$500 but less than \$600
C. at least \$600 but less than \$700
D. at least \$700 but less than \$800
E. at least \$800

19.2 (1 point) Assume you have a distribution $F(x) = 1 - e^{-x/666}$.
What is the Mean Excess Loss at \$10,000?
A. less than \$500
B. at least \$500 but less than \$600
C. at least \$600 but less than \$700
D. at least \$700 but less than \$800
E. at least \$800

19.3 (3 points) The random variables X and Y have joint density function

$$f(x, y) = 60{,}000{,}000\ x \exp(-10x^2) / (100 + y)^4,\ 0 < x < \infty,\ 0 < y < \infty.$$

Determine the Mean Excess Loss function for the marginal distribution of Y evaluated at Y = 1000.
A. less than 200
B. at least 200 but less than 300
C. at least 300 but less than 400
D. at least 400 but less than 500
E. at least 500

19.4 (1 point) Which of the following distributions would be most useful for modeling the age at death of humans?
A. Gamma B. Inverse Gaussian C. LogNormal D. Pareto E. Weibull

19.5 (1 point) Given the following empirical mean excess losses for 500 claims:

x	0	5	10	15	25	50	100	150	200	250	500	1000
e(x)	15.6	16.7	17.1	17.4	17.6	18.0	18.2	18.3	18.3	18.4	18.5	18.5

Which of the following distributions would be most useful for modeling this data?
A. Gamma with $\alpha > 1$ B. Gamma with $\alpha < 1$ C. Pareto
D. Weibull with $\tau > 1$ E. Weibull with $\tau < 1$

19.6 (2 points) You have a Pareto distribution with parameters α and θ.
If e(1000) / e(100) = 2.5, what is θ?
A. 100 B. 200 C. 300 D. 400 E. 500

For the following three questions, assume you have a LogNormal distribution with parameters μ = 11.6, σ = 1.60.

19.7 (3 points) What is the Mean Excess Loss at $100,000?
A. less than $500,000
B. at least $500,000 but less than $600,000
C. at least $600,000 but less than $700,000
D. at least $700,000 but less than $800,000
E. at least $800,000

19.8 (1 point) What is the average size of those losses greater than $100,000?
A. less than $500,000
B. at least $500,000 but less than $600,000
C. at least $600,000 but less than $700,000
D. at least $700,000 but less than $800,000
E. at least $800,000

19.9 (2 points) What percent of the total loss dollars are represented by those losses greater than $100,000?
A. less than 0.91
B. at least 0.91 but less than 0.92
C. at least 0.92 but less than 0.93
D. at least 0.93 but less than 0.94
E. at least 0.94

19.10 (2 points) You observe the following 35 losses: 6, 7, 11, 14, 15, 17, 18, 19, 25, 29, 30, 34, 40, 41, 48, 49, 53, 60, 63, 78, 85, 103, 124, 140, 192, 198, 227, 330, 361, 421, 514, 546, 750, 864, 1638.
What is the (empirical) Mean Excess Loss at 500?
A. less than 350
B. at least 350 but less than 360
C. at least 360 but less than 370
D. at least 370 but less than 380
E. at least 380

Use the following information for the next two questions:

- The annual frequency of ground up losses is Negative Binomial with $r = 4$ and $\beta = 1.3$.
- The sizes of ground up losses follow a Pareto Distribution with $\alpha = 3$ and $\theta = 5000$.
- There is a franchise deductible of 1000.

19.11 (2 points) Determine the insurer's average payment per nonzero payment.
(A) 2500 (B) 3000 (C) 3500 (D) 4000 (E) 4500

19.12 (2 points) Determine the insurer's expected annual payments.
(A) 8,000 (B) 9,000 (C) 10,000 (D) 11,000 (E) 12,000

19.13 (3 points) F is a continuous size of loss distribution on $(0, \infty)$.
LER(x) is the corresponding loss elimination ratio at x.
Which of the following are true?
A. $F(x) \geq LER(x)$ for all $x > 0$.
B. $F(x) \geq LER(x)$ for all $x > c$, for some $c > 0$.
C. $F(x) \geq LER(x)$ for all $x > 0$, if and only if $e(x) \geq e(0)$.
D. $F(x) \geq LER(x)$ for all $x > 0$, if and only F is an Exponential Distribution.
E. None of A, B, C or D is true.

19.14 (2 points) The size of loss follows an Exponential Distribution with $\theta = 5$.
The largest integer contained in each loss is the amount paid for that loss.
For example, a loss of size 3.68 results in a payment of 3.
What is the expected payment?
A. 4.46 B. 4.48 C. 4.50 D. 4.52 E. 4.54

19.15 (2 points) Globetrotter Insurance sells policies all with the same size of deductible.
Individual losses, prior to the effect of the deductible follow a Pareto distribution with $\alpha = 5$ and $\theta = 3000$. The average payment per payment is 875.
Determine the size of the deductible.
A. 250 B. 500 C. 750 D. 1000 E. 1250

19.16 (2 points) Losses follow a Single Parameter Pareto distribution with $\theta = 1000$ and $\alpha > 1$.
Determine the ratio of the Mean Excess Loss function at x = 3000 to the Mean Excess Loss function at x = 2000.
A. 1 B. 4/3 C. 3/2 D. 2
E. Cannot be determined from the given information.

19.17 (3 points) For a Gamma Distribution with $\alpha = 2$, what is the behavior of the mean excess loss e(x) as x approaches infinity?

19.18 (1 point) For a Pareto Distribution with $\alpha > 1$:
$E[X - 1000 \mid X > 1000] = 1.3\ E[X - 700 \mid X > 700]$.
Determine θ.

19.19 (4, 5/86, Q.58) (2 points) For a certain machine part, the Mean Excess Loss e(x) varies as follows with the age (x) of the part:

Age x	e(x)
5 months	12.3 months
10	18.6
20	34.3
50	69.1

Which of the following continuous distributions best fits this pattern of Mean Excess Loss?
A. Exponential B. Gamma ($\alpha > 1$) C. Pareto
D. Weibull ($\tau > 1$) E. Normal

19.20 (160, 5/87, Q.1) (2.1 points) You are given the following survival function:
$S(x) = (b - x/a)^{1/2}$, $0 \le x \le k$. The median age is 75. Determine e(75).
(A) 8.3 (B) 12.5 (C) 16.7 (D) 20.0 (E) 33.3

19.21 (4B, 5/92, Q.14) (1 point) Which of the following statements are true about the Mean Excess Loss function e(x)?
1. If e(x) increases linearly as x increases, this suggests that a Pareto model may be appropriate.
2. If e(x) decreases as x increases, this suggests that a Weibull model may be appropriate.
3. If e(x) remains constant as x increases, this suggests that an exponential model may be appropriate.
A. 1 only B. 2 only C. 1 and 3 only D. 2 and 3 only E. 1, 2, and 3

19.22 (4B, 5/93, Q.24) (2 points) The underlying distribution function is assumed to be the following: $F(x) = 1 - e^{-x/10}$, $x \ge 0$
Calculate the value of the Mean Excess Loss function e(x), for x = 8.
A. less than 7.00
B. at least 7.00 but less than 9.00
C. at least 9.00 but less than 11.00
D. at least 11.00 but less than 13.00
E. at least 13.00

19.23 (4B, 5/94, Q.4) (2 points) You are given the following information from an unknown size of loss distribution for random variable X:

Size k ($000s)	1	3	5	7	9
Count of X ≥ k	180	118	75	50	34
Sum of X ≥ k	990	882	713	576	459

If you are using the empirical Mean Excess Loss function to help you select a distributional family for fitting the empirical data, which of the following distributional families should you attempt to fit first?

A. Pareto B. Gamma C. Exponential D. Weibull E. Lognormal

19.24 (4B, 5/95, Q.21) (3 points) Losses follow a Pareto distribution, with parameters θ and $\alpha > 1$. Determine the ratio of the Mean Excess Loss function at $x = 2\theta$ to the Mean Excess Loss function at $x = \theta$.

A. 1/2 B. 1 C. 3/2 D. 2

E. Cannot be determined from the given information.

19.25 (4B, 11/96, Q.22) (2 points) The random variable X has the density function

$f(x) = e^{-x/\lambda}/\lambda$, $0 < x < \infty$, $\lambda > 0$.

Determine $e(\lambda)$, the Mean Excess Loss function evaluated at λ.

A. 1 B. λ C. $1/\lambda$ D. λ/e E. e/λ

19.26 (4B, 5/97, Q.13) (1 point) Which of the following statements are true?

1. Empirical Mean Excess Loss functions are continuous.
2. The Mean Excess Loss function of an exponential distribution is constant.
3. If it exists, the Mean Excess Loss function of a Pareto distribution is decreasing.

A. 2 B. 1, 2 C. 1, 3D. 2, 3 E. 1, 2, 3

19.27 (4B, 5/98, Q.3) (3 points) The random variables X and Y have joint density function

$f(x, y) = \exp(-2x - y/2)$ $0 < x < \infty$, $0 < y < \infty$.

Determine the Mean Excess Loss function for the marginal distribution of X evaluated at X = 4.

A. 1/4 B. 1/2 C. 1 D. 2 E. 4

19.28 (4B, 11/98, Q.6) (2 points) Loss sizes follow a Pareto distribution, with parameters $\alpha = 0.5$ and $\theta = 10,000$. Determine the Mean Excess Loss at 10,000.

A. 5,000 B. 10,000 C. 20,000 D. 40,000 E. ∞

19.29 (4B, 11/99, Q.25) (2 points) You are given the following:

- The random variable X follows a Pareto distribution, as per Loss Models, with parameters $\theta = 100$ and $\alpha = 2$.
- The mean excess loss function, $e_X(k)$, is defined to be $E[X - k \mid X \geq k]$.

Determine the range of $e_X(k)$ over its domain of $[0, \infty)$.

A. [0, 100] B. $[0, \infty)$ C. 100 D. $[100, \infty)$ E. ∞

19.30 (4B, 11/99, Q.27) (2 points) You are given the following:

- The random variable X follows a Pareto distribution, as per Loss Models, with parameters $\theta = 100$ and $\alpha = 2$.
- The mean excess loss function, $e_X(k)$, is defined to be E[X - k | X ≥ k].

$Z = \min(X, 500)$.

Determine the range of $e_Z(k)$ over its domain of [0, 500].

A. [0, 150] B. [0, ∞) C. [100, 150] D. [100, ∞) E. [150, ∞)

19.31 (SOA3, 11/04, Q.24) (2.5 points)
The future lifetime of (0) follows a two-parameter Pareto distribution with $\theta = 50$ and $\alpha = 3$.
Calculate $\mathring{e}_{20}$.

(A) 5 (B) 15 (C) 25 (D) 35 (E) 45

19.32 (CAS3, 5/05, Q.4) (2.5 points) Well-Traveled Insurance Company sells a travel insurance policy that reimburses travelers for any expenses incurred for a planned vacation that is canceled because of airline bankruptcies. Individual claims follow a Pareto distribution with $\alpha = 2$ and $\theta = 500$. Because of financial difficulties in the airline industry, Well-Traveled imposes a limit of $1,000 on each claim. If a policyholder's planned vacation is canceled due to airline bankruptcies and he or she has incurred more than $1,000 in expenses, what is the expected non-reimbursed amount of the claim?

A. Less than $500
B. At least $500, but less than $1,000
C. At least $1,000, but less than $1,500
D. At least $1,500, but less than $2,000
E. $2,000 or more

19.33 (SOA M, 5/05, Q.9 & 2009 Sample Q.162) (2.5 points) A loss, X, follows a 2-parameter Pareto distribution with $\alpha = 2$ and unspecified parameter θ. You are given:

E[X - 100 | X > 100] = (5/3) E[X - 50 | X > 50].

Calculate E[X - 150 | X > 150].

(A) 150 (B) 175 (C) 200 (D) 225 (E) 250

19.34 (CAS3, 11/05, Q.10) (2.5 points)
You are given the survival function s(x) as described below:

- $s(x) = 1 - x/40$ for $0 \le x \le 40$.
- s(x) is zero elsewhere.

Calculate $\mathring{e}_{25}$, the complete expectation of life at age 25.

A. Less than 7.7
B. At least 7.7 , but less than 8.2
C. At least 8.2, but less than 8.7
D. At least 8.7, but less than 9.2
E. At least 9.2

19.35 (CAS3, 5/06, Q.38) (2.5 points) The number of calls arriving at a customer service center follows a Poisson distribution with λ = 100 per hour. The length of each call follows an exponential distribution with an expected length of 4 minutes.
There is a $3 charge for the first minute or any fraction thereof and a charge of $1 per minute for each additional minute or fraction thereof.
Determine the total expected charges in a single hour.
A. Less than $375
B. At least $375, but less than $500
C. At least $500, but less than $625
D. At least $625, but less than $750
E. At least $750

Solutions to Problems:

19.1. D. For the Pareto, $e(x) = (\theta + x) / (\alpha - 1) = 3000 / 4 =$ **$750**.
Alternately, a Pareto truncated and shifted at 2000, is another Pareto with $\alpha = 5$ and $\theta = 1000 + 2000 = 3000$. e(2000) is the mean of this new Pareto: 3000/(5 - 1) = **$750**.

Alternately, $e(2000) = \int_0^\infty {}_tp_{2000}\, dt = \int_0^\infty S(2000+t)/S(2000)\, dt =$

$$\int_0^\infty (2000+\theta)^\alpha/(2000+\theta+t)^\alpha\, dt = (2000 + \theta) / (\alpha - 1) = 3000 / 4 = \mathbf{\$750}.$$

Alternately, $E[X \wedge 2000] = (1000/4) \{ 1- (1000/(1000+2000))^{5-1}\} = 246.91$.
$e(2000) = \{\text{mean} - E[X \wedge 2000]\} / S(2000) = (250 - 246.91) / 0.004115 = \$751$.

19.2. C. For the exponential distribution the mean excess loss is a constant; it is equal to the mean. The mean in this case is $\theta =$ **$666**.

19.3. E. X and Y are independent since the support doesn't depend on x or y and the density can be factored into a product of terms each just involving x and y.
$f(x, y) = 60{,}000{,}000\ x \exp(-10x^2) / (100 + y)^4 = \{20\ x \exp(-10x^2)\} \{3{,}000{,}000 / (100 + y)^4\}$.
The former is the density of a Weibull Distribution with $\theta = 1/\sqrt{10}$ and $\tau = 2$. The latter is the density of a Pareto Distribution with $\alpha = 3$ and $\theta = 100$. When one integrates from $x = 0$ to ∞ in order to get the marginal distribution of y, one is left with just a Pareto, since the Weibull integrates to unity and the Pareto is independent of x. Thus the marginal distribution is just a Pareto, with parameters $\alpha = 3$ and $\theta = 100$. Thus $e(y) = (\theta + y)/(\alpha - 1) = (100 + y)/(3 - 1)$.
e(1000) = 1100/2 = **550.**

19.4. E. Of these distributions, only the Weibull (for $\tau > 1$) has mean residual lives decline to zero. The Weibull (for $\tau > 1$) has the force of mortality increase as the age approaches infinity, as is observed for humans. The other distributions have the force of mortality decline or approach a positive constant as the age increases.

19.5. B. The empirical mean residual lives seem to be increasing towards a limit of about 18.5 as x approaches infinity. This is the behavior of a Gamma with alpha less than 1.
The other distributions given all exhibit different behaviors than this.

19.6. E. For the Pareto Distribution: $e(x) = (E[X] - E[X \wedge x])/S(x) = (x + \theta)/(\alpha - 1)$.
$e(1000)/e(100) = (1000 + \theta)/(100 + \theta) = 2.5. \Rightarrow \theta =$ **500**.
Comment: One can not determine α from the given information.

19.7. C. $E[X \wedge x] = \exp(\mu + \sigma^2/2)\ \Phi[(\ln x - \mu - \sigma^2)/\sigma] + x\ \{1 - \Phi[(\ln x - \mu)/\sigma]\}$.
$E[X \wedge 100{,}000] = \exp(12.88)\Phi[-1.65] + (100{,}000)\ \{1 - \Phi[-0.05]\} =$
(392,385)(1 - 0.9505) + (100,000)(0.5199) = 71,413. For the LogNormal,
$E[X] = \exp(\mu + \sigma^2/2) = \exp(12.88) = 392{,}385$. For the LogNormal,
$F(x) = \Phi[\{\ln(x) - \mu\}/\sigma]$. $F(100{,}000) = \Phi[-0.05] = (1 - 0.5199)$. Therefore,
$e(100{,}000) = \{E[X] - E[X \wedge 100{,}000]\} / \{1 - F(100{,}000)\} = (392{,}385 - 71{,}413) / 0.5199 \cong$ **$617,000.**
Alternately, for the LogNormal distribution,
$e(x) = \exp(\mu + \sigma^2/2)\{1 - \Phi[(\ln x - \mu - \sigma^2)/\sigma] / \{1 - \Phi[(\ln x - \mu)/\sigma]\} - x$.
For $\mu = 11.6$, $\sigma = 1.60$, $e(100{,}000) = \exp(12.88)(1 - \Phi[-1.65]) / \{1 - \Phi[-0.05]\} - 100{,}000 =$
(392,385)(0.9505) / 0.5199 - 100,000 = **$617 thousand**.

19.8. D. The size of those claims greater than $100,000 = $100,000 + e(100,000).
But from the previous question e(100,000) ≅ $617,000. Therefore, the solution ≅ **$717,000**.

19.9. E. Use the results from the previous two questions. $F(100{,}000) = \Phi[-0.05] = 1 - 0.5199$.
Thus, S(100,000) = 0.5199. $E[X] = \exp(\mu + \sigma^2/2) = \exp(12.88) = 392{,}385$.
Percent of the total loss dollars represented by those losses greater than $100,000 =
S(100,000) {size of those claims greater than $100,000} / E[X]
= (0.5199)(717,000)/392,385 = **0.95**.
Alternately, the losses represented by the small losses are:
$E[X \wedge 100{,}000] - S(100{,}000)(100{,}000) = 71{,}413 - 51{,}990 = 19{,}423$.
Divide by the mean of 392,385 and get 0.049 of the losses are from small claims.
Thus the percentage of losses from large claims is: 1 - 0.049 = **0.95**.

19.10. C. Each claim above 500 contributes its excess above 500 and then divide by the number of claims greater than 500. e(500) = {14 + 46 + 250 + 364 + 1138}/ 5 = **362.4**.

19.11. D. With a franchise deductible the insurer pays the full value of every large loss and pays nothing for small losses. Therefore, the Pareto Distribution has been truncated from below at 1000. The mean of a distribution truncated and shifted from below at 1000 is e(1000) ⇒ the mean of a distribution truncated from below at 1000 is: e(1000) + 1000.
For a Pareto Distribution $e(x) = (E[X] - E[X \wedge x])/S(x) = (x + \theta)/(\alpha - 1)$.
e(1000) = (1000 + 5000)/(3 - 1) = 3000. e(1000) + 1000 = **4000**.

19.12. E. For the Pareto Distribution, $S(1000) = \{5000/(5000 + 1000)\}^3 = 0.5787$.
Mean frequency = $r\beta$ = (4)(1.3) = 5.2. Expected # of nonzero payments = (0.5787)(5.2) = 3.009.
From the previous solution, average nonzero payment is 4000.
Expected annual payments = (3.009)(4000) = **12,036**.
Alternately, with a franchise deductible of 1000 the payment is 1000 more than that for an ordinary deductible for each large loss, and thus the average payment per loss is:
$E[X] - E[X \wedge 1000] + 1000S(1000) =$
$(5000/2) - (5000/2)\{1 - (5000/6000)^2\} + (1000)(5000/6000)^3 = 2315$.
Expected annual payments = (5.2)(2315) = **12,038**.

19.13. C. F(x) - LER(x) = 1 - S(x) - {1 - S(x)e(x)/E[X]} = {S(x)/E[X]} {e(x) - E[X]} = {S(x)/E[X]} {e(x) - e(0)}. Therefore, F(x) ≥ LER(x) ⇔ e(x) ≥ e(0).
Alternately, e(x)/e(0) = {E[(X - x)$_+$]/S(x)}/E[X] = {E[(X - x)$_+$]/E[X]}/S(x) = R(x)/S(x).
Therefore, e(x) ≥ e(0). ⇔ R(x) ≥ S(x). ⇔ LER(x) = 1 - R(x) ≤ 1 - F(x) = S(x).
Comment: For an Exponential Distribution, e(x) = e(0) = θ, and therefore F(x) = LER(x).
For a Pareto Distribution with α > 1, e(x) increases linearly, and therefore F(x) > LER(x).

19.14. D. The expected payment is the curtate expectation of life at zero.

$$e_0 = \sum_{t=1}^{\infty} S(t) = S(1) + S(2) + S(3) + ... = e^{-1/5} + e^{-2/5} + e^{-3/5} + ...$$

$= e^{-1/5}/(1 - e^{-1/5}) = 1/(e^{0.2} - 1) =$ **4.517**.
Comment: Approximately 1/2 less than the mean of 5.

19.15. B. 875 = e(d) = (θ+d)/(α-1) = (3000 + d)/4. ⇒ d = **500**.

19.16. C. $e(x) = \{E[X] - E[X \wedge x]\} / S(x) = (\alpha\theta/(\alpha-1) - \{\alpha\theta/(\alpha-1) - \theta^\alpha/((\alpha-1)x^{\alpha-1})\}) / \{(\theta/x)^\alpha\}$
= x/(α-1). e(3000)/e(2000) = 3000/2000 = **3/2**.

19.17. The value of the scale parameter θ does not affect the behavior, for simplicity set θ = 1.

$f(x) = x\, e^{-x}$, x > 0. $S(x) = \int_x^\infty f(t)\, dt = x\, e^{-x} + e^{-x}$.

$$e(x) = \int_x^\infty S(t)\, dt \;/\; S(x) = (x\, e^{-x} + 2e^{-x}) / (x\, e^{-x} + e^{-x}) = 1 + 1/(1+x).$$

Thus, as x approaches infinity, e(x) decreases to a constant.
Comment: In this case the limit of e(x) is one, while in general it is θ.
In general, for α > 1, e(x) decreases to a constant, while h(x) increases to a constant.
For α < 1, e(x) increases to a constant, while h(x) decreases to a constant.
For α = 1, we have an Exponential, and e(x) and h(x) are each constant.
For α = 2 and θ = 1, $h(x) = f(x) / S(x) = x\, e^{-x} / (x\, e^{-x} + e^{-x}) = x / (x + 1)$.

19.18. E[X - x | X > x] = e(x). For the Pareto Distribution, e(x) = (x+θ)/(α-1).
e(1000) / e(700) = (1000 + θ) / (700 + θ) = 1.3. ⇒ θ = **300**.
Comment: Similar to SOA M, 5/05, Q.9 (2009 Sample Q.162).

19.19. C. The mean residual life increases approximately linearly, which indicates a Pareto.
Comment: The Pareto has a mean residual life that increases linearly. The Exponential has a constant mean residual life. For a Gamma with α > 1 the mean residual life decreases towards a horizontal asymptote. For a Weibull with τ > 1 the mean residual life decreases to zero.
For a Normal Distribution the mean residual life decreases to zero.

19.20. C. We want S(0) = 1. b = 1. ⇒ We want S(k) = 0. ⇒ k = a.
0.5 = S(75) = $(1 - 75/a)^{1/2}$. ⇒ a = 100.

$$e(75) = \int_{75}^{100} S(x)\,dx \,/ S(75) = \int_{75}^{100} (1 - x/100)^{1/2}\,dx \,/ 0.5 = (25/3)/0.5 = \mathbf{16.7}.$$

19.21. E. 1. T. Mean Residual Life of the Pareto Distribution increases linearly.
2. T. The Weibull Distribution for $\tau > 1$ has the mean residual life decrease (to zero.)
3. T. The mean residual life for the Exponential Distribution is constant.

19.22. C. For the Exponential Distribution, e(x) = mean = θ = **10**.

19.23. C. The empirical mean residual life is calculated as:
e(k) = ($ excess of k) / (# claims > k) = {($ on claims > k) / (# claims > k)} - k =
(average size of those claims of size greater than k) - k.

Size k ($000)	1	3	5	7	9
# claims ≥ k	180	118	75	50	34
Sum of X ≥ k	990	882	713	576	459
average size of those claims of size > k	5.500	7.475	9.507	11.520	13.500
e(k)	**4.500**	**4.475**	**4.507**	**4.520**	**4.500**

Since the mean residual life is approximately constant, one would attempt first to fit an exponential distribution, since it has a constant mean residual life.

19.24. C. For the Pareto, e(x) = $(x+\theta) / (\alpha-1)$. e(2θ) = $3\theta / (\alpha-1)$. e(θ) = $2\theta / (\alpha-1)$.
e(2θ) / e(θ) = **3/2**.
Comment: If one doesn't remember the formula for the mean residual life of the Pareto, it is a longer question. In that case, one can compute: e(x) = (mean - E[X∧x]) / S(x).

19.25. B. The mean residual life of the Exponential is a constant equal to its mean, here λ.

19.26. A. 1. False. The empirical mean residual life is the ratio of the observed losses excess of the limit divided by the number of observed claims greater than the limit. While the numerator is continuous, the denominator is not. For example, assume you observe 3 claims of sizes 2, 6 and 20. Then e(5.999) = {(20-5.999) + (6-5.999)}/2 = 7.001, while e(6.001) = (20-6.001)/1 = 13.999. The limit of e(x) as x approaches 6 from below is 7, while the limit of e(x) as x approaches 6 from above is 14. Thus the empirical mean residual life is discontinuous at 6. 2. True. 3. False. For the Pareto Distribution, the mean residual life increases (linearly).
Comment: A function is continuous at a point x, if and only if the limits approaching x from below and above both exist and are each equal to the value of the function at x. The empirical mean residual life is discontinuous at points at which there are observed claims, since so are the Empirical Distribution Function and the tail probability. In contrast, the empirical Excess Ratio and empirical Limited Expected Value are continuous. The numerator of the Excess Ratio is the observed losses excess of the limit; the denominator is the total observed losses. This numerator is continuous, while this denominator is independent of x. Thus the empirical Excess Ratio is continuous. The numerator of the empirical Limited Expected Value is the observed losses limited by the limit; the denominator is the total number of observed claims. This numerator is continuous, while this denominator is independent of x. Thus the empirical Limited Expected Value is continuous.

19.27. B. The marginal distribution of X is obtained by integrating with respect to y:

$$f(x) = \int_0^\infty \exp(-2x - y/2)\ dy = e^{-2x} \int_0^\infty \exp(-y/2)\ dy\, y = e^{-2x}\ (-2e^{-y}/2) \Big]_{y=0}^{y=\infty} = 2e^{-2x}.$$

Thus the marginal distribution is an Exponential with a mean of 1/2.
It has a mean residual life of **1/2**, regardless of x.

19.28. E. The mean excess loss for the Pareto only exists for $\alpha > 1$.
For $\alpha \le 1$ the relevant integral is **infinite**.

$$e(x) = \int_x^\infty t\ f(t)\ dt\ /S(x) - x.$$

For a Pareto with $\alpha = 0.5$: $\int_x^\infty t\ f(t)\ dt = \int_x^\infty t\ (0.5\theta^{0.5})\ (\theta + t)^{-1.5}\ dt$.

For large t, the integrand is proportional to $t\ t^{-1.5} = t^{-0.5}$, whose integral approaches infinity as the upper limit of the integral approaches infinity. (The integral of $t^{-0.5}$ is $2t^{0.5}$.)
Alternately, e(x) = (E[X] - E[X∧x]) / S(x). The limited expected value E[X∧x] is finite (it is less than x), as is S(x). However, for $\alpha \le 1$, the mean E[X] (does not exist or) is infinite. Therefore, so is the mean excess loss.
Comment: While choice E is the best of those available, in my opinion a better answer might have been that the mean excess loss does not exist.

19.29. D. For the Pareto Distribution $e(k) = (k+\theta)/(\alpha-1) = k + 100$.
Therefore, as k goes from zero to infinity, e(k) goes from **100 to infinity**.

19.30. A. Z is for data censored at 500, corresponding to a maximum covered loss of 500.
$e_Z(k)$ = (dollars of loss excess of k) / S(k) = (E[X∧ 500] - E[X∧k])/ S(k).
E[X∧ x] = {θ/(α-1)} {1-(θ/(θ+x))$^{\alpha-1}$}, for the Pareto.
Thus (E[X∧500] - E[X∧ k]) / S(k) = {100/(100+k) - 100/600} {(100+k)/100}^{2} =
(100 + k) {600 - (100 + k)} /600 = (100 + k)(500 - k)/600.
$e_Z(0)$ = 83.33. $e_Z(500)$ = 0.
Setting the derivative equal to zero: (400-2k)/ 600 = 0. k = 200. $e_Z(200)$ = 150.
Thus the maximum over the interval is 150, while the minimum is 0.
Therefore, as k goes from zero to 500, $e_Z(k)$ is in the interval **[0, 150].**

19.31. D. S(20) = {50/(50 + 20)}^{3} = 0.3644. E[X] = θ/(α-1) = 50/(3-1) = 25.
E[X ∧ 20] = {θ/(α-1)}{1 - (θ/(θ+x))$^{\alpha-1}$} = (25){1 - (50/(50 + 20))^2} = 12.245.
e(20) = (E[X] - E[X ∧ 20]) / S(20) = (25 - 12.245) / 0.3644 = **35**.
Alternately, for the Pareto, e(x) = (x + θ)/(α - 1). e(20) = (20 + 50)/(3 - 1) = **35**.

19.32. D. Given someone incurred more than $1,000 in expenses, the expected non-reimbursed amount of the claim is the mean residual life at e(1000).
For the Pareto, e(x) = (x + θ) / (α - 1). e(1000) = (1000 + 500)/(2 - 1) = **1500**.
Alternately, (E[X] - E[X ∧ 1000]) / S(1000) = {500 - 500(1 - 500/1500)} / (500/1500)^2 = **1500**.
Alternately, a Pareto truncated and shifted from below is another Pareto, with parameters α and θ + d. Therefore, the unreimbursed amounts follow a Pareto Distribution with parameters α = 2 and θ = 500 + 1000 = 1500, with mean 1500/(2 - 1) = **1500**.

19.33. B. e(d) = E[X - d | X > d] = (E[X] - E[X ∧ d]) / S(d) = {θ - θ(1 - θ/(θ + d))} / {θ/(θ + d)}^{2}
= θ + d. The given equation states e(100) = (5/3)e(50). ⇒ 100 + θ = (5/3)(50 + θ). ⇒ θ = 25.
E[X - 150 | X > 150] = e(150) = 150 + 25 = **175**.
Comment: A Pareto truncated and shifted from below is another Pareto, with parameters α and θ + d. ⇒ e(x) = (x + θ)/(α - 1).

19.34. A. $e(25) = \int_{25}^{40} S(x)dx \ / S(25) = \int_{25}^{40} (1 - x/40)\ dx \ / (1 - 25/40) = 2.8125 / 0.375 =$ **7.5**.

Alternately, the given survival function is a uniform distribution on 0 to 40.
At age 25, the future lifetime is uniform from 0 to 15, with an average of **7.5**.
Comment: DeMoivre's Law with ω = 40.

19.35. D. The charge per call of length t is: 3 + 1(if t > 1) + 1(if t > 2) + 1(if t > 3) + 1(if t > 4) + ...
The expected charge per call is: 3 + S(1) + S(2) + S(3) + ... = 3 + $e^{-1/4} + e^{-2/4} + e^{-3/4}$ + ...
= 3 + $e^{-1/4}/(1 - e^{-1/4})$ = 6.521. (100)(6.521) = **652.1**.
Comment: Ignore the possibility that a call lasts exactly an integer, since the Exponential is a continuous distribution. Then the cost of a call is: 3 + curtate lifetime of a call.
For example, if the call lasted 4.6 minutes, the cost is: 3 + 4 = 7.
The expected cost of a call is: 3 + curtate expected lifetime of a call.

$$e_0 = \sum_{t=1}^{\infty} S(t) = \sum_{t=1}^{\infty} e^{-t/4} = e^{-1/4}/(1 - e^{-1/4}) = 1/(e^{1/4} - 1) = 3.521. \quad (100)(3 + 3.521) = 652.1.$$

$e_0 \cong e(0) - 1/2 = E[X] - 1/2 = 4 - 1/2 = 3.5$. (100)(3 + 3.5) = 650, close to the exact answer.

Section 20, Hazard Rate

The **hazard rate**, force of mortality, or failure rate, is defined as: $\mathbf{h(x) = \frac{f(x)}{S(x)}}$ **x ≥ 0.**

h(x) can be thought of as the failure rate of machine parts. The hazard rate can also be interpreted as the force of mortality = probability of death / chance of being alive.
For a given age x, the hazard rate is the density of the deaths, divided by the probability of still being alive at age x.

Exercise: $F(x) = 1 - e^{-x/10}$. What is the hazard rate?
[Solution: $h(x) = f(x)/S(x) = (e^{-x/10}/10)/e^{-x/10} = 1/10$.]

The hazard rate determines the survival (distribution) function and vice versa.

$$\frac{d\ \ln(S(x))}{dx} = \frac{dS(x)}{dx} / S(x) = -f(x) / S(x) = -h(x).$$

Thus $h(x) = -\frac{d\ \ln(S(x))}{dx}$.

$$S(x) = \exp\left[-\int_0^x h(t)\ dt\right].$$ [122]

$S(x) = \exp[-H(x)]$, where $H(x) = \int_0^x h(t)\ dt$. H is called the cumulative hazard rate.[123]

Note that $h(x) = f(x)/S(x) \geq 0$.

$$S(\infty) = \exp[-H(\infty)] = 0. \Leftrightarrow H(\infty) = \int_0^\infty h(t)\ dt = \infty.$$

$h(x) \geq 0$, thus H(x) is nondecreasing and therefore, $S(x) = \exp[-H(x)]$ is nonincreasing.
H(x) usually increases, while S(x) decreases, although H(x) and S(x) can be constant on an interval. Since $H(0) = 0$, $S(0) = \exp[-0] = 1$.

A function h(x) defined for $x > 0$ is a legitimate hazard rate, in other words it corresponds to a legitimate survival function, if and only if $h(x) \geq 0$ and the integral of h(x) from 0 to infinity is infinite, in other words $H(\infty) = \infty$.

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[122] The lower limit of the integral should be the lower end of the support of the distribution.
[123] The name "cumulative hazard rate" is not on the syllabus of this exam.

As in Life Contingencies, one can write the distribution function and the density function in terms of the force of mortality h(t):

$$F(x) = 1 - \exp\left[-\int_0^x h(t)\, dt\right]. \qquad f(x) = h(x)\, \exp\left[-\int_0^x h(t)\, dt\right].$$

Exercise: h(x) = 1/10. What is the distribution function?
[Solution: $F(x) = 1 - e^{-x/10}$, an Exponential Distribution with $\theta = 10$.]

h constant ⇔ the Exponential Distribution, with constant hazard rate of 1/θ = 1/mean.
The Exponential is the only continuous distribution with a constant hazard rate, and therefore constant mean excess loss.

The Force of Mortality for various distributions is given below: [124] [125] [126]

Distribution	Force of Mortality or Hazard Rate	Behavior as x approaches ∞
Exponential	$1/\theta$	h(x) constant
Weibull	$\frac{\tau x^{\tau-1}}{\theta^\tau}$	$\tau < 1$, $h(x) \to 0$. $\tau > 1$, $h(x) \to \infty$.
Pareto	$\frac{\alpha}{\theta + x}$	$h(x) \to 0$, as $x \to \infty$
Burr	$\frac{\alpha\gamma\, x^{\gamma-1}}{\theta^\gamma + x^\gamma}$	$h(x) \to 0$, as $x \to \infty$
Single Parameter Pareto	α/x	$h(x) \to 0$, as $x \to \infty$
Gompertz's Law	Bc^x	$h(x) \to \infty$, as $x \to \infty$
Makeham's Law	$A + Bc^x$	$h(x) \to \infty$, as $x \to \infty$

Exercise: h(x) = 3 / (10 + x). What is the distribution function?
[Solution: $F(x) = 1 - \exp\left[-\int_0^x 3/(10 + t)\, dt\right] = 1 - \exp[-3\,\{\ln(10 + x) - \ln(10)\}] = 1 - \left(\frac{10}{10 + x}\right)^3$.
A Pareto Distribution with $\alpha = 3$ and $\theta = 10$.

[124] The Loglogistic is a special case of the Burr with $\alpha = 1$.
[125] Gompertz's Law is as per Life Contingencies. See Actuarial Mathematics Section 3.7.
[126] Makeham's Law is as per Life Contingencies. See Actuarial Mathematics Section 3.7.

Relationship to Other Items of Interest:

One can obtain the Mean Excess Loss from the Hazard Rate:

$$S(t) / S(x) = \exp[-\int_0^t h(s)\ ds] / \exp[-\int_0^x h(s)\ ds] = \exp[\int_0^x h(s)\ ds - \int_0^t h(s)\ ds] = \exp[\int_t^x h(s)\ ds].$$

$$e(x) = \int_x^\infty S(t)\ dt \ / \ S(x) = \int_x^\infty S(t)/S(x)\ dt = \int_x^\infty \exp[\int_t^x h(s)\ ds]\ dt.$$

Thus, $e(x) = \int_x^\infty \exp[H(x) - H(t)]\ dt = \exp[H(x)] \int_x^\infty \exp[-H(t)]\ dt$.[127]

Exercise: Given a hazard rate of h(x) = 4 / (100+x), what is the mean excess loss, e(x)?

[Solution: $H(x) = \int_0^x h(t)\ dt = 4 \ln(100+x)$.

$$e(x) = (100+x)^4 \int_x^\infty 1/(100+t)^4\ dt = -(100+x)^4 / \{3(100+t)\}^3 \Big]_{t=x}^{t=\infty} = (100 + x) / 3.$$

Comment: This is Pareto Distribution with $\alpha = 4$ and $\theta = 100$; $e(x) = (\theta+x) / (\alpha-1)$,
$h(x) = \alpha / (\theta + x)$.]

One can obtain the Hazard Rate from the Mean Excess Loss as follows:

$$e(x) = \int_x^\infty S(t)\ dt \ / \ S(x).$$

$$\text{Thus, } e'(x) = \frac{-S^2(x) + f(x)\int_x^\infty S(t)\ dt}{S^2(x)} = -1 + f(x)\ e(x) / S(x) = -1 + e(x)h(x).$$

$$\text{Thus, } h(x) = \frac{1 + e'(x)}{e(x)}.$$

[127] $H(x) = \int_0^x h(t)\ dt$. H is called the cumulative hazard rate and is used in Survival Analysis. S(x) = Exp[-H(x)].

Exercise: Given a mean excess loss of e(x) = (100+x) / 3, what is the hazard rate, h(x)?
[Solution: e'(x) = 1/3 . h(x) = {1 + e'(x)} / e(x) = (4/3) {3/(100+x)} = 4 / (100+x).
<u>Comment</u>: This is Pareto Distribution with α = 4 and θ = 100.
It has e(x) = (θ+x) / (α-1) and, h(x) = α / (θ + x).]

Finally, one can obtain the Survival Function from the Mean Excess Loss as follows:

$$h(x) = \frac{1 + e'(x)}{e(x)}.$$

$$H(x) = \int_0^x h(t)\, dt = \int_0^x \{1/e(t) + e'(t) / e(t)\}\, dt = \int_0^x \frac{1}{e(t)}\, dt + \ln[e(x)/e(0)].$$

$$\text{Thus, } S(x) = \exp[-H(x)] = \{e(0)/e(x)\} \exp[-\int_0^x \frac{1}{e(t)}\, dt].$$

For example, for the Pareto, e(x) = (θ + x) / (α - 1).

$$\int_0^x \frac{1}{e(t)}\, dt = (\alpha - 1) \int_0^x \frac{1}{\theta + t}\, dt = (\alpha - 1) \ln[(\theta + x)/\theta].$$

$$\text{Thus, } S(x) = \{e(0)/e(x)\} \exp[-\int_0^x \frac{1}{e(t)}\, dt] = \frac{(\theta + x)/(\alpha - 1)}{(\theta + 0)/(\alpha - 1)} \left(\frac{\theta + x}{\theta}\right)^{-(\alpha-1)} = \left(\frac{\theta}{\theta + x}\right)^{\alpha}.$$

<u>Tail Behavior of the Hazard Rate</u>:

As was derived previously, the limit as x approaches infinity of e(x) is equal to the limit as x approaches infinity of: S(x) / f(x) = 1/h(x).

$$\lim_{x \to \infty} e(x) = \lim_{x \to \infty} 1/h(x).$$

Thus an increasing mean excess loss, e(x), is equivalent to a decreasing hazard or failure rate, h(x), and vice versa.

Since the force of mortality for the Pareto, $\alpha / (\theta+x)$, decreases with age,
the Mean Excess Loss increases with age.[128]
The more quickly the hazard rate declines, the faster the Mean Excess Loss increases and the heavier the righthand tail of the distribution.

For the Weibull, if $\tau > 1$ then the hazard rate, $\tau x^{\tau-1}/\theta^\tau$, increases and thus the Mean Excess Loss decreases.
For the Weibull with $\tau < 1$, the hazard rate decreases and thus the Mean Excess Loss increases.

Lighter Righthand Tail	e(x) decreases	h(x) increases
Heavier Righthand Tail	e(x) increases	h(x) decreases

<u>Hazard Rate of the LogNormal Distribution</u>:

$S(x) = 1 - \Phi\left[\frac{\ln(x) - \mu}{\sigma}\right]$.

For very large y, $1 - \Phi[y] \cong \phi[y] / y = \frac{\exp[-y^2/2]}{y\sqrt{2\pi}}$.[129]

Therefore, taking $y = \frac{\ln(x) - \mu}{\sigma}$, for very large x, $S(x) \cong \frac{\exp\left[-\frac{(\ln(x) - \mu)^2}{2\sigma^2}\right]}{\sqrt{2\pi}} \frac{\sigma}{\ln(x) - \mu}$.

For the LogNormal Distribution, $f(x) = \frac{\exp\left[-\frac{(\ln(x) - \mu)^2}{2\sigma^2}\right]}{x\,\sigma\sqrt{2\pi}}$.

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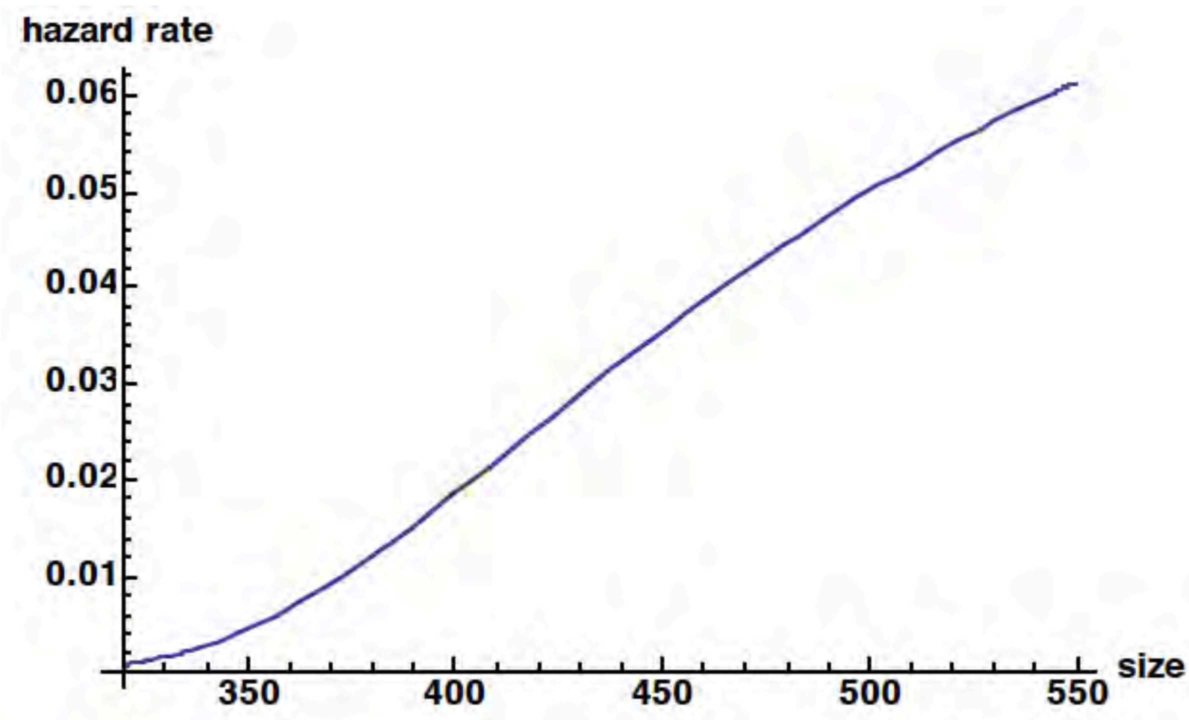
128 Unlike the situation for mortality of humans. For Gompertz's or Makeham's Law with B > 0 and c > 0, the force of mortality increases with age, so the Mean Excess Loss decreases with age. For the Pareto, if $\alpha \le 1$, then the force of mortality is sufficiently small so that there exists no mean; for $\alpha \le 1$ the mean lifetime is infinite.
129 See the Handbook of Mathematical Functions, by Abramowitz, et. al., p. 932.

Therefore, for very large x, $h(x) = \frac{f(x)}{S(x)} \cong \frac{1}{x\,\sigma}\,\frac{\ln(x) - \mu}{\sigma} = \frac{\ln(x) - \mu}{x\,\sigma^2}$.

Thus, as x approaches infinity, the hazard rate h(x) approaches zero.[130] However, this behavior may start to take over only in the extreme righthand tail, beyond most of the probability. Prior to the extreme righthand tail, the behavior of h(x) depends on σ.

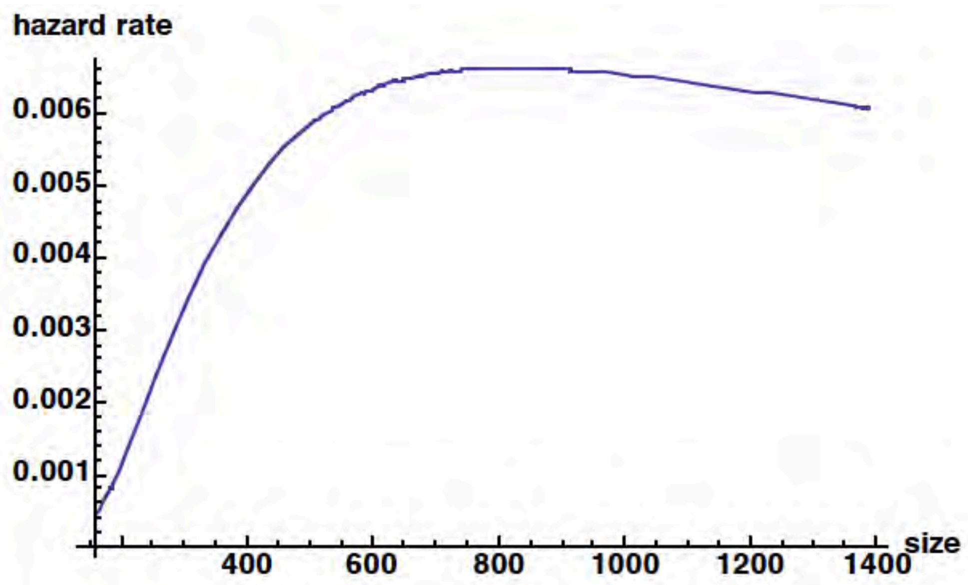
For μ = 6 and σ = 0.1, here is a graph of h(x), out to the 99.9th percentile:



For small σ, h(x) increases, prior to the extreme righthand tail.

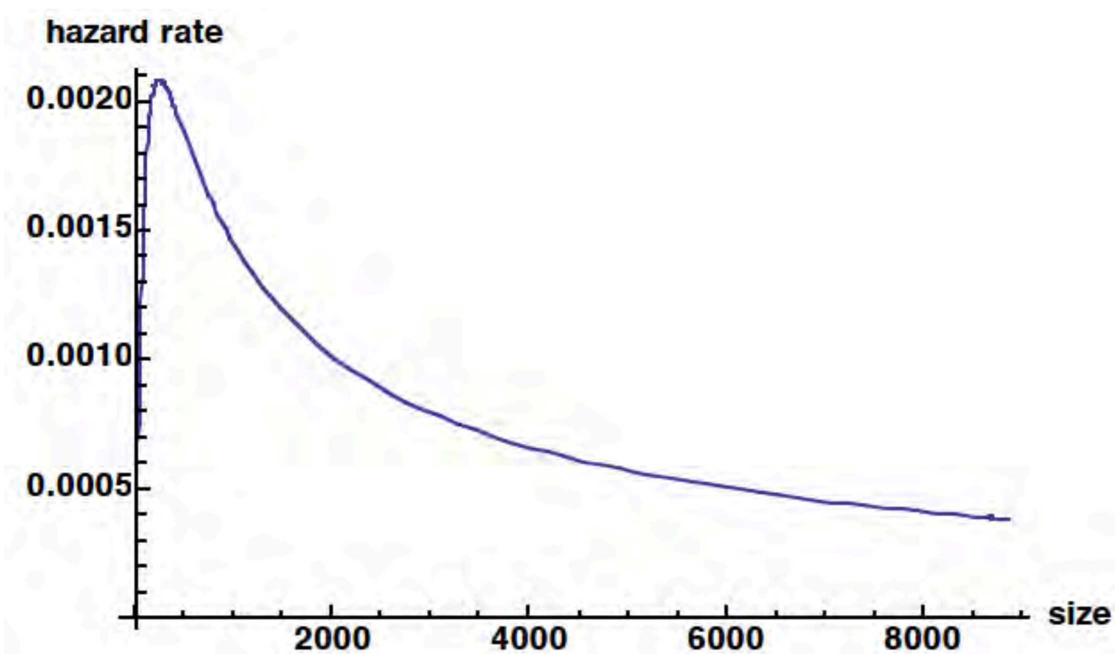
[130] x increases more quickly than ln(x).
As discussed previously, for the LogNormal Distribution, the mean excess loss e(x) approaches to infinity as x approaches infinity. Therefore, it follows that h(x) approaches zero as x approaches infinity.

For $\mu = 6$ and $\sigma = 0.4$, here is a graph of h(x), out to the 99.9th percentile:



For medium σ, h(x) is relatively flat above the median, prior to the extreme righthand tail.[131]

For $\mu = 6$ and $\sigma = 1$, here is a graph of h(x), out to the 99.9th percentile:



For large σ, beyond the very low values of x, h(x) decreases as x increases.

[131] The median is: $\exp[\mu] = \exp[6] = 403$.

Problems:

20.1 (1 point) You are given the following three loss distributions.
1. Gamma $\alpha = 1.5$, $\theta = 0.5$
2. LogNormal $\mu = 0.1$, $\sigma = 0.6$
3. Weibull $\theta = 1.4$, $\tau = 0.8$
For which of these distributions does the hazard rate increase?
A. 1 B. 2 C. 3 D. 1,2,3 E. None of A, B, C, or D

Use the following information for the next 5 questions:
$e(x) = 72 - 0.8x$, $0 < x < 90$.

20.2 (2 points) What is the force of mortality at 50?
A. less than 0.003
B. at least 0.003 but less than 0.004
C. at least 0.004 but less than 0.005
D. at least 0.005 but less than 0.006
E. at least 0.006

20.3 (3 points) What is the Survival Function at 60?
A. 72% B. 74% C. 76% D. 78% E. 80%

20.4 (2 points) What is ${}_{50}p_{30}$?
A. 62% B. 64% C. 66% D. 68% E. 70%

20.5 (1 point) What is the mean lifetime?
A. 72 B. 74 C. 76 D. 78 E. 80

20.6 (2 points) What is the probability density function at 40?
A. less than 0.002
B. at least 0.002 but less than 0.003
C. at least 0.003 but less than 0.004
D. at least 0.004 but less than 0.005
E. at least 0.005

20.7 (2 points) For a LogNormal distribution with parameters $\mu = 11.6$, $\sigma = 1.60$, what is the hazard rate at \$100,000?
A. less than 4×10^{-6}
B. at least 4×10^{-6} but less than 5×10^{-6}
C. at least 5×10^{-6} but less than 6×10^{-6}
D. at least 6×10^{-6} but less than 7×10^{-6}
E. at least 7×10^{-6}

20.8 (2 points) The hazard rate $h(x) = 0.002 + 1.1^x / 10{,}000$, $x > 0$. What is S(50)?
(A) 0.76 (B) 0.78 (C) 0.80 (D) 0.82 (E) 0.84

<u>20.9</u> (1 point) If the hazard rate of a certain machine part is a constant 0.10 for t > 0, what is the Mean Excess Loss at age 25?
A. less than 10
B. at least 10 but less than 15
C. at least 15 but less than 20
D. at least 20 but less than 25
E. at least 25

<u>20.10</u> (2 points) Losses follow a Weibull Distribution with $\theta = 25$ and $\tau = 1.7$.
What is the hazard rate at 100?
A. less than 0.05
B. at least 0.05 but less than 0.10
C. at least 0.10 but less than 0.15
D. at least 0.15 but less than 0.20
E. at least 0.20

<u>20.11</u> (2 points) For a loss distribution where $x \geq 10$, you are given:
i) The hazard rate function: $h(x) = z/x$, for $x \geq 10$.
ii) A value of the survival function: S(20) = 0.015625.
Calculate z.
A. 2 B. 3 C. 4 D. 5 E. 6

20.12 (2 points) For a loss distribution where $x \geq 0$, you are given:
i) The hazard rate function: $h(x) = z\,x^2$, for $x \geq 0$.
ii) A value of the distribution function: F(5) = 0.1175.
Calculate z.
A. 0.002 B. 0.003 C. 0.004 D. 0.005 E. 0.006

Use the following information for the next four questions:
Ground up losses follow a Weibull Distribution with $\tau = 2$ and $\theta = 10$.

20.13 (3 points) There is an ordinary deductible of 5.
What is the hazard rate of the per <u>loss</u> variable?

20.14 (3 points) There is an ordinary deductible of 5.
What is the hazard rate of the per <u>payment</u> variable?

20.15 (3 points) There is a franchise deductible of 5.
What is the hazard rate of the per <u>loss</u> variable?

20.16 (3 points) There is a franchise deductible of 5.
What is the hazard rate of the per <u>payment</u> variable?

20.17 (3 points) X follows a Gamma Distribution with parameters $\alpha = 3$ and θ.
Determine the form of the hazard rate h(x).
What is the behavior of h(x) as x approaches infinity?

Hint: $\Gamma(n; x) = 1 - \sum_{j=0}^{j=n-1} x^j e^{-x}/j!$, for n a positive integer.

<u>20.18</u> (2 points) The hazard rate h(x) = 4/(100 + x), x > 0. What is S(50)?
(A) 0.18 (B) 0.20 (C) 0.22 (D) 0.24 (E) 0.26

20.19 (2 points)
Determine the hazard rate at 300 for a Loglogistic Distribution with $\gamma = 2$ and $\theta = 100$.
(A) 0.005 (B) 0.006 (C) 0.007 (D) 0.008 (E) 0.009

20.20 (1 point) F(x) is a Pareto Distribution.
If the hazard rate h(x) is doubled for all x, what is the new distribution function?

20.21 (2 points) You are using a Weibull Distribution to model the length of time workers remain unemployed. Briefly discuss the implications of different values of the parameter τ.

20.22 (2 points) S(0) = 1. S(x) = h(x). Determine the form of S(x).

20.23 (2 points) $F(x) = \frac{1}{1 + \exp[-(x-\mu)/s]}$, $-\infty < x < \infty$, $s > 0$.
Find the form of the hazard rate as a function of x.

20.24 (5 points)
The following data is from the mortality study of Edmond Halley published in 1693.

x	0	5	10	15	20	25	30
S(x)	1	0.710	0.653	0.622	0.592	0.560	0.523
x	35	40	45	50	55	60	65
S(x)	0.481	0.436	0.387	0.335	0.282	0.232	0.182
x	70	75	80	85			
S(x)	0.131	0.078	0.034	0			

Using this data, graph the hazard rate as a function of age.

20.25 (2 points) Robots can fail due to two independent decrements: Internal and External.
(Internal includes normal wear and tear. External includes accidents.)
Assuming no external events, a robot's time until failure is given by a Pareto DIstribution with $\alpha = 2$ and $\theta = 10$.
Assuming no internal events, a robot's time until failure is given by a Pareto Distribution with $\alpha = 4$ and $\theta = 10$.
At time = 5, what is the hazard rate of the robot's time until failure?
A. 0.35 B. 0.40 C. 0.45 D. 0.50 E. 0.55

20.26 (1 point) F(x) is a Weibull Distribution.
If the hazard rate h(x) is doubled for all x, what is the new distribution function?

20.27 (2 point) Two independent variables X and Y have hazard rates as a function of time of $h_X(t)$ and $h_Y(t)$. Given that the minimum of X and Y is m, what is the probability that X < Y?

20.28 (2 points) For a Gamma Distribution, determine the behavior of the hazard rate h(x) as x approaches infinity.

20.29 (160, 5/87, Q.2) (2.1 points)
The exponential distribution defined by $S(t) = e^{-t/2}$, $t \geq 0$, is truncated from above at t = 4.
Calculate the hazard rate of the truncated distribution at t = 2.

(A) $\frac{1}{2\ (e-1)}$ (B) $\frac{e^2}{2\ (e^2-1)}$ (C) 1/2 (D) $\frac{e}{2\ (e-1)}$ (E) $\frac{e^2}{2\ (e-1)}$

20.30 (160, 11/87, Q.7) (2.1 points) Which of the following are true for all values of x > 0?
I. For every exponential survival model $h(x) = \dfrac{S(x) - S(x+1)}{\int_x^{x+1} S(t)dt}$.
II. For every survival model $f(x) \leq h(x)$.
III. For every survival model $f(x) \leq f(x + 1)$.
(A) I and II only (B) I and III only (C) II and III only (D) I, II and III
(E) The correct answer is not given by (A), (B), (C), or (D).

20.31 (160, 11/87, Q.8) (2.1 points) The force of mortality for a survival distribution is given by:
$h(x) = \dfrac{1}{2\ (100-x)}$, $0 < x < 100$. Determine e(64).
(A) 16 (B)18 (C) 20 (D) 22 (E) 24

20.32 (160, 11/87, Q.15) (2.1 points) For a Weibull distribution as per <u>Loss Models</u>,
the hazard rate at the median age is 0.05. Determine the median age.
(A) $\tau \ln(2)$ (B) $\tau \ln(20)$ (C) $20\tau \ln(2)$ (D) $2\ln(\tau)$ (E) $2\tau \ln(20)$

20.33 (160, 11/88, Q.2) (2.1 points) A survival model is represented by the following probability density function: $f(t) = (0.1)(25 - t)^{-1/2}$; $0 \le t \le 25$. Calculate the hazard rate at 20.
(A) 0.05 (B) 0.10 (C) 0.15 (D) 0.20 (E) 0.25

20.34 (160, 11/89, Q.1) (2.1 points) For a survival model, you are given:
(i) The hazard rate is $h(t) = 2/(w - t)$, $0 \le t < w$.
(ii) T is the random variable denoting time of failure.
Calculate Var(T).
(A) $w^2/18$ (B) $w^2/12$ (C) $w^2/9$ (D) $w^2/6$ (E) $w^2/3$

20.35 (160, 11/89, Q.2) (2.1 points) $S(x) = 0.1(100 - x)^{1/2}$, $0 \le x \le 100$.
Calculate the hazard rate at 84.
(A) 1/32 (B) 1/24 (C) 1/16 (D) 1/8 (E) 1/4

20.36 (160, 5/90, Q.4) (2.1 points) You are given that y is the median age for the survival function $S(x) = 1 - (x/100)^2$, $0 \le x \le 100$. Calculate the hazard rate at y.
(A) 0.013 (B) 0.014 (C) 0.025 (D) 0.026 (E) 0.028

20.37 (Course 160 Sample Exam #1, 1996, Q.2) (1.9 points)
X has a uniform distribution from 0 to 10.
$Y = 4X^2$. Calculate the hazard rate of Y at 4.
(A) 0.007 (B) 0.014 (C) 0.021 (D) 0.059 (E) 0.111

20.38 (Course 160 Sample Exam #2, 1996, Q.2) (1.9 points) You are given:
(i) A survival model has a hazard rate $h(x) = \frac{1}{3\,(\omega - x)}$, $0 \le x \le \omega$.
(ii) The median age is 63.
Calculate the mean residual life at 63, e(63).
(A) 4.5 (B) 6.8 (C) 7.9 (D) 9.0 (E) 13.5

20.39 (Course 160 Sample Exam #3, 1997, Q.1) (1.9 points) You are given:
(i) For a Weibull distribution with parameters θ and τ, the median age is 22.
(ii) At the median age, the value of the Hazard Rate Function is 1.26.
Calculate τ.
(A) 37 (B) 38 (C) 39 (D) 40 (E) 41

20.40 (Course 160 Sample Exam #1, 1999, Q.19) (1.9 points)
Losses follow a Loglogistic Distribution, with parameters $\gamma = 3$ and $\theta = 0.1984$.
For what value of x is the hazard rate, h(x), a maximum?
(A) 0.18 (B) 0.20 (C) 0.22 (D) 0.25 (E) 0.28

20.41 (Course 160 Sample Exam #1, 1999, Q.20) (1.9 points)
A sample of 10 batteries in continuous use is observed until all batteries fail. You are given:
(i) The times to failure (in hours) are 14.1, 21.3, 23.2, 26.2, 29.8, 31.3, 35.7, 39.4, 39.2, 45.3.
(ii) The composite hazard rate function for these batteries is defined by

$h(t) = \lambda,\ 0 \le t < 27.9,$

$h(t) = \lambda + \beta(t - 27.9)^2,\ t \ge 27.9.$

(iii) S(15) = 0.7634, S(30) = 0.5788.
Calculate the absolute difference between the cumulative hazard rate at 34, H(34), based on the assumed hazard rate, and the cumulative hazard rate at 34, $H^O(34)$, based on the observed data.
(A) 0.03 (B) 0.06 (C) 0.08 (D) 0.11 (E) 0.14

20.42 (CAS3, 11/03, Q.19) (2.5 points) For a loss distribution where $x \ge 2$, you are given:
i) The hazard rate function: $h(x) = z^2 / (2x)$, for $x \ge 2$.
ii) A value of the distribution function: F(5) = 0.84.
Calculate z.
A. 2 B. 3 C. 4 D. 5 E. 6

20.43 (CAS3, 11/04, Q.7) (2.5 points)
Which of the following formulas could serve as a force of mortality?
1. $\mu_x = BC^x$, $B > 0,\ C > 1$
2. $\mu_x = a(b+x)^{-1}$, $a > 0,\ b > 0$
3. $\mu_x = (1+x)^{-3}$, $x \ge 0$
A. 1 only B. 2 only C. 3 only D. 1 and 2 only E. 1 and 3 only

20.44 (CAS3, 11/04, Q.27) (2.5 points) You are given:
- X has density f(x), where $f(x) = 500{,}000 / x^3$, for x > 500 (single-parameter Pareto with $\alpha = 2$).
- Y has density g(y), where $g(y) = y\, e^{-y/500} / 250{,}000$ (gamma with $\alpha = 2$ and $\theta = 500$).

Which of the following are true?
1. X has an increasing mean residual life function.
2. Y has an increasing hazard rate.
3. X has a heavier tail than Y based on the hazard rate test.
A. 1 only. B. 2 only. C. 3 only. D. 2 and 3 only. E. All of 1, 2, and 3.
<u>Note</u>: I have rewritten this exam question.

20.45 (CAS3, 5/05, Q.30) (2.5 points)
Acme Products will offer a warranty on their products for x years, where x is the largest integer for which there is no more than a 1% probability of product failure.
Acme introduces a new product with a hazard function for failure at time t of 0.002t.
Calculate the length of the warranty that Acme will offer on this new product.
A. Less than 3 years B. 3 years C. 4 years D. 5 years E. 6 or more years

20.46 (CAS3, 11/05, Q.11) (2.5 points) Individuals with Flapping Gum Disease are known to have a constant force of mortality μ. Historically, 10% will die within 20 years.
A new, more serious strain of the disease has surfaced with a constant force of mortality equal to 2μ.
Calculate the probability of death in the next 20 years for an individual with this new strain.
A. 17% B. 18% C. 19% D. 20% E. 21%

20.47 (SOA M, 11/05, Q.13) (2.5 points) The actuarial department for the SharpPoint Corporation models the lifetime of pencil sharpeners from purchase using a generalized DeMoivre model with $s(x) = (1 - x/\omega)^{\alpha}$, for $\alpha > 0$ and $0 < x \le \omega$.
A senior actuary examining mortality tables for pencil sharpeners has determined that the original value of α must change. You are given:
(i) The new complete expectation of life at purchase is half what it was previously.
(ii) The new force of mortality for pencil sharpeners is 2.25 times the previous force of mortality for all durations.
(iii) ω remains the same.
Calculate the original value of α.
(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

20.48 (CAS3, 5/06, Q.10) (2.5 points) The force of mortality is given as:

$\mu(x) = 2 / (110 - x)$, for $0 \le x < 110$.

Calculate the expected future lifetime for a life aged 30.
A. Less than 20
B. At least 20, but less than 30
C. At least 30, but less than 40
D. At least 40, but less than 50
E. At least 50

20.49 (CAS3, 5/06, Q.11) (2.5 points)
Eastern Digital uses a single machine to manufacture digital widgets.
The machine was purchased 10 years ago and will be used continuously until it fails.
The failure rate of the machine, u(x), is defined as:

$u(x) = x^2 / 4000$, for $x \le \sqrt{4000}$, where x is the number of years since purchase.

Calculate the probability that the machine will fail between years 12 and 14, given that the machine has not failed during the first 10 years.
A. Less than 1.5%
B. At least 1.5%, but less than 3.5%
C. At least 3.5%, but less than 5.5%
D. At least 5.5%, but less than 7.5%
E. At least 7.5%

20.50 (CAS3, 5/06, Q.16) (2.5 points) The force of mortality is given as:

$\mu(x) = 1 / (100 - x)$, for $0 \le x < 100$.

Calculate the probability that exactly one of the lives (40) and (50) will survive 10 years.
A. 9/30 B. 10/30 C. 19/30 D. 20/30 E. 29/30

Solutions to Problems:

20.1. A. For a Gamma with $\alpha > 1$, hazard rate increases (toward a horizontal asymptote given by an exponential.) For $\alpha > 1$ the Gamma is lighter-tailed than an Exponential.
For a LogNormal the hazard rate decreases.
For a Weibull with $\tau < 1$, the hazard rate decreases; for $\tau < 1$ the Weibull is heavier-tailed than an Exponential.
Alternately, the hazard rate increases if and only if the mean excess loss decreases.
For a Gamma with $\alpha > 1$, mean excess loss decreases (toward a horizontal asymptote given by an exponential.)
For a LogNormal mean excess loss increases.
For a Weibull with $\tau < 1$, mean excess loss increases.

20.2. E. $h(x) = \{1 + e'(x)\} / e(x) = (1 - 0.8)/(72 - 0.8x) = 1/(360 - 4x)$. $h(50) = \mathbf{0.00625}$.

20.3. C. $S(x) = \exp[-\int_0^x h(t)\ dt] = \exp[-\int_0^x \frac{1}{360 - 4t}\ dt] = \exp[\ln(360 - 4t)/4]\Big]_{t=0}^{t=x} =$

$\exp[\ln(360 - 4x)/4 - \ln(360)/4] = \exp[(1/4)\ln(1 - x/90)] = (1 - x/90)^{1/4}$. $S(60) = \mathbf{0.760}$.

20.4. B. ${}_{50}p_{30}$ = probability that a life aged 30 lives at least 50 more years = $S(80)/S(30) = (1/9)^{1/4} / (2/3)^{1/4} = \mathbf{0.639}$.

20.5. A. mean lifetime = e(0) = 72. Alternately,

$$E[X] = \int_0^{90} S(t)\ dt = \int_0^{90} (1 - x/90)^{1/4}\ dt = -72(1 - x/90)^{5/4}\Big]_{x=0}^{x=90} = \mathbf{72}.$$

20.6. D. $f(x) = -\frac{dS(x)}{dx} = (1 - x/90)^{-3/4} / 360$. $f(40) = \mathbf{0.0043}$.

20.7. B. For the LogNormal, $F(x) = \Phi[\{\ln(x) - \mu\} / \sigma]$. $F(100{,}000) = \Phi[-0.0544] = 1 - 0.5217$.
Therefore, $S(100{,}000) = 0.5217$. $f(x) = \exp[-0.5\ (\{\ln(x)-\mu\} /\sigma)^2] / \{x\sigma\sqrt{2\pi}\}$.
$f(100{,}000) = \exp[-0.5(\{\ln(100{,}000)-11.6\}/1.6)^2)]/\{160{,}000\sqrt{2\pi}\} = 2.490 \times 10^{-6}$.
$h(100{,}000) = f(100{,}000) / S(100{,}000) = 2.490 \times 10^{-6} / 0.5217 = \mathbf{4.77 \times 10^{-6}}$.

20.8. C. $S(x) = \exp(-\int_0^x h(t)\, dt) = \exp(-\int_0^x 0.002 + 1.1t/10000\, dt)$

$= \exp[-0.002x + (1 - 1.1^x) / (10{,}000 \ln[1.1])]$.
$S(50) = \exp(-0.1 - 0.1221) =$ **0.80**.
Comment: This is an example of Makeham's Law of mortality.

20.9. B. A constant rate of hazard implies an Exponential Distribution, with $\theta = 1$ / the hazard rate. The mean excess loss is θ at all ages. Thus the mean excess loss at age 25 (or any other age) is: 1 / 0.10 = **10**.
Comment: Note, one can write down the equation:
hazard rate = chance of failure / probability of still working $= F'(x) / S(x) = -S'(x) / S(x) = 0.10$
and solve the resulting differential equation:
$S'(x) = -0.1\, S(x)$, for $S(x) = e^{-0.1x}$ or $F(x) = 1 - e^{-0.1x}$.

20.10. D. $h(x) = f(x) / S(x) = \{\tau(x/\theta)^\tau \exp[-(x/\theta)^\tau] / x\} / \exp[-(x/\theta)^\tau] = \tau x^{\tau-1}/\theta^\tau$.
$h(100) = (1.7)(100^{0.7})/(25^{1.7}) =$ **0.179**.

20.11. E. $S(x) = \exp(-\int_{10}^x h(t)\, dt = \exp[-z\{\ln(x) - \ln(10)\}] = (10/x)^z$, for $x \geq 10$.

$0.015625 = S(20) = (1/2)^z$. $\Rightarrow z =$ **6**.
Comment: $S(x) = (10/x)^6$, for $x \geq 10$. A Single Parameter Pareto, with $\alpha = 6$ and $\theta = 10$.
Similar to CAS3, 11/03, Q.19.

20.12. B. $S(x) = \exp(-\int_0^x h(t)\, dt) = \exp[-z\, x^3/3]$, for $x \geq 0$.

$0.8825 = S(5) = \exp[-z\, 5^3/3]$. $\Rightarrow z = -\ln(0.8825)\, 3/125 =$ **0.0030**.
Comment: $S(x) = \exp[-(x/10)^3]$, for $x \geq 0$. A Weibull Distribution, with $\theta = 10$ and $\tau = 3$.

20.13. For the Weibull, $f(x) = \tau(x/\theta)^\tau \exp(-(x/\theta)^\tau) / x = x \exp(-(x/10)^2)/ 50$.
$S(x) = \exp(-(x/\theta)^\tau) = \exp(-(x/10)^2)$. $h(x) = f(x)/S(x) = x/50$.
The per loss variable Y is 0 for $x \le 5$, and is X - 5 for $x > 5$.
Y has a point mass of F(5) at 0. Thus $f_Y(y)$ is undefined at zero. $f_Y(y) = f_X(y+5)$ for $y > 0$.
$S_Y(0) = S_X(5)$. $S_Y(y) = S_X(y+5)$ for $y > 0$.
$h_Y(y)$ undefined at zero. $h_Y(y) = f_Y(y)/S_Y(y) = f_X(y+5)/S_X(y+5) = h_X(y+5) = (y+5)/50$ for $y > 0$.
Comment: Similar to Example 8.1 in Loss Models. Loss Models uses the notation Y^P for the per payment variable and Y^L for the per loss variable.

20.14. The per payment variable Y is undefined for $x \le 5$, and is X - 5 for $x > 5$.
$f_Y(y) = f_X(y+5)/S_X(5)$ for $y > 0$. $S_Y(y) = S_X(y+5)/S_X(5)$ for $y > 0$.
$h_Y(y) = f_Y(y)/S_Y(y) = f_X(y+5)/S_X(y+5) = h_X(y+5) = (y+5)/50$ for $y > 0$.

20.15. The per loss variable Y is 0 for $x \le 5$, and is X for $x > 5$.
Y has a point mass of $F_X(5)$ at 0. Thus $f_Y(y)$ is undefined at zero.
$f_Y(y) = 0$ for $0 < y \le 5$. $f_Y(y) = f_X(y)$ for $y > 5$. $S_Y(y) = S_X(5)$ for $0 < y \le 5$. $S_Y(y) = S_X(x)$ for $y > 5$.
$h_Y(y)$ undefined at zero. $h_Y(y) = 0$ for $0 < y \le 5$.
$h_Y(y) = f_Y(y)/S_Y(y) = f_X(x)/S_X(x) = h_X(y) = y/50$ for $y > 5$.

20.16. The per payment variable Y is undefined for $x \le 5$, and is X for $x > 5$.
$f_Y(y) = f_X(x)/S_X(5)$ for $y > 5$. $S_Y(y) = S_X(x)/S_X(5)$ for $y > 5$.
$h_Y(y) = f_Y(y)/S_Y(y) = f_X(y)/S_X(y) = h_X(y) = y/50$ for $y > 5$.
Comment: Similar to Example 8.2 in Loss Models.

20.17. $f(x) = 0.5\, x^2 e^{-x/\theta} / \theta^3$. $S(x) = 1 - \Gamma(3 ; x/\theta) = e^{-x/\theta} + (x/\theta)e^{-x/\theta} + (x/\theta)^2 e^{-x/\theta}/2$.
$h(x) = f(x)/S(x) = x^2/(2\theta^3 + 2\theta^2 x + \theta x^2)$.
$h(x) = 1/(2\theta^3/x^2 + 2\theta^2/x + \theta)$, which increases to $1/\theta$ as x approaches infinity.
Comment: I have used Theorem A.1 in Appendix A of Loss Models, in order to write out the incomplete Gamma Function for an integer parameter. One can also verify that $\frac{dS(x)}{dx} = -f(x)$.

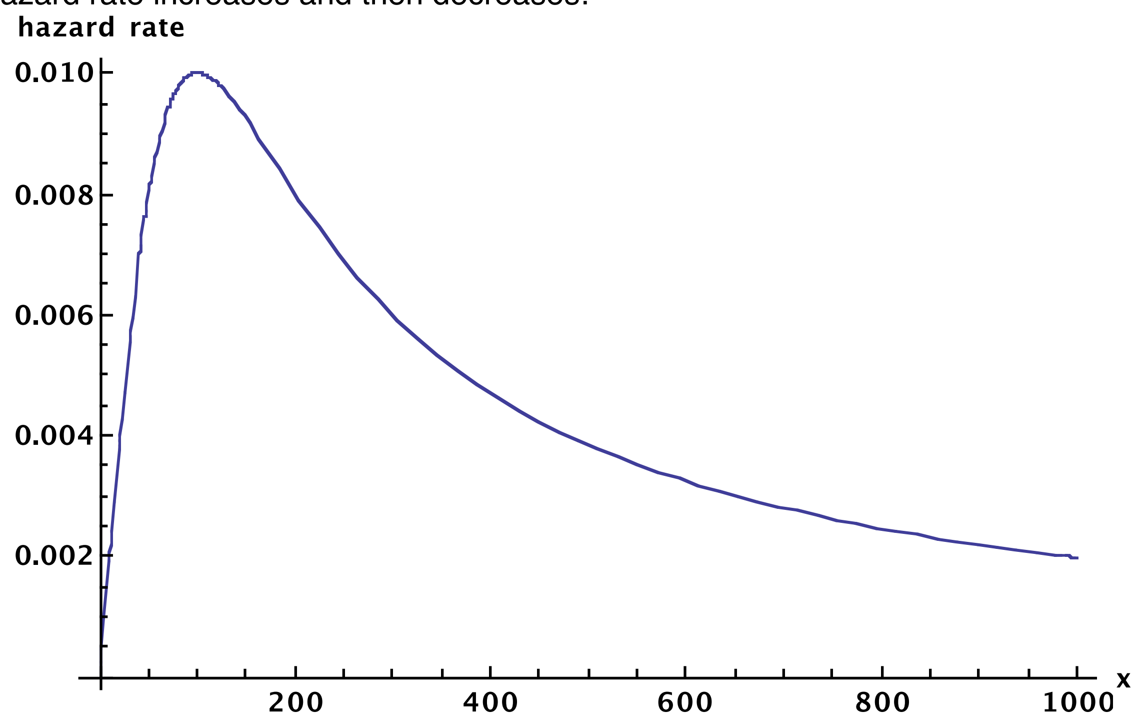
A Gamma Distribution for $\alpha > 1$ is lighter tailed than an Exponential ($\alpha = 1$), and the hazard rate increases to $1/\theta$, while the mean excess loss decreases to θ.

20.18. B. $S(x) = \exp[-\int_0^x \lambda(t)\, dt] = \exp[-\int_0^x \frac{4}{100+t}\, dt] =$

$\exp[-4\{\ln(100+x) - \ln(100)\}] = \left(\frac{100}{100+x}\right)^4$. $S(50) = (100/150)^4 =$ **0.198**.

Comment: This is a Pareto Distribution, with $\alpha = 4$ and $\theta = 100$.

20.19. B. $F(x) = (x/\theta)^\gamma / \{1 + (x/\theta)^\gamma\}$. $F(300) = 3^2 / (1 + 3^2) = 0.9$. $S(300) = 0.1$.
$f(x) = \gamma(x/\theta)^\gamma / (x\{1 + (x/\theta)^\gamma\}^2)$. $f(300) = (2)\ (3^2) / \{(300)(1 + 3^2)^2\} = 0.0006$.
$h(300) = f(300)/S(300) = 0.0006/0.1 =$ **0.006**.
Comment: For the Loglogistic, $h(x) = f(x)/S(x) = \gamma x^{\gamma-1}\ \theta^{-\gamma} / \{1 + (x/\theta)^\gamma\}$.
For $\gamma = 2$ and $\theta = 100$: $h(x) = 0.0002\ x / \{1 + (x/100)^2\}$.
The hazard rate increases and then decreases:



20.20. For the Pareto the hazard rate is: $h(x) = f(x) / S(x) = \frac{\alpha}{\theta + x}$. $\Rightarrow 2\,h(x) = \frac{2\,\alpha}{\theta + x}$.

This is the hazard rate for another Pareto Distribution with parameters 2α and θ.

20.21. If $\tau = 1$, then we have an Exponential with constant hazard rate.
The probability of the period of unemployment ending is independent of how long the worker has been out of work.
If $\tau < 1$, then we have decreasing hazard rate. As a worker remains out of work for longer periods of time, his chance of finding a job declines. This could be due to exhaustion of possible employment opportunities, some employers being unwilling to hire the longterm unemployed, or the worker becoming discouraged.
If $\tau > 1$, then we have increasing hazard rate. As a worker remains out of work for longer periods of time, his chance of going back to work increases. This could be due to worry about exhaustion of unemployment benefits, the worker becoming more willing to settle for a less than desired job, or the worker being more willing to relocate.
Comment: A Weibull with $\tau < 1$ has a heavier righthand tail than an Exponential.
A Weibull with $\tau > 1$ has a lighter righthand tail than an Exponential.

20.22. $S(x) = h(x) = f(x) / S(x). \Rightarrow S(x)^2 = f(x) = -\frac{dS}{dx}$.

Let $y = S(x)$. Then: $y^2 = -\frac{dy}{dx}. \Rightarrow -dy / y^2 = dx. \Rightarrow 1/y = x + c. \Rightarrow S(x) = 1/(x+c)$.

Since $S((0) = 1$, $S(x) = \mathbf{1/(x+1)}$.
Comment: A Pareto Distribution with $\alpha = 1$ and $\theta = 1$.

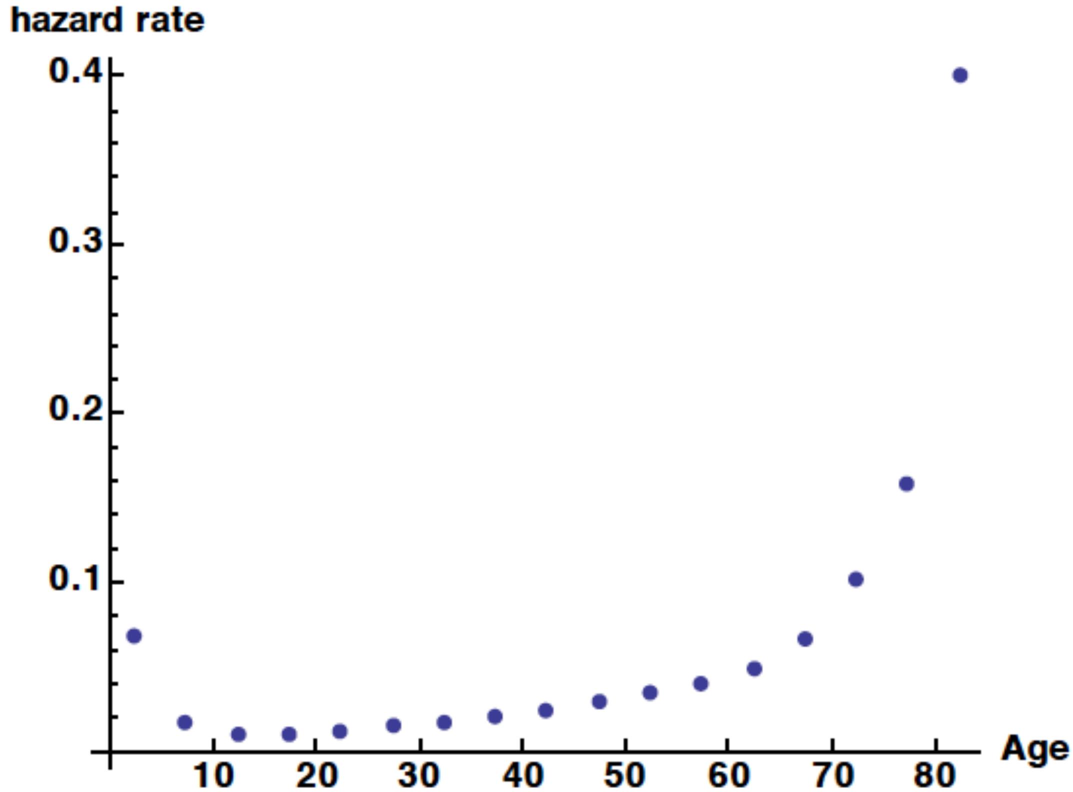
20.23. $S(x) = 1 - F(x) = \frac{\exp[-(x-\mu)/s]}{1 + \exp[-(x-\mu)/s]}$.

$f(x) = F'(x) = \frac{\exp[-(x-\mu)/s] / s}{\{1 + \exp[-(x-\mu)/s]\}^2}$.

$h(x) = f(x) / S(x) = \mathbf{\frac{1/s}{1 + \exp[-(x-\mu)/s]}}$.

Comment: Logistic Distribution with scale parameter s and location parameter μ, not on the syllabus.

20.24. Take the difference of survival functions to get the probability in each interval.
For example, the height of the rectangle from 5 to 10 years old is: 0.710 - 0.653 = 0.057.
The estimate of the density at 7.5 years is: 0.057/5 = 0.0114.
The estimate of S(7.5) is: (0.710 + 0.653)/2 = 0.6815.
The estimate of hazard rate at 7.5 is: 0.0114/0.6815 = 0.0167.
h(12.5) = (2/5) (0.653 - 0.622) / (0.653 + 0.622) = 0.0097.
The vector of estimated hazard rates: 0.0678, 0.0167, 0.0097, 0.0099, 0.0111, 0.0137, 0.0167, 0.0196, 0.0238, 0.0288, 0.0344, 0.0389, 0.0483, 0.0652, 0.1014, 0.1571, 0.4000.



Comment: This was the first published mortality study.
Note the higher hazard rates for young children and old people.
"A few products show a decreasing failure rate in the early life and an increasing failure rate in later life. Reliability engineers call such a hazard function a bathtub curve. ... Some products, such as high-reliability capacitors and semiconductor devices, are subjected to a burn-in to weed out infant mortality before they are put into service, and they are removed from service before wear out starts. Thus they are in service only in the low failure rate portion of their life. This increases their reliability in service. While in service, such products may have an essentially constant failure rate, and the exponential distribution may adequately describe their lives."
Quoted from Applied Life Data Analysis by Wayne Nelson, not on the syllabus.

20.25. B. For the Pareto Distribution, $h(x) = f(x)/S(x) = \frac{\alpha}{\theta + x}$.

Thus $h_1(x) = \frac{2}{10 + x}$, and $h_2(x) = \frac{4}{10 + x}$.

Since the decrements are independent, the hazard rates add, and $h(x) = \frac{6}{10 + x}$.

$h(5) = 6/15 =$ **0.4**.
Alternately, the probability of surviving past age x is the product of the probabilities of surviving both of the independent decrements:
$S(x) = S_1(x)\ S_2(x) = \left(\frac{10}{10 + x}\right)^2 \left(\frac{10}{10 + x}\right)^4 = \left(\frac{10}{10 + x}\right)^6$.

This is a Pareto DIstribution with $\alpha = 6$ and $\theta = 10$. ⇒
$h(x) = \frac{6}{10 + x}$. ⇒ $h(5) = 6/15 =$ **0.4**.

20.26. For the Weibull the hazard rate is: $h(x) = f(x) / S(x) = \frac{\tau\ x^{\tau-1}}{\theta^\tau}$. ⇒

$2\ h(x) = 2 \frac{\tau\ x^{\tau-1}}{\theta^\tau} = \frac{\tau\ x^{\tau-1}}{(\theta/2^{1/\tau})^\tau}$.

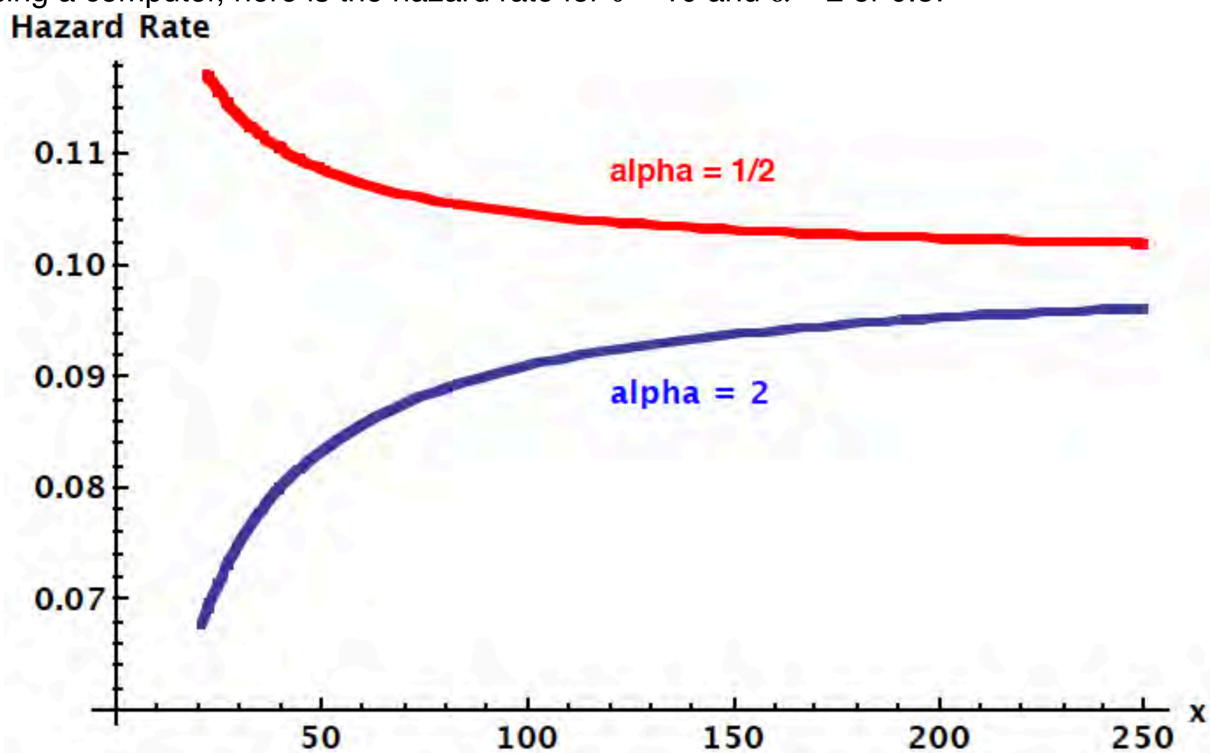
This is the hazard rate for another Weibull Distribution with parameters τ and $\theta / 2^{1/\tau}$.
Comment: If $\tau = 1$ we have an Exponential, and if the hazard rate is doubled then the mean is halved.

20.27. If the minimum is m and Y > X, then X = m and Y > m; this has probability: $f_X(m)\ S_Y(m)$.
If the minimum is m and Y < X, then Y = m and X > m; this has probability: $f_Y(m)\ S_X(m)$.
Thus the desired probability is:
$$\frac{f_X(m)\ S_Y(m)}{f_X(m)\ S_Y(m) + f_Y(m)\ S_X(m)} = \frac{f_X(m) / S_X(m)}{f_X(m) / S_X(m) + f_Y(m) / S_Y(m)} = \frac{h_X(m)}{h_X(m) + h_Y(m)}.$$
Comment: See Exercise 5.7 in Introduction to Probability Models by Ross.
If both X and Y are Exponential, then $\frac{h_X(m)}{h_X(m) + h_Y(m)} = \frac{\lambda_X}{\lambda_X + \lambda_Y}$.

20.28. For $\alpha = 1$, we have an Exponential with constant hazard rate.
For $\alpha > 1$, the Gamma has a lighter righthand tail than the Exponential, and thus the hazard rate increases. For $\alpha < 1$, the Gamma has a heavier righthand tail than the Exponential, and thus the hazard rate decreases.
Alternately, for $\alpha > 1$, the mean excess loss decreases to a constant, and thus the hazard rate increases to a constant. For $\alpha < 1$, the mean excess loss increases to a constant, and thus the hazard rate decreases to a constant.
Alternately, $h(x) = f(x) / S(x)$. As $x \to \infty$, both the numerator and the denominator approach 0.
Thus using L'Hopital's rule, h(x) approaches:

$$f'(x) / \{-f(x)\} = -\frac{(\alpha-1)\, x^{\alpha-2}\, e^{-x/\theta} - x^{\alpha-1}\, e^{-x/\theta} / \theta}{\theta^{\alpha}\, \Gamma(\alpha)} \Big/ \frac{x^{\alpha-1}\, e^{-x/\theta}}{\theta^{\alpha}\, \Gamma(\alpha)} = 1/\theta - (\alpha-1)/x.$$

Therefore, for $\alpha > 1$, the hazard increases to $1/\theta$ as x approaches infinity.
For $\alpha < 1$, the hazard decreases to $1/\theta$ as x approaches infinity.
Comment: As a approaches ∞, the Gamma approaches a Normal, which has a lighter righthand tail. Using a computer, here is the hazard rate for $\theta = 10$ and $\alpha = 2$ or 0.5:



In both cases, the hazard rate approaches $1/\theta = 0.1$ as x approaches infinity.

20.29. D. For truncation from above at 4, the truncated distribution is F(x) / F(4).
$F(2) / F(4) = (1 - e^{-1}) / (1 - e^{-2})$.
For truncation from above at 4, the truncated density is f(x) / F(4).
$f(2) / F(4) = (e^{-1}/2) / (1 - e^{-2})$.
Thus the hazard rate at 2 of the truncated distribution is the ratio of the density to the survival function:

$$\frac{(e^{-1}/2) / (1 - e^{-2})}{1 - (1 - e^{-1}) / (1 - e^{-2})} = \frac{e^{-1}/2}{(1 - e^{-2}) - (1 - e^{-1})} = \mathbf{\frac{e}{2(e - 1)}}.$$

20.30. A. For the Exponential, $h(x) = 1/\theta$ and $S(x) = e^{-x/\theta}$. $\{S(x) - S(x+1)\} / \int_x^{x+1} S(t)dt =$

$\{e^{-x/\theta} - e^{-(x+1)/\theta}\} / \{e^{-x/\theta}/\theta - e^{-(x+1)/\theta}/\theta\} = 1/\theta$. Statement I is true.
$h(x) = f(x)/S(x) \geq f(x)$, since $S(x) \leq 1$. Therefore, Statement II is true.
While the density must go to zero as x approaches infinity, the density can either increase or decrease over short periods. Statement III is not true.

Comment: $\{S(x) - S(x+1)\} / \int_x^{x+1} S(t)dt = m_x$ = central death rate.

See page 70 of Actuarial Mathematics.

20.31. E. $S(x) = \exp[-\int_0^x h(t)\ dt] = \exp[\ln(100-x)/2 - \ln(100)/2] = \sqrt{1 - x/100}$.

$e(64) = \int_{64}^{100} S(t)dt / S(64) = (200/3)(1 - 64/100)^{3/2} / (1 - 64/100)^{1/2} = \mathbf{24}$.

20.32. C. For the Weibull, $S(x) = \exp(-(x/\theta)^\tau)$, and $f(x) = \tau x^{\tau-1}\exp(-(x/\theta)^\tau)/\theta^\tau$.
$\Rightarrow h(x) = f(x)/S(m) = \tau x^{\tau-1}/\theta^\tau$.
Let m be the median. S(m) = 0.5. $\Rightarrow \ln S(m) = -(m/\theta)^\tau = -\ln(2)$. h(m) = 0.05. $\Rightarrow \tau m^{\tau-1}/\theta^\tau = 0.05$.
Dividing the two equations: $m/\tau = \ln(2)/0.05$. $\Rightarrow$ m = $\mathbf{20\tau \ln(2)}$.

20.33. B. Integrating f(t) from t to 25, $S(t) = (0.2)(25 - t)^{1/2}$.
$h(t) = f(t)/S(t) = 1/(50 - 2t)$. h(20) = **1/10**.

20.34. A. h(t) = 2/(w - t). $H(t) = \int_0^x h(t)\, dt = 2\ln(w) - 2\ln(2 - t)$.

$S(t) = \exp[-H(t)] = \{(w - t)/w\}^2 = (1 - t/w)^2$. $f(t) = 2(1 - t/w)/w. = 2/w - 2t/w^2$.

$\int_0^t x\, f(x)dx = w/3$. $\int_0^t x^2\, f(x)dx = w^2/6$. $Var(T) = w^2/6 - (w/3)^2 =$ $\mathbf{w^2/18}$.

20.35. A. $S(x) = 0.1(100 - x)^{1/2}$. $f(x) = 0.05(100 - x)^{-1/2}$. h(x) = f(x)/S(x) = 0.5/(100 - x).
h(84) = 0.5/16 = **1/32.**

20.36. E. $0.5 = (y/100)^2$. ⇒ y = 70.71. f(x) = x/50,000.
f(y) = f(70.71) = 70.71/50,000 = 0.01414. h(y) = f(y)/S(y) = 0.01414/.5 = **0.0283**.

20.37. B. $Y = 4X^2$. ⇒ $X = \sqrt{Y}\ / 2$.

$S_X(x) = 1 - x/10,\ 0 \le x \le 10$. ⇒ $S_Y(y) = 1 - \sqrt{y}\ / 20,\ 0 \le y \le 400$.

$f_Y(y) = 1/(40\sqrt{y})$. $f_Y(4) = 1/80$. $S_Y(4) = 0.9$. $h_Y(4) = (1/80)/0.9 = 1/72 =$ **0.0139**.

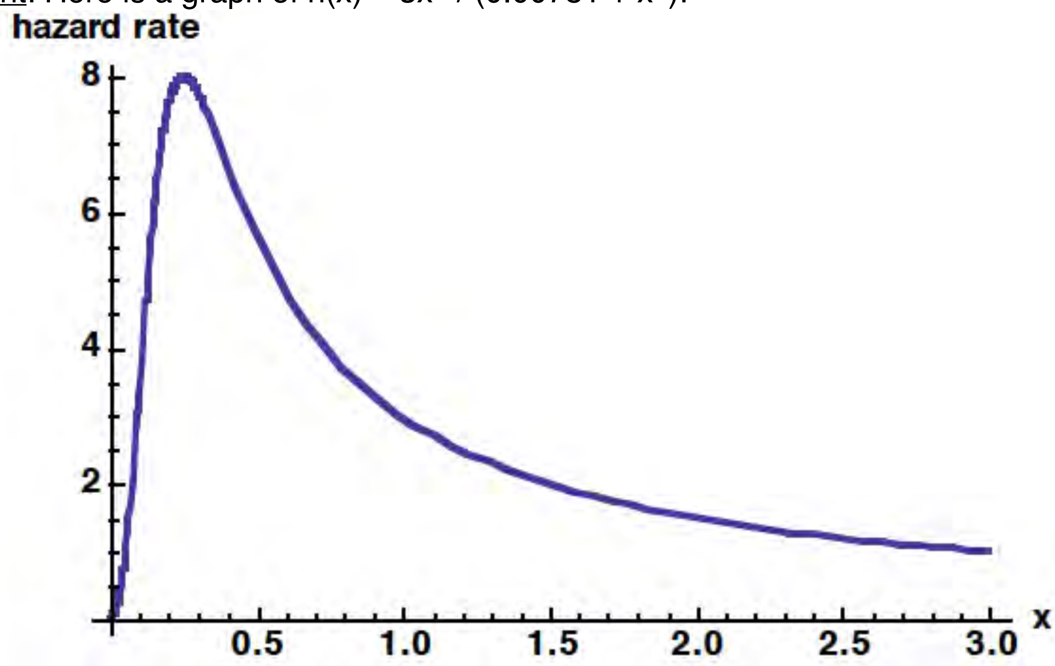
20.38. B. $H(t) = \int_0^t h(x)dx = \ln(\omega)/3 - \ln(\omega - t)/3$.

$S(x) = \exp[-H(t)] = \exp[\ln(\omega - x)/3 - \ln(\omega)/3] = (\omega - x)^{1/3}/\omega^{1/3} = (1 - x/\omega)^{1/3}$.
Median age is 63. ⇒ $0.5 = S(63) = (1 - 63/\omega)^{1/3}$. ⇒ ω = 63/0.875 = 72. ⇒ $S(t) = (1 - t/72)^{1/3}$.

$e(63) = \int_{63}^{72} S(t)dt\ / S(63) = \{(3/4)(72)(1 - 63/72)^{4/3}\}/0.5 =$ **6.75**.

20.39. D. $0.5 = S(22) = \exp[-(22/\theta)^\tau]$. ⇒ $0.69315 = (22/\theta)^\tau$.
$h(x) = f(x)/S(x) = \tau x^{\tau-1}/\theta^\tau$. We are given: $1.26 = h(22) = \tau 22^{\tau-1}/\theta^\tau$.
Dividing the two equations: τ/22 = 1.8178. ⇒ τ = **40**.
Comment: θ = 22.2.

20.40. D. $S(x) = 1/(1 + (x/\theta)^\gamma)$. $f(x) = \gamma x^{\gamma-1} \theta^{-\gamma} / (1+ (x/\theta)^\gamma)^2$.
$h(x) = f(x)/S(x) = \gamma x^{\gamma-1} \theta^{-\gamma} / (1+ (x/\theta)^\gamma) = \{(3)x^2/(0.1984^3)\}/\{1 + (x/0.1984)^3\} = 3x^2/(0.00781 + x^3)$.
$0 = h'(x) = \{6x(0.00781 + x^3) - (3x^2)(3x^2)\} / (0.00781 + x^3)^2$. $\Rightarrow x^3 = 0.01562$. $\Rightarrow x =$ **0.25**.
Comment: Here is a graph of $h(x) = 3x^2 / (0.00781 + x^3)$:



20.41. E. For $t < 27.9$, $H(t) = \int_0^t h(x)dx = \lambda t$. $S(t) = e^{-\lambda t}$. $S(15) = 0.7634$. $\Rightarrow \lambda = 0.018$.

For $t \ge 27.9$, $H(t) = H(27.9) + \int_{27.9}^t h(x)dx = (27.9)(0.018) + (t - 27.9)(0.018) + \beta(t - 27.9)^3/3$.

$S(t) = \exp[-0.018t - \beta(t - 27.9)^3/3]$. $0.5788 = S(30) = \exp[-(0.018)(30) - \beta(30 - 27.9)^3/3]$.
$\Rightarrow (0.018)(30) + \beta(30 - 27.9)^3/3 = 0.5468$. $\Rightarrow \beta = 0.00222$.
$H(34) = -\ln S(34) = (0.018)(34) + (0.00222)(34 - 27.9)^3/3 = 0.780$.
For the observed data, $S^O(34) = 4/10 = 0.4$. $H^O(34) = -\ln(0.4) = 0.916$.
$|H(34) - H^O(34)| = 0.916 - 0.780 =$ **0.14**.

20.42. A. $S(x) = \exp(-\int_2^\infty h(t)\, dt) = \exp[-(z^2/2)\{\ln(x) - \ln(2)\}] = (x/2)^{-z^2/2}$, for $x \ge 2$.

$0.16 = S(5) = 2.5^{-z^2/2}$. $\Rightarrow \ln 0.16 = (-z^2/2) \ln 2.5$. $\Rightarrow z^2 = -2 \ln 0.16 / \ln 2.5 = 4$. $\Rightarrow z =$ **2**.
Comment: $S(x) = (2/x)^2$, for $x \ge 2$. A Single Parameter Pareto, with $\alpha = 2$ and $\theta = 2$.

20.43. D. μ_x corresponds to a legitimate survival function, if and only if it is nonnegative and its integral from 0 to infinity is infinite. All the candidates are nonnegative.

$$\int_0^\infty BC^x \, dx = B \ln C \; C^x \Big]_{x=0}^{x=\infty} = \infty, \text{ since } C > 1.$$

$$\int_0^\infty a(b+x)^{-1} \, dx = a \ln(b+x) \Big]_{x=0}^{x=\infty} = \infty. \qquad \int_0^\infty (1+x)^{-3} \, dx = -(1+x)^{-2} / 2 \Big]_{x=0}^{x=\infty} = 1/2 \neq \infty.$$

Thus 1 and 2 could serve as a force of mortality.
Comment: #1 is Gompertz Law. #2 is a Pareto Distribution with $a = \alpha$ and $b = \theta$.
The Pareto is not useful for modeling human lives, since its force of mortality decreases to zero as x approaches infinity.

20.44. E. 1. Single Parameter Pareto is heavy tailed with an increasing mean residual life.
2. A Gamma with $\alpha > 1$ is lighter tailed than an Exponential; it has a decreasing mean residual life and an increasing hazard rate.
3. Single Parameter Pareto has a heavier tail than a Gamma.
Comments: The mean residual life of a Single Parameter Pareto increases linearly as x goes to infinity, $e(x) = x/(\alpha-1)$. The hazard rate of a Single Parameter Pareto goes to zero as x goes to infinity, $h(x) = \alpha/x$. For this Gamma Distribution, $h(x) = f(x)/S(x) =$
$\{y\, e^{-y/500} / 250{,}000\} / \{1 - \Gamma[2 ; y/500]\} = \{y\, e^{-y/500} / 250{,}000\} / \{e^{-y/500} + (y/500)e^{-y/500}\} =$
$1/(250{,}000/y + 500)$, where I have used Theorem A.1 to write out the incomplete Gamma function for integer parameter. h(x) increases to $1/500 = 1/\theta$ as x approaches infinity.
A Single Parameter Pareto, which has a decreasing hazard rate, has a heavier righthand tail than a Gamma, which has an increasing hazard rate.
A Gamma with $\alpha = 1$ is an Exponential with constant hazard rate. For α integer, a Gamma is a sum of α independent, identically distributed Exponentials. Therefore, as $\alpha \to \infty$, the Gamma Distribution approaches a Normal Distribution. The Normal Distribution is very light-tailed and has an increasing hazard rate. This is one way to remember that for $\alpha > 1$, the Gamma Distribution has an increasing hazard rate. For $\alpha < 1$, the Gamma Distribution has a decreasing hazard rate.

20.45. B. $h(t) = 0.002t$. $H(t) = \int_0^t h(x)dx = 0.001t^2$. $S(t) = \exp[-H(t)] = \exp[-0.001t^2]$.

We want $F(t) \le 1\%$. $F(3) = 1 - \exp[-0.009] = 0.009 \le 1\%$, so **3** is OK.
$F(4) = 1 - \exp[-0.016] = 0.016 > 1\%$, so 4 is not OK.
Comment: A Weibull Distribution with $\tau = 2$. $99\% = S(t) = \exp[-0.001t^2]$. $\Rightarrow t = 3.17$.

20.46. C. For a constant force of mortality (hazard rate) one has an Exponential Distribution.
For the original strain of the disease: $10\% = 1 - e^{-20\mu}$. $\Rightarrow \mu = 0.005268$.
For the new strain, the probability of death in the next 20 years is:
$1 - \exp[-(20)(2\mu)] = 1 - \exp[-(20)(2)(0.005268)] = 1 - e^{-0.21072} =$ **19.0%**.
Alternately, for twice the hazard rate, the survival function is squared.
For the original strain, $S(20) = 1 - 0.010 = 0.90$.
For the new strain, $S(20) = 0.9^2 = 0.81$.
For the new strain, the probability of death in the next 20 years is: $1 - 0.81 =$ **19%**.

20.47. D. $e(x) = \int_x^\infty S(t)\,dt \,/\, S(x) = \int_x^\infty (1 - t/\omega)^\alpha\,dt \,/\, (1 - x/\omega)^\alpha = \{\omega(1 - x/\omega)^\alpha/(\alpha + 1)\} / (1 - x/\omega)^\alpha$

$= (\omega - x)/(\alpha + 1)$. $\Rightarrow e(0) = \omega/(\alpha + 1)$.

By differentiating, $f(x) = -\frac{dS(x)}{dx} = \alpha(1 - x/\omega)^{\alpha-1}/\omega$.

$h(x) = f(x)/S(x) = \{\alpha(1 - x/\omega)^{\alpha-1}/\omega\}/(1 - x/\omega)^\alpha = \alpha/(\omega - x)$.
Let α be the original value and α' be the new value of this parameter.
From bullet i: $\omega/(\alpha' + 1) = 0.5\omega/(\alpha + 1)$. $\Rightarrow \alpha' = 2\alpha + 1$.
From bullet ii: $\alpha'/(\omega - x) = 2.25\alpha/(\omega - x)$. $\Rightarrow \alpha' = 2.25\alpha$.
Therefore, $2.25\alpha = 2\alpha + 1$. $\Rightarrow \alpha =$ **4**.
Alternately, $H(x) = -\ln S(x) = -\alpha \ln(1 - x/\omega)$.

$h(x) = \frac{dH(x)}{dx} = (\alpha/\omega)/(1 - x/\omega) = \alpha/(\omega - x)$. Proceed as before.

Comment: If $\alpha = 1$, then one has DeMoivre's Law, the uniform distribution.
A Modified DeMoivre model has α times the hazard rate of DeMoivre's Law for all ages.

20.48. B. $H(x) = \int_0^x h(t)\,dt = \int_0^x 2/(110 - t)\,dt = -2\{\ln(110 - x) - \ln(110)\}$.

$S(x) = \exp[-H(t)] = \{(110 - x)/110\}^2 = (1 - x/100)^2$, for $0 \le x < 110$.

$e(30) = \int_{30}^{110} S(t)\,dt \ / \ S(30) = \int_{30}^{110} (1 - x/100)^2\,dt \ / \ (1 - 30/100)^2$

$= (110/3)(1 - 30/110)^3 / (1 - 30/100)^2 = (110 - 30)/3 =$ **26.67**.

Comment: Generalized DeMoivre's Law with $\omega = 110$ and $\alpha = 2$. $\mu(x) = \alpha/(\omega - x)$, $0 \le x < \omega$.
$e(x) = (\omega - x)/(\alpha+1) = (110 - x)/3$.
The remaining lifetime at age 30 is a Beta Distribution with a = 1, b = a = 2, and $\theta = \omega - 30 = 80$.

20.49. E. $H(x) = \int_0^x h(t)\,dt = \int_0^x t^2/4000\,dt = x^3/12{,}000$.

$S(x) = \exp[-H(t)] = \exp[-x^3/12{,}000]$, for $0 \le x < \sqrt{4000}$.
S(10) = 0.9200. S(12) = 0.8659. S(14) = 0.7956.
Prob[fail between 12 and 14 | survive until 10] = {S(12) - S(14)} / S(10) =
(0.8659 - 0.7956) / 0.9200 = **0.0764**.
Comment: Without the restriction, $x \le \sqrt{4000}$, this would be a Weibull Distribution with $\tau = 3$.

20.50. A. $H(x) = \int_0^x h(t)\,dt = \ln(100) - \ln(100 - x)$. $S(x) = \exp[-H(x)] = (100 - x)/100$.

Prob[life aged 40 survives at least 10 years] = S(50)/S(40) = 0.5/0.6 = 5/6.
Prob[life aged 50 survives at least 10 years] = S(60)/S(50) = 0.4/0.5 = 4/5.
Prob[exactly one survives 10 years] = (5/6)(1 - 4/5) + (1 - 5/6)(4/5) = **9/30**.
Comment: DeMoivre's Law with $\omega = 100$.

Section 21, Loss Elimination Ratios and Excess Ratios

As discussed previously, the Loss Elimination Ratio (LER) is defined as the ratio of the losses eliminated by a deductible to the total losses prior to imposition of the deductible. The losses eliminated by a deductible d, are $E[x \wedge d]$, the Limited Expected Value at d.[132]

$$\textbf{LER(x)} = \frac{E[X \wedge x]}{E[X]}.$$

The excess ratio R(x), is defined as the ratio of loss dollars excess of x divided by the total loss dollars.[133] It is the complement of the Loss Elimination Ratio; they sum to unity.

$$R(x) = \frac{E[X] - E[X \wedge x]}{E[X]} = 1 - \frac{E[X \wedge x]}{E[X]} = 1 - LER(x).$$

Using the formulas in Appendix A of Loss Models for the Limited Expected Value one can use the relationship: $R(x) = 1 - \frac{E[X \wedge x]}{E[X]}$ to compute the Excess Ratio.

For various distributions, here are the resulting formulas for the excess ratios, R(x):

Distribution	Excess Ratio, R(x)
Exponential	$e^{-x/\theta}$
Pareto	$\left(\frac{\theta}{\theta + x}\right)^{\alpha - 1}$, $\alpha > 1$
LogNormal	$1 - \Phi\left[\frac{\ln[x] - \mu - \sigma^2}{\sigma}\right] - x\,\frac{1 - \Phi\left[\frac{\ln(x) - \mu}{\sigma}\right]}{\exp[\mu + \sigma^2/2]}$
Gamma	$1 - \Gamma(\alpha+1\ ; x/\theta) - x\{1 - \Gamma(\alpha\ ; x/\theta)\}/(\alpha\theta)$
Weibull	$1 - \Gamma[1 + 1/\tau\ ; (x/\theta)^\tau] - (x/\theta)\exp[-(x/\theta)^\tau] / \Gamma[1 + 1/\tau]$
Single Parameter Pareto	$(1/\alpha)\ (x/\theta)^{1-\alpha}$, $\alpha > 1$, $x > \theta$

[132] The losses eliminated are paid by the insured rather than the insurer. *The insured would generally pay less for its insurance in exchange for accepting a deductible. By estimating the percentage of losses eliminated the insurer can price how much of a credit to give the insured for selecting various deductibles. How the LER is used to price deductibles is beyond the scope of this exam, but generally the higher the loss elimination ratio, the greater the deductible credit.*

[133] *The excess ratio is used by actuaries to price reinsurance, workers compensation excess loss factors, etc.*

Recall that the mean and thus the Excess Ratio fails to exist for: Pareto with $\alpha \le 1$, Generalized Pareto with $\alpha \le 1$, and Burr with $\alpha\gamma \le 1$. Except where the formula could be simplified, there is a term in the Excess Ratio which is: -x(S(x)) / mean.[134]

Due to the computational length, exam questions involving the computation of Loss Elimination or Excess Ratios are most likely to involve the Exponential, Pareto, Single Parameter Pareto, or LogNormal Distributions.[135]

Exercise: Compute the excess ratios at $1 million and $5 million for a Pareto with parameters $\alpha = 1.702$ and $\theta = 240,151$.
[Solution: For the Pareto $R(x) = \{\theta/(\theta+x)\}^{\alpha-1}$. R($1 million) = $(240,151/1,240,151)^{0.702}$ = 0.316.
R($5 million) = $(240,151/5,240,151)^{0.702}$ = 0.115.]

Since LER(x) = 1 - R(x), one can use the formulas for the Excess Ratio to get the Loss Elimination Ratio and vice-versa.

Exercise: Compute the loss elimination ratio at 10,000 for the Pareto with parameters: $\alpha = 1.702$ and $\theta = 240,151$.
[Solution: For the Pareto, $R(x) = \{\theta/(\theta+x)\}^{\alpha-1}$. Therefore, $LER(x) = 1 - \{\theta/(\theta+x)\}^{\alpha-1}$.
LER(10,000) = 1 - $(240,151/250,151)^{0.702}$ = 2.8%.
<u>Comment</u>: One could get the same result by using LER(x) = E[X∧x] / mean.]

[134] This term comes from the second part of -E[X∧x] in the numerator, -xS(x). For example for the Gamma Distribution, the excess ratio has a term $-x\{1-\Gamma(\alpha; x/\theta)\}/(\alpha\theta)$ = -x S(x)/mean.
[135] The name “Excess Ratio” is <u>not</u> on the syllabus.

Loss Elimination Ratio and Excess Ratio in Terms of the Survival Function:

As discussed previously, for a distribution with support starting at zero, the Limited Expected Value can be written as an integral of the Survival Function from 0 to the limit:

$$E[X \wedge x] = \int_0^x S(t)\, dt.$$

LER(x) = E[X∧x] / E[X], therefore: $LER(x) = \dfrac{\int_0^x S(t)\, dt}{E[X]} = \dfrac{\int_0^x S(t)\, dt}{\int_0^\infty S(t)\, dt}$.

Thus, **for a distribution with support starting at zero, the Loss Elimination Ratio is the integral from zero to the limit of S(x) divided by the mean**.

Since R(x) = 1 - LER(x) = (E[X] - E[X∧x]) / E[X], the Excess Ratio can be written as:

$$R(x) = \frac{\int_x^\infty S(t)\, dt}{E[X]} = \frac{\int_x^\infty S(t)\, dt}{\int_0^\infty S(t)\, dt}.$$

So the excess ratio is the integral of the survival from the limit to infinity, divided by the mean.[136]

For example, for the Pareto Distribution, $S(x) = \theta^\alpha (\theta+x)^{-\alpha}$. So that:

$$R(x) = \frac{\theta^\alpha\ (\theta+x)^{1-\alpha} / (\alpha-1)}{\theta / (\alpha-1)} = \{\theta/(\theta+x)\}^{\alpha-1}.$$

This matches the formula given above for the Excess Ratio of the Pareto Distribution.

$$LER(x) = \frac{\int_0^x S(t)\, dt}{E[X]}. \Rightarrow \frac{d\ LER(x)}{dx} = \frac{S(x)}{E[X]}.$$

136 This result is used extremely often by property/casualty actuaries. See for example, “The Mathematics of Excess of Loss Coverage and Retrospective Rating -- A Graphical Approach,” by Y.S. Lee, PCAS LXXV, 1988.

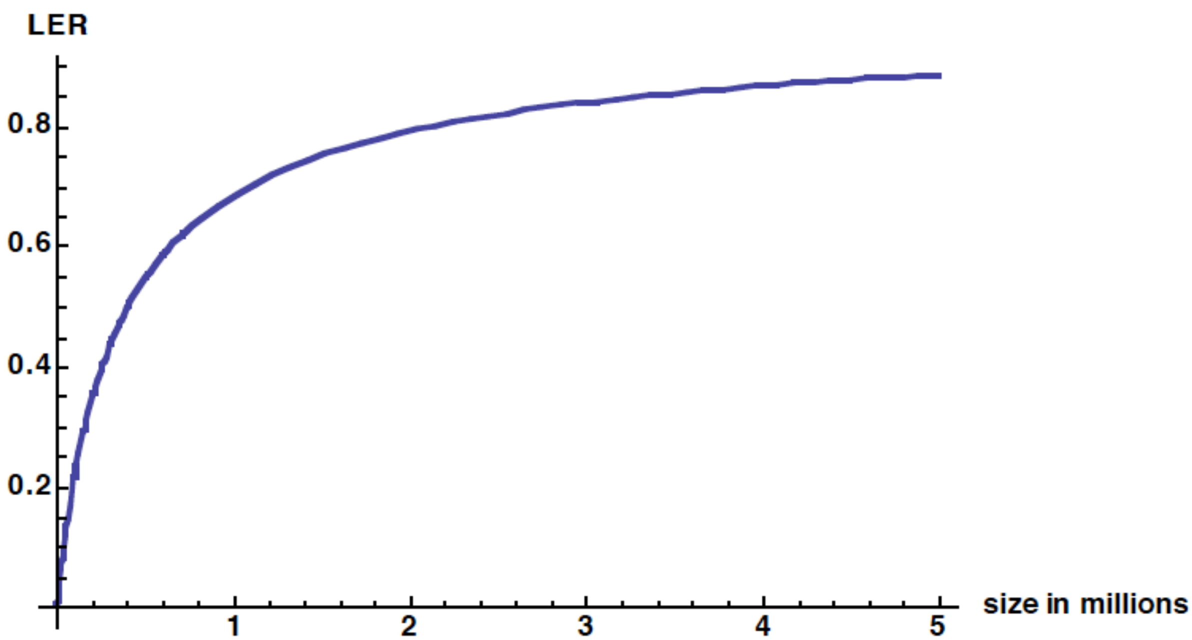
Since $\frac{S(x)}{E[X]} \geq 0$, the loss elimination ratio is a increasing function of x.[137]

$$\frac{d\ LER(x)}{dx} = \frac{S(x)}{E[X]}. \Rightarrow \frac{d^2\ LER(x)}{dx^2} = -\frac{f(x)}{E[X]}.$$

Since $\frac{f(x)}{E[X]} \geq 0$, the loss elimination ratio is a concave downwards function of x.

The loss elimination ratio as a function of x is increasing, concave downwards, and approaches one as x approaches infinity.

For example, here is a graph of the loss elimination ratio for a Pareto Distribution with parameters α = 1.702 and θ = 240,151:[138]



Since the loss elimination ratio is increasing and concave downwards, the excess ratio is decreasing and concave upwards (convex).

[137] If S(x) = 0, in other words there is no possibility of a loss of size greater than x, then the loss elimination is a constant 1, and therefore, more precisely the loss elimination is nondecreasing.
For example, if S(1000) = 0, then there are no losses of size greater than 1000.
Thus a deductible of 1000 would eliminate all of the loss dollars, and LER(1000) = 1.
[138] As x approaches infinity, the loss elimination ratio approaches one. In this case it approaches the limit slowly.

For a distribution with support starting at zero:

$$\frac{d\ \text{LER}(x)}{dx} = \frac{S(x)}{E[X]}. \Rightarrow \frac{d\ \text{LER}(0)}{dx} = \frac{1}{E[X]}. \Rightarrow S(x) = \frac{d\ \text{LER}(x)}{dx} \ / \ \frac{d\ \text{LER}(0)}{dx}.$$

Therefore, the loss elimination ratio function determines the distribution function, as well as vice-versa.

Layers of Loss:

As discussed previously, layers can be thought of in terms of the difference of loss elimination ratio or the difference of excess ratios in the opposite order.

Exercise: Compute the percent of losses in the layer from $1 million to $5 million for a Pareto Distribution with parameters $\alpha = 1.702$ and $\theta = 240{,}151$.
[Solution: For this Pareto Distribution, R($1 million) - R($5 million) = 0.316 - 0.115 = 0.201.
Thus for this Pareto, 20.1% of the losses are in the layer from $1 million to $5 million.]

Moments in Terms of Integrals of Excess Ratios:[139]

$$R(x) = \frac{\int_x^\infty S(t)\ dt}{E[X]} = \frac{\int_x^\infty S(t)\ dt}{\int_0^\infty S(t)\ dt}.$$

Thus, $R'(x) = -S(x) / E[X]$.

As has been discussed previously: $\int_0^\infty R(x)\ dx = \frac{E[X^2]}{2\ E[X]}$.

Assuming they exist, the higher moments can be written in terms of integrals of $R(x)\ x^k$.
Using integration by parts:[140]

[139] See for example "The Mathematics of Excess Losses," by Leigh J. Halliwell, Variance Volume 6 Issue 1.
[140] Where the terms at infinity vanish, since R(x) and S(x) must go to zero sufficiently quickly as x approaches infinity in order for the third moment to exist.

$$\int_0^\infty R(x)\, x\, dx = R(x) x^2/2 \Big]_{x=0}^{x=\infty} - \int_0^\infty \{-S(x) / E[X]\}\, x^2/2\, dx = \frac{1}{2\, E[X]} \int_0^\infty S(x)\, x^2\, dx =$$

$$\frac{1}{2\, E[X]} \left\{ S(x)\, x^3/3 \Big]_{x=0}^{x=\infty} - \int_0^\infty -f(x)\, x^3/3\, dx \right\} = \frac{E[X^3]}{6\, E[X]}.$$

In a similar manner one can show that provided the moments exist:

$$\int_0^\infty R(x)\, x^k\, dx = \frac{E[X^{k+2}]}{(k+2)\,(k+1)\, E[X]},\ k \ge 0.$$

Exercise: Verify that the above equation holds for the Exponential Distribution.

[Solution: $R(x) = e^{-x/\theta}$. $\int_0^\infty R(x)\, x^k\, dx = \int_0^\infty e^{-x/\theta}\, x^k\, dx$.

This is a Gamma type integral with $\alpha = k + 1$.

Thus, $\int_0^\infty R(x)\, x^k\, dx = \int_0^\infty e^{-x/\theta}\, x^k\, dx = \theta^{k+1}\, \Gamma[k+1] = \theta^{k+1}\, k!$.

$$\frac{E[X^{k+2}]}{(k+2)\,(k+1)\, E[X]} = \frac{(k+2)!\, \theta^{k+2}}{(k+2)\,(k+1)\, \theta} = k!\, \theta^{k+1}.$$

Showing that in this case: $\int_0^\infty R(x)\, x^k\, dx = \frac{E[X^{k+2}]}{(k+2)\,(k+1)\, E[X]}$.]

Rewriting this equation: $E[X^{k+2}] = (k+1)\,(k+2)\, E[X] \int_0^\infty R(x)\, x^k\, dx$, $k \ge 0$.

Problems:

21.1 (2 points) Assume you have a Pareto distribution with $\alpha = 5$ and $\theta = \$1000$.
What is the Loss Elimination Ratio for \$500?
A. less than 78%
B. at least 78% but less than 79%
C. at least 79% but less than 80%
D. at least 80% but less than 81%
E. at least 81%

21.2 (2 points) Assume you have Pareto distribution with $\alpha = 5$ and $\theta = \$1000$.
What is the Excess Ratio for \$2000?
A. less than 1%
B. at least 1% but less than 2%
C. at least 2% but less than 3%
D. at least 3% but less than 4%
E. at least 4%

21.3 (3 points) You observe the following 35 losses: 6, 7, 11, 14, 15, 17, 18, 19, 25, 29, 30, 34, 40, 41, 48, 49, 53, 60, 63, 78, 85, 103, 124, 140, 192, 198, 227, 330, 361, 421, 514 ,546, 750, 864, 1638.
What is the (empirical) Loss Elimination Ratio at 50?
A. less than 0.14
B. at least 0.14 but less than 0.16
C. at least 0.16 but less than 0.18
D. at least 0.18 but less than 0.20
E. at least 0.20

21.4 (2 points) The size of losses follows a LogNormal distribution with parameters $\mu = 11$ and $\sigma = 2.5$. What is the Excess Ratio for 100 million?
A. less than 5%
B. at least 5% but less than 10%
C. at least 10% but less than 15%
D. at least 15% but less than 20%
E. at least 20%

21.5 (2 points) The size of losses follows a Gamma distribution with parameters $\alpha = 3$, $\theta = 100{,}000$. What is the excess ratio for 500,000?
Hint: Use Theorem A.1 in Appendix A of Loss Models:

$$\Gamma(n; x) = 1 - \sum_{j=0}^{j=n-1} x^j e^{-x} / j! \text{ , for n a positive integer.}$$

A. less than 5.6%
B. at least 5.6% but less than 5.8%
C. at least 5.8% but less than 6.0%
D. at least 6.0% but less than 6.2%
E. at least 6.2%

21.6 (2 points) The size of losses follows a LogNormal distribution with parameters $\mu = 10$, $\sigma = 3$. What is the Loss Elimination Ratio for 7 million?
A. less than 10%
B. at least 10% but less than 15%
C. at least 15% but less than 20%
D. at least 20% but less than 25%
E. at least 25%

Use the following information for the next two questions:
- Accident sizes for Risk 1 follow an Exponential distribution, with mean θ.
- Accident sizes for Risk 2 follow an Exponential distribution, with mean 1.2θ.
- The insurer pays all losses in excess of a deductible of d.
- 10 accidents are expected for each risk each year.

21.7 (1 point) Determine the expected amount of annual losses paid by the insurer for Risk 1.
A. $10d\theta$ B. $10 / (d\theta)$ C. 10θ D. $10\theta e^{-d/\theta}$ E. $10e^{-d/\theta}$

21.8 (1 point) Determine the limit as d goes to infinity of the ratio of the expected amount of annual losses paid by the insurer for Risk 2 to the expected amount of annual losses paid by the insurer for Risk 1.
A. 0 B. 1/1.2 C. 1 D. 1.2 E. ∞

21.9 (2 points) For a Beta Distribution with $b = 1$ and $\theta = 1$, determine the form of the Loss Elimination Ratio as function of x, for $0 \le x \le 1$.

21.10 (1 point) You have the following estimates of integrals of the Survival Function.

$$\int_0^{1000} S(x)\,dx \cong 400. \qquad \int_{1000}^{\infty} S(x)\,dx \cong 2300.$$

Estimate the Loss Elimination Ratio at 1000.
A. less than 15%
B. at least 15% but less than 16%
C. at least 16% but less than 17%
D. at least 17% but less than 18%
E. at least 18%

21.11 (4 points) For a LogNormal distribution with coefficient of variation equal to 3, what is the Loss Elimination Ratio at twice the mean?
A. less than 50%
B. at least 50% but less than 55%
C. at least 55% but less than 60%
D. at least 60% but less than 65%
E. at least 65%

21.12 (3 points) The loss elimination ratio at $1 \geq x \geq 0$ is: $\dfrac{\ln[a + b^x] - \ln[a+1]}{\ln[a + b] - \ln[a+1]}$, $1 > b > 0$, $a > 0$.

Determine the form of the distribution function.

21.13 (3 points) $f(x) = \begin{cases} 0.0001\,x, & 0 < x \leq 100 \\ 0.0001\,(200 - x), & 100 < x \leq 200 \\ 0, & x > 200 \end{cases}$.

Calculate the loss elimination ratio for an ordinary deductible of 50.
A. 44% B. 46% C. 48% D. 50% E. 52%

21.14 (3 points) Sizes of loss follow a Poisson Distribution with mean 6.
What is the loss elimination ratio at 5?
A. 55% B. 60% C. 65% D. 70% E. 75%

21.15 (3 points) For a Beta Distribution with $a = 1$ and $\theta = 1$, determine the form of the Loss Elimination Ratio as function of x, for $0 \leq x \leq 1$.

21.16 (4, 5/86, Q.59) (2 points) Assume that losses follow the probability density function $f(x) = x/18$ for $0 \leq x \leq 6$ with $f(x) = 0$ otherwise.
What is the loss elimination ratio (LER) for a deductible of 2?
A. Less than 0.35
B. At least 0.35, but less than 0.40
C. At least 0.40, but less than 0.45
D. At least 0.45, but less than 0.50
E. 0.50 or more

21.17 (4, 5/87, Q.57) (2 points) Losses are distributed with a probability density function $f(x) = 2/x^3$, $1 < x < \infty$. What is the expected loss eliminated by a deductible of d = 5?
A. Less than 0.5
B. At least 0.5, but less than 1
C. At least 1, but less than 1.5
D. At least 1.5, but less than 2
E. 2 or more.

21.18 (4B, 5/92, Q.25) (2 points) You are given the following information:

Deductible	$250
Expected size of loss with no deductible	$2,500
Probability of a loss exceeding deductible	0.95
Mean Excess Loss value of the deductible	$2,375

Determine the loss elimination ratio.
A. Less than 0.035
B. At least 0.035 but less than 0.070
C. At least 0.070 but less than 0.105
D. At least 0.105 but less than 0.140
E. At least 0.140

21.19 (4B, 11/92, Q.18) (2 points) You are given the following information:

Deductible, d	$500
Expected value limited to d, $E[x \wedge d]$	$465
Probability of a loss exceeding deductible, 1-F(d)	0.86
Mean Excess Loss value of the deductible, e(d)	$5,250

Determine the loss elimination ratio.
A. Less than 0.035
B. At least 0.035 but less than 0.055
C. At least 0.055 but less than 0.075
D. At least 0.075 but less than 0.095
E. At least 0.095

21.20 (4B, 5/94, Q.10) (2 points) You are given the following:
- The amount of a single loss has a Pareto distribution with parameters $\alpha = 2$ and $\theta = 2000$.

Calculate the Loss Elimination Ratio (LER) for a $500 deductible.
A. Less than 0.18
B. At least 0.18, but less than 0.23
C. At least 0.23, but less than 0.28
D. At least 0.28, but less than 0.33
E. At least 0.33

21.21 (4B, 5/96, Q.9 & Course 3 Sample Exam, Q.17) (2 points)
You are given the following:

- Losses follow a lognormal distribution, with parameters $\mu = 7$ and $\sigma = 2$.
- There is a deductible of 2,000.
- 10 losses are expected each year.
- The number of losses and the individual loss amounts are independent.

Determine the loss elimination ratio (LER) for the deductible.
A. Less than 0.10
B. At least 0.10, but less than 0.15
C. At least 0.15, but less than 0.20
D. At least 0.20, but less than 0.25
E. At least 0.25

21.22 (4B, 11/96, Q.13) (2 points) You are given the following:

- Losses follow a Pareto distribution, with parameters $\theta = k$ and $\alpha = 2$, where k is a constant.
- There is a deductible of 2k.

What is the loss elimination ratio (LER)?
A. 1/3 B. 1/2 C. 2/3 D. 4/5 E. 1

21.23 (4B, 5/97, Q.19) (3 points) You are given the following:

- Losses follow a distribution with density function
  $f(x) = (1/1000)\, e^{-x/1000}$, $0 < x < \infty$.
- There is a deductible of 500.
- 10 losses are expected to exceed the deductible each year.

Determine the amount to which the deductible would have to be raised to double the loss elimination ratio (LER).
A. Less than 550
B. At least 550, but less than 850
C. At least 850, but less than 1,150
D. At least 1,150, but less than 1,450
E. At least 1,450

Use the following information for the next two questions:
- Loss sizes for Risk 1 follow a Pareto distribution, with parameters θ and α, $\alpha > 2$.
- Loss sizes for Risk 2 follow a Pareto distribution, with parameters θ and 0.8α, $\alpha > 2$.
- The insurer pays losses in excess of a deductible of k.
- 1 loss is expected for each risk each year.

21.24 (4B, 11/97, Q.22) (2 points)
Determine the expected amount of annual losses paid by the insurer for Risk 1.

A. $\frac{\theta+k}{\alpha-1}$ B. $\frac{\theta^\alpha}{(\theta+k)^\alpha}$ C. $\frac{\alpha\theta^\alpha}{(\theta+k)^{\alpha+1}}$

D. $\frac{\theta^{\alpha+1}}{(\alpha-1)\ (\theta+k)^\alpha}$ E. $\frac{\theta^\alpha}{(\alpha-1)\ (\theta+k)^{\alpha-1}}$

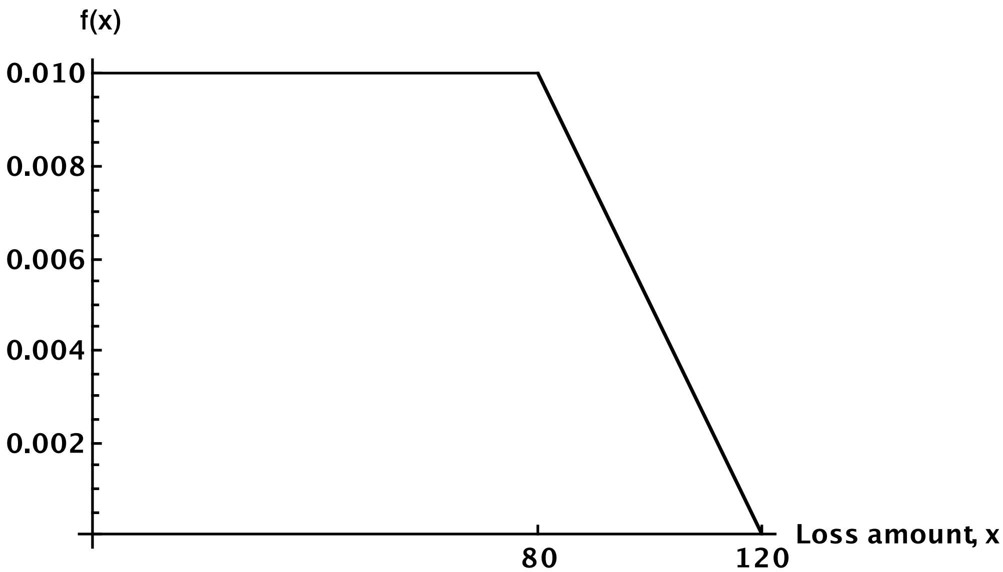
21.25 (4B, 11/97, Q.23) (1 point) Determine the limit of the ratio of the expected amount of annual losses paid by the insurer for Risk 2 to the expected amount of annual losses paid by the insurer for Risk 1 as k goes to infinity.

A. 0 B. 0.8 C. 1 D. 1.25 E. ∞

21.26 (4B, 5/99, Q.20) (2 points) Losses follow a lognormal distribution, with parameters $\mu = 6.9078$ and $\sigma = 1.5174$. Determine the ratio of the loss elimination ratio (LER) at 10,000 to the loss elimination ratio (LER) at 1,000.

A. Less than 2
B. At least 2, but less than 4
C. At least 4, but less than 6
D. At least 6, but less than 8
E. At least 8

21.27 (SOA3, 11/03, Q.29 & 2009 Sample Q.87) (2.5 points)
The graph of the density function for losses is:



Calculate the loss elimination ratio for an ordinary deductible of 20.
(A) 0.20 (B) 0.24 (C) 0.28 (D) 0.32 (E) 0.36

21.28 (SOA M, 11/06, Q.29 & 2009 Sample Q.284) (2.5 points)
A risk has a loss amount which has a Poisson distribution with mean 3.
An insurance covers the risk with an ordinary deductible of 2.
An alternative insurance replaces the deductible with coinsurance α, which is the proportion of the loss paid by the insurance, so that the expected insurance cost remains the same.
Calculate α.
(A) 0.22 (B) 0.27 (C) 0.32 (D) 0.37 (E) 0.42

Solutions to Problems:

21.1. D. LER(x) = { E[X∧x] / mean }, for the Pareto: mean = θ/(α-1) = $250 and
E[X∧x] = {θ/(α-1)} {1-(θ/θ+x)$^{\alpha-1}$} = $200.6. LER(x) = 200.6/250 = **80.24%**.

21.2. B. Excess Ratio = 1 - { E[X∧x] / mean } = {θ/(θ+x)}$^{\alpha-1}$ = **1.23%.**
Comment: E[X∧x] = 246.925, mean = 250.

21.3. D. E[X∧50] =
{6 + 7 + 11+ 14 +15 + 17 + 18 + 19 + 25+ 29+ 30+ 34 + 40 + 41+ 48 + 49 + (19)(50)} /35 =
(403 + 950)/35 = 38.66.
E[X] = {6 + 7 + 11 +14 + 15 + 17 + 18 +19 + 25 + 29 + 30 + 34 + 40 + 41+ 48 + 49 + 53 + 60 + 63 + 78 + 85 + 103 + 124 + 140 + 192 + 198 + 227 + 330 + 361 + 421 + 514 + 546 + 750 + 864 + 1638}/35 = 7150 /35 = 204.29.
LER(50) = E[X∧50] / E[X] = 38.66 / 204.29 = **0.189**.

21.4. E. mean = exp(μ + σ^2/2) = exp(11 + 2.5^2/2) = 1362729.
E{X∧ x] = exp(μ + σ^2/2)Φ[(lnx - μ − σ^2)/σ] + x {1 -Φ[(lnx - μ)/σ]}.
E{X∧100 million] = 1,362,729Φ[(18.421 - 11 - 2.5^2)/2.5] - (100 million){1 - Φ[(18.421 - 11)/2.5]} =
1,362,729Φ[0.47] - (100 million){1 - Φ[2.97]} =1,362,729(0.6808) - (100 million){1 - 0.9985) =
1,077,745. Then R(100 million) = 1 - 1,077,745 / 1,362,729 = 1 - 0.791 = **20.9%**.

21.5. B. Γ[3 ; 5] = 1 - e^{-5}(1 + 5 + 5^2/2) = 0.875.
Γ[4 ; 5] = 1 - e^{-5}(1 + 5 + 5^2/2 +5^3/6) = 0.735.
For Gamma E[X] = αθ = 300,000.
E[X ∧ 500,000] = (αθ)Γ[α+1 ; 500,000/θ] + 500,000 (1 - Γ[α ; 500,000/θ]) =
300,000Γ[4 ; 5] + 500,000(1 - Γ[3 ; 5]) = 283,000.
Excess Ratio = 1 - { E[X ∧ 500,000] / E[X] } = 1 - 283,000/300,000 = 1 - 0.943 = **5.7%**.

21.6. D. mean = exp(μ + σ^2/2) = exp(10 + 3^2/2) = 1,982,759.
E{X∧ x] = exp(μ + σ^2/2)Φ[(lnx − μ − σ^2)/σ] + x {1 - Φ[(lnx - μ)/σ]}.
E{X∧ 7 million] = 1,982,759Φ[(15.761 - 10 - 3^2)/3] + (7 million){1 - Φ[(15.761 - 10)/3]}
= 1,982,759 Φ[-1.08] + (7 million){1 - Φ[1.92]} =
1,982,759(0.1401) + (7 million)(1 - 0.9726) =469,585.
Then LER(7 million) = E{X∧ 7 million] / E[X] = 469,585 / 1,982,759 = **0.237**.

21.7. D. The expected amount paid by the insurer is: 10{E[X] - E[X∧d]} =
10{θ - θ(1-e$^{-d/\theta}$)} = **10 θ e$^{-d/\theta}$**.
Alternately, per claim the losses excess of the limit d are: e(k) (1-F(k)) = θ e$^{-d/\theta}$.
Thus for 10 claims we expect the insurer to pay: **10 θ e$^{-d/\theta}$**.
Alternately, per claim the losses excess of the limit k are: R(k)E[X] = e$^{-d/\theta}$ θ = θ e$^{-d/\theta}$.
Thus for 10 claims we expect the insurer to pay: **10 θ e$^{-d/\theta}$**.

21.8. E. Using the solution to the previous question, the expected amount paid by the insurer for Risk 1 is: $10\theta\ e^{-d/\theta}$.
Similarly, the expected amount paid by the insurer for Risk 2 is: $12\theta\ e^{-d/(1.2\theta)}$.
Therefore, the ratio of the expected amount of annual losses paid by the insurer for Risk 2 to the expected amount of annual losses paid by the insurer for Risk 1 is:
$\{12\theta\ e^{-d/(1.2\theta)}\} / \{10\theta\ e^{-d/\theta}\} = 1.2\ e^{0.167d/\theta}$. As d goes to infinity this goes to **infinity**.

21.9. E[X] = a/(a+b) = a/(a+1).
$f(x) = ax^{a-1}$, $0 \le x \le 1$. $\Rightarrow$ $S(x) = 1 - x^a$.

$$E[X \wedge x] = \int_0^x t\ f(t)\ dt + x\ S(x) = \int_0^x t\ at^{a-1}\ dt + x(1 - x^a) = x^{a+1}\ a/(a+1) + x - x^{a+1} = x - x^{a+1} / (a+1).$$

LER[x] = E[X ∧ x] / E[X] = $\mathbf{x\ (a+1)/a - x^{a+1}/a}$.

$$\textbf{21.10. A.}\ \ \text{LER}(1000) = \frac{\int_0^{1000} S(t)\ dt}{\int_0^{\infty} S(t)\ dt} \cong 400/(400+2300) = \mathbf{14.8\%}.$$

<u>Comment</u>: The estimated mean is 400 + 2300 = 2700.
The estimated limited expected value at 1000 is 400.

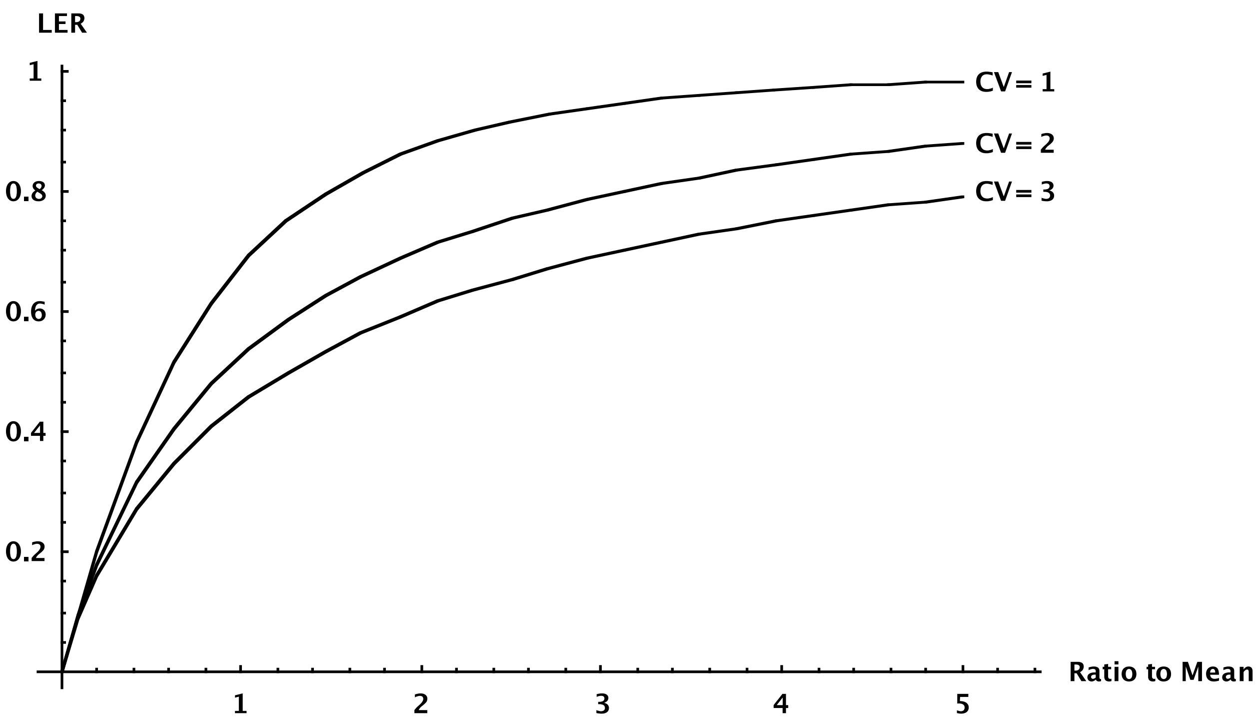
21.11. D. $1 + CV^2 = E[X^2]/E[X]^2 = \exp[2\mu + 2\sigma^2]/\exp[\mu + \sigma^2/2]^2 = \exp[\sigma^2]$.
$1 + 3^2 = \exp[\sigma^2]$. $\Rightarrow \sigma = \sqrt{\ln(10)} = 1.5174$.

$LER[x] = E[X \wedge x]$ / mean = $\{\exp(\mu+\sigma^2/2)\Phi[(\ln x - \mu - \sigma^2)/\sigma] + x\{1 - \Phi[(\ln x - \mu)/\sigma]\}\} / \exp(\mu + \sigma^2/2)$ =
$\Phi[(\ln x - \mu)/\sigma - \sigma]$ + (x/mean) $\{1 - \Phi[(\ln x - \mu)/\sigma]\}$. x = 2 mean. $\Rightarrow \ln(x) = \ln(2) + \mu + \sigma^2/2$.

$\Rightarrow (\ln x - \mu)/\sigma = \ln(2)/\sigma + \sigma/2 = 0.69315/1.5174 + 1.5174/2 = 1.2155$.

$LER[x] = \Phi[1.2155 - 1.5174] + (2)\{1 - \Phi[1.2155]\} = \Phi[-0.30] + 2(1 - \Phi[1.22]) =$
$0.3821 + 2(1 - 0.8888) =$ **60.5%**.

Comment: See Table I in “Estimating Pure Premiums by Layer - An Approach” by Robert J. Finger, PCAS 1976. Finger calculates excess ratios, which are one minus the loss elimination ratios. Here is a graph of Loss Elimination Ratios as a function of the ratio to the mean, for LogNormal Distributions with some different values of the coefficient of variation:



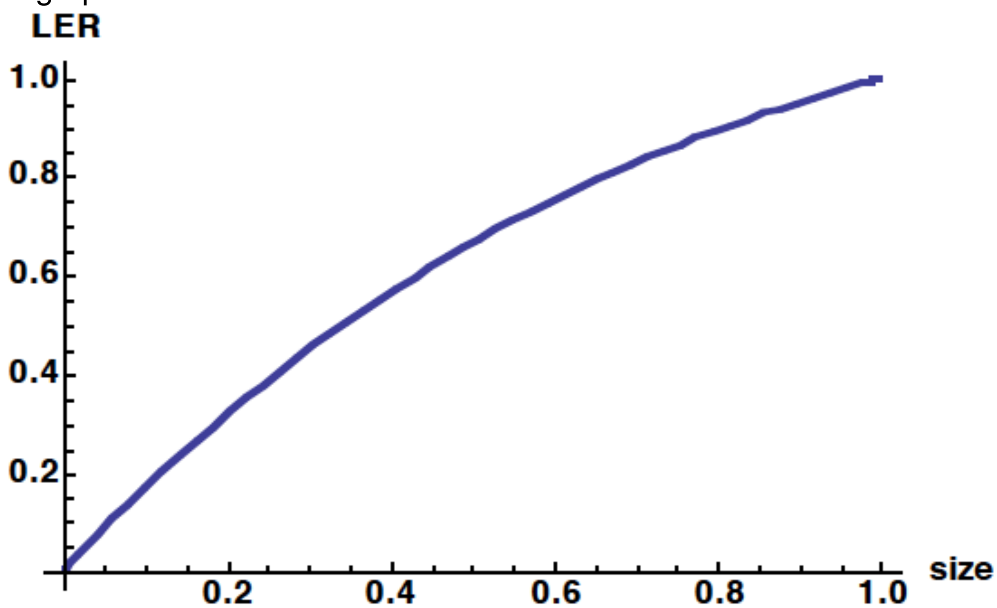
21.12. $\text{LER}(x) = \dfrac{\int_0^x S(t)\,dt}{E[X]}$. $\Rightarrow \dfrac{d\,\text{LER}(x)}{dx} = \dfrac{S(x)}{E[X]}$.

$\Rightarrow \dfrac{d\,\text{LER}(0)}{dx} = \dfrac{1}{E[X]}$. $\Rightarrow S(x) = \dfrac{d\,\text{LER}(x)}{dx} \Big/ \dfrac{d\,\text{LER}(0)}{dx}$. $\Rightarrow F(x) = 1 - \dfrac{d\,\text{LER}(x)}{dx} \Big/ \dfrac{d\,\text{LER}(0)}{dx}$.

$\dfrac{d\,\text{LER}(x)}{dx} = \dfrac{1}{\ln[a + b] - \ln[a+1]} \dfrac{\ln[b]\, b^x}{a + b^x}$. $\dfrac{d\,\text{LER}(0)}{dx} = \dfrac{1}{\ln[a + b] - \ln[a+1]} \dfrac{\ln[b]}{a + 1}$.

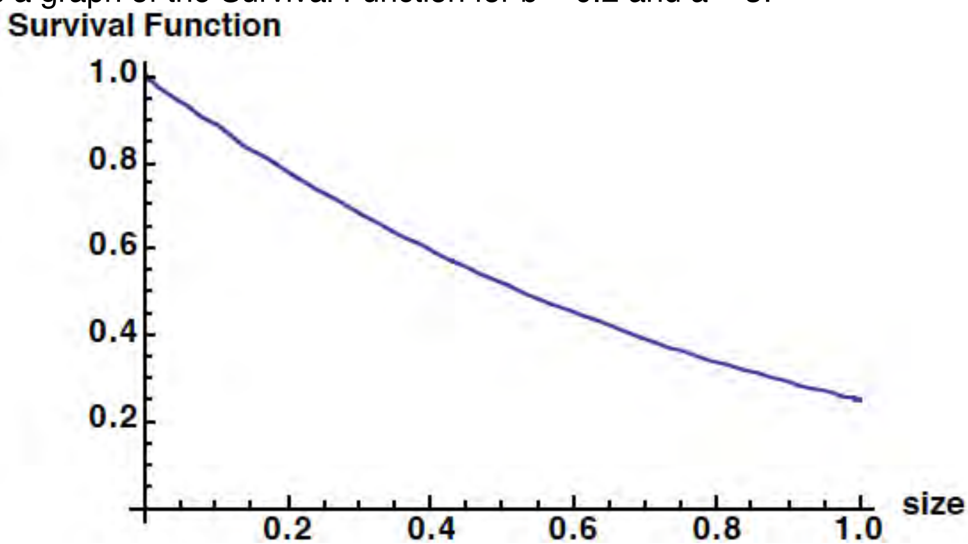
$\Rightarrow F(x) = 1 - (a+1) \dfrac{b^x}{a + b^x}$, $0 \le x < 1$.

$S(1^-) = (a+1)b / (a + b) > 0$.
Thus there is a point mass of probability at 1 of size: (a+1)b / (a + b).
Comment: Note that $F(1^-) = a\,(1 - b) / (a + b) < 1$.
This is a member of the MBBEFD Distribution Class, not on the syllabus of your exam.
See “Swiss Re Exposure Curves and the MBBEFD Distribution Class,” by Stefan Bernegger, ASTIN Bulletin, Vol. 27, No. 1, May 1997, pp. 99-111.
Here is a graph of the loss elimination ratio for b = 0.2 and a = 3:



As it should, the LER is increasing, concave downwards, and approaches 1 as x approaches 1.

Here is a graph of the Survival Function for b = 0.2 and a = 3:



There is a point mass of probability at x = 1 of size: (a+1)b / (a + b) = (4)(0.2)/3.2 = 25%.

21.13. C. $E[X] = \int_0^{100} x\ 0.0001\ x\ dx + \int_{100}^{200} x\ 0.0001\ (200 - x)\ dx =$

$0.0001 \int_0^{100} x^2\ dx + 0.0001 \int_{100}^{200} 200x - x^2\ dx =$

$(0.0001)\ \{100^3/3 + (100)(200^2 - 100^2) - (200^3/3 - 100^3/3)\} = 100.$

Alternately, the density is symmetric around 100, so the mean is 100.

$E[X \wedge 50] = \int_0^{50} x\ 0.0001\ x\ dx + 50 \int_{50}^{100} 0.0001\ x\ dx + 50 \int_{100}^{200} 0.0001\ (200 - x)\ dx =$

$0.0001 \int_0^{50} x^2\ dx + 0.005 \int_{50}^{100} x\ dx + 0.005 \int_{100}^{200} (200 - x)\ dx =$

$(0.0001)(50^3/3) + (0.005)\ (100^2/2 - 50^2/2) + (0.005)\ \{(100)(200) - (200^2/2 - 100^2/2)\} = 47.9167.$

Loss Elimination Ratio at 50 is: $E[X \wedge 50]/E[X]$ = 47.9167/100 = **47.9%**.

Comment: Similar to SOA3, 11/03, Q.29 & (2009 Sample Q.87).

21.14. E. $f(0) = e^{-6} = 0.0025$. $f(1) = 6e^{-6} = 0.0149$. $f(2) = 6^2e^{-6}/2 = 0.0446$.
$f(3) = 6^3e^{-6}/6 = 0.0892$. $f(4) = 6^4e^{-6}/24 = 0.1339$.
$1 - f(0) - f(1) - f(2) - f(3) - f(4) =$
$1 - 0.0025 - 0.0149 - 0.0446 - 0.0892 - 0.1339 = 0.7149$.
$E[X \wedge 5] = 0\ f(0) + 1\ f(1) + 2\ f(2) + 3\ f(3) + 4\ f(4) + 5\{1 - f(0) - f(1) - f(2) - f(3) - f(4)\} =$
$0.0149 + (2)(0.0446) + (3)(0.0892) + (4)(0.1339) + (5)(0.7149) = 4.4818$.
Loss Elimination Ratio at 5 is: $E[X \wedge 5]/E[X] = 4.4818/6 =$ **74.7%**.
Comment: Similar to SOA M, 11/06, Q.29 (2009 Sample Q.284).

21.15. $E[X] = a/(a+b) = 1/(1+b)$.
$f(x) = b\ (1-x)^{b-1},\ 0 \le x \le 1. \Rightarrow S(x) = (1-x)^b$.

$$E[X \wedge x] = \int_0^x t\ f(t)\ dt + x\ S(x) = \int_0^x t\ b\ (1-t)^{b-1}\ dt + x(1-x)^b =$$

$$b \int_0^x (1-t)^{b-1}\ dt - b \int_0^x (1-t)^b\ dt + x\ (1-x)^b = -\{(1-x)^b - 1\} + \{(1-x)^{b+1} - 1\}b/(b+1) + x\ (1-x)^b.$$

$LER[x] = E[X \wedge x] / E[X] = -\{(1-x)^b - 1\}(b+1) + \{(1-x)^{b+1} - 1\}b + x\ (1-x)^b(b+1) =$
$1 + b\ (1-x)^{b+1} - (b+1)(1-x)^b + x\ (1-x)^b(b+1) = 1 + b\ (1-x)^{b+1} - (b+1)(1-x)^{b+1} = \mathbf{1 - (1-x)^{b+1}}$.

21.16. D. $F(x) = x^2 / 36$.

$$LER(2) = \{\int_0^2 f(x)\ x\ dx + 2S(2)\} / \int_0^6 f(x)\ x\ dx = \{(2^3)/(54) + 2\ (1 - 2^2 / 36)\} / \{(6^3)/54\} =$$

$1.926 / 4 =$ **0.481**.

21.17. D. Integrating f(x) from 1 to x, $F(x) = 1 - 1/x^2$.
A deductible of d eliminates the size of the loss for small losses and d per large loss.
The expected losses eliminated by a deductible of d is:

$$\int_1^d f(x)\ x\ dx + d\ S(d) = \int_1^d 2x^{-2}\ dx + d(1/d^2) = (2 - 2/d) + 1/d = 2 - 1/d\ .$$

For d = 5, the expected losses eliminated are: 2 - 1/5 = **1.8**.
Comment: A Single Parameter Pareto with $\alpha = 2$ and $\theta = 1$.

21.18. C. $e(x) = \{\text{mean} - E[X \wedge x]\} / S(x)$. Therefore: $2375 = (2500 - E[X \wedge x]) / 0.95$.
Thus $E[X \wedge x] = 243.75$. Then, $LER(x) = E[X \wedge x] / E[X] = 243.75 / 2500 =$ **0.0975.**
Alternately, $LER(x) = 1 - e(x)\ S(x) / E[X] = 1 - (2375)(1 - 0.05)/2500 = 0.0975$.

21.19. D. LER(d) = E[X∧d] / Mean. e(d) = (Mean - E[X∧d]) / S(d).
Therefore, 5250 = (Mean - 465) / 0.86. Therefore Mean = 4515 + 465 = 4980.
Thus LER(d) = 465/4980 = **0.093**.
Comment: One does not use the information that the deductible amount is $500.
However, note that E[X∧d] ≤ d, as it should.

21.20. B. For the Pareto distribution LER(x) = $1 - (1 + x / \theta)^{1 - \alpha}$.
For α = 2 and θ = 2800, LER(500) = 1 - 1/1.25 = **0.2**.
Alternately, E[X∧x] = $\{\theta/(\alpha-1)\}\{1-(\theta/\theta+x)^{\alpha-1}\}$. E[X∧500] = (2000)(1 - 0.8) = 400.
The mean is: $\theta/(\alpha-1)$ = 2000. LER(x) = E[X∧x] / mean = 400 / 2000 = **0.2**.

21.21. B. Mean = $\exp(\mu + 0.5\sigma^2)$ = exp(7 + 2) = e^9 = 8103.
E[X∧x] = $\exp(\mu + \sigma^2/2)\,\Phi[(\ln x - \mu - \sigma^2)/\sigma] + x\{1 - \Phi[(\ln x - \mu)/\sigma]\}$.
E[X∧2000] = 8103Φ[(ln2000 - 7 - 4)/2] + 2000 {1 - Φ[(ln2000 - 7)/2]} =
8103 Φ(-1.7) + 2000{1 - Φ(0.3)} = (8103)(1 - 0.9554) + (2000)(1 - 0.6179) = 361 + 764 = 1125.
LER(2000) = E[X∧2000] / Mean = 1125 / 8103 = **0.139**.

21.22. C. For the Pareto Distribution, the Loss Elimination Ratio is: $1 - (\theta/(\theta+x))^{\alpha-1}$.
For θ = k and α = 2, LER(x) = 1 - (k/(k+x)) = x / (k+x). Thus LER(2k) = 2k/ 3k = **2/3**.
Comment: If one does not remember the formula for the LER for the Pareto, one can use the formula for the limited expected value and the fact that LER(x) = E[X∧x] / E[X].

21.23. E. For the Exponential Distribution: LER(x) = $1 - e^{-x/\theta}$. For θ = 1000,
LER(500) = $1 - e^{-0.5}$ = 0.3935. In order to double the LER, then (2)(0.3935) = $1 - e^{-x/1000}$.
Thus $e^{-x/1000}$ = 0.214. ⇒ x = -1000 ln(0.213) = **1546**.
Comment: For the Exponential, e(x) = θ, and thus R(x) = e(x) S(x) / mean = $(\theta)(e^{-x/\theta})/\theta = e^{-x/\theta}$.
Thus LER(X) = 1 - R(x) = $1 - e^{-x/\theta}$.

21.24. E. Since there is 1 loss expected per risk per year, the expected amount paid by the insurer is: E[X] - E[X∧k] = $\theta/(\alpha-1) - \{\theta/(\alpha-1)\}\{1 - \theta^{\alpha-1}/(\theta+k)^{\alpha-1}\}$ =
$\{\theta/(\alpha-1)\}\theta^{\alpha-1}/(\theta+k)^{\alpha-1}$ = $\boldsymbol{\theta^{\alpha} / \{(\alpha-1)(\theta + k)^{\alpha-1}\}}$.
Alternately, the losses excess of the limit k are
e(k)S(k) = $\{(k + \theta)/(\alpha-1)\}\,\theta^{\alpha}/(\theta+k)^{\alpha}$ = $\boldsymbol{\theta^{\alpha} / \{(\alpha-1)(\theta + k)^{\alpha-1}\}}$.
Alternately, the losses excess of the limit k are
R(k)E[X] = $\{\theta/(\theta+k)\}^{\alpha-1}\{\theta/(\alpha-1)\}$ = $\boldsymbol{\theta^{\alpha} / \{(\alpha-1)(\theta + k)^{\alpha-1}\}}$.

21.25. E. Using the solution to the prior question, but with 0.8α rather than α, the expected amount of annual losses paid by the insurer for Risk 2 is: $\theta^{0.8\alpha} / \{(0.8\alpha - 1)(\theta+k)^{0.8\alpha-1}\}$.
That for Risk 1 is: $\theta^{\alpha} / \{(\alpha-1)(\theta+k)^{\alpha-1}\}$. The ratio is: $\{(\alpha-1)/(0.8\alpha - 1)\} (\theta + k)^{0.2\alpha} / \theta^{0.2\alpha}$.
As k goes to infinity, this ratio goes to **infinity**.
Comment: The loss distribution of Risk 2 has a heavier tail than Risk 1. The pricing of very large deductibles is very sensitive to the value of the Pareto shape parameter, α.

21.26. B. LER = $E[X \wedge x] / E[X]$. $\Rightarrow$ LER(10,000) / LER (1000) = $E[X \wedge 10{,}000] / E[X \wedge 1000]$.
$E[X \wedge x] = \exp(\mu + \sigma^2/2)\, \Phi[(\ln x - \mu - \sigma^2)/\sigma] + x\, \{1 - \Phi[(\ln x - \mu)/\sigma]\}$.
$E[X \wedge 10{,}000] = e^{8.059}\Phi[0] + 10{,}000\, \{1 - \Phi[1.52]\} = (3162)(0.5) + 10{,}000(1 - 0.9357) = 2224$.
$E[X \wedge 1000] = e^{8.059}\Phi[-1.52] + 1000\, \{1 - \Phi[0]\} = (3162)(0.0643) + 1000(0.5) = 703$.
$E[X \wedge 10{,}000] / E[X \wedge 1000] = 2224 / 703 =$ **3.16**.

21.27. E. F(20) = (20)(0.010) = 0.2. S(20) = 1 - 0.2 = 0.8.
$f(x) = 0.01,\ 0 \le x \le 80$.
From 80 to 120 the graph is linear, and it is 0 at 120 and 0.010 at 80.
$\Rightarrow f(x) = (0.01)(120 - x)/40 = 0.03 - 0.00025x,\ 80 \le x \le 120$.

$$E[X] = \int_0^{80} 0.01x\ dx + \int_{80}^{120} x\,(0.03 - 0.00025x)\ dx =$$

$$0.01x^2/2 \Big]_{x=0}^{x=80} + 0.03x^2/2 \Big]_{x=80}^{x=120} - 0.00025x^3/3 \Big]_{x=80}^{x=120} = 50.67.$$

$$E[X \wedge 20] = \int_0^{20} 0.01x\ dx + 20S(20) = 2 + (20)(0.8) = 18.$$

LER(20) = $E[X \wedge 20]/E[X]$ = 18 / 50.67 = **35.5%**.

21.28. E. $E[X \wedge 2] = 0f(0) + 1f(1) + 2\{1 - f(0) - f(1)\} = 2 - 2f(0) - f(1) = 2 - 2e^{-3} - 3e^{-3} = 2 - 5e^{-3}$.
Loss Elimination Ratio = $E[X \wedge 2]/E[X] = (2 - 5e^{-3})/3 = 0.584$.
We told that, α = 1 - LER = 1 - 0.584 = **0.416**.
Comment: α is set equal to the excess ratio for a deductible of 2.

Section 22, The Effects of Inflation

Inflation is a very important consideration when pricing Health Insurance and Property/Casualty Insurance. Important ideas include the effect of inflation when there is a maximum covered loss and/or deductible, in particular the effect on the average payment per loss and the average payment per payment, the effect on other quantities of interest, and the effect on size of loss distributions. **Memorize the formulas for the average sizes of payment including inflation, discussed in this section**!

On this exam, we deal with the effects of uniform inflation, meaning that a single inflation factor is applied to all sizes of loss.[141] For example, if there is 5% inflation from 1999 to the year 2000, we assume that a loss of size x in 1999 would have been of size 1.05x if it had instead occurred in the year 2000.

Effect of a Maximum Covered Loss:

Exercise: You are given the following:
- For 1999 the amount of a single loss has the following distribution:

Amount	Probability
$500	20%
$1,000	30%
$5,000	25%
$10,000	15%
$25,000	10%

- An insurer pays all losses after applying a $10,000 maximum covered loss to each loss.
- Inflation of 5% impacts all losses uniformly from 1999 to 2000.

Assuming no change in the maximum covered loss, what is the inflationary impact on dollars paid by the insurer in the year 2000 as compared to the dollars the insurer paid in 1999?
[Solution: One computes the average amount paid by the insurer per loss in each year:

Probability	1999 Amount of Loss	1999 Insurer Payment	2000 Amount of Loss	2000 Insurer Payment
0.20	500	500	525	525
0.30	1,000	1,000	1,050	1,050
0.25	5,000	5,000	5,250	5,250
0.15	10,000	10,000	10,500	10,000
0.10	25,000	10,000	26,250	10,000
Average	5650.00	4150.00	5932.50	4232.50

4232.50 / 4150 = 1.020, therefore the insurer’s payments increased 2.0%.]

[141] *Over a few years inflation can often be assumed to be approximately uniform by size of loss. However, over longer periods of time the larger losses often increase at a different rate than the smaller losses.*

Inflation on the limited losses is 2%, less than that of the total losses. Prior to the application of the maximum covered loss, the average loss increased by the overall inflation rate of 5%, from 5650 to 5932.5. In general, **for a fixed limit, limited losses increase more slowly than the overall rate of inflation.**

Effect of a Deductible:

Exercise: You are given the following:
- For 1999 the amount of a single loss has the following distribution:

Amount	Probability
$500	20%
$1,000	30%
$5,000	25%
$10,000	15%
$25,000	10%

- An insurer pays all losses after applying a $1000 deductible to each loss.
- Inflation of 5% impacts all losses uniformly from 1999 to 2000.

Assuming no change in the deductible, what is the inflationary impact on dollars paid by the insurer in the year 2000 as compared to the dollars the insurer paid in 1999?
[Solution: One computes the average amount paid by the insurer per loss in each year:

Probability	1999 Amount of Loss	1999 Insurer Payment	2000 Amount of Loss	2000 Insurer Payment
0.20	500	0	525	0
0.30	1,000	0	1,050	50
0.25	5,000	4,000	5,250	4,250
0.15	10,000	9,000	10,500	9,500
0.10	25,000	24,000	26,250	25,250
Average	5650.00	4750.00	5932.50	5027.50

5027.5 / 4750 = 1.058, therefore the insurer's payments increased 5.8%.]

Inflation on the losses excess of the deductible is 5.8%, greater than that of the total losses. Prior to the application of the deductible, the average loss increased by the overall inflation rate of 5%, from 5650 to 5932.5. In general, **for a fixed deductible, losses paid excess of the deductible increase more quickly than the overall rate of inflation.**

The Loss Elimination Ratio in 1999 is: (5650 - 4750) / 5650 = 15.9%.
The Loss Elimination Ratio in 2000 is: (5932.5 - 5027.5) / 5932.5 = 15.3%.

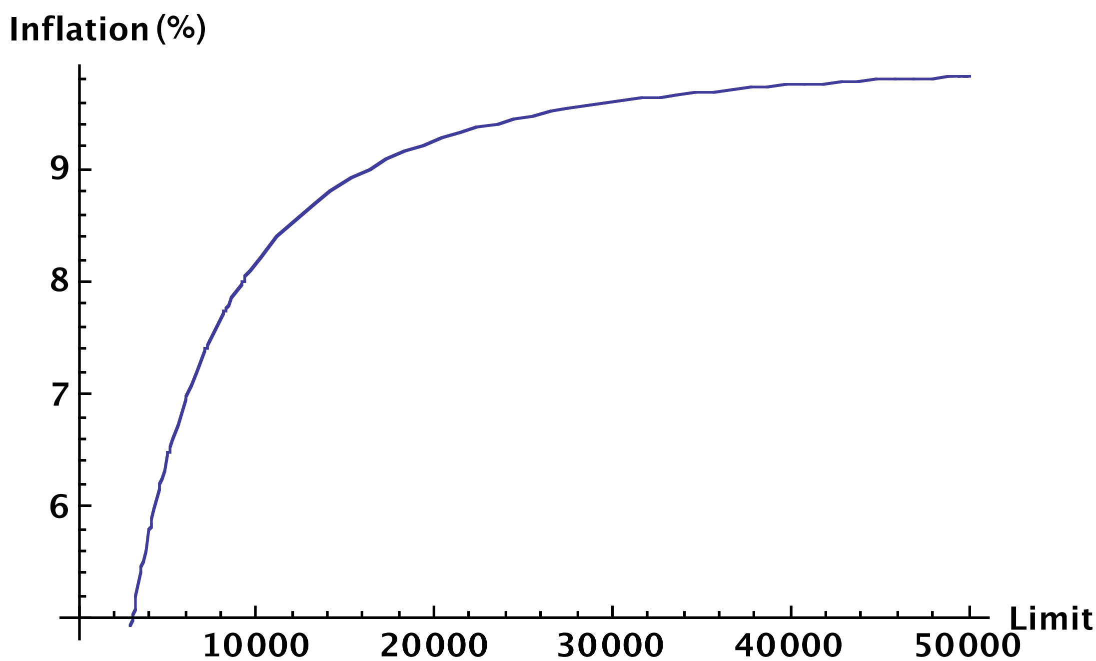
In general, under uniform inflation for a fixed deductible amount the LER declines.
The effect of a fixed deductible decreases over time.

Similarly, under uniform inflation the Excess Ratio over a fixed amount increases.[142]
If a reinsurer were selling reinsurance excess of a fixed limit such as $1 million, then over time the losses paid by the reinsurer would be expected to increase faster than the overall rate of inflation, in some cases much faster.

Limited Losses increase slower than the total losses.
Excess Losses increase faster than total losses.
Limited Losses plus Excess Losses = Total Losses.

Graphical Examples:

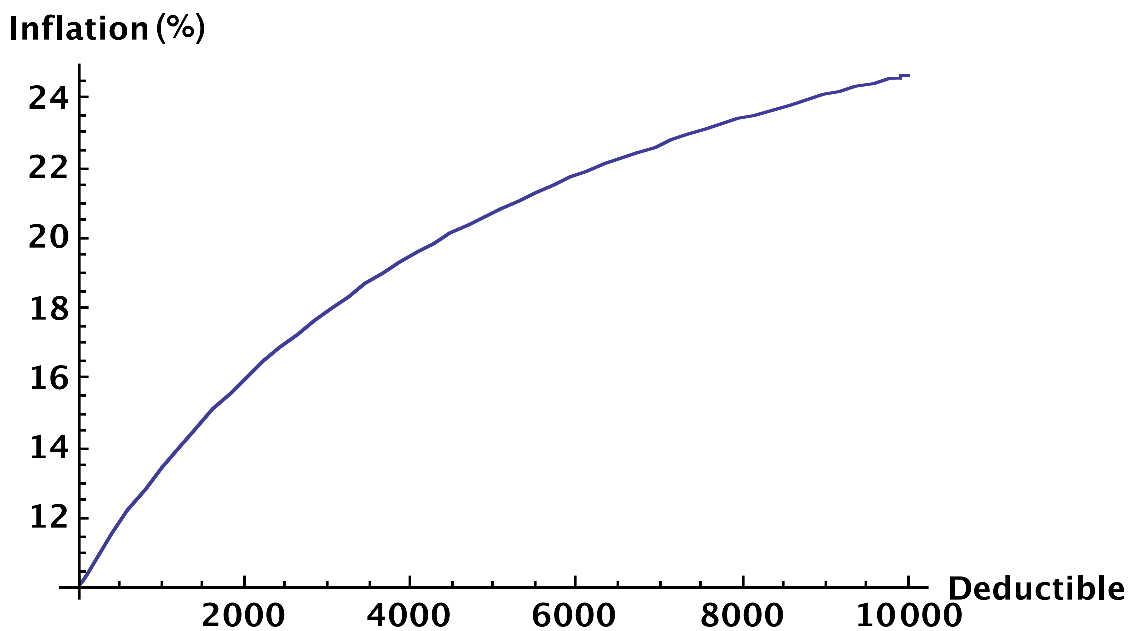
Assume for example that losses follow a Pareto Distribution with α = 3 and θ = 5000 in the earlier year.[143] Assume that there is 10% inflation and the same limit in both years. Then the increase in limited losses as a function of the limit is:



As the limit increases, so does the rate of inflation. For no limit, the rate is 10%.

[142] See 3, 11/00, Q.42.
[143] How to work with the Pareto and other continuous size of loss distributions under uniform inflation will be discussed subsequently.

If instead there were a fixed deductible, then the increase in losses paid excess of the deductible as a function of the deductible is:



For no deductible the rate of inflation is 10%. As the size of the deductible increases, the losses excess of the deductible becomes "more excess", and the rate of inflation increases.

<u>Effect of a Maximum Covered Loss and Deductible, Layers of Loss</u>:

Exercise: You are given the following:

- For 1999 the amount of a single loss has the following distribution:

Amount	Probability
\$500	20%
\$1,000	30%
\$5,000	25%
\$10,000	15%
\$25,000	10%

- An insurer pays all losses after applying a \$10,000 maximum covered loss and then a \$1000 deductible to each loss.
- Inflation of 5% impacts all loss uniformly from 1999 to 2000.

Assuming no change in the deductible or maximum covered loss, what is the inflationary impact on dollars paid by the insurer in the year 2000 as compared to 1999?

[Solution: One computes the average amount paid by the insurer per loss in each year:

Probability	1999 Amount of Loss	1999 Insurer Payment	2000 Amount of Loss	2000 Insurer Payment
0.20	500	0	525	0
0.30	1,000	0	1,050	50
0.25	5,000	4,000	5,250	4,250
0.15	10,000	9,000	10,500	9,000
0.10	25,000	9,000	26,250	9,000
Average	5650.00	3250.00	5932.50	3327.50

3327.5 / 3250 = 1.024, therefore the insurer's payments increased 2.4%.]

In this case, the layer of loss from 1000 to 10,000 increased more slowly than the overall rate of inflation. However, there were two competing effects. The deductible made the rate of increase larger, while the maximum covered loss made the rate of increase smaller. Which effect dominates depends on the particulars of a given situation.

For example, for the ungrouped data in Section 1, the dollars of loss in various layers are as follows for the original data and the revised data after 50% uniform inflation:

LAYER ($ million)			Dollars of Loss ($000)			Ratio
Bottom	Top		Original Data	Revised Data		
0	0.5		24,277	30,174		1.24
0.5	1		6,424	10,239		1.59
1	2		4,743	8,433		1.78
2	3		2,441	4,320		1.77
3	4		1,961	2,661		1.36
4	5		802	2,000		2.49
5	6		0	1,942		infinite
6	7		0	1,000		infinite
7	8		0	203		infinite
0	infinity		40,648	60,972		1.50

We observe that the inflation rate for higher layers is usually higher than the uniform inflation rate, but not always. For the layer from 3 to 4 million dollars the losses increase by 36%, which is less than the overall rate of 50%.

A layer (other than the very first) "gains" dollars as loss sizes increase and are thereby pushed above the bottom of the layer. For example, a loss of size 0.8 million would contribute nothing to the layer from 1 to 2 million prior to inflation, while after 50% inflation it would be of size 1.2 million, and would contribute 0.2 million. In addition, losses which were less than the top of the layer and more than the bottom of the layer, now contribute more dollars to the layer. For example, a loss of size 1.1 million would contribute 0.1 million to the layer from 1 to 2 million prior to inflation, while after 50% inflation it would be of size 1.65 million, and would contribute 0.65 million to this layer. Either of these two types of increases can be very big compared to the dollars that were in the layer prior to the effects of inflation.

On the other hand, a loss whose size was bigger than the top of a given layer, contributes no more to that layer no matter how much it grows. For example, a loss of size 3 million would contribute 1 million to the layer from 1 to 2 million prior to inflation, while after 50% inflation it would be of size 4.5 million, and would still contribute 1 million. A loss of size 3 million has already contributed the width of the layer, and that is all that any single loss can contribute to that layer. So for such losses there is no increase to this layer.

Thus for an empirical sample of losses, how inflation impacts a particular layer depends how the varying effects from the various sizes of losses contribute to the combined effect.

Manners of Expressing the Amount of Inflation:

Uniform Inflation ⇔ Every size of loss increases by a factor of 1+r.

There are a number of different ways to express the amount of inflation:
1. State the total amount of inflation from the earlier year to the later year.
2. Give a constant annual inflation rate.
3. Give the different amounts of inflation during each annual period between the earlier and later year.
4. Give the value of some consumer price index in the earlier and later year.

In all cases, you want to determine the total inflation factor, (1+r), to get from the earlier year to the later year.

For example, from the year 2021 to 2024, inflation might be:

1. A total of 15%; $1 + r = 1.15$.

2. 4% per year; $1 + r = (1.04)^3 = 1.125$.

3. 7% between 2021 and 2022, 4% between 2022 and 2023, and 5% between 2023 and 2024;
 $1 + r = (1.07)(1.04)(1.05) = 1.168$.

4. The CPI (Consumer Price Index) was 327 in 2021 and is 366 in 20214;
 $1 + r = 366/327 = 1.119$.

Moments, etc.:

If one multiplies all of the loss sizes by 1.1, then the mean is also multiplied by 1.1.
$E[1.1X] = 1.1\ E[X]$.

Since each loss is multiplied by the inflation factor, (1+r), so are the Mean, Mode and Median of the distribution.

Any percentile of the distribution is also multiplied by (1+r); in fact this is the definition of inflation uniform by size of loss.

Any quantity in dollars is expected to be multiplied by the inflation factor, 1+r.

If one multiplies all of the loss sizes by 1.1, then the second moment is multiplied by 1.1^2.
$E[(1.1X)^2] = 1.1^2\ E[X^2]$.

$E[\{(1+r)X\}^n] = (1+r)^n\ E[X^n]$.
In general, under uniform inflation the nth moment is multiplied by $(1+r)^n$.

Exercise: In 2023 the mean loss is 100 and the second moment is 50,000. Between 2023 and 2024 there is 5% inflation. What is the variance of the losses in 2024?
[Solution: In 2024, the mean is: (1.05)(100) = 105, and the second moment is: $(1.05^2)(50000)$ = 55,125. Thus in 2024, the variance is: 55,125 - 105^2 = 44,100.]

The variance in 2023 was 50,000 - 100^2 = 40,000. The variance increased by a factor of:
44,100/40,000 = 1.1025 = $1.05^2 = (1+r)^2$.
$Var[(1+r)X] = E[\{(1+r)X\}^2] - E[(1+r)X]^2 = (1+r)^2 E[X^2] - (1+r)^2 E[X]^2 = (1+r)^2 Var[X]$.

Under uniform inflation, the Variance is multiplied by $(1+r)^2$. Any quantity in dollars squared is expected to the multiplied by the square of the inflation factor, $(1+r)^2$.

Since the Variance is multiplied by $(1+r)^2$, the Standard Deviation is multiplied by (1+r).

CV, Skewness, and Kurtosis:

Exercise: In 2023 the mean loss is 100 and the second moment is 50,000. Between 2023 and 2024 there is 5% inflation. What is the coefficient of variation of the losses in 2024?
[Solution: In 2024, the mean is 105, and the standard deviation is $\sqrt{44{,}100}$ = 210.
The Coefficient of Variation is: 210 / 105 = 2.]

In this case, the CV for 2023 is: $\sqrt{40{,}000}$ / 100 = 2. Thus the coefficient of variation remained the same. CV = standard deviation / mean, and in general both the numerator and denominator are multiplied by (1+r), and therefore the CV remains the same.

Skewness = (3rd central moment)/ standard deviation^3.
Both the numerator and denominator are in dollars cubed, and under uniform inflation they are each multiplied $(1+r)^3$. Thus the skewness is unaffected by uniform inflation.

Kurtosis = (4th central moment)/ standard deviation^4.
Both the numerator and denominator are in dollars to the fourth power, and under uniform inflation they are each multiplied $(1+r)^4$. Thus the kurtosis is unaffected by uniform inflation.

The Coefficient of Variation, the Skewness, and the Kurtosis are each unaffected by uniform inflation. Each is a dimensionless quantity, which helps to describe the shape of a distribution and is independent of the scale of the distribution.

Limited Expected Values:

As discussed previously, losses limited by a fixed limit increase slower than the rate of inflation. For example, if the expected value limited to $1 million is $300,000 in the prior year, then after uniform inflation of 10%, the expected value limited to $1 million is less than $330,000 in the later year.

Exercise: You are given the following:
- For 1999 the amount of a single loss has the following distribution:

Amount	Probability
$500	20%
$1,000	30%
$5,000	25%
$10,000	15%
$25,000	10%

- Inflation of 5% impacts all losses uniformly from 1999 to 2000.

An insurer pays all losses after applying a maximum covered loss to each loss.
The maximum covered loss in 1999 is $10,000.
The maximum covered loss in 2000 is $10,500, 5% more than that in 1999.
What is the inflationary impact on dollars paid by the insurer in the year 2000 as compared to the dollars the insurer paid in 1999?
[Solution: One computes the average amount paid by the insurer per loss in each year:

Probability	1999 Amount of Loss	1999 Insurer Payment	2000 Amount of Loss	2000 Insurer Payment
0.20	500	500	525	525
0.30	1,000	1,000	1,050	1,050
0.25	5,000	5,000	5,250	5,250
0.15	10,000	10,000	10,500	10,500
0.10	25,000	10,000	26,250	10,500
Average	5650.00	4150.00	5932.50	4357.50

4357.50 / 4150 = 1.050, therefore the insurer's payments increased 5.0%.]

On exam questions, the maximum covered loss would usually be the same in the two years. In that case, as discussed previously, the insurer's payments would increase at 2%, less than the overall rate of inflation. In this exercise, instead the maximum covered loss was increased in order to keep up with inflation. The result was that the insurer's payments, the limited expected value, increased at the overall rate of inflation.

Provided the limit keeps up with inflation, the Limited Expected Value is multiplied by the inflation factor.[144] If we increase the limit at the rate of inflation, then the Limited Expected Value, which is in dollars, also keeps up with inflation.

[144] As discussed previously, if rather than being increased in order to keep up with inflation the limit is kept fixed, then the limited losses increase slower than the overall rate of inflation.

Exercise: The expected value limited to \$1 million is \$300,000 in the 2007.
There is 10% uniform inflation between 2007 and 2008.
What is the expected value limited to \$1.1 million in 2008?
[Solution: Since the limit kept up with inflation, (\$300,000)(1.1) = \$330,000.]

Proof of the Result for Limited Expected Values:

The Limited Expected Value is affected for two reasons by uniform inflation. Each of the losses entering into its computation is multiplied by (1+r), but in addition the relative effect of the limit has been affected. Due to the combination of these two effects it turns out that if Z = (1+r) X, then $E[Z \wedge u(1+r)] = (1+r)\ E[X \wedge u]$.

In terms of the definition of the Limited Expected Value:

$$E[Z \wedge u(1+r)] = \int_0^{u(1+r)} z\ f_Z(z)\ dz\ + \{S_Z(u(1+r))\}\{u(1+r)\} =$$

$$\int_0^{u} x\ f_X(x)\ dx\ + \{S_X(u)\}\{L(1+u)\} = (1+r)\ E[X \wedge u].$$

Where we have applied the change of variables, z = (1+r) x and thus $F_Z(L(1+r)) = F_X(L)$, and $f_X(x)\ dx = f_Z(z)\ dz$.

We have shown that $E[(1+r)X \wedge u(1+r)] = (1+r)\ E[X \wedge u]$. The left hand side is the Limited Expected Value in the later year, with a limit of u(1+r); we have adjusted u, the limit in the prior year, in order to keep up for inflation via the factor 1+r. This yields the Limited Expected Value in the prior year, except multiplied by the inflation factor to put them in terms of the subsequent year's dollars, which is the right hand side.

<u>Mean Excess Loss</u>:

Exercise: You are given the following:
- For 1999 the amount of a single loss has the following distribution:

Amount	Probability
$500	20%
$1,000	30%
$5,000	25%
$10,000	15%
$25,000	10%

- Inflation of 5% impacts all losses uniformly from 1999 to 2000.

Compute the mean excess loss at $3000 in 1999.
Compute the mean excess loss at $3000 in 2000.
Compute the mean excess loss at $3150 in 2000.
[Solution: In 1999, e(3000) = {(2000)(.25) + (7000)(.15) + (22,000)(.1)}/(.25 + .15 + .1) = 7500.
In 2000, e(3000) = {(5250 - 3000)(.25) + (10,500 - 3000)(.15) + (26,250 - 3000)(.1)}/.5 = 8025.
In 2000, e(3150) = {(5250 - 3150)(.25) + (10,500 - 3150)(.15) + (26,250 - 3150)(.1)}/.5 = 7875.]

In this case, if the limit is increased for inflation, from 3000 to (1.05)(3000) = $3150 in 2000, then the mean excess loss increases by the rate of inflation; (1.05)(7500) = 7875.

The mean excess loss in the later year is multiplied by the inflation factor, provided the limit has been adjusted to keep up with inflation.

Exercise: The mean excess loss beyond $1 million is $3 million in 2007.
There is 10% uniform inflation between 2007 and 2008.
What is the mean excess loss beyond $1.1 million in 2008?
[Solution: Since the limit kept up with inflation, ($3 million)(1.1) = $3.3 million.]

If the limit is fixed, then the behavior of the mean excess loss, depends on the particular size of loss distribution.[145]

<u>Proof of the Result for the Mean Excess Loss:</u>

The Mean Excess Loss or Mean Residual Life at L in the prior year is given by
$e_X(L) = \{E[X] - E[X \wedge L]\} / S(L)$.
Letting Z = (1+r)X, the mean excess loss at L(1+r) in the latter year is given by
$e_Z(L(1+r)) = \{ E[Z] - E[Z \wedge L(1+r)] \} / S_Z(L(1+r)) =$
$\{(1+r)E[X] - (1+r)E[X \wedge L] \} / \{S_X(L)\} = (1+r)e_X(L)$.

[145] As was discussed in a previous section, different distributions have different behaviors of the mean excess loss as a function of the limit.

Loss Elimination Ratio:

As discussed previously, for a fixed deductible, the Loss Elimination Ratio declines under uniform inflation. For example, if the LER(1000) = 13% in the prior year, then after uniform inflation, LER(1000) is less than 13% in the latter year.

Exercise: You are given the following:
- For 1999 the amount of a single loss has the following distribution:

Amount	Probability
$500	20%
$1,000	30%
$5,000	25%
$10,000	15%
$25,000	10%

- Inflation of 5% impacts all losses uniformly from 1999 to 2000.

An insurer pays all losses after applying a deductible to each loss.
The deductible in 1999 is $1000.
The deductible in 2000 is $1050, 5% more than that in 1999.
Compare the loss elimination ratio in the year 2000 to that in the year 1999.
[Solution: One computes the average amount paid by the insurer per loss in each year:

Probability	1999 Amount of Loss	1999 Insurer Payment	2000 Amount of Loss	2000 Insurer Payment
0.20	500	0	525	0
0.30	1,000	0	1,050	0
0.25	5,000	4,000	5,250	4,200
0.15	10,000	9,000	10,500	9,450
0.10	25,000	24,000	26,250	25,200
Average	5650.00	4750.00	5932.50	4987.50

The Loss Elimination Ratio in 1999 is: 1 - 4750/5650 = 15.9%.
The Loss Elimination Ratio in 2000 is: 1 - 4987.5/5932.5 = 15.9%.
Comment: 4987.50 / 4750 = 1.050, therefore the insurer's payments increased 5.0%.]

On exam questions, the deductible would usually be the same in the two years. In that case, as discussed previously, the loss elimination ratio would decrease from 15.9% to 15.3%. In this exercise, instead the deductible was increased in order to keep up with inflation. The result was that the insurer's payments increased at the overall rate of inflation, and the loss elimination ratio stayed the same.

The Loss Elimination Ratio in the later year is unaffected by uniform inflation, provided the deductible has been adjusted to keep up with inflation.[146]

[146] As discussed above, for a fixed deductible the Loss Elimination Ratio decreases under uniform inflation.

Exercise: The Loss Elimination Ratio for a deductible of $1000 is 13% in 2007.
There is 10% uniform inflation between 2007 and 2008.
What is the Loss Elimination ratio for a deductible of $1100 in 2008?
[Solution: Since the deductible keeps up with inflation, the Loss Elimination Ratio is the same in 2008 as in 2007, 13%.]

Since the Excess Ratio is just unity minus the LER, the Excess Ratio in the latter year is unaffected by uniform inflation, provided the limit has been adjusted to keep up with inflation.[147]

Proof of the Result for Loss Elimination Ratios:

The Loss Elimination Ratio at d in the prior year is given by $LER_X(d) = E[X \wedge d] / E[X]$.
Letting Z = (1+r)X, the Loss Elimination Ratio at d(1+r) in the latter year is given by $LER_Z(d(1+r))$ $= E[Z \wedge d(1+r)] / E[Z] = (1+r)E[X \wedge d] / \{(1+r)E[X]\} = E[X \wedge d] / E[X] = LER_X(d)$.

Using Theoretical Loss Distributions:

It would also make sense to use continuous distributions, obtained perhaps from fitting to a data set, in order to estimate the impact of inflation. We could apply a factor of 1+r to every loss in the data set and then fit a distribution to the altered data. In most cases, it would be a waste of time fitting new distributions to the data modified by the uniform effects of inflation. For most size of loss distributions, after uniform inflation one gets the same type of distribution with the scale parameter revised by the inflation factor. For example, for a Pareto Distribution with parameters $\alpha = 1.702$ and $\theta = 240,151$, under uniform inflation of 50% one would get another Pareto Distribution with parameters: $\alpha = 1.702$, $\theta = (1.5)(240,151) = 360,360$.[148]

Behavior of Specific Distributions under Uniform Inflation of (1+r):

For the Pareto, **θ becomes θ(1+r).** The *Burr and Generalized Pareto have the same behavior.* Not coincidentally, for these distributions the mean is proportional to θ. As discussed in a previous section, theta is the scale parameter for these distributions; everywhere x appears in the Distribution Function it is divided by θ. In general under inflation, scale parameters are transformed under inflation by being multiplied by (1+r). For the Pareto the shape parameter α remains the same. *For the Burr the shape parameters α and γ remain the same. For the Generalized Pareto the shape parameters α and τ remain the same.*

Similarly, **for the Gamma, and Weibull**, **θ becomes θ(1+r).** *The Transformed Gamma has the same behavior.* As parameterized in Loss Models, theta is the scale parameter for the Gamma, Weibull, and Transformed Gamma distributions. For the Gamma the shape parameter α remains the same. For the Weibull the shape parameter τ remains the same. *For the Transformed Gamma the shape parameters α and τ remain the same.* Since the Exponential is a special case of the Gamma, **for the Exponential θ becomes θ(1+r),** under uniform inflation of 1+r.

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[147] As discussed above, for a fixed limit the Excess Ratio increases under uniform inflation.
[148] *Prior to inflation, this the Pareto fit by maximum likelihood to the ungrouped data in Section 1.*

Exercise: In 2001 losses follow a Gamma Distribution with parameters $\alpha = 2$ and $\theta = 100$.
There is 10% inflation in total between 2001 and 2004. What is loss distribution in 2004?
[Solution: Gamma with $\alpha = 2$ and $\theta = (1.1)(100) = 110$.]

The behavior of the LogNormal under uniform inflation is explained by noting that multiplying each loss by a factor of (1+r) is the same as adding a constant amount ln(1+r) to the log of each loss. Adding a constant amount to a Normal distribution, gives another Normal Distribution, with the same variance but with the mean shifted. $\mu' = \mu + \ln(1+r)$, and $\sigma' = \sigma$.
$X \sim \text{LogNormal}(\mu, \sigma)$. $\Leftrightarrow \ln(X) \sim \text{Normal}(\mu, \sigma)$. $\Rightarrow$
$\ln[(1+r)X] = \ln(X) + \ln(1+r) \sim \text{Normal}(\mu, \sigma) + \ln(1+r) = \text{Normal}(\mu + \ln(1+r), \sigma)$.
$\Leftrightarrow (1+r)X \sim \text{LogNormal}(\mu + \ln(1+r), \sigma)$. Thus **under uniform inflation for the LogNormal, μ becomes μ + ln(1+r)**. The other parameter, σ, remains the same.

The behavior of the LogNormal under uniform inflation can also be explained by the fact that e^{μ} is the scale parameter and σ is a shape parameter. Therefore, e^{μ} is multiplied by (1+r); e^{μ} becomes $e^{\mu}(1+r) = e^{\mu + \ln(1+r)}$. Therefore, μ becomes μ + ln(1+r).

Exercise: In 2001 losses follow a LogNormal Distribution with parameters $\mu = 5$ and $\sigma = 2$.
There is 10% inflation in total between 2001 and 2004. What is loss distribution in 2004?
[Solution: LogNormal with $\mu = 5 + \ln(1.1) = 5.095$, and $\sigma = 2$.]

Note that in each case, the behavior of the parameters under uniform inflation depends on the particular way in which the distribution is parameterized. For example, in Loss Models the Exponential distribution is given as: $F(x) = 1 - e^{-x/\theta}$. Thus in this parameterization of the Exponential, θ acts as a scale parameter, and under uniform inflation θ becomes θ(1+r). This contrasts with the parameterization of the Exponential in Life Contingencies, $F(x) = 1 - e^{-\lambda x}$, where 1/λ acts as a scale parameter, and under uniform inflation λ becomes λ/(1+r). $\theta \Leftrightarrow 1/\lambda$.

Conveniently, most of the distributions in Loss Models have a scale parameter, which is multiplied by (1+r), while the shape parameters are unaffected. Exceptions are the LogNormal Distribution and the Inverse Gaussian.[149]

Note that all of the members of "Transformed Beta Family" all act similarly.[150] The scale parameter θ is multiplied by the inflation factor and all of the shape parameters remain the same. All of the members of the "Transformed Gamma Family" all act similarly.[151] The scale parameter θ is multiplied by the inflation factor and all of the shape parameters remain the same.

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[149] This is discussed along with the behavior under uniform inflation of the LogNormal and Inverse Gaussian, in Appendix A of Loss Models. However it is not included in the Tables attached to the exam.
[150] See Figures 5.2 and 5.4, and Appendix A of Loss Models.
[151] See Figure 5.3 and Appendix A of Loss Models

Distribution	Parameters Prior to Inflation				Parameters After Inflation			
Pareto	α	θ			α	θ(1+r)		
Generalized Pareto	α	θ	τ		α	θ(1+r)	τ	
Burr	α	θ	γ		α	θ(1+r)	γ	
Inverse Burr	τ	θ	γ		τ	θ(1+r)	γ	
Transformed Beta	α	θ	γ	τ	α	θ(1+r)	γ	τ
Inverse Pareto	τ	θ			τ	θ(1+r)		
Loglogistic	γ	θ			γ	θ(1+r)		
Paralogistic	α	θ			α	θ(1+r)		
Inverse Paralogistic	τ	θ			τ	θ(1+r)		
Exponential		θ				θ(1+r)		
Gamma	α	θ			α	θ(1+r)		
Weibull		θ	τ			θ(1+r)	τ	
Inverse Gamma	α	θ			α	θ(1+r)		
Inverse Weibull		θ	τ			θ(1+r)	τ	
Trans. Gamma	α	θ	τ		α	θ(1+r)	τ	
Inverse Trans. Gamma	α	θ	τ		α	θ(1+r)	τ	
Normal		μ	σ			μ(1+r)	σ(1+r)	
LogNormal		μ	σ			μ + ln(1+r)	σ	
Inverse Gaussian		μ	θ			μ(1+r)	θ(1+r)	
Single Par. Pareto	α	θ			α	θ(1+r)		
Uniform Distribution	a	b				a(1+r)	b(1+r)	
Beta Distribution	a	b	θ		a	b	θ(1+r)	
Generalized Beta Distribution	a	b	θ	τ	a	b	θ(1+r)	τ

Note that all the distributions in the above table are preserved under uniform inflation.

After uniform inflation, we get the same type of distributions, but some or all of the parameters have changed. If X follows a type of distribution implies that cX, for any c > 0, also follows the same type of distribution, then that is defined as a **scale family**.

So for example, the Inverse Gaussian is a scale family of distributions, even though it does not have scale parameter. If X follows an Inverse Gaussian, then Y = cX also follows an Inverse Gaussian. Any distribution with a scale parameter is a scale family.

In order to compute the effects of uniform inflation on a loss distribution, one can adjust the parameters as in the above table. Then one can work with the loss distribution revised by inflation in the same manner one would work with any loss distribution.

Exercise: Losses prior to inflation follow a Pareto Distribution with parameters α = 1.702 and θ = 240,151. Losses increase uniformly by 50%.
What are the means prior to and subsequent to inflation?
[Solution: For the Pareto Distribution, E[X] = $\theta/(\alpha-1)$.
Prior to inflation, E[X] = 240,151 / 0.702 = 342,095.
After inflation, with parameters α = 1.702 and θ = (1.5)(240,151) = 360,227.
After inflation, E[X] = 360,227 / 0.702 = 513,143.
Alternately, inflation increases the mean by 50% to: (1.5)(342,095) = 513,143.]

Exercise: Losses prior to inflation follow a Pareto Distribution with parameters α = 1.702 and θ = 240,151. Losses increase uniformly by 50%.
What are the limited expected values at 1 million prior to and subsequent to inflation?
[Solution: For the Pareto Distribution, $E[X \wedge x] = \frac{\theta}{\alpha - 1} \left\{1 - \left(\frac{\theta}{\theta + x}\right)^{\alpha - 1}\right\}$.

Prior to inflation, E[X$\wedge$1 million] = (240,151 / 0.702) {1 - (240,151/1,240,151)$^{0.702}$} = 234,044.
After inflation, E[X$\wedge$1 million] = (360,227 / 0.702) {1 - (360,227/1,360,227)$^{0.702}$} = 311,232.]

Exercise: Losses prior to inflation follow a Pareto Distribution with parameters α = 1.702 and θ = 240,151. Losses increase uniformly by 50%. Excess Ratio = 1 - LER.
What are the excess ratios at 1 million prior to and subsequent to inflation?
[Solution: Excess ratio = R(x) = (E[X] - E[X $\wedge$ x]) / E[X] = 1 - E[X $\wedge$ x] / E[X].
Prior to inflation, R(1 million) = 1 - 234,044 / 342,095 = 31.6%.
After inflation, R(1 million) = 1 - 311,232 / 513,143 = 39.3%.
<u>Comment</u>: As expected, for a fixed limit the Excess Ratio increases under uniform inflation.
For the Pareto the excess ratio is given by $R(x) = \left(\frac{\theta}{\theta + x}\right)^{\alpha - 1}$.]

Behavior in General of Distributions under Uniform Inflation of (1+r):

For distributions in general, including those not discussed in Loss Models, one can determine the behavior under uniform inflation as follows. One makes the change of variables Z = (1+r) X. For the Distribution Function one just sets $F_Z(z) = F_X(x)$; one substitutes x = z / (1+r). Alternately, for the density function $f_Z(z) = f_X(x) / (1+r)$.[152]

For example, for the Normal Distribution $f(x) = \frac{\exp[-\frac{(x-\mu)^2}{2\sigma^2}]}{\sigma\sqrt{2\pi}}$. Under uniform inflation, x = z/(1+r)

and $f_Z(z) = f_X(x) / (1+r) = \frac{\exp[-\frac{(z/(1+r)-\mu)^2}{2\sigma^2}]}{(1+r)\sigma\sqrt{2\pi}} = \frac{\exp[-\frac{\{z-\mu(1+r)\}^2}{2\{\sigma(1+r)\}^2}]}{(1+r)\sigma\sqrt{2\pi}}$.

This is a Normal density function with sigma and mu each multiplied by (1+r). Thus under inflation, for the Normal m becomes $\mu(1+r)$ and σ becomes $\sigma(1+r)$. The location parameter μ has been multiplied by the inflation factor, as has the scale parameter σ.

Alternately, the Distribution Function for the Normal is $\Phi[(x-\mu)/\sigma]$.
Therefore, $F_Z(z) = F_X(x) = \Phi[(x-\mu)/\sigma] = \Phi[(\{z/(1+r)\}-\mu)/\sigma] = \Phi[\{z- \mu(1+r)\}/\{\sigma(1+r)\}]$.
This is the Distribution Function for a Normal with sigma and mu each multiplied by (1+r), which matches the previous result.

Exercise: What is the behavior under inflation of the distribution function:
$F(x) = x^a / (x^a + b^a)$, x > 0.
[Solution: Under uniform inflation, $F_Z(z) = F_X(x) = x^a / (x^a + b^a) = \{z/(1+r)\}^a / (\{z/(1+r)\}^a + b^a) =$
$z^a/(z^a + \{b(1+r)\}^a)$. This is the same type of distribution, where b has become b (1+r).
The scale parameter b has been multiplied by the inflation factor (1+r).
Alternately, one can work with the density function $f(x) = a\, b^a x^{a-1} / (x^a + b^a)^2 =$
$(a/b)(x/b)^{a-1} / (1+ (x/b)^a)^2$. Then under uniform inflation: x = z/(1+r) and $f_Z(z) = f_X(x) / (1+r) =$
$(a/b)(x/b)^{a-1} / \{(1+r)(1+ (x/b)^a)^2\} = (a / \{b (1+r)\})(z / \{b (1+r)\})^{a-1} / \{ (1+ (z / \{b (1+r)\})^a)^2\}$,
which is same type of density, where b has become b(1+r), as was shown previously.
Alternately, you can recognize b is a scale parameter, since $F(x) = (x/b)^a / \{(x/b)^a + 1\}$.
Or alternately, you can recognize that this is Loglogistic Distribution with a = γ and b = θ.]

[152] Under change of variables you need to divide by dz/dx = 1+ r,
since $f_Z(z) = dF/dz = (dF/dx) / (dz/dx) = f_X(x) / (1+r)$.

Exercise: What is the behavior under uniform inflation of the density function:

$$f(x) = \sqrt{\frac{\theta}{2\pi}}\ \frac{\exp\left[-\frac{\theta\left(\frac{x}{\mu}-1\right)^2}{2x}\right]}{x^{1.5}}.$$

[Solution: In general one substitutes for x = z / (1+r),
and for the density function $f_Z(z) = f_X(x) / (1+r)$.

$$f_Z(z) = f_X(x) / (1+r) = \sqrt{\frac{\theta}{2\pi}}\ \frac{\exp\left[-\frac{\theta\left(\frac{z}{\mu(1+r)}-1\right)^2}{2\{z/(1+r)\}}\right]}{(1+r)\ \{z/(1+r)\}^{1.5}} = \sqrt{\frac{\theta(1+r)}{2\pi}}\ \frac{\exp\left[-\frac{\theta(1+r)\left(\frac{z}{\mu(1+r)}-1\right)^2}{2z}\right]}{z^{1.5}}.$$

This is of the same form, but with parameters (1+r)μ and (1+r)θ, rather than μ and θ.]

Thus we have shown that under uniform inflation for the Inverse Gaussian Distribution μ and θ become (1+r)μ and (1+r)θ.

Behavior of the Domain of Distributions:

For all of the distributions discussed so far, the domain has been 0 to ∞.
For the Single Parameter Pareto distribution the domain is x > θ.
Under uniform inflation the domain becomes x > (1+r)θ.

In general, the domain [a, b] becomes under uniform inflation [(1+r)a, (1+r)b].
If a = 0, multiplying by 1+r has no effect; if b = ∞, multiplying by 1+r has no effect.
So for distributions like the Gamma, the domain remains (0, ∞) after uniform inflation.

For the **Single Parameter Pareto** $F(x) = 1 - (x / \theta)^{-\alpha}$, x > θ,
under uniform inflation **α is unaffected and θ becomes (1+r)θ**.

The uniform distribution on [a , b] becomes under uniform inflation the uniform distribution on [a(1+r) , b(1+r)].

Working in Either the Earlier or Later Year:

Exercise: Losses prior to inflation follow a Pareto Distribution with parameters α = 1.702 and θ = 240,151. Losses increase uniformly by 50%. What is the average contribution per loss to the layer from 1 million to 5 million, both prior to and subsequent to inflation?

[Solution: For the Pareto Distribution, $E[X \wedge x] = \frac{\theta}{\alpha - 1} \left\{1 - \left(\frac{\theta}{\theta + x}\right)^{\alpha - 1}\right\}$.

Prior to inflation, $E[X \wedge 5 \text{ million}] = (240{,}151 / 0.702)\{1 - (240{,}151/5{,}240{,}151)^{0.702}\} = 302{,}807$.
After inflation, losses follow a Pareto with α = 1.702 and θ = (1.5)(240,151) = 360,227, and $E[X \wedge 5 \text{ million}] = (360{,}227 / 0.702)\{1 - (360{,}227/5{,}360{,}227)^{0.702}\} = 436{,}041$.
Prior to inflation, the average loss contributes: $E[X \wedge 5 \text{ million}] - E[X \wedge 1 \text{ million}]$ = 302,807 - 234,045 = 68,762, to this layer.
After inflation, the average loss contributes: 436,041 - 311,232 = 124,809, to this layer.]

The contribution to this layer has increased by 82%, in this case more than the overall rate of inflation.

There are two alternative ways to solve many problems involving inflation. In the above solution, one adjusts the size of loss distribution in the earlier year to the later year based on the amount of inflation. Then one calculates the quantity of interest in the later year. However, there is an alternative, which many people will prefer. Instead one calculates the quantity of interest in the earlier year at its deflated value, and then adjusts it to the later year for the effects of inflation. Here's how this alternate method works for this example.

A limit of 1 million in the later year corresponds to a limit of 1 million /1.5 = 666,667 in the earlier year. Similarly, a limit of 5 million in the later year corresponds to 5 million /1.5 = 3,333,333 in the earlier year. Using the Pareto in the earlier year, with α = 1.702 and θ = 240,151,
$E[X \wedge 666{,}667] = (240{,}151 / 0.702)\{1 - (240{,}151/906{,}818)^{0.702}\} = 207{,}488$, and
$E[X \wedge 3{,}333{,}333] = (240{,}151 / 0.702)\{1 - (240{,}151/3{,}573{,}484)^{0.702}\} = 290{,}694$. In terms of the earlier year dollars, the contribution to the layer is: 290,694 - 207,488 = 83,206.
However, one has to inflate back up to the level of the later year: (1.5)(83,206) = 124,809, matching the previous solution.

This type of question can also be answered using the formula discussed subsequently for the average payment per loss. This formula for the average payment per loss is just an application of the technique of working in the earlier year, by deflating limits and deductibles. However, this technique of working in the earlier year is more general, and also applies to other quantities of interest, such as the survival function.

Exercise: Losses in 2003 follow a LogNormal Distribution with parameters $\mu = 3$ and $\sigma = 5$.
Between 2003 and 2009 there is a total of 35% inflation.
Determine the percentage of the total number of losses in 2009 that would be expected to exceed a deductible of 1000.
[Solution: The losses in year 2009 follow a LogNormal Distribution with parameters
$\mu = 3 + \ln(1.35) = 3.300$ and $\sigma = 5$. Thus in 2009, S(1000) = 1 - F(1000) =
$1 - \Phi[\{\ln(1000) - 3.300\} / 5] = 1 - \Phi[0.72] = 1 - 0.7642 = 0.2358$.
Alternately, we first deflate to 2003. A deductible of 1000 in 2009 is equivalent to a deductible of 1000/1.35 = 740.74 in 2003. The losses in 2003 follow a LogNormal Distribution with parameters $\mu = 3$ and $\sigma = 5$. Thus in 2003, S(740.74) = 1 - F(740.47) =
$1 - \Phi[\{\ln(740.47) - 3\}/ 5] = 1 - \Phi[0.72] = 1 - 0.7642 = 0.2358$.]

Of course both methods of solution produce the same answer. One can work either in terms of 2003 or 2009 dollars. In this case, the survival function is a dimensionless quantity.
However, when working with quantities in dollars, such as the limited expected value, if one works in the earlier year, in this case 2003, one has to remember to reinflate the final answer back to the later year, in this case 2009.

<u>Formulas for Average Payments</u>:

The ideas discussed above can be put in terms of formulas:[153]

Given uniform inflation, with **inflation factor of 1+r, Deductible Amount d, Maximum Covered Loss u, and coinsurance factor c, then in terms of the values in the earlier year, the insurer's average payment per <u>loss</u> in the later year is:**

$$(1+r)\ c\ \{E[X \wedge \frac{u}{1+r}] - E[X \wedge \frac{d}{1+r}]\}.$$

Given uniform inflation, with **inflation factor of 1+r, Deductible Amount d, Maximum Covered Loss u, and coinsurance factor c, then in terms of the values in the earlier year, the average payment per (non-zero) <u>payment</u> by the insurer in the later year is:**

$$(1+r)\ c\ \frac{E[X \wedge \frac{u}{1+r}] - E[X \wedge \frac{d}{1+r}]}{S\left(\frac{d}{1+r}\right)}.$$

In each case we have deflated the Maximum Covered Loss and the Deductible back to the earlier year, computed the average payment in the earlier year, and then reinflated back to the later year.
Important special cases are: d = 0 ⇔ no deductible, L = ∞ ⇔ no maximum covered loss,
c = 1 ⇔ no coinsurance, r = 0 ⇔ no inflation or prior to the effects of inflation.

For example, assume losses in 2001 follow an Exponential distribution with θ = 1000.
There is a total of 10% inflation between 2001 and 2004. In 2004 there is a deductible of 500, a maximum covered loss of 5000, and a coinsurance factor of 80%. Then the average payment per (non-zero) payment in 2004 is computed as follows, using that for the Exponential Distribution, $E[X \wedge x] = \theta(1 - e^{-x/\theta})$.

Take d = 500, u = 5000, c = 0.8, and r = 0.1.
average payment per (non-zero) payment in 2004 =
(1+r) c (E[X ∧ u/(1+r)] - E[X ∧ d/(1+r)]) / S(d/(1+r)) =
(1.1)(0.8)(E[X ∧ 4545] - E[X ∧ 455]) / S(455) =
$(0.88)\{1000(1 - e^{-4545/1000}) - 1000(1 - e^{-455/1000})\}/e^{-455/1000}$ = (0.88)(989 - 366)/0.634 = 865.
Note that all computations use the original Exponential Distribution in 2001, with θ = 1000.

[153] See Theorem 8.7 in <u>Loss Models</u>.

Exercise: For a LogNormal Distribution with parameters $\mu = 3$ and $\sigma = 5$, determine
E[X∧ 100,000], E[X∧ 1,000,000], E[X∧ 74,074], and E[X∧ 740,740].
[Solution: E[X∧ 100,000] = $\exp(\mu + \sigma^2/2)\ \Phi[(\ln x - \mu - \sigma^2)/\sigma] + x\ \{1 - \Phi[(\ln x - \mu)/\sigma]\}$ =
exp(3 + 25/2)Φ[(ln(100,000) - 3 - 25)/5] + (100,000){1 - Φ[{ln(100,000) - 3} / 5]} =
5,389,670Φ[-3.30] + (740,740){1 - Φ[1.70]} = 5,389,670(0.0005) + (100,000)(1 - 0.9554) = 7155.
E[X∧ 1,000,000] =
exp(3 + 25/2)Φ[(ln(1,000,000) - 3 - 25)/5] + (1,000,000) {1 - Φ[{ln(1,000,000) - 3} / 5]} =
5,389,670Φ[-2.84] + (1,000,000){1 - Φ[2.16]} = 5,389,670(0.0023) + (1,000,000)(1 - 0.9846) =
27,796.
E[X∧ 74,074] = exp(3 + 25/2)Φ[(ln(74,074) - 3 - 25)/5] + (74,074){1 - Φ[{ln(74,074) - 3} / 5]) =
5,389,670Φ[-3.36] + (74,074){1 - Φ[1.64]} = 5,389,670(0.0004) + (74,074)(1 - 0.9495) = 5897.
E[X∧ 740,740] = exp(3 + 25/2)Φ[(ln(740,740) - 3 - 25)/5] + (740,740){1 - Φ[{ln(740,740) - 3}/ 5]}
= 5,389,670Φ[-2.90] + (740,740){1 - Φ[2.10]} = 5,389,670(0.0019) + (740,740)(1 - 0.9821) =
23,500.]

Exercise: Losses in 2003 follow a LogNormal Distribution with parameters $\mu = 3$ and $\sigma = 5$.
Between 2003 and 2009 there is a total of 35% inflation.
In 2009 there is a deductible of $100,000 and a maximum covered loss of $1 million. Determine the increase between 2003 and 2009 in the insurer's average payment per <u>loss</u> to the insured.
[Solution: In 2003, take r = 0, d = 100,000, u = 1 million, and c = 1.
Average payment per loss = E[X∧1 million] - E[X∧100,000] = 27,796 - 7155 = 20,641.
In 2009, take r = 0.35, d = 100,000, u = 1 million, and c = 1.
Average payment per loss = 1.35 (E[X∧1 million/1.35] - E[X∧100,000/1.35]) =
1.35 (E[X∧ 740,740] - E[X∧ 74,074]) = 1.35 (23,500 - 5897) = 23,764.
The increase is: 23,764/20,641 - 1 = 15%.
<u>Comment</u>: Using a computer, the exact answer without rounding is: 23,554/20,481 - 1 = 15.0%.
Using the formula in order to get the average payment per loss in 2009 is equivalent to deflating to 2003, working in the year 2003, and then reinflating to the year 2009. The 2009 LogNormal has parameters $\mu = 3 + \ln(1.35) = 3.300$ and $\sigma = 5$. For this LogNormal, E[X∧ 100,000] =
exp(3.3 + 25/2)Φ[(ln(100,000) - 3.3 - 25)/5] + (100,000){1 - Φ[{ln(100,000) - 3.3} / 5]} =
7,275,332Φ[-3.36] + (100,000){1 - Φ[1.64]} = 7,275,332(0.0004) + (100,000)(1 - 0.9495) = 7960.
For this LogNormal, E[X∧ 1,000,000] =
exp(3.3 + 25/2)Φ[(ln(1,000,000) - 3.3 - 25)/5] + (1,000,000){1 - Φ[{ln(1,000,000) - 3.3} / 5]} =
7,275,332Φ[-2.90] + (1,000,000){1 - Φ[2.10]} = 7,275,332(0.0019) + (1,000,000)(1 - 0.9821) =
31,723. 31,723 - 7960 = 23,763, matching the 23,764 obtained above except for rounding.]

Exercise: Losses in 2003 follow a LogNormal Distribution with parameters $\mu = 3$ and $\sigma = 5$.
Between 2003 and 2009 there is a total of 35% inflation.
In 2009 there is a deductible of \$100,000 and a maximum covered loss of \$1 million.
Determine the increase between 2003 and 2009 in the insurer's average payment per (non-zero) payment to the insured.
[Solution: In 2003, take r = 0, d = 100,000, u = 1 million, and c = 1.
Average payment per non-zero payment = (E[X∧1 million] - E[X∧100,000])/S(100,000) =
(27,796 - 7155) / {1 - Φ[{ln(100,000) - 3} / 5]} = 20,641 / {1 - Φ[1.70]} = 20,641/0.0446 = 462,803.
In 2009, take r = 0.35, d = 100,000, d = 1 million, and c = 1.
Average payment per non-zero payment =
1.35 (E[X∧1 million/1.35] - E[X∧100,000/1.35])/S(100,000/1.35) =
1.35 (E[X∧ 740,740] - E[X∧ 74,074])/S(74,074) = 1.35 (23,500 - 5897)/{1 - Φ[1.64]} =
23,764/0.0505 = 470,574. The increase is: 470,574/462,803 - 1 = 1.7%.
Comment: Using a computer, the exact answer without rounding is:
468,852/462,085 - 1 = 1.5%.]

Formulas for Second Moments:[154]

We have previously discussed second moments of layers. We can incorporate inflation in a manner similar to the formulas for first moments. However, since the second moment is in dollars squared, we reinflate back by multiplying by $(1+r)^2$. Also we multiply by the coinsurance factor squared.

Given uniform inflation, with **inflation factor of 1+r, Deductible Amount d, Maximum Covered Loss u, and coinsurance factor c, then in terms of the values in the earlier year, the second moment of the insurer's payment per loss in the later year is:**

$$(1+r)^2 c^2 \left\{E[(X \wedge \frac{u}{1+r})^2] - E[(X \wedge \frac{d}{1+r})^2] - 2\frac{d}{1+r}\left(E[X \wedge \frac{u}{1+r}] - E[X \wedge \frac{d}{1+r}]\right)\right\}.$$

Given uniform inflation, with **inflation factor of 1+r, Deductible Amount d, Maximum Covered Loss u, and coinsurance factor c, then in terms of the values in the earlier year, the second moment of the insurer's payment by the insurer in the later year is:**

$$(1+r)^2 c^2 \frac{E[(X \wedge \frac{u}{1+r})^2] - E[(X \wedge \frac{d}{1+r})^2] - 2\frac{d}{1+r}\{E[X \wedge \frac{u}{1+r}] - E[X \wedge \frac{d}{1+r}]\}}{S\left(\frac{d}{1+r}\right)}.$$

One can combine the formulas for the first and second moments in order to calculate the variance.

[154] See Theorem 8.8 in Loss Models. If r = 0, these reduce to formulas perviously discussed.

Exercise: Losses in 2005 follow a Single Parameter Pareto Distribution with $\alpha = 3$ and $\theta = 200$.
Between 2005 and 2010 there is a total of 25% inflation.
In 2010 there is a deductible of 300, a maximum covered loss of 900, and a coinsurance of 90%. In 2010, determine the variance of Y^P, the per payment variable.
[Solution: From the Tables attached to the exam, for the Single Parameter Pareto, for $x \geq \theta$:

$$E[X \wedge x] = \frac{\alpha\, \theta}{\alpha - 1} - \frac{\theta^\alpha}{(\alpha - 1)\, x^{\alpha-1}}.$$

$$E[(X \wedge x)^2] = \frac{\alpha\, \theta^2}{\alpha - 2} - \frac{2\, \theta^\alpha}{(\alpha - 2)\, x^{\alpha-2}}.$$

Thus $[X \wedge 300/1.25] = [X \wedge 240] = (3)(200) / 2 - 200^3 / \{(2)\ (240^2)\} = 230.556$.

$E[X \wedge 900/1.25] = E[X \wedge 720] = (3)(200) / 2 - 200^3 / \{(2)\ (720^2)\} = 292.284$.

$S(300/1.25) = S(240) = (200/240)^3 = 0.5787$.
Thus the mean payment per payment is: (1.25) (90%) (292.284 - 230.556) / 0.5787 = 120.00.
$E[(X \wedge 240)^2] = (3)(200^2) / 1 - (2)(200^3) / 240 = 53,333$.

$E[(X \wedge 720)^2] = (3)(200^2) / 1 - (2)(200^3) / 720 = 97,778$.

Since the second moment is in dollars squared, we multiply by the square of the coinsurance factor, and the square of the inflation factor.
Thus the second moment of the non-zero payments is:
$(1.25^2)\ (90\%)^2\ \{97,778 - 53,333 - (2)(240)(292.284 - 230.556)\} / 0.5787 = 32,402$.
Thus the variance of the non-zero payments is: $32,402 - 120.00^2 = 18,002$.
Alternately, work with the 2010 Single Parameter Pareto with $\alpha = 3$, and $\theta = (200)(1.25) = 250$.
$E[X \wedge 300] = (3)(250) / 2 - 250^3 / \{(2)\ (300^2)\} = 288.194$.

$E[X \wedge 900] = (3)(250) / 2 - 250^3 / \{(2)\ (900^2)\} = 365.355$.

$S(300) = (250/300)^3 = 0.5787$.
Thus the mean payment per payment is: (90%) (365.355 - 288.194) / 0.5787 = 120.00.
$E[(X \wedge 300)^2] = (3)(250^2) / 1 - (2)(250^3) / 300 = 83,333$.

$E[(X \wedge 900)^2] = (3)(250^2) / 1 - (2)(250^3) / 900 = 152,778$.

Since the second moment is in dollars squared, we multiply by the square of the coinsurance factor. Thus the second moment of the non-zero payments is:
$(90\%)^2\ \{152,778 - 83,333 - (2)(300)(365.355 - 288.194)\} / 0.5787 = 32,401$.
Thus the variance of the non-zero payments is: $32,401 - 120.00^2 = 18,001$.]

Mixed Distributions:

If one has a mixed distribution, then under uniform inflation each of the component distributions acts as it would under uniform inflation.

Exercise: The size of loss distribution is: $F(x) = 0.7\{1 - e^{-x/130}\} + 0.3\{1 - (250/(250+x))^2\}$.
After uniform inflation of 20%, what is the size of loss distribution?
[Solution: After uniform inflation of 20%, we get another Exponential Distribution, but with $\theta = (1.2)(130) = 156$: $1 - e^{-x/156}$. After uniform inflation of 20%, we get another Pareto Distribution, but with $\alpha = 2$ and $\theta = (1.2)(250) = 300$: $1 - \{300/(300+x)\}^2$.
Therefore, the mixed distribution becomes: $0.7\{1 - e^{-x/156}\} + 0.3\{1 - (300/(300+x))^2\}$.]

Non-Uniform Rates of Inflation by Size of Loss:

On the exam, inflation is assumed to be uniform by size of loss. What would one expect to see if for example large losses were inflating at a higher rate than smaller losses? Then we would expect for example the 90th percentile to increase at a faster rate than the median.[155]

Exercise: In 2001 the losses follow a Pareto Distribution with parameters $\alpha = 3$ and $\theta = 1000$.
In 2004 the losses follow a Pareto Distribution with parameters $\alpha = 2.5$ and $\theta = 1100$.
What is the increase from 2001 to 2004 in the median (50th percentile)?
Also, what is the increase from 2001 to 2004 in the 90th percentile?
[Solution: For the Pareto, at the 90th percentile: $0.9 = 1 - \{\theta/(\theta + x)\}^{\alpha}$. $\Rightarrow x = \theta\{10^{1/\alpha} - 1\}$.
In 2001 the 90th percentile is: $1000\{10^{1/3} - 1\} = 1154$.
In 2004 the 90th percentile is: $1100\{10^{1/2.5} - 1\} = 1663$.
For the Pareto, the median is: $\theta\{2^{1/\alpha} - 1\}$.
In 2001 the median is: $1000\{2^{1/3} - 1\} = 260$.
In 2004 the median is: $1100\{2^{1/2.5} - 1\} = 351$.
The median increased by: (351/260) - 1 = 35.0%,
while the 90th percentile increased by: (1663/1154) - 1 = 44.1%.]

In this case, the 90th percentile increased more than the median did. The shape parameter of the Pareto decreased, resulting in a heavier-tailed distribution in 2004 than in 2001. If the higher percentiles increase at a different rate than the lower percentiles, then inflation is not uniform by size of loss. When inflation is uniform by size of loss, all percentiles increase at the same rate.[156]

[155] If the larger losses are inflating at a lower rate than the smaller losses then the situation is reversed and the higher percentiles will inflate more slowly than the lower percentiles. *Which situation applies may be determined by graphing selected percentiles over time, with the size of loss on a log scale. In practical applications this analysis would be complicated by taking into account the impacts of any deductible and/or maximum covered loss.*
[156] In practical situations, the estimated rates of increase in different percentiles based on data will differ somewhat, even if the underlying inflation is uniform by size of loss.

Over the past few decades, the median wage in the United States has been has been increasing much more slowly than the 90th percentile wage. Therefore, the distribution of wages has been changing shape; this is not an example of uniform inflation.

Fixed Exchange Rates of Currency:[157]

Finally it is useful to note that the mathematics of changes in currency is the same as that for inflation. Thus if loss sizes are expressed in dollars and you wish to convert to some other currency one multiplies each loss size by the appropriate exchange rate.

Assuming each loss is paid at (approximately) the same time one can apply (approximately) the same exchange rate to each loss. This is mathematically identical to applying the same inflation factor under uniform inflation.

If the exchange rate is 80 yen per dollar, then the Loss Elimination Ratio at 80,000 yen is the same as that at $1000.

Exercise: The Limited Expected Value at $1000 is $600.
The exchange rate is 80 yen per dollar.
Determine the Limited Expected Value at 80,000 yen.
[Solution: 80,000 yen ⇔ 1000.
Limited Expected Value at 80,000 yen is: (600)(80) = 48,000 yen.
Comment: The Loss Elimination Ratio is dimensionless, while the Limited Expected Value is in dollars or yen. Thus here we need to multiply by the exchange rate of 80 yen per dollar.]

The Coefficient of Variation, Skewness, and Kurtosis, which describe the shape of the size of loss distribution, are unaffected by converting to yen.

Exercise: The size of loss distribution in dollars is Gamma with $\alpha = 3$ and $\theta = 2000$.
The exchange rate is 0.80 euros per dollar.
Determine the size of loss distribution in euros.
[Solution: Gamma with $\alpha = 3$ and $\theta = (0.80)(2000) = 1600$.
Comment: The mean in euros is: (3)(1600) = €4800.
The mean in dollars is: (3)(2000) = $6000. ⇔ (0.8)(6000) = €4800.
0.80 euros per dollar. ⇔ 1.25 dollars per euro.
Going from euros to dollars would be mathematically equivalent to 25% inflation.
Going from dollars to euros is mathematically equivalent to deflating back to the earlier year from the later year with 25% inflation. $6000/1.25 = €4800.]

[157] See CAS3, 5/06, Q.26.

Problems:

22.1 (1 point) The size of losses in 1994 follow a Pareto Distribution,
with parameters $\alpha = 3$, $\theta = 5000$.
Assume that inflation uniformly increases the size of losses between 1994 and 1997 by 20%.
What is the average size of loss in 1997?
A. 2500 B. 3000 C. 3500 D. 4000 E. 4500

22.2 (1 point) The size of losses in 2004 follow an Exponential Distribution: $F(x) = 1 - e^{-x/\theta}$, with $\theta = 200$. Assume that inflation uniformly increases the size of losses between 2004 and 2009 by 3% per year. What is the variance of the loss distribution in 2009?
A. 48,000 B. 50,000 C. 52,000 D. 54,000 E. 56,000

22.3 (2 points) The size of losses in 1992 follows a Burr Distribution, $F(x) = 1 - \left(\frac{1}{1 + (x/\theta)^\gamma}\right)^\alpha$,

with parameters $\alpha = 2$, $\theta = 19{,}307$, $\gamma = 0.7$.
Assume that inflation uniformly increases the size of losses between 1992 and 1996 by 30%.
What is the probability of a loss being greater than 10,000 in 1996?
A. 39% B. 41% C. 43% D. 45% E. 47%

22.4 (1 point) The size of losses in 1994 follow a Gamma Distribution, with parameters $\alpha = 2$, $\theta = 100$.
Assume that inflation uniformly increases the size of losses between 1994 and 1996 by 10%.
What are the parameters of the loss distribution in 1996?
A. $\alpha = 2, \theta = 100$ B. $\alpha = 2, \theta = 110$ C. $\alpha = 2, \theta = 90.9$ D. $\alpha = 2, \theta = 82.6$
E. None of A, B, C, or D.

22.5 (2 points) The size of losses in 1995 follow a Pareto Distribution, with $\alpha = 1.5$, $\theta = 15{,}000$.
Assume that inflation uniformly increases the size of losses between 1995 and 1999 by 25%.
In 1999, what is the average size of the non-zero payments excess of a deductible of 25,000?
A. 72,000 B. 76,000 C. 80,000 D. 84,000 E. 88,000

22.6 (2 points) The size of losses in 1992 follow the density function $f(x) = 2.5x^{-2}$ for $2 < x < 10$.
Assume that inflation uniformly increases the size of losses between 1992 and 1996 by 20%.
What is the probability of a loss being greater than 6 in 1996?
A. 23% B. 25% C. 27% D. 29% E. 31%

Use the following information for the next 4 questions:
A size of loss distribution has been fit to certain data in terms of dollars. The loss sizes have been converted to yen. Assume the exchange rate is 80 yen per dollar.

22.7 (1 point) In terms of dollars the sizes of loss are given by a Loglogistic, with parameters $\gamma = 4$, and $\theta = 100$.
Which of the following are the parameters of the distribution in terms of yen?
A. $\gamma = 4$, and $\theta = 100$ B. $\gamma = 320$, and $\theta = 100$ C. $\gamma = 4$, and $\theta = 8000$
D. $\gamma = 320$, and $\theta = 8000$ E. None of A, B, C, or D.

22.8 (1 point) In terms of dollars the sizes of loss are given by a LogNormal Distribution, with parameters $\mu = 10$ and $\sigma = 3$.
Which of the following are the parameters of the distribution in terms of yen?
A. $\mu = 10$ and $\sigma = 3$ B. $\mu = 800$ and $\sigma = 3$ C. $\mu = 10$ and $\sigma = 240$
D. $\mu = 800$ and $\sigma = 240$ E. None of A, B, C, or D.

<u>22.9</u> (1 point) In terms of dollars the sizes of loss are given by a Weibull Distribution, with parameters $\theta = 625$ and $\tau = 0.5$.
Which of the following are the parameters of the distribution in terms of yen?
A. $\theta = 625$ and $\tau = 0.5$ B. $\theta = 69.9$ and $\tau = 0.5$ C. $\theta = 5{,}590$ and $\tau = 0.5$
D. $\theta = 50{,}000$ and $\tau = 0.5$ E. None of A, B, C, or D.

22.10 (1 points) In terms of dollars the sizes of loss are given by a Paralogistic Distribution, with $\alpha = 4, \theta = 100$. Which of the following are the parameters of the distribution in terms of yen?
A. $\alpha = 4$, and $\theta = 100$ B. $\alpha = 320$, and $\theta = 100$ C. $\alpha = 4$, and $\theta = 8000$
D. $\alpha = 320$, and $\theta = 8000$ E. None of A, B, C, or D.

22.11 (1 point) The size of losses in 1994 follows a distribution $F(x) = \Gamma[\alpha; \lambda \ln(x)]$, $x > 1$, with parameters $\alpha = 40$, $\lambda = 10$. Assume that inflation uniformly increases the size of losses between 1994 and 1996 by 10%. What are the parameters of the loss distribution in 1996?
A. $\alpha = 40$, $\lambda = 10$ B. $\alpha = 40$, $\lambda = 9.1$ C. $\alpha = 40$, $\lambda = 11$
D. $\alpha = 40$, $\lambda = 12.1$ E. None of A, B, C, or D.

22.12 (1 point) $X_1, X_2, \ldots X_{50}$, are independent, identically distributed variables, each with an Exponential Distribution with mean 800. What is the distribution of $\bar{X}$ their average?

<u>22.13</u> (1 point) Assume that inflation uniformly increases the size of losses between 1994 and 1996 by 10%. Which of the following statements is true regarding the size of loss distribution?
1. If the skewness in 1994 is 10, then the skewness in 1996 is 13.31.
2. If the 70th percentile in 1994 is \$10,000, then the 70th percentile in 1996 is \$11,000.
3. If in 1994 the Loss Elimination Ratio for a deductible of \$1000 is 10%,
then in 1996 the Loss Elimination Ratio for a deductible of \$1100 is 11%.
A. 1 B. 2 C. 3 D. 1, 2, 3 E. None of A, B, C, or D

22.14 (3 points) The size of losses in 1995 follow the density function:
$f(x) = 375x^2 e^{-10x} + 20x^3\exp(-20x^4)$.
Assume that inflation uniformly increases the size of losses between 1995 and 1999 by 25%.
Which of the following is the density in 1999?
A. $468.75x^2 e^{-12.5x} + 16x^3\exp(-16x^4)$
B. $192x^2 e^{-8x} + 8.192x^3\exp(-8.192x^4)$
C. $468.75x^2 e^{-12.5x} + 8.192x^3\exp(-8.192x^4)$
D. $192x^2 e^{-8x} + 16x^3\exp(-16x^4)$
E. None of the above.

22.15 (3 points) You are given the following:
- Losses follow a distribution with density function
$$f(x) = \frac{\exp\left[-0.5\left(\frac{\ln(x) - 7}{3}\right)^2\right]}{3x\sqrt{2\pi}},\ 0 < x < \infty.$$
- There is a deductible of 1000.
- 173 losses are expected to exceed the deductible each year.

Determine the expected number of losses that would exceed the deductible each year if all loss amounts increased by 40%, but the deductible remained at 1000.
A. Less than 175
B. At least 175, but less than 180
C. At least 180, but less than 185
D. At least 185, but less than 190
E. At least 190

<u>**22.16**</u> (3 points) Losses in the year 2001 have a Pareto Distribution with parameters $\alpha = 3$ and $\theta = 40$. Losses are uniformly 6% higher in the year 2002 than in the year 2001. In both 2001 and 2002, an insurance policy has a deductible of 5 and a maximum covered loss of 25.
What is the ratio of expected payments in 2002 over expected payments in the year 2001?
(A) 104% (B) 106% (C) 108% (D) 110% (E) 112%

22.17 (2 points) You are given the following:
- 1000 observed losses occurring in 1993 for a group of risks have been recorded and are grouped as follows:

Interval	Number of Losses
(0, 100]	341
100, 500]	202
(500, 1000]	131
(1000, 5000]	151
(5000, 10,000]	146
(10,000, ∞)	29

- Inflation of 8% per year affects all losses uniformly from 1993 to 2002.

What is the expected proportion of losses for this group of risks that will be greater than 1000 in the year 2002?
A. 38% B. 40% C. 42% D. 44% E. 46%

22.18 (2 points) The probability density function of losses in 1996 is:

$$f(x) = \mu \frac{\exp\left[-\frac{(x-\mu)^2}{2\beta x}\right]}{\sqrt{2\beta\pi\, x^3}}, \ x > 0.$$

Between 1996 and 2001 there is a total of 20% inflation. What is the density function in 2001?
A. Of the same form, but with parameters 1.2μ and β, rather than μ and β.
B. Of the same form, but with parameters μ and 1.2β, rather than μ and β.
C. Of the same form, but with parameters 1.2μ and 1.2β, rather than μ and β.
D. Of the same form, but with parameters μ/1.2 and β, rather than μ and β.
E. Of the same form, but with parameters μ and β/1.2, rather than μ and β.

Use the following information for the next two questions:
- The losses in 1998 prior to any deductible follow a Distribution: $F(x) = 1 - e^{-x/5000}$.
- Assume that losses increase uniformly by 40% between 1998 and 2007.
- In 1998, an insurer pays for losses excess of a 1000 deductible.

22.19 (2 points) If in 2007 this insurer pays for losses excess of a 1000 deductible, what is the increase between 1998 and 2007 in the dollars of losses that this insurer expects to pay?
A. 44% B. 46% C. 48% D. 50% E. 52%

22.20 (2 points) If in 2007 this insurer pays for losses excess of a 1400 deductible, what is the increase between 1998 and 2007 in the dollars of losses that this insurer expects to pay?
A. 38% B. 40% C. 42% D. 44% E. 46%

22.21 (3 points) You are given the following:
- In 1990, losses follow a LogNormal Distribution, with parameters 3 and σ.
- Between 1990 and 1999 there is uniform inflation at an annual rate of 4%.
- In 1990, 5% of the losses exceed the mean of the losses in 1999.

Determine σ.
A. 0.960 or 2.960
B. 0.645 or 2.645
C. 0.546 or 3.374
D. 0.231 or 3.059
E. None of the above

22.22 (3 points) You are given the following:
- The losses in 1995 follow a Weibull Distribution with parameters θ = 1 and τ = 0.3.
- A relevant Consumer Price Index (CPI) is 170.3 in 1995 and 206.8 in 2001.
- Assume that losses increase uniformly by an amount based on the increase in the CPI.

What is the increase between 1995 and 2001 in the expected number of losses exceeding a 1000 deductible?
A. 45% B. 48% C. 51% D. 54% E. 57%

22.23 (3 points) You are given the following:
- The losses in 1994 follow a LogNormal Distribution, with parameters $\mu = 3$ and $\sigma = 4$.
- Assume that losses increase by 5% from 1994 to 1995, 3% from 1995 to 1996, 7% from 1996 to 1997, and 6% from 1997 to 1998.
- In both 1994 and 1998, an insurer sells policies with a \$25,000 maximum covered loss.

What is the increase due to inflation between 1994 and 1998 in the dollars of losses that the insurer expects to pay?
A. 9% B. 10% C. 11% D. 12% E. 13%

22.24 (3 points) You are given the following:
- The losses in 1994 follow a Distribution: $F(x) = 1 - (100{,}000/x)^3$ for $x > \$100{,}000$.
- Assume that inflation is a total of 20% from 1994 to 1999.
- In each year, a reinsurer pays for the layer of loss from \$500 thousand to \$2 million.

What is the increase due to inflation between 1994 and 1999 in the dollars that the reinsurer expects to pay?
A. 67% B. 69% C. 71% D. 73% E. 75%

22.25 (2 points) You are given the following:
- The losses in 2001 follow an Inverse Gaussian Distribution, with parameters $\mu = 3$ and $\theta = 10$.
- There is uniform inflation from 2001 to 2009 at an annual rate of 3%.

What is the variance of the distribution of losses in 2009?
A. Less than 2
B. At least 2, but less than 3
C. At least 3, but less than 4
D. At least 4, but less than 5
E. At least 5

22.26 (1 point) In the year 2002 the size of loss distribution is a Pareto with $\alpha = 3$ and $\theta = 5000$.
During the year 2002 what is the median of those losses of size greater than 10,000?
A. 13,900 B. 14,000 C. 14,100 D. 14,200 E. 14,300

22.27 (2 points) In the year 2002 the size of loss distribution is a Pareto with $\alpha = 3$ and $\theta = 5000$.
You expect a total of 15% inflation between the years 2002 and 2006.
During the year 2006 what is the median of those losses of size greater than 10,000?
A. 13,900 B. 14,000 C. 14,100 D. 14,200 E. 14,300

22.28 (1 point) The size of losses in 1992 follow the density function $f(x) = 1000e^{-1000x}$.
Assume that inflation uniformly increases the size of losses between 1992 and 1998 by 25%.
Which of the following is the density in 1998?
A. $800e^{-800x}$ B. $1250e^{-1250x}$ C. $17841\ x^{.5}\ e^{-1000x}$
D. $1000e^{-1000x}$ E. None of the above.

22.29 (2 points) You are given the following:
- For 2003 the amount of a single loss has the following distribution:

Amount	Probability
\$1,000	1/6
\$2,000	1/3
\$5,000	1/3
\$10,000	1/6

- An insurer pays all losses after applying a \$2000 deductible to each loss.
- Inflation of 4% per year impacts all claims uniformly from 2003 to 2006.

Assuming no change in the deductible, what is the inflationary impact on losses paid by the insurer in 2006 as compared to the losses the insurer paid in 2003?
A. 9% B. 12% C. 15% D. 18% E. 21%

22.30 (2 points) You are given the following:
- The size of loss distribution in 2007 is LogNormal Distribution with $\mu = 5$ and $\sigma = 0.7$.
- Assume that losses increase by 4% per year from 2007 to 2010.

What is the second moment of the size of loss distribution in 2010?
A. Less than 70,000
B. At least 70,000, but less than 75,000
C. At least 75,000, but less than 80,000
D. At least 80,000, but less than 85,000
E. At least 85,000

22.31 (3 points) In 2005 sizes of loss follow a distribution F(x), with survival function S(x) and density f(x). Between 2005 and 2008 there is a total of 10% inflation.
In 2008 there is a deductible of 1000.
Which of the following does not represent the expected payment per loss in 2008?
A. $1.1\ E[X] - 1.1\ E[X \wedge 909]$

B. $1.1 \int_{909}^{\infty} x\, f(x)\, dx - 1000 \int_{909}^{\infty} f(x)\, dx$

C. $1.1 \int_{909}^{\infty} S(x)\, dx$

D. $1.1 \int_{909}^{\infty} (x-1000)\, f(x)\, dx$

E. $1.1 \int_{909}^{\infty} x\, f(x)\, dx + 1.1 \int_{0}^{909} \{x\, f(x) - S(x)\}\, dx$

22.32 (3 points) For Actuaries Professional Liability insurance, severity follows a Pareto Distribution with $\alpha = 2$ and $\theta = 500{,}000$.
Excess of loss reinsurance covers the layer from R to $1 million.
Annual unlimited ground up inflation is 10% per year.
Determine R, less than $1 million, such that the annual loss trend for the reinsured layer is exactly equal to the overall inflation rate of 10%.

22.33 (3 points) In 2011, the claim severity distribution is exponential with mean 5000.
In 2013, an insurance company will pay the amount of each claim in excess of a deductible of 1000.
There is a total of 8% inflation between 2011 and 2013.
In 2013, calculate the variance of the amount paid by the insurance company for one claim, including the possibility that the amount paid is 0.
(A) 24 million (B) 26 million (C) 28 million (D) 30 million (E) 32 million

22.34 (3 points) In 2005 sizes of loss follow a certain distribution, and you are given the following selected values of the distribution function and limited expected value:

x	F(x)	Limited Expected Value at x
3000	0.502	2172
3500	0.549	2409
4000	0.590	2624
4500	0.624	2820
5000	0.655	3000
5500	0.681	3166
6000	0.705	3319
6500	0.726	3462
7000	0.744	3594

Between 2005 and 2010 there is a total of 25% inflation.
In both 2005 and 2010 there is a deductible of 5000.
In 2010 the average payment per payment is 15% more than it was in 2005.
Determine E[X] in 2005.
A. 5000 B. 5500 C. 6000 D. 6500 E. 7000

Use the following information for the next 2 questions:
- In 2010, losses follow a Pareto Distribution with $\alpha = 5$ and $\theta = 40$.
- There is a total of 25% inflation between 2010 and 2015.
- In 2015, there is a deductible of 10.

22.35 (2 points) In 2015, determine the variance of Y^P, the per-payment variable.
A. 300 B. 325 C. 350 D. 375 E. 400

22.36 (3 points) In 2015, determine the variance of Y^L, the per-loss variable.
A. 160 B. 180 C. 200 D. 220 E. 240

Use the following information for the next four questions:
- Losses in 2002 follow a LogNormal Distribution with parameters $\mu = 9.7$ and $\sigma = 0.8$.
- In 2007, the insured has a deductible of 10,000, maximum covered loss of 50,000, and a coinsurance factor of 90%.
- Inflation is 3% per year from 2002 to 2007.

22.37 (3 points) In 2007, what is the average payment per loss?
A. less than 12,100
B. at least 12,100 but less than 12,200
C. at least 12,200 but less than 12,300
D. at least 12,300 but less than 12,400
E. at least 12,400

22.38 (1 point) In 2007, what is the average payment per non-zero payment?
A. less than 15,500
B. at least 15,500 but less than 15,600
C. at least 15,600 but less than 15,700
D. at least 15,700 but less than 15,800
E. at least 15,800

22.39 (3 points) In 2007, what is the standard deviation of Y^L, the per-loss variable?
A. less than 12,100
B. at least 12,100 but less than 12,200
C. at least 12,200 but less than 12,300
D. at least 12,300 but less than 12,400
E. at least 12,400

22.40 (1 point) In 2007, what is the standard deviation of Y^P, the per-payment variable?
A. less than 12,100
B. at least 12,100 but less than 12,200
C. at least 12,200 but less than 12,300
D. at least 12,300 but less than 12,400
E. at least 12,400

22.41 (1 point) Determine the distribution followed by the average of n independent, identically distributed Gamma Distributions.

22.42 (3 points) In 2011, losses prior to the effect of a deductible follow a Pareto Distribution with $\alpha = 2$ and $\theta = 250$.
There is deductible of 100 in both 2011 and 2016.
The ratio of the expected aggregate payments in 2016 to 2011 is 1.26.
Determine the total amount of inflation between 2011 and 2016.
A. 19% B. 20% C. 21% D. 22% E. 23%

22.43 (3 points) In 2004 losses follow a LogNormal Distribution with mean 30,000 and coefficient of variation 4. Inflation is 5% per year.
In 2015, what percent of total losses are excess of 200,000?

22.44 (3 points) In 2012 losses follow an Exponential Distribution with mean 1000.
Inflation is 3% per year.
In 2016, there is deductible of size 500, maximum covered loss of 2000,
and a coinsurance of 80%.
Determine the average payment per payment in 2016.
A. less than 500
B. at least 500 but less than 550
C. at least 550 but less than 600
D. at least 600 but less than 650
E. at least 650

22.45 (2 points) The changes in the hourly real wages of men from 1973 to 2012 in the United States, at different percentiles of the wage distribution:

Percentile	Change
10th	-10.9%
20th	-17.9%
30th	-13.7%
40th	-10.5%
50th	-5.8%
60th	-1.9%
70th	6.3%
80th	16.9%
90th	25.0%
95th	35.5%

Briefly discuss what this data implies.

22.46 (2 points) Severity in 2014 follows an Inverse Gamma with $\alpha = 6$ and $\theta = 10$.
There is a total of 20% inflation between 2014 and 2020.
What is the fourth moment of the distribution of severity in 2020?
A. less than 100
B. at least 110 but less than 130
C. at least 130 but less than 150
D. at least 150 but less than 170
E. at least 170

Use the following information for the next two questions:
- In 2015 a risk has a two-parameter Pareto distribution with $\alpha = 3$ and $\theta = 200$.
- An insurance on the risk has a deductible of 50 in each year.
- P_i, the insurance premium in year i, equals 1.3 times the expected losses excess of the deductible of 50.
- The risk is reinsured with a deductible that stays the same in each year. The reinsurer pays for any payments by the insurer excess of 250 per claim; in other words the reinsurer pays for the ground up layer excess of 300.
- R_i, the reinsurance premium in year i, equals 1.1 times the expected reinsured claims.

22.47 (2 points) Determine the ratio R_{2015} / P_{2015}.
(A) 0.21 (B) 0.22 (C) 0.23 (D) 0.24 (E) 0.25

22.48 (2 points) In 2020 losses inflate by 25%.
Calculate R_{2020} / P_{2020}.
(A) 0.21 (B) 0.22 (C) 0.23 (D) 0.24 (E) 0.25

22.49 (2 points) In 2014 losses follow an Exponential Distribution with mean 3000.
Inflation is 6% per year.
In 2015, there is deductible of size d, and the average payment per payment is 3180.
Determine d.
(A) 100 (B) 250 (C) 500 (D) 1000 (E) Cannot be determined.

22.50 (3 points) In 2010, prior to the application of any deductible, losses follow a Pareto Distribution with $\alpha = 4$ and $\theta = 500$.
Between 2010 and 2020 there is a total of 30% inflation. In 2020, there is a deductible of 200.
In 2020, what is the variance of amount paid by the insurer for one loss, including the possibility that the amount paid is zero?
A. less than 70,000
B. at least 70,000 but less than 75,000
C. at least 75,000 but less than 80,000
D. at least 80,000 but less than 85,000
E. at least 85,000

22.51 (2 points) A group health policy has a mean frequency of 70.
Ground-up severity is given by the following table:

Severity	Probability
100	0.50
250	0.40
500	0.10

If severity increases by 20% with no change in frequency, and there is a per claim deductible of 100, calculate the expected total claim payment.
(A) 9,600 (B) 9,800 (C) 10,000 (D) 10,200 (E) 10.400

22.52 (3 points) An insurer has excess-of-loss reinsurance. You are given:
(i) In the year 2016 individual losses have a Pareto distribution with

$$F(x) = 1 - \left(\frac{10,000}{10,000 + x}\right)^{1.5}, \ x > 0.$$

(ii) Reinsurance will pay the excess of each loss over 50,000.
(iii) Individual losses increase 5% each year due to inflation.
(iv) The frequency distribution does not change.

Calculate $\frac{\text{expected losses paid by the reinsurer in 2020}}{\text{expected losses paid by the reinsurer in 2016}}$.

(A) 1.24 (B) 1.26 (C) 1.28 (D) 1.30 (E) 1.32

22.53 (4 points) In 2023, an insurance policy has no deductible and no policy limit.
Claim severity is assumed to follow a Pareto distribution with parameters of $\alpha = 3$ and $\theta = 2000$.
The number of claims and cost of each claim are assumed to be mutually independent.
Between 2023 and 2026, the cost of each claim is expected to increase uniformly by 9% due to inflation.
(a) (2 points) The insurer is considering imposing an ordinary deductible per claim, d, that would result in the insurer's expected aggregate payments being the same in 2026 as in 2023. Determine d.
(b) (2 points) The insurer is instead considering imposing an ordinary deductible of 50 along with a maximum covered loss of L on each claim payment in 2026.
Calculate the value of L that would result in the insured's expected aggregate payments being the same in 2026 as in 2023.

22.54 (4, 5/86, Q.61 & 4, 5/87, Q.59) (1 point) Let there be a 10% rate of inflation over the period of concern. Let X be the uninflated losses and Z be the inflated losses.
Let F_X be the distribution function (d.f.) of X, and f_X be the probability density function (p.d.f.) of X. Similarly, let F_Z and f_Z be the d.f. and p.d.f. of Z.
Then which of the following statements are true?
1. $f_Z(Z) = f_X(Z / 1.1)$
2. If F_X is a Pareto, then F_Z is also a Pareto.
3. If F_X is a LogNormal, then F_Z is also a LogNormal.
A. 2 B. 3 C. 1, 2 D. 1, 3 E. 2, 3

22.55 (4, 5/89, Q.58) (2 points) The random variable X with distribution function $F_X(x)$ is distributed according to the Burr distribution, $F(x) = 1 - \left(\frac{1}{1 + (x/\theta)^\gamma}\right)^\alpha$,

with parameters $\alpha > 0$, $\theta > 0$, and $\gamma > 0$.
If Z = (1 + r)X where r is an inflation rate over some period of concern, find the parameters for the distribution function $F_Z(z)$ of the random variable z.
A. α, θ, γ B. $\alpha(1+r), \theta, \gamma$ C. $\alpha, \theta(1+r), \gamma$
D. $\alpha, \theta, \gamma(1+r)^\gamma$ E. None of the above

22.56 (4, 5/90, Q.37) (2 points) Liability claim severity follows a Pareto distribution with a mean of \$25,000 and parameter $\alpha = 3$. If inflation increases all claims by 20%, the probability of a claim exceeding \$100,000 increases by:
A. less than 0.02
B. at least 0.02 but less than 0.03
C. at least 0.03 but less than 0.04
D. at least 0.04 but less than 0.05
E. at least 0.05

22.57 (4, 5/91, Q.27) (3 points) The Pareto distribution with parameters $\theta = 12{,}500$ and $\alpha = 2$ appears to be a good fit to 1985 policy year liability claims.
Assume that inflation has been a constant 10% per year.
What is the estimated claim severity for a policy issued in 1992 with a \$200,000 limit of liability?
A. Less than 22,000
B. At least 22,000 but less than 23,000
C. At least 23,000 but less than 24,000
D. At least 24,000 but less than 25,000
E. At least 25,000

22.58 (4, 5/91, Q.44) (2 points) Inflation often requires one to modify the parameters of a distribution fitted to historical data. If inflation has been at the same rate for all sizes of loss, which of the sets of parameters shown in final column would be correct?
The form of the distributions is as given in the Appendix A of Loss Models.

Distribution Family	Distribution Function	Parameters of z = (1+r)(x)
1. Inverse Gaussian	$\Phi\left[\left(\frac{x}{\mu}-1\right)\sqrt{\frac{\theta}{x}}\right] + e^{2\theta/\mu}\,\Phi\left[-\left(\frac{x}{\mu}+1\right)\sqrt{\frac{\theta}{x}}\right]$	μ, $\theta(1+r)$
2. Generalized Pareto	$\beta[\tau, \alpha; x/(\theta+x)]$	α, $\theta/(1+r)$, τ
3. Weibull	$1 - \exp\left[-\left(\frac{x}{\theta}\right)^{\tau}\right]$	$\theta(1+r)$, τ

A. 1 B. 2 C. 3 D. 1, 2, 3 E. None of the above

22.59 (4B, 5/92, Q.7) (2 points) The random variable X for claim amounts with distribution function $F_X(x)$ is distributed according to the Erlang distribution with parameters b and c.
The density function for X is as follows: $f(x) = (x/b)^{c-1}\, e^{-x/b} / \{ b\,(c-1)! \}$; $x > 0$, $b > 0$, $c > 1$.
Inflation of 100r% acts uniformly over a one year period.
Determine the distribution function $F_Z(Z)$ of the random variable $Z = (1+r)X$.

A. Erlang with parameters b and c(1+r)
B. Erlang with parameters b(1+r) and c
C. Erlang with parameters b/(1+r) and c
D. Erlang with parameters b/(1+r) and c/(1+r)
E. No longer an Erlang distribution

22.60 (4B, 11/92, Q.20) (3 points) Claim severity follows a Burr distribution,
$F(x) = 1 - \left(\frac{1}{1 + (x/\theta)^{\gamma}}\right)^{\alpha}$, with parameters $\alpha = 3$, $\gamma = 0.5$ and θ. The mean is 10,000.

If inflation increases all claims uniformly by 44%, determine the probability of a claim exceeding $40,000 after inflation.
Hint: The nth moment of a Burr Distribution is: $\theta^n\, \Gamma(1 + n/\gamma)\, \Gamma(\alpha - n/\gamma) / \Gamma(\alpha)$, $\alpha\gamma > n$.

A. Less than 0.01
B. At least 0.01 but less than 0.03
C. At least 0.03 but less than 0.05
D. At least 0.05 but less than 0.07
E. At least 0.07

22.61 (4B, 5/93, Q.11) (1 point) You are given the following:
- The underlying distribution for 1992 losses is given by a lognormal distribution with parameters $\mu = 17.953$ and $\sigma = 1.6028$.
- Inflation of 10% impacts all claims uniformly the next year.

What is the underlying loss distribution after one year of inflation?
A. lognormal with $\mu' = 19.748$ and $\sigma' = 1.6028$
B. lognormal with $\mu' = 18.048$ and $\sigma' = 1.6028$
C. lognormal with $\mu' = 17.953$ and $\sigma' = 1.7631$
D. lognormal with $\mu' = 17.953$ and $\sigma' = 1.4571$
E. no longer a lognormal distribution

22.62 (4B, 5/93, Q.12) (3 points) You are given the following:
- The underlying distribution for 1992 losses is given by $f(x) = e^{-x}$, $x > 0$, where losses are expressed in millions of dollars.
- Inflation of 10% impacts all claims uniformly from 1992 to 1993.
- Under a basic limits policy, individual losses are capped at $1.0 (million).

What is the inflation rate from 1992 to 1993 on the capped losses?
A. less than 2%
B. at least 2% but less than 3%
C. at least 3% but less than 4%
D. at least 4% but less than 5%
E. at least 5%

22.63 (4B, 5/93, Q.28) (3 points) You are given the following:
- The underlying loss distribution function for a certain line of business in 1991 is: $F(x) = 1 - x^{-5}$, $x > 1$.
- From 1991 to 1992, 10% inflation impacts all claims uniformly.

Determine the 1992 Loss Elimination Ratio for a deductible of 1.2.
A. Less than 0.850
B. At least 0.850 but less than 0.870
C. At least 0.870 but less than 0.890
D. At least 0.890 but less than 0.910
E. At least 0.910

22.64 (4B, 11/93, Q.5) (3 points) You are given the following:
- The underlying distribution for 1993 losses is given by $f(x) = e^{-x}$, $x > 0$, where losses are expressed in millions of dollars.
- Inflation of 5% impacts all claims uniformly from 1993 to 1994.
- Under a basic limits policy, individual losses are capped at $1.0 million in each year.

What is the inflation rate from 1993 to 1994 on the capped losses?
A. Less than 1.5%
B. At least 1.5%, but less than 2.5%
C. At least 2.5%, but less than 3.5%
D. At least 3.5%, but less than 4.5%
E. At least 4.5%

22.65 (4B, 11/93, Q.15) (3 points) You are given the following:
- X is the random variable for claim severity with probability distribution function F(x).
- During the next year, uniform inflation of r% impacts all claims.

Which of the following are true of the random variable Z = X(1+r), the claim severity one year later?
1. The coefficient of variation for Z equals (1+r) times the coefficient of variation for X.
2. For all values of d > 0, the mean excess loss of Z at d(1+r) equals (1+r) times the mean excess loss of X at d.
3. For all values of d > 0, the limited expected value of Z at d equals (1+r) times the limited expected value of X at d.

A. 2 B. 3 C. 2, 3 D. 1, 2, 3 E. None of A, B, C or D

22.66 (4B, 11/93, Q.27) (3 points) You are given the following:
- Losses for 1991 are uniformly distributed on [0, 10,000].
- Inflation of 5% impacts all losses uniformly from 1991 to 1992 and from 1992 to 1993 (5% each year).

Determine the 1993 Loss Elimination Ratio for a deductible of $500.
A. Less than 0.085
B. At least 0.085, but less than 0.090
C. At least 0.090, but less than 0.095
D. At least 0.095, but less than 0.100
E. At least 0.100

22.67 (4B, 5/94, Q.16) (1 point) You are given the following:
- Losses in 1993 follow the density function

$$f(x) = 3x^{-4},\ x \geq 1,$$

where x = losses in millions of dollars.
- Inflation of 10% impacts all claims uniformly from 1993 to 1994.

Determine the probability that losses in 1994 exceed $2.2 million.
A. Less than 0.05
B. At least 0.05, but less than 0.10
C. At least 0.10, but less than 0.15
D. At least 0.15, but less than 0.20
E. At least 0.20

22.68 (4B, 5/94, Q.21) (2 points) You are given the following:
- For 1993 the amount of a single claim has the following distribution:

Amount	Probability
$1,000	1/6
$2,000	1/6
$3,000	1/6
$4,000	1/6
$5,000	1/6
$6,000	1/6

- An insurer pays all losses AFTER applying a $1,500 deductible to each loss.
- Inflation of 5% impacts all claims uniformly from 1993 to 1994.

Assuming no change in the deductible, what is the inflationary impact on losses paid by the insurer in 1994 as compared to the losses the insurer paid in 1993?
A. Less than 5.5%
B. At least 5.5%, but less than 6.5%
C. At least 6.5%, but less than 7.5%
D. At least 7.5%, but less than 8.5%
E. At least 8.5%

22.69 (4B, 5/94, Q.24) (3 points) You are given the following:
- X is a random variable for 1993 losses, having the density function $f(x) = 0.1e^{-0.1x}$, $x > 0$.
- Inflation of 10% impacts all losses uniformly from 1993 to 1994.
- For 1994, a deductible, d, is applied to all losses.
- P is a random variable representing payments of losses truncated and shifted by the deductible amount.

Determine the value of the cumulative distribution function at p = 5, $F_P(5)$, in 1994.
A. $1 - e^{-0.1(5+d)/1.1}$
B. $\{e^{-0.1(5/1.1)} - e^{-0.1(5+d)/1.1}\} / \{1 - e^{-0.1(5/1.1)}\}$
C. 0
D. At least 0.25, but less than 0.35
E. At least 0.35, but less than 0.45

22.70 (4B, 11/94, Q.8) (3 points) You are given the following:
In 1993, an insurance company's underlying loss distribution for an individual claim amount is a lognormal distribution with parameters $\mu = 10.0$ and $\sigma^2 = 5.0$.
From 1993 to 1994, an inflation rate of 10% impacts all claims uniformly.
In 1994, the insurance company purchases excess-of-loss reinsurance that caps the insurer's loss at $2,000,000 for any individual claim. Determine the insurer's 1994 expected net claim amount for a single claim after application of the $2,000,000 reinsurance cap.
A. Less than $150,000
B. At least $150,000, but less than $175,000
C. At least $175,000, but less than $200,000
D. At least $200,000, but less than $225,000
E. At least $225,000

22.71 (4B, 11/94, Q.28) (2 points) You are given the following:
In 1993, the claim amounts for a certain line of business were normally distributed with mean $\mu = 1000$ and variance $\sigma^2 = 10,000$: $f(x) = \frac{\exp[-\frac{(x-\mu)^2}{2\sigma^2}]}{\sigma\sqrt{2\pi}}$.
Inflation of 5% impacted all claims uniformly from 1993 to 1994.
What is the distribution for claim amounts in 1994?
A. No longer a normal distribution
B. Normal with $\mu = 1000.0$ and $\sigma = 102.5$
C. Normal with $\mu = 1000.0$ and $\sigma = 105.0$
D. Normal with $\mu = 1050.0$ and $\sigma = 102.5$
E. Normal with $\mu = 1050.0$ and $\sigma = 105.0$

22.72 (CAS9, 11/94, Q.37) (2 points)
The following first-dollar claims have been observed for a certain class of business:

Claim Number	Claim Amount
500	26
501	115
502	387
503	449
504	609
505	774
506	2,131
507	5,791
508	7,499
509	12,526
Total	30,307

a. (1/2 point) What are the empirical loss elimination ratios for deductibles of $1,000 and $5,000?
b. (1/2 point) Assume that losses can be modeled by an exponential distribution,
with hazard rate = 0.00033.
What are the indicated loss elimination ratios from the model for deductibles of $1,000 and $5,000?
c. (1 point) Assume that losses can be modeled by an exponential distribution,
with hazard rates = 0.00033.
For a fixed deductible of $5,000, what is the leveraged inflation rate under the model, if first dollar inflation is 10% per year?

22.73 (Course 160 Sample Exam #3, 1994, Q.2) (1.9 points) You are given:
(i) The random variable X has an exponential distribution.
(ii) p_x is the probability of life aged x surviving one year. $p_x = 0.95$, for all x.
(iii) Y = 2X.
(iv) $f_Y(Y)$ is the probability density function of the random variable Y.
Calculate $f_Y(1)$.
(A) 0.000 (B) 0.025 (C) 0.050 (D) 0.075 (E) 0.100

22.74 (4B, 5/95, Q.6) (3 points) You are given the following:
- For 1994, loss sizes follow a uniform distribution on [0, 2500].
- In 1994, the insurer pays 100% of all losses.
- Inflation of 3.0% impacts all losses uniformly from 1994 to 1995.
- In 1995, a deductible of $100 is applied to all losses.

Determine the Loss Elimination Ratio (L.E.R.) of the $100 deductible on 1995 losses.
A. Less than 7.3%
B. At least 7.3%, but less than 7.5%
C. At least 7.5%, but less than 7.7%
D. At least 7.7%, but less than 7.9%
E. At least 7.9%

22.75 (4B, 5/95, Q.23) (2 points) You are given the following:
- Losses follow a Pareto distribution, with parameters $\theta = 1000$ and $\alpha = 2$.
- 10 losses are expected each year.
- The number of losses and the individual loss amounts are independent.
- For each loss that occurs, the insurer's payment is equal to the **entire** amount of the loss if the loss is greater than 100. The insurer makes no payment if the loss is less than or equal to 100.

Determine the insurer's expected number of annual payments if all loss amounts increased uniformly by 10%.
A. Less than 7.9
B. At least 7.9, but less than 8.1
C. At least 8.1, but less than 8.3
D. At least 8.3, but less than 8.5
E. At least 8.5

22.76 (4B, 11/95, Q.6) (2 points) You are given the following:
- In 1994, losses follow a Pareto distribution, with parameters $\theta = 500$ and $\alpha = 1.5$.
- Inflation of 5% impacts all losses uniformly from 1994 to 1995.

What is the median of the portion of the 1995 loss distribution above 200?
A. Less than 600
B. At least 600, but less than 620
C. At least 620, but less than 640
D. At least 640, but less than 660
E. At least 660

22.77 (CAS9, 11/95, Q.11) (1 point) Which of the following are true?
1. A franchise deductible provides incentive for the insured to reduce the magnitude of losses.
2. The expected pure premium for a policy with a straight deductible of d is the expected pure premium prior to the deductible times the loss elimination ratio at d.
3. Pure premiums on policies with a constant deductible increase faster than the rate of inflation.

A. 1 only B. 3 only C. 1 and 2 D. 2 and 3 E. 1, 2, and 3

22.78 (CAS9, 11/95, Q.35) (4 points)
Using the information given below on a claim size distribution, compute the following:
a. (0.5 point) Pure premium for a $1,000,000 policy limit.
b. (1.5 points) Frequency, severity, and pure premium for a $1,000,000 maximum covered loss with a deductible of $1,000.
c. (2 points) The proportional increase in pure premium caused by 10% inflation, assuming the maximum covered loss remains at $1,000,000 and the deductible remains at $1,000.

Claim frequency is 0.15

Claim Size x	Distribution Function F(x)	Limited Expected Value E[X∧x]
909	0.0672	878
1,000	0.0734	962
1,100	0.0801	1,055
909,091	0.9731	73,493
1,000,000	0.9751	75,845
1,100,000	0.9769	78,243

Show all work.

22.79 (4B, 5/96, Q.10 & Course 3 Sample Exam, Q.18) (2 points)
You are given the following:
- Losses follow a lognormal distribution, with parameters $\mu = 7$ and $\sigma = 2$.
- There is a deductible of 2,000.
- 10 losses are expected each year.
- The number of losses and the individual loss amounts are independent.

Determine the expected number of annual losses that exceed the deductible if all loss amounts increased uniformly by 20%, but the deductible remained the same.
A. Less than 4.0
B. At least 4.0, but less than 5.0
C. At least 5.0, but less than 6.0
D. At least 6.0, but less than 7.0
E. At least 7.0

22.80 (4B, 11/96, Q.1) (1 point) Using the information in the following table, determine the total amount of losses from 1994 and 1995 in 1996 dollars.

Year	Actual Losses	Cost Index
1994	10,000,000	0.8
1995	9,000,000	0.9
1996	---	1.0

A. Less than 16,000,000
B. At least 16,000,000, but less than 18,000,000
C. At least 18,000,000, but less than 20,000,000
D. At least 20,000,000, but less than 22,000,000
E. At least 22,000,000

22.81 (4B, 11/96, Q.14) (2 points) You are given the following:
- Losses follow a Pareto distribution, with parameters $\theta = k$ and $\alpha = 2$, where k is a constant.
- There is a deductible of 2k.

Over a period of time, inflation has uniformly affected all losses, causing them to double, but the deductible remains the same. What is the new loss elimination ratio (LER)?
A. 1/6 B. 1/3 C. 2/5 D. 1/2 E. 2/3

22.82 (4B, 11/96, Q.25) (1 point)
The random variable X has a lognormal distribution, with parameters μ and σ.
If the random variable Y is equal to 1.10X what is the distribution of Y?
A. Lognormal with parameters 1.10μ and σ
B. Lognormal with parameters μ and 1.10σ
C. Lognormal with parameters $\mu + \ln 1.10$ and σ
D. Lognormal with parameters μ and $\sigma + \ln 1.10$
E. Not lognormal

22.83 (4B, 5/97, Q.17) (2 points) You are given the following:
- The random variable X has a Weibull distribution, with parameters $\theta = 625$ and $\tau = 0.5$.
- Z is defined to be 0.25X.

Determine the distribution of Z.
A. Weibull with parameters $\theta = 10,000$ and $\tau = 0.5$
B. Weibull with parameters $\theta = 2500$ and $\tau = 0.5$
C. Weibull with parameters $\theta = 156.25$ and $\tau = 0.5$
D. Weibull with parameters $\theta = 39.06$ and $\tau = 0.5$
E. Not Weibull

22.84 (4B, 5/97, Q.20) (2 points) You are given the following:

- Losses follow a distribution with density function $f(x) = (1/1000)\ e^{-x/1000}$, $0 < x < \infty$.
- There is a deductible of 500.
- 10 losses are expected to exceed the deductible each year.

Determine the expected number of losses that would exceed the deductible each year if all loss amounts doubled, but the deductible remained at 500.

A. Less than 10
B. At least 10, but less than 12
C. At least 12, but less than 14
D. At least 14, but less than 16
E. At least 16

22.85 (4B, 11/97, Q.4) (1 point) You are given the following:

- The random variable X has a distribution that is a mixture of a Burr distribution,

$$F(x) = 1 - \left(\frac{1}{1 + (x/\theta)^{\gamma}}\right)^{\alpha}, \text{ with parameters } \theta = \sqrt{1{,}000}\ , \alpha = 1 \text{ and } \gamma = 2,$$

and a Pareto distribution, with parameters θ = 1,000 and α = 1.

- Each of the two distributions in the mixture has equal weight.

Y is defined to be 1.10 X, which is also a mixture of a Burr distribution and a Pareto distribution.
Determine θ for the Burr distribution in this mixture.

A. Less than 32
B. At least 32, but less than 33
C. At least 33, but less than 34
D. At least 34, but less than 35
E. At least 35

22.86 (4B, 11/97, Q.26) (3 points) You are given the following:

- In 1996, losses follow a lognormal distribution, with parameters μ and σ.
- In 1997, losses follow a lognormal distribution with parameters μ + ln k and σ, where k is greater than 1.
- In 1996, 100p% of the losses exceed the mean of the losses in 1997.

Determine σ.
Note: z_p is the 100pth percentile of a normal distribution with mean 0 and variance 1.

A. 2 ln k B. $-z_p \pm \sqrt{z_p^2 - 2 \ln k}$ C. $z_p \pm \sqrt{z_p^2 - 2 \ln k}$

D. $\sqrt{-z_p \pm \sqrt{z_p^2 - 2 \ln k}}$ E. $\sqrt{z_p \pm \sqrt{z_p^2 - 2 \ln k}}$

22.87 (4B, 5/98, Q.25) (2 points) You are given the following:
● 100 observed claims occurring in 1995 for a group of risks have been recorded and are grouped as follows:

Interval	Number of Claims
(0, 250)	36
[250, 300)	6
[300, 350)	3
[350, 400)	5
[400, 450)	5
[450, 500)	0
[500, 600)	5
[600, 700)	5
[700, 800)	6
[800, 900)	1
[900, 1000)	3
[1000, ∞)	25

● Inflation of 10% per year affects all claims uniformly from 1995 to 1998.
Using the above information, determine a range for the expected proportion of claims for this group of risks that will be greater than 500 in 1998.
A. Between 35% and 40%
B. Between 40% and 45%
C. Between 45% and 50%
D. Between 50% and 55%
E. Between 55% and 60%

22.88 (4B, 11/98, Q.13) (2 points) You are given the following:
● Losses follow a distribution (prior to the application of any deductible) with cumulative distribution function and limited expected values as follows:

Loss Size (x)	F(x)	E[X∧x]
10,000	0.60	6,000
15,000	0.70	7,700
22,500	0.80	9,500
∞	1.00	20,000

● There is a deductible of 15,000 per loss and no maximum covered loss.
● The insurer makes a nonzero payment p.
After several years of inflation, all losses have increased in size by 50%, but the deductible has remained the same. Determine the expected value of p.
A. Less than 15,000
B. At least 15,000, but less than 30,000
C. At least 30,000, but less than 45,000
D. At least 45,000, but less than 60,000
E. At least 60,000

22.89 (4B, 5/99, Q.17) (2 points) You are given are following:
- In 1998, claim sizes follow a Pareto distribution, with parameters θ (unknown) and α = 2.
- Inflation of 6% affects all claims uniformly from 1998 to 1999.
- r is the ratio of the proportion of claims that exceed d in 1999 to the proportion of claims that exceed d in 1998.

Determine the limit of r as d goes to infinity.
A. Less than 1.05
B. At least 1.05, but less than 1.10
C. At least 1.10, but less than 1.15
D. At least 1.15, but less than 1.20
E. At least 1.20

22.90 (4B, 5/99, Q.21) (2 points) Losses follow a lognormal distribution, with parameters μ = 6.9078 and σ = 1.5174. Determine the percentage increase in the number of losses that exceed 1,000 that would result if all losses increased in value by 10%.
A. Less than 2%
B. At least 2%, but less than 4%
C. At least 4%, but less than 6%
D. At least 6%, but less than 8%
E. At least 8%

<u>22.91</u> **(CAS6, 5/99, Q.39)** (2 points) Use the information shown below to determine the one-year severity trend for the loss amounts in the following three layers of loss:
$0 - $50 $50 - $100 $100 - $200
- Losses occur in multiples of $40, with equal probability, up to $200, i.e., if a loss occurs, it has an equal chance of being $40, $80, $120, $160, or $200.
- For the next year, the severity trend will uniformly increase all losses by 10%.

22.92 (4B, 11/99, Q.26) (1 point) You are given the following:
- The random variable X follows a Pareto distribution, as per <u>Loss Models</u>, with parameters θ = 100 and α = 2.
- The mean excess loss function, $e_X(k)$, is defined to be E[X - k | X ≥ k].
- Y = 1.10 X.

Determine the range of the function $e_Y(k)/e_X(k)$ over its domain of [0, ∞).
A. (1, 1.10] B. (1, ∞) C. 1.10 D. [1.10, ∞) E. ∞

22.93 (**CAS9, 11/99, Q.38**) (1.5 points) Assume a ground-up claim frequency of 0.05.
Based on the following claim size distribution, answer the following questions. Show all work.

Claim size (d)	$F_X(d)$	$E(X \wedge d)$
\$909	0.075	\$870
\$1,000	0.090	\$945
\$1,100	0.100	\$1,040
Unlimited	1.000	\$10,000

a. (1 point) For a \$1,000 franchise deductible, what is the frequency of payments and the average payment per payment?
b. (0.5 point) Assuming a constant annual inflation rate of 10% across all loss amounts, what is the pure premium one year later if there is a \$1,000 franchise deductible?

22.94 (Course 151 Sample Exam #1, Q.7) (1.7 points)
For a certain insurance, individual losses in 1994 were uniformly distributed over (0, 1000).
A deductible of 100 is applied to each loss.
In 1995, individual losses have increased 5%, and are still uniformly distributed.
A deductible of 100 is still applied to each loss.
Determine the percentage increase in the standard deviation of amount paid.
(A) 5.00% (B) 5.25% (C) 5.50% (D) 5.75% (E) 6.00%

22.95 (Course 1 Sample Exam, Q.17) (1.9 points) An actuary is reviewing a study she performed on the size of claims made ten years ago under homeowners insurance policies.
In her study, she concluded that the size of claims followed an exponential distribution and that the probability that a claim would be less than \$1,000 was 0.250.
The actuary feels that the conclusions she reached in her study are still valid today with one exception: every claim made today would be twice the size of a similar claim made ten years ago as a result of inflation.
Calculate the probability that the size of a claim made today is less than \$1,000.
A. 0.063 B. 0.125 C. 0.134 D. 0.163 E. 0.250

22.96 (3, 5/00, Q.30) (2.5 points) X is a random variable for a loss.
Losses in the year 2000 have a distribution such that:
$E[X \wedge d] = -0.025d^2 + 1.475d - 2.25$, d = 10, 11, 12, ..., 26
Losses are uniformly 10% higher in 2001.
An insurance policy reimburses 100% of losses subject to a deductible of 11 up to a maximum reimbursement of 11.
Calculate the ratio of expected reimbursements in 2001 over expected reimbursements in the year 2000.
(A) 110.0% (B) 110.5% (C) 111.0% (D) 111.5% (E) 112.0%

Use the following information for the next two questions:
An insurer has excess-of-loss reinsurance on auto insurance. You are given:
(i) Total expected losses in the year 2001 are 10,000,000.
(ii) In the year 2001 individual losses have a Pareto distribution with

$$F(x) = 1 - \left(\frac{2000}{2000 + x}\right)^2, \ x > 0.$$

(iii) Reinsurance will pay the excess of each loss over 3000.
(iv) Each year, the reinsurer is paid a ceded premium, C_{year}, equal to 110% of
the expected losses covered by the reinsurance.
(v) Individual losses increase 5% each year due to inflation.
(vi) The frequency distribution does not change.

22.97 (3, 11/00, Q.41) (1.25 points) Calculate C_{2001}.
(A) 2,200,000 (B) 3,300,000 (C) 4,400,000 (D) 5,500,000 (E) 6,600,000

22.98 (3, 11/00, Q.42 & 2009 Sample Q.120) (1.25 points) Calculate C_{2002} / C_{2001}.
(A) 1.04 (B) 1.05 (C) 1.06 (D) 1.07 (E) 1.08

22.99 (3, 11/01, Q.6 & 2009 Sample Q.97) (2.5 points) A group dental policy has a negative binomial claim count distribution with mean 300 and variance 800.
Ground-up severity is given by the following table:

Severity	Probability
40	0.25
80	0.25
120	0.25
200	0.25

You expect severity to increase 50% with no change in frequency.
You decide to impose a per claim deductible of 100.
Calculate the expected total claim payment after these changes.
(A) Less than 18,000
(B) At least 18,000, but less than 20,000
(C) At least 20,000, but less than 22,000
(D) At least 22,000, but less than 24,000
(E) At least 24,000

22.100 (CAS3, 5/04, Q.17) (2.5 points) Payfast Auto insures sub-standard drivers.

- Each driver has the same non-zero probability of having an accident.
- Each accident does damage that is exponentially distributed with $\theta = 200$.
- There is a $100 per accident deductible and insureds only "report" claims that are larger than the deductible.
- Next year each individual accident will cost 20% more.
- Next year Payfast will insure 10% more drivers.

What will be the percentage increase in the number of "reported" claims next year?
A. Less than 15%
B. At least 15%, but less than 20%
C. At least 20%, but less than 25%
D. At least 25%, but less than 30%
E. At least 30%

22.101 (CAS3, 5/04, Q.29) (2.5 points) Claim sizes this year are described by a 2-parameter Pareto distribution with parameters $\theta = 1{,}500$ and $\alpha = 4$. What is the expected claim size per loss next year after 20% inflation and the introduction of a $100 deductible?
A. Less than $490
B. At least $490, but less than $500
C. At least $500, but less than $510
D. At least $510, but less than $520
E. At least $520

22.102 (CAS3, 5/04, Q.34) (2.5 points) Claim severities are modeled using a continuous distribution and inflation impacts claims uniformly at an annual rate of i.
Which of the following are true statements regarding the distribution of claim severities after the effect of inflation?
1. An Exponential distribution will have scale parameter $(1+i)\theta$.
2. A 2-parameter Pareto distribution will have scale parameters $(1+i)\alpha$ and $(1+i)\theta$.
3. A Paralogistic distribution will have scale parameter $\theta/(1+i)$.
A. 1 only B. 3 only C. 1 and 2 only D. 2 and 3 only E. 1, 2, and 3

22.103 (CAS3, 11/04, Q.33) (2.5 points)
Losses for a line of insurance follow a Pareto distribution with $\theta = 2{,}000$ and $\alpha = 2$.
An insurer sells policies that pay 100% of each loss up to $5,000. The next year the insurer changes the policy terms so that it will pay 80% of each loss after applying a $100 deductible. The $5,000 limit continues to apply to the original loss amount. That is, the insurer will pay 80% of the loss amount between $100 and $5,000. Inflation will be 4%.
Calculate the decrease in the insurer's expected payment per loss.
A. Less than 23%
B. At least 23%, but less than 24%
C. At least 24%, but less than 25%
D. At least 25%, but less than 26%
E. At least 26%

22.104 (SOA3, 11/04, Q.18 & 2009 Sample Q.127) (2.5 points)
Losses in 2003 follow a two-parameter Pareto distribution with $\alpha = 2$ and $\theta = 5$.
Losses in 2004 are uniformly 20% higher than in 2003.
An insurance covers each loss subject to an ordinary deductible of 10.
Calculate the Loss Elimination Ratio in 2004.
(A) 5/9 (B) 5/8 (C) 2/3 (D) 3/4 (E) 4/5

22.105 (CAS3, 11/05, Q.21) (2.5 points) Losses during the current year follow a Pareto distribution with $\alpha = 2$ and $\theta = 400,000$. Annual inflation is 10%.
Calculate the ratio of the expected proportion of claims that will exceed $750,000 next year to the proportion of claims that exceed $750,000 this year.
A Less than 1.105
B. At least 1.105, but less than 1.115
C. At least 1.115, but less than 1.125
D. At least 1.125, but less than 1.135
E. At least 1.135

22.106 (CAS3, 11/05, Q.33) (2.5 points)
In year 2005, claim amounts have the following Pareto distribution: $F(x) = 1 - \left(\frac{800}{800 + x}\right)^3$.
The annual inflation rate is 8%. A franchise deductible of 300 will be implemented in 2006.
Calculate the loss elimination ratio of the franchise deductible.
A. Less than 0.15
B. At least 0.15, but less than 0.20
C. At least 0.20, but less than 0.25
D. At least 0.25, but less than 0.30
E. At least 0.30

22.107 (SOA M, 11/05, Q.28 & 2009 Sample Q.209) (2.5 points)
In 2005 a risk has a two-parameter Pareto distribution with $\alpha = 2$ and $\theta = 3000$.
In 2006 losses inflate by 20%. An insurance on the risk has a deductible of 600 in each year.
P_i, the premium in year i, equals 1.2 times the expected claims.
The risk is reinsured with a deductible that stays the same in each year.
R_i, the reinsurance premium in year i, equals 1.1 times the expected reinsured claims.
$R_{2005}/P_{2005} = 0.55$. Calculate R_{2006}/P_{2006}.
(A) 0.46 (B) 0.52 (C) 0.55 (D) 0.58 (E) 0.66

22.108 (CAS3, 5/06, Q.26) (2.5 points) The aggregate losses of Eiffel Auto Insurance are denoted in euro currency and follow a Lognormal distribution with $\mu = 8$ and $\sigma = 2$.
Given that 1 euro = 1.3 dollars, which set of lognormal parameters describes the distribution of Eiffel's losses in dollars?
A. $\mu = 6.15$, $\sigma = 2.26$ B. $\mu = 7.74$, $\sigma = 2.00$ C. $\mu = 8.00$, $\sigma = 2.60$
D. $\mu = 8.26$, $\sigma = 2.00$ E. $\mu = 10.40$, $\sigma = 2.60$

22.109 (CAS3, 5/06, Q.39) (2.5 points) Prior to the application of any deductible, aggregate claim counts during 2005 followed a Poisson distribution with $\lambda = 14$. Similarly, individual claim sizes followed a Pareto distribution with $\alpha = 3$ and $\theta = 1000$. Annual severity inflation is 10%.
If all policies have a $250 ordinary deductible in 2005 and 2006, calculate the expected increase in the number of claims that will exceed the deductible in 2006.
A. Fewer than 0.41 claims
B. At least 0.41, but fewer than 0.45
C. At least 0.45, but fewer than 0.49
D. At least 0.49, but fewer than 0.53
E. At least 0.53

22.110 (CAS3, 11/06, Q.30) (2.5 points) An insurance company offers two policies.
Policy R has no deductible and no limit. Policy S has a deductible of $500 and a limit of $3,000; that is, the company will pay the loss amount between $500 and $3,000.
In year t, severity follows a Pareto distribution with parameters $\alpha = 4$ and $\theta = 3{,}000$.
The annual inflation rate is 6%.
Calculate the difference in expected cost per loss between policies R and S in year t+4.
A. Less than $500
B. At least $500, but less than $550
C. At least $550, but less than $600
D. At least $600, but less than $650
E. At least $650

22.111 (CAS5, 5/07, Q.46) (2.0 points) You are given the following information:

Claim	Ground-up Uncensored Loss Amount
A	$35,000
B	125,000
C	180,000
D	206,000
E	97,000

If all claims experience an annual ground-up severity trend of 8.0%, calculate the effective trend in the layer from $100,000 to $200,000 ($100,000 in excess of $100,000.) Show all work.

Solutions to Problems:

22.1. B. For the Pareto, the new theta is the old theta multiplied by the inflation factor of 1.2. Thus the new theta = (1.2)(5000) = 6000. Alpha is unaffected.
The average size of claim for the Pareto is: $\theta/(\alpha-1)$. In 1997, this is: 6000/(3-1) = **3000**.
Alternately, the mean in 1994 is 5000 / (3-1) = 2500. The mean increases by the inflation factor of 1.2; therefore the mean in 1997 is (1.2)(2500) = 3000.

22.2. D. The inflation factor is: 1.03^5 = 1.1593. For the Exponential, the new θ is the old θ multiplied by the inflation factor. Thus the new θ is: (200)(1.1593) = 231.86.
The variance for the Exponential Distribution is: θ^2, which in 2009 is: 231.86^2 = **53,758**.
Alternately, the variance in 2004 is: 200^2 = 40,000. The variance increases by the square of the inflation factor; therefore the variance in 2009 is: $(1.1593^2)(40{,}000)$ = 53,759.

22.3. C. For a Burr, the new theta = $\theta(1+r)$ = (19,307)(1.3) = 25,099. (Alpha and gamma are unaffected.) Thus in 1996, 1 - F(10,000) = $\{1/(1 + (10{,}000/25{,}099)^{0.7})\}^2$ = **43.0%.**
Alternately, \$10,000 in 1996 corresponds to \$10,000 / 1.3 = \$7692 in 1992.
Then in 1992, 1 - F(7692) = $\{1/(1 + (7692/19{,}307)^{0.7})\}^2$ = 43.0%.

22.4. B. For the Gamma Distribution, θ is multiplied by the inflation factor of 1.1, while α is unaffected. Thus the parameters in 1996 are: $\alpha = 2$, $\theta = 110$.

22.5. E. For the Pareto, the new theta is the old theta multiplied by the inflation factor of 1.25. Thus the new theta = (1.25)(15,000) = 18,750. Alpha is unaffected. The average size of claim for data truncated and shifted at 25,000 in 1999 is the mean excess loss, e(25,000), in 1999.
For the Pareto e(x) = $(x+\theta) / (\alpha-1)$.
In 1999, e(25,000) = (25,000 + 18,750) / (1.5 - 1) = **87,500**.
Alternately, \$25,000 in 1999 corresponds to \$25,000 / 1.25 = \$20,000 in 1995.
The average size of claim for data truncated and shifted at 20,000 in 1995 is the mean excess loss, e(20,000), in 1995. For the Pareto e(x) = $(x+\theta) / (\alpha-1)$.
In 1995, e(20,000) = (20,000 + 15,000) / (1.5 - 1) = 70,000. However, we need to inflate this back up to get the average size in 1999 dollars: (70,000)(1.25) = **87,500**.
Comment: The alternate solution uses the fact that the effect of a deductible keeps up with inflation provided the limit keeps up with inflation, or equivalently if the limit keeps up with inflation, then the mean excess loss increases by the inflation rate.

22.6. B. Integrating the density function, $F_X(x) = (2.5)(1/2 - 1/x)$, $2 < x < 10$.
$F_Z(z) = F_X(x) = (2.5)(1/2 - 1/x) = (2.5)(1/2 - 1.2/z)$.
Alternately, $F_Z(z) = F_X(z/(1+r)) = F_X(z/1.2) = (2.5)(1/2 - 1.2/z)$.
$F_Z(6) = (2.5)(1/2 - 1.2/6) = 0.75$. $1 - F_Z(6) =$ **0.25**.
Alternately, 6 in 1996 is 6 /1.2 = 5 in terms of 1992.
In 1992, $F_X(5) = (2.5)(1/2 - 1/5) = 0.75$. $1 - F_X(5) = 0.25$.
Comment: Note that the domain becomes [2.4, 12] in 1996.

22.7. C. For the Loglogistic, θ is multiplied by 80 (the inflation factor), while the other parameter γ is unaffected.

22.8. E. For the LogNormal Distribution μ has ln(80) added to it, while σ is unaffected.
New μ = 10 + ln(80) = 14.38 and σ = 3.

22.9. D. For the Weibull Distribution, θ is multiplied by 80, while τ is unaffected.
New θ = (625) (80) = 50,000.

22.10. C. For the Paralogistic, θ is multiplied by 80 (the inflation factor), while the other parameter α is unaffected.

22.11. E. In 1996 x becomes 1.1x. z = 1.1x. x = z/1.1. ln(x) = ln(z/1.1) = ln(z) - ln(1.1).
Thus in 1996 the distribution function is: $F(z) = \Gamma[\alpha; \lambda\ln(x)] = \Gamma[\alpha; \lambda\{\ln(z) - \ln(1.1)\}]$.
This is not of the same form, so the answer is none of the above.
Comment: This is called the LogGamma Distribution. If ln(x) follows a Gamma Distribution, then x follows a LogGamma Distribution. Under uniform inflation, ln(x) becomes ln(x) + ln(1+r). If you add a constant amount to a Gamma distribution, then you no longer have a Gamma distribution. Which is why under uniform inflation a LogGamma distribution is not reproduced.

22.12. The sum of 50 independent, identically distributed Exponentials each with θ = 800 is Gamma with α = 50 and θ = 800. The average is 1/50 times the sum, and has a Gamma Distribution with α = 50 and θ = 800/50 = 16.

22.13. B. 1. F. The skewness is unaffected by uniform inflation. (The numerator of the skewness is the third central moment which would be multiplied by 1.1^3.)
2. T. Since each claim size is increased by 10%, the place where 70% of the claims are less and 30% are more is also increased by 10%. Under uniform inflation, each percentile is increased by the inflation factor.
3. F. Under uniform inflation, if the deductible increases to keep up with inflation, then the Loss Elimination Ratio is unaffected. So in 1996 the Loss Elimination Ratio at $1100 is 10% not 11%.

22.14. B. This is a mixed Gamma-Weibull Distribution.
The Gamma has parameters $\alpha = 3$ and $\theta = 1/10$, and density: $\theta^{-\alpha}x^{\alpha-1} e^{-x/\theta} / \Gamma(\alpha) = 500x^2 e^{-10x}$.
The Weibull has parameters $\theta = 1/(20^{1/4})$ and $\tau = 4$, and density:
$\tau(x/\theta)^\tau \exp[-(x/\theta)^\tau] / x = 80x^3\exp(-20x^4)$. In the mixed distribution, the Gamma is given a weight of 0.75, while the Weibull is given a weight of 0.25. Note that (0.75)(500) = 375, and (0.25)(80) = 20. Under uniform inflation of 25%, the Gamma has parameters:
$\alpha = 3$ and $\theta = (1/10)1.25 = 1/8$, and density: $\theta^{-\alpha}x^{\alpha-1} e^{-x/\theta} / \Gamma(\alpha) = 256x^2 e^{-8x}$.
Under uniform inflation of 25%, the Weibull has parameters: $\theta = 1.25/(20^{1/4})$.
$\theta^{-1/4} = 20/(1.25)^4 = 8.192$ and $\tau = 4$, and density: $\tau(x/\theta)^\tau\exp(-(x/\theta)^\tau) / x$
$= 32.768x^3\exp(-8.192x^4)$. Therefore, the mixed distribution after inflation has a density of:
$(0.75)\{256x^2e^{-8x}\} + (0.25)\{32.768x^3\exp(-8.192x^4)\} =$ $\mathbf{192x^2e^{-8x} + 8.192x^3\exp(-8.192x^4)}$.
Comment: For a mixed distribution under uniform inflation, the weights are unaffected, while the each separate distribution is affected as usual.

22.15. D. This is a LogNormal Distribution with parameters (prior to inflation) of $\mu = 7$ and $\sigma = 3$.
Thus posterior to inflation of 40%, one has a LogNormal Distribution with parameters of:
$\mu = 7 + \ln(1.4) = 7.336$ and $\sigma = 3$. For the LogNormal, $S(x) = 1 - \Phi[(\ln(x) - \mu)/\sigma]$. Prior to inflation, $S(1000) = 1 - \Phi[(\ln(x) - \mu)/\sigma] = 1 - \Phi[(\ln(1000) - 7)/3] = 1 - \Phi(-0.31) = \Phi(0.03) = 0.5120$.
After inflation, $S(1000) = 1 - \Phi[(\ln(x) - \mu)/\sigma] = 1 - \Phi[(\ln(1000) - 7.336)/3] =$
$1 - \Phi(-0.143) = \Phi(0.14) = 0.5557$. Prior to inflation, 173 losses are expected to exceed the deductible each year. The survival function increased from 0.5120 to 0.5557 after inflation.
Thus after inflation one expects to exceed the deductible per year:
173(0.5557)/0.5120 = **187.8** claims.
Alternately, a limit of 1000 after inflation is equivalent to 1000/1.4 = 714.29 prior to inflation.
Thus the tail probability after inflation at 1000 is the same as the tail probability at 714.29 prior to inflation.
Prior to inflation, $1 - F(714.29) = 1 - \Phi[(\ln(x) - \mu)/\sigma] = 1 - \Phi[(\ln(714.3) - 7)/3] = 1 - \Phi[-0.14] =$
$\Phi[0.14] = 0.5557$. Proceed as before.
Comment: The expected number of claims over a fixed deductible increases under uniform inflation.

22.16. A. For the Pareto Distribution, $E[X \wedge x] = \{\theta/(\alpha-1)\}\{1-(\theta/(\theta+x))^{\alpha-1}\} = 20\{1 - (1 + x/40)^{-2}\}$.
$E[X \wedge 5/1.06] = 20\{1 - (1 + 4.717/40)^{-2}\} = 3.997$.
$E[X \wedge 5] = 20\{1 - (1+ 5/40)^{-2}\} = 4.198$. $E[X \wedge 25/1.06] = 20\{1 - (1 + 23.585/40)^{-2}\} = 12.085$.
$E[X \wedge 25] = 20\{1 - (1+ 25/40)^{-2}\} = 12.426$.
In 2001 the expected payments are: $E[X \wedge 25] - E[X \wedge 5] = 12.426 - 4.198 = 8.228$.
A deductible of 5 and maximum covered loss of 25 in the year 2002, when deflated back to the year 2001, correspond to a deductible of: 5/1.06 = 4.717, and a maximum covered loss of: 25/1.06 = 23.585. Therefore, reinflating back to the year 2002, the expected payments in the year 2002 are: $(1.06)(E[X \wedge 23.585] - E[X \wedge 4.717]) = (1.06)(12.085 - 3.997) = 8.573$.
The ratio of expected payments in 2002 over the expected payments in the year 2001 is:
8.573/ 8.228 = **1.042**.
Alternately, the insurer's average payment per loss is: $(1+r)\ c\ (E[X \wedge u/(1+r)] - E[X \wedge d/(1+r)])$.
c = 100%, u = 25, d = 5. r = 0.06 for the year 2002 and r = 0 for the year 2001.
Then proceed as previously.

22.17. E. Inflation is 8% per year for 9 years, thus the inflation factor is $1.08^9 = 1.999$.
Thus 1000 in the year 2002 is equivalent to 1000/1.999 = 500 in 1993. There are 457 claims excess of 500 in 1993; this is 457/1000 = **45.7%**.
Comment: Note the substantial increase in the proportion of claims over a fixed limit. In 1993 there are 32.6% of the claims excess of 1000, while in 2002 there are 45.7%.

22.18. C. In general one substitutes for x = z / (1+r), and for the density function
$f_Z(z) = f_X(x) / (1+r)$. In this case, 1 + r = 1.2.
Thus, $f_Z(z) = f_X(x) / 1.2 = \mu \exp[- (x-\mu)^2 /(2\beta x)] / \{1.2 \sqrt{2\beta\pi\ x^3}\} =$

$\mu \exp[- (\{z/1.2\}-\mu)^2 /(2\beta\{z/1.2\})] / \{1.2 \sqrt{2\beta\pi\ \{z/1.2\}^3}\} =$

$(1.2\mu) \exp[- (z-(1.2\mu))^2 /\{2(1.2\beta)z\}] / \sqrt{2(1.2\beta)\pi\ z^3}$.

This is of the same form, but with parameters 1.2μ and 1.2β, rather than μ and β.
Comment: This is an Inverse Gaussian Distribution. Let $\beta = \mu^2 / \theta$ and one has the parameterization in Loss Models, with parameters μ and θ. Since under uniform inflation, for the Inverse Gaussian each of μ and θ are multiplied by the inflation factor, so is $\beta = \mu^2 / \theta$.
Recall that that under change of variables when working with the density you need to divide by $\frac{dz}{dx} = 1+r$, since $\frac{dF}{dz} = \frac{dF}{dx} / \frac{dz}{dx}$.

22.19. C. This is an Exponential Distribution with θ = 5000. $\Rightarrow$ new θ = (5000)(1.4) = 7000.
For the Exponential Distribution, $E[X \wedge x] = \theta(1-e^{-x/\theta})$. The mean is θ.
The losses excess of a limit are proportional to $E[X] - E[X \wedge x] = \theta e^{-x/\theta}$.
In 1998 this is for x = 1000: $5000e^{-1000/5000}$ = 4094.
In 2007 this is for x = 1000: $7000e^{-1000/7000}$ = 6068.
The increase is: (6068/4094) - 1 = **48.2%**.
Comment: In general excess losses over a fixed limit, increase faster than the rate of inflation.
Note that $E[X] - E[X \wedge x] = S(x)e(x) = R(x)E[X] = \theta e^{-x/\theta}$.

22.20. B. This is an Exponential Distribution with θ = 5000.
Therefore, the new theta = (5000) 1.4 = 7000.
For the Exponential Distribution, $E[X \wedge x] = \theta(1-e^{-x/\theta})$. The mean is θ.
The losses excess of a limit are proportional to $E[X] - E[X \wedge x] = \theta e^{-x/\theta}$.
In 1998 this is for x = 1000: $5000e^{-1000/5000} = 5000e^{-0.2}$.
In 2007 this is for x = 1400: $7000e^{-1400/7000} = 7000e^{-0.2}$.
The increase is: (7000/5000) - 1 = **40.0%**.
Comment: If the limit keeps up with inflation, then excess losses increase at the rate of inflation.

22.21. D. The inflation factor from 1990 to 1999 is: $(1.04)^9$ = 1.423. Thus the parameters of the 1999 LogNormal are: 3 + ln(1.423) and σ. Therefore, the mean of the 1999 LogNormal is:
$\text{Mean}_{99} = \exp(3 + \ln(1.423) + \sigma^2/2) = 1.423 \exp(3 + \sigma^2/2) = 1.423\ \text{Mean}_{90}$.
Therefore, $(\ln(\text{Mean}_{99}) = 3 + \ln(1.423) + \sigma^2/2$. $F_{90}(\text{Mean}_{99}) = \Phi[(\ln(\text{Mean}_{99})) - \mu)/\sigma] =$
$\Phi[(3 + \ln 1.423 + \sigma^2/2 - 3)/\sigma] = \Phi[(\ln 1.423 + \sigma^2/2)/\sigma]$.
We are given that in 1990 5% of the losses exceed the mean of the losses in 1999.
Thus, $F_{90}(\text{Mean}_{99}) = 0.95$. Therefore, $\Phi((\ln 1.423 + \sigma^2/2)/\sigma) = 0.95$.
$\Phi(1.645) = 0.95$. $\Rightarrow (\ln 1.423 + \sigma^2/2)/\sigma = 1.645$. $\Rightarrow \sigma^2/2 - 1.645\sigma + \ln 1.423 = 0$. $\Rightarrow$
$\sigma = 1.645 \pm \sqrt{1.645^2 - 2 \ln 1.423} = 1.645 \pm \sqrt{2.000}$ = **0.231 or 3.059**.

22.22. E. The inflation factor from 1995 to 2001 is 206.8/170.3 = 1.214.
For the Weibull Distribution, θ is multiplied by 1 + r, while τ is unaffected.
Thus in 2001 the new θ is: (1) (1.214) = 1.214.
The survival function of the Weibull is $S(x) = \exp[-(x/\theta)^\tau]$.
In 1995, $S(10{,}000) = \exp[-(1000^{0.3})) = 0.000355$.
In 2001, $S(10{,}000) = \exp[-(1000/1.214)^{0.3}] = 0.000556$.
The ratio of survival functions is: 0.000556/0.000355 = 1.57 or a **57%** increase in the expected number of claims excess of the deductible.
Comment: Generally for the Weibull, the ratio of the survival functions at x is:
$\exp[-(x/ (1+r)\theta)^\tau] / \exp[-(x/\theta)^\tau] = \exp[(x/\theta)^\tau \{1 - 1/(1+r)\}]$.

22.23. B. The inflation factor from 1994 to 1998 is: (1.05)(1.03)(1.07)(1.06) = 1.2266.
For the LogNormal Distribution, μ has ln(1+r) added, while σ is unaffected.
Thus in 1998 the new μ is: 3 + ln(1.2266) = 3.2042.
The Limited Expected Value for the LogNormal Distribution is:
$E[X \wedge x] = \exp(\mu + \sigma^2/2)\Phi[(\ln x - \mu - \sigma^2)/\sigma] + x\{1 - \Phi[(\ln x - \mu)/\sigma]\}$.
In 1994, $E[X \wedge 25{,}000] = e^{11}\Phi[(\ln(25{,}000) - 19)/4] + 25{,}000\{1 - \Phi[(\ln((25{,}000) - 3)/4]\} =$
$59{,}874\Phi[-2.22] + 25{,}000\{1 - \Phi[1.78]\} = 59{,}874(1 - 0.9868) + 25{,}000(1 - 0.9625) = 1728$.
In 1998, $E[X \wedge 25{,}000] =$
$e^{11.2042}\Phi[(\ln(25{,}000) - 19.2042)/4] + 25{,}000\{1 - \Phi[(\ln((25{,}000) - 3.2043)/4]\} =$
$73{,}438\Phi[-2.27] + 25{,}000\{1 - \Phi[1.73]\} = 73{,}438(1 - 0.9884) + 25{,}000\{1 - 0.9582) = 1897$.
The ratio of Limited Expected Values is: 1897/1728 = 1.098 or a **9.8%** increase in the expected dollars of claims between 1994 and 1998.
Alternately, in 1994, $E[X \wedge 20{,}382] = e^{11}\Phi[(\ln(20{,}382) - 19)/4] + 20{,}382\{1 - \Phi[(\ln((20{,}382) - 3)/4]\}$
$= 59{,}874\Phi[-2.27] + 20{,}382\{1 - \Phi[1.73]\} = 59{,}874(1 - 0.9884) + 20{,}382(1 - 0.9582) = 1547$.
In 1998 the average payment per loss is:
$(1+r)\ c\ (E[X \wedge L/(1+r)] - E[X \wedge d/(1+r)]) = 1.2266\ E[X \wedge 25{,}000/1.2266] =$
$1.2266\ E[X \wedge 20{,}382] = (1.2266)(1547) = 1898$. Proceed as before.
<u>Comment</u>: For a fixed limit, basic limit losses increase at less than the overall rate of inflation.
Here unlimited losses increase 22.7%, but limited losses increase only 9.8%.
When using the formula for the average payment per loss, use the original LogNormal for 1994.

22.24. D. In general a layer of loss is proportional to the integral of the survival function.
In 1994, $S(x) = 10^{15}\ x^{-3}$. The integral from 500,000 to 2,000,000 of S(x) is:
$10^{15}(500{,}000^{-2} - 2{,}000{,}000^{-2})/2 = 1875$.
In 1999, the Distribution Function is gotten by substituting x = z/1.2.
$F(z) = 1 - (100{,}000/(z/1.2))^3 = 1 - (120{,}000/z)^3$ for z > \$120,000.
Thus in 1999, the integral from 500,000 to 2,000,000 of the survival function is:
$(120{,}000)^3(500{,}000^{-2} - 2{,}000{,}000^{-2})/2 = 3240$.
3240 / 1875 = 1.728, representing a **72.8%** increase.
Alternately, this is a Single Parameter Pareto Distribution, with α = 3 and θ = 100,000.
Under uniform inflation of 20%, θ becomes 120,000 while α is unaffected.
$E[X \wedge x] = \theta\ [\{\alpha - (x/\theta)^{1-\alpha}\} / (\alpha - 1)]$ for $\alpha > 1$.
Then the layer from 500,000 to 2,000,000 is proportional to:
$E[X \wedge 2{,}000{,}000] - E[X \wedge 500{,}000] = (\theta/(\alpha-1))\ \{(\alpha - (2{,}000{,}000/\theta)^{1-\alpha}) - (\alpha - (500{,}000/\theta)^{1-\alpha})\} =$
$(\theta/(\alpha - 1))\{(500{,}000/\theta)^{1-\alpha} - (2{,}000{,}000/\theta)^{1-\alpha}\}$.
In 1994, $E[X \wedge 2{,}000{,}000] - E[X \wedge 500{,}000] = (100{,}000/2)\{5^{-2} - 20^{-2}) = 1875$.
In 1999, $E[X \wedge 2{,}000{,}000] - E[X \wedge 500{,}000] = (120{,}000/2)\{4.1667^{-2} - 16.6667^{-2}) = 3240$.
3240 / 1875 = 1.728, representing a **72.8%** increase.
<u>Comment</u>: As shown in “A Practical Guide to the Single Parameter Pareto Distribution,” by Stephen W. Philbrick, PCAS LXXII, 1985, pp. 44, for the Single Parameter Pareto Distribution, a layer of losses is multiplied by $(1+r)^\alpha$. In this case $1.2^3 = 1.728$.

22.25. D. The total inflation factor is $(1.03)^8 = 1.2668$.
Under uniform inflation both parameters of the Inverse Gaussian are multiplied by 1 + r = 1.2668. Thus in 2009 the parameters are: $\mu = 3(1.26668) = 3.8003$ and $\theta = 10(1.2668) = 12.668$.
Thus the variance in 2009 is: $\mu^3 / \theta = 3.8003^3 / 12.668 =$ **4.33.**
Alternately, the variance in 2001 is: $\mu^3 / \theta = 3^3 / 10 = 2.7$. Under uniform inflation, the variance is multiplied by $(1+r)^2$. Thus in 2009 the variance is: $(2.7)(1.2668^2) = 4.33$.

22.26. A. $S(10{,}000) = (5000/(5000 + 10{,}000))^3 = 1/27$.
Thus we want x such that $S(x) = (1/2)(1/27) = 1/54$.
$(5000/(5000 + x))^3 = 1/54. \Rightarrow x = 5000(54^{1/3} - 1) =$ **13,899**.

22.27. C. During the year 2006, the losses are Pareto with $\alpha = 3$ and $\theta = (1.15)(5000) = 5750$.
$S(10{,}000) = \{5750/(5750 + 10{,}000)\}^3 = 0.04866$.
Thus we want x such that $S(x) = (1/2)(0.04866) = 0.02433$.
$\{5750/(5750 + x)\}^3 = 0.02433. \Rightarrow x = 5750(1/0.02433^{1/3} - 1) =$ **14,094**.

22.28. A. The new theta = (1/1000)1.25 = 1/800.
Thus in 1998 the density is: $e^{-x/\theta}/\theta = 800e^{-800x}$.

22.29. E. The inflation factor is $1.04^3 = 1.1249$.

Probability	2003 Amount Loss Amount	2003 Insurer Payment	2006 Amount Loss Amount	2006 Insurer Payment
0.1667	1000	0	1124.9	0.0
0.3333	2000	0	2249.7	249.7
0.3333	5000	3000	5624.3	3624.3
0.1667	10000	8000	11248.6	9248.6
Average	4166.7	2333.3	4686.9	2832.8

2832.8 / 2333.3 = 1.214, therefore the insurer's expected payments increased **21.4%**.
Comment: Similar to 4B, 5/94, Q.21.

22.30. B. The second moment of the LogNormal in 2007 is $\exp[(2)(5) + (2)(0.7^2)] = 58{,}689$.
The second moment increases by the square of the inflation factor: $(1.04^3)^2$ (58,689) = **74,260**.
Alternately, the LogNormal in 2010 has parameters of: $\mu = 5 + \ln[1.04^3] = 5.1177$, and $\sigma = 0.7$.
The second moment of the LogNormal in 2010 is $\exp[(2)(5.1177) + (2)(0.7^2)] =$ **74,265.**

22.31. D. Deflating, the 1000 deductible in 2005 is equivalent to a deductible of 1000/1.1 = 909 in 2005. Work in 2005 and then reinflate back up to the 2008 level by multiplying by 1.1.
Average payment per loss is: 1.1{E[X] - E[X ∧ 909]}

$$= 1.1\{ \int_0^\infty x\, f(x)\, dx - \int_0^{909} x\, f(x)\, dx - 909\, S(909) \} = 1.1 \int_{909}^\infty x\, f(x)\, dx - 1000 \int_{909}^\infty f(x)\, dx.$$

Average payment per loss is:

$$1.1\, \{E[X] - E[X \wedge 909]\} = 1.1\{ \int_0^\infty S(x)\, dx - \int_0^{909} S(x)\, dx \} = 1.1 \int_{909}^\infty S(x)\, dx.$$

Average payment per loss is: $1.1\, E[(X - 909)_+] = 1.1 \int_0^{909} \{x\, f(x) - S(x)\}\, dx.$

Average payment per loss is: $1.1\, \{E[X] - E[X \wedge 909]\} = 1.1 \int_0^\infty x\, f(x)\, dx - 1.1 \int_0^{909} S(x)\, dx$

$$= 1.1 \int_{909}^\infty x\, f(x)\, dx + 1.1 \int_0^{909} \{x\, f(x) - S(x)\}\, dx.$$

22.32. For convenience put everything in millions of dollars.
Prior to inflation, $E[X \wedge x] = 0.5 - 0.5^2 / (0.5 + x)$.
Thus prior to inflation the average payment per loss is:
$E[X \wedge 1] - E[X \wedge R] = 0.5^2 / (0.5 + R) - 0.5^2 / (0.5 + 1) = 0.5^2 / (0.5 + R) - 0.166667$.
After inflation the average payment per loss is:
$1.1(E[X \wedge 1/1.1] - E[X \wedge R/1.1]) =$
$(1.1)(0.5^2) / (0.5 + R/1.1) - (1.1)(0.5^2) / (0.5 + 1/1.1) = 0.55^2 / (0.55 + R) - 0.195,161$.
Setting the ratio of the two average payments per loss equal to 1.1:
$(1.1)\{0.5^2 / (0.5 + R) - 0.166667\} = 0.55^2 / (0.55 + R) - 0.195161. \Rightarrow$
$0.011827(0.5 + R)\, (0.55 + R) + (1.1)(0.5^2)(0.55 + R) - 0.55^2(0.5 + R) = 0. \Rightarrow$
$0.011827R^2 - 0.015082R + 0.0032524 = 0.$

$$R = \frac{0.015082 \pm \sqrt{0.015082^2 - (4)(0.011827)(0.0032524)}}{(2)(0.011827)} = 0.6376 \pm 0.3627.$$

R = 0.275 or 1.000. ⇒ R = **$275,000**.
Comment: A rewritten version of CAS9, 11/99, Q.39.

22.33. C. During 2013, the losses follow an Exponential with mean: (1.08)(5000) = 5400.
An Exponential distribution truncated and shifted from below is the same Exponential Distribution, due to the memoryless property of the Exponential. Thus the nonzero payments are Exponential with mean 5400. The probability of a nonzero payment is the probability that a loss is greater than the deductible of 1000; $S(1000) = e^{-1000/5400} = 0.8310$. Thus the payments of the insurer can be thought of as a compound distribution, with Bernoulli frequency with mean 0.8310 and Exponential severity with mean 5400. The variance of this compound distribution is:
(Mean Freq.)(Var. Sev.) + (Mean Sev.)^2(Var. Freq.) =
$(0.8310)(5400^2) + (5400)^2 \{(0.8310)(1 - 0.8310)\}$ = **28.3 million**.
Equivalently, the payments of the insurer in this case are a two point mixture of an Exponential with mean 5400 and a distribution that is always zero, with weights 0.8310 and 0.1690.
This has a first moment of: (5400)(0.8310) + (0)(0.1690) = 4487.4,
and a second moment of: $\{(2)(5400^2)\}(0.8310) + (0^2)(0.1690)$ = 48,463,920.
Thus the variance is: 48,463,920 - 4487.4^2 = **28.3 million**.
Comment: Similar to 3, 11/00, Q.21, which does not include inflation.

22.34. E. In 2005, the average payment per payment is:
$(E[X] - E[X \wedge 5000])/S(5000) = (E[X] - 3000)/(1 - 0.655) = 2.8986E[X] - 8695.7$.
In 2005, the average payment per payment is:
$1.25(E[X] - E[X \wedge 5000/1.25])/S(5000/1.25) = 1.25(E[X] - E[X \wedge 4000])/(1 - F(4000))$
$= 1.25(E[X] - 2624)/(1 - 0.590) = 3.0488E[X] - 8000$.
Set $1.15(2.8986E[X] - 8695.7) = 3.0488E[X] - 8000$. $\Rightarrow$ E[X] = 2000/.2846 = **7027**.

22.35. D. In 2015, the losses are Pareto with $\alpha = 5$ and $\theta = (1.25)(40) = 50$.
With a deductible of 10, the non-zero payments are Pareto with $\alpha = 5$ and $\theta = 50 + 10 = 60$.
The mean of this Pareto is: 60/4 = 15.
The second moment of this Pareto is: $\frac{(2)\,(60^2)}{(5 - 1)\,(5 - 2)} = 600$.
The variance of this Pareto is: $600 - 15^2$ = **375**.

22.36. C. The non-zero payments are Pareto with $\alpha = 5$ and $\theta = 50 + 10 = 60$,
with mean: 15, second moment: 600, and variance: $600 - 15^2 = 375$.
The probability of a non-zero payment is the survival function at 10 of the original Pareto:
$\left(\frac{50}{50 + 10}\right)^5 = 0.4019$.

Thus Y^L is a two-point mixture of a Pareto distribution $\alpha = 5$ and $\theta = 60$, and a distribution that is always zero, with weights 0.4019 and 0.5981.
The mean of the mixture is: (0.4019)(15) + (0.5981)(0) = 6.029.
The second moment of the mixture is: $(0.4019)(600) + (0.5981)(0^2) = 241.14$.
The variance of this mixture is: $241.14 - 6.029^2 =$ **205**.
Alternately, Y^L can be thought of as a compound distribution,
with Bernoulli frequency with mean 0.4019 and Pareto distribution $\alpha = 5$ and $\theta = 60$.
The variance of this compound distribution is:
(Mean Freq.)(Var. Sev.) + $(\text{Mean Sev.})^2$(Var. Freq.) =
$(0.4019)(375) + (15)^2 \{(0.4019)(0.5981)\} =$ **205**.

22.37. B. inflation factor = $(1+r) = 1.03^5 = 1.1593$. coinsurance factor = c = 0.90.
Maximum Covered Loss = u = 50,000. Deductible amount = d = 10,000.
L/(1+r) = 43,130. d/(1+r) = 8626.
$E[X \wedge x] = \exp(\mu + \sigma^2/2)\ \Phi[(\ln x - \mu - \sigma^2)/\sigma] + x\ \{1 - \Phi[(\ln x - \mu)/\sigma]\}$.
$E[X \wedge u/(1+r)] = E[X \wedge 43{,}130] =$
$\exp(10.02)\Phi[(\ln(43{,}130) - 9.7 - 0.64)/0.8] + (43{,}130)\ \{1 - \Phi[(\ln(43{,}130) - 9.7)/0.8]\} =$
$(22{,}471)\Phi[0.41] + (43{,}130)\{1 - \Phi[1.21]\} = (22{,}471)(0.6591) + (43{,}130)\{1 - 0.8869\} = 19{,}689$.
$E[X \wedge d/(1+r)] = E[X \wedge 8626] =$
$\exp(10.02)\Phi[(\ln(8626) - 9.7 - 0.64)/0.8] + (8626)\ \{1 - \Phi[(\ln(8626) - 9.7)/0.8]\} =$
$(22{,}471)\Phi[-1.60] + (8626)\{1 - \Phi[-0.80]\} = (22{,}471)(0.0548) + (8626)\ (0.7881) = 8030$.
The average payment per loss is: $(1+r)\ c\ (E[X \wedge L/(1+r)] - E[X \wedge d/(1+r)]) =$
(1.1593)(0.9)(19,689 - 8030) = **12,165.**
<u>Comment</u>: In 2002, the average payment per loss is: $(0.9)(E[X \wedge 50{,}000] - E[X \wedge 10{,}000]) \cong$ (0.9)(20,345 - 9073) = 10,145. Thus it increased from 2002 to 2007 by: 12,165/10,145 - 1 = 19.9%. The maximum covered loss would cause the increase to be less than the rate of inflation of 15.9%, while the deductible would cause it to be greater. In this case the deductible had a bigger impact than the maximum covered loss on the rate of increase. When using the formula for the average payment for loss, use the parameters of the original LogNormal for 2002. This formula is equivalent to deflating the 2007 values back to 2002, working in 2002, and then reinflating back up to 2007. One could instead inflate the LogNormal to 2007 and work in 2007.

22.38. A. Inflation factor = (1+r) = 1.03^5 = 1.1593. coinsurance factor = c = 0.90.
Maximum Covered Loss = u = 50,000. Deductible amount = d = 10,000.
u/(1+r) = 43,130. d/(1+r) = 8626. S(d/(1+r)) = 1 - Φ[(ln(8626) -9.7)/0.8] = 1 - Φ[-0.80] = 0.7881.
The average payment per non-zero payment is:
(1+r)c(E[X∧u/(1+r)] - E[X∧d/(1+r)]) / S(d/(1+r)) = (1.1593)(0.9)(19,689 - 8030)/0.7881 = **15,435.**
Comment: The average payment per non-zero payment is:
average payment per loss / S(d/(1+r)) = 12,165 /0.7881 = 15,436.

22.39. D. $E[(X \wedge x)^2] = \exp[2\mu + 2\sigma^2]\, \Phi\left[\frac{\ln(x) - \mu - 2\sigma^2}{\sigma}\right] + x^2 \{1 - \Phi\left[\frac{\ln(x) - \mu}{\sigma}\right]\}$.

$E[(X \wedge u/(1+r))^2] = E[(X \wedge 43{,}130)^2] =$
$\exp(20.68)\Phi[\{\ln(43{,}130) - 9.7 - (2)(0.8^2)\}/0.8] + (43{,}130)^2 \{1 - \Phi[(\ln(43{,}130) - 9.7)/0.8]\} =$
$e^{20.68}\, \Phi[-0.39] + (43{,}130^2) \{1 - \Phi[1.21]\} = (e^{20.68})(0.3483) + (43{,}130^2) \{(1 - 0.8869) =$
543,940,124.
$E[(X \wedge d/(1+r))^2] = E[(X \wedge 8626)^2] =$
$\exp(20.68)\Phi[\{\ln(8626) - 9.7 - (2)(0.8^2)\}/0.8] + (8626^2) \{1 - \Phi[(\ln(8626) - 9.7)/0.8]\} =$
$(e^{20.68})\Phi[-2.40] + (8626^2) \{1 - \Phi[-0.80]\} = (e^{20.68})(0.0082) + (8626^2) (0.7881) = 66{,}493{,}633.$
From previous solutions: E[X∧43,130] = 19,689, E[X∧8626] = 8030, S(8626) = 0.7881.
Thus the second moment of the per-loss variable is:
(1.1593^2) $(90\%^2)$ {543,940,124 - 66,493,633 - (2)(8626)(19,689 - 8030)} = 300,791,874.
From a previous solution, the average payment per loss is 12,165.
Thus the variance of the per-loss variable is: 300,791,874 - $12{,}165^2$ = 152,804,649.
The standard deviation of the per-loss variable is **12,361**.
Comment: One could instead inflate the LogNormal to 2007 and work in 2007.
The 2007 LogNormal has parameters $\mu = 9.7 + \ln[1.03^5] = 9.848$, and $\sigma = 0.8$

22.40. A. The second moment of the per-payment variable is:
(second moment of the per-loss variable) / S(d/(1+r)) = 300,791,874 / 0.7881 = 381,667,141.
From a previous solution, the average payment per payment is 15,435.
Thus the variance of the per-payment variable is: 381,667,141 - $15{,}435^2$ = 143,427,916.
The standard deviation of the per-payment variable is **11,976**.

22.41. For independent Gammas with the same θ, the alphas add.
Thus the sum of n independent, identically distributed Gamma Distributions is
a Gamma Distribution with θ and nα.
The average is the sum multiplied by 1/n.
Multiplying a Gamma Distribution by a constant gives another Gamma Distribution with theta multiplied by that constant.
Thus the average of n independent, identically distributed Gamma Distributions is
a **Gamma Distribution with θ/n and nα**.
Comment: The average has mean αθ, the same as the original Gamma, as it should be.
The average has variance: $(n\alpha)(\theta/n)^2 = \alpha\theta/n$, 1/n times the variance of the original Gamma, as it should be.

22.42. B. In 2016 the ground up losses are Pareto with $\alpha = 2$ and $\theta = (1+r)250$.
The average payment per loss is: $E[X] - E[X \wedge 100]$.

For a Pareto with $\alpha = 2$: $E[X] - E[X \wedge 100] = \theta - \alpha\,(1 - \frac{\theta}{\theta + 100}) = \frac{\theta^2}{\theta + 100}$.

In 2011, $E[X] - E[X \wedge 100] = 178.57$.

In 2016, $E[X] - E[X \wedge 100] = \frac{\{250(1+r)\}^2}{250(1+r) + 100}$.

Setting this ratio equal to 1.26:

$(1.26)(178.57) = \frac{\{250(1+r)\}^2}{250(1+r) + 100}$.

$\Rightarrow (225)(250)(1+r) + 22{,}500 = 62{,}500\,(1 + 2r + r^2)$.

$\Rightarrow r^2 + 1.1r - 0.26 = 0. \Rightarrow r = 0.2$, taking the positive root of the quadratic equation.
In other words, there is a total of **20% inflation** between 2011 and 2016.
Comment: The mean ground up loss increases by 20%, but the losses excess of the deductible increase at a faster rate of 26%.
The average payment per loss is 2011: $250^2 / (250 + 100) = 178.57$.
The average payment per loss is 2016: $300^2 / (300 + 100) = 225.00$.
Their ratio is: 225.00/178.57 = 1.260.
The average payment per payment is: $e(100) = (\theta + 100) / (\alpha - 1) = q + 100$.
In 2011, $e(100) = 250 + 100 = 350$. In 2016, $e(100) = (1.2)(250) + 100 = 400$.
Their ratio is: 400/350 = 1.143.

22.43. $1 + CV^2 = E[X^2]/E[X]^2 = \exp[2\mu + 2\sigma^2] / \exp[\mu + \sigma^2/2]^2 = \exp[\sigma^2]$.

$1 + 4^2 = \exp[\sigma^2]. \Rightarrow \sigma = \sqrt{\ln(17)} = 1.6832$.

$30{,}000 = \exp[\mu + \sigma^2/2] = \exp[\mu + 1.6832^2/2]. \Rightarrow \mu = 8.8924$.

Deflating the limit: $200{,}000 / 1.05^{11} = 116{,}936$.

$E[X \wedge x] = \exp(\mu + \sigma^2/2)\Phi[(\ln x - \mu - \sigma^2)/\sigma] + x\,\{1 - \Phi[(\ln x - \mu)/\sigma]\}$.

$E[X \wedge 116{,}936] =$

$30{,}000\ \Phi[(\ln 116{,}936 - 8.8924 - 1.6832^2) / 1.6832]$
$\quad + 116{,}936\ \{1 - \Phi[(\ln 116{,}936 - 8.8924) / 1.6832]\}$

$= 30{,}000\ \Phi[-0.03] + 116{,}936\ \{1 - \Phi[1.65]\} = (30{,}000)(0.4880) + (116{,}936)(1 - 0.9505) = 20{,}428$.

Excess ratio is: 1 - 20,428/30,000 = **31.9**%.

Alternately, if the 2004 LogNormal has $\mu = 8.8924$, and $\sigma = 1.6832$,
then the 2015 LogNormal has $\mu = 8.8924 + 11\ln(1.05) = 9.4291$, and $\sigma = 1.6832$.

Mean in 2014 is: $(30{,}000)\,(1.05^{11}) = 51{,}310$.

$E[X \wedge 200{,}000] =$

$51{,}310\ \Phi[(\ln 200{,}000 - 9.4291 - 1.6832^2) / 1.6832]$
$\quad + 200{,}000\ \{1 - \Phi[(\ln 116{,}936 - 9.4291) / 1.6832]\}$

$= 51{,}310\ \Phi[-0.03] + 200{,}000\ \{1 - \Phi[1.65]\} = (51{,}310)(0.4880) + (200{,}000)(1 - 0.9505) = 34{,}939$.

Excess ratio is: 1 - 34,939/51,310 = **31.9**%.

22.44. E. In 2016 the losses are Exponential with $\theta = (1.03^4)(1000) = 1125.51$.
$E[X \wedge 500] = (1125.51)(1 - e^{-500/1125.51}) = 403.71$.
$E[X \wedge 2000] = (1125.51)(1 - e^{-2000/1125.51}) = 935.13$.
Average payment per loss is:
$(0.8)\ (E[X \wedge 2000] - E[X \wedge 500]) = (0.8)\ (935.13 - 403.71) = 425.14$.
$S(500) = e^{-500/1125.51} = 0.6413$.
Average payment per payment is: 425.14 / 0.6413 = **663**.
Alternately use the original Exponential and the formula for the average payment per payment:
$(1+r)\ c\ (E[X \wedge u/(1+r)] - E[X \wedge d/(1+r)]) / S(d/(1+r)) =$
$(1.03^4)\ (0.8)\ (E[X \wedge 2000/1.03^4] - E[X \wedge 500/1.03^4]) / S(500/1.03^4) =$
$(1.12551)(0.8)\ \{1000(1 - e^{-1776.97/1000}) - 1000(1 - e^{-444.24/1000})\} / e^{-444.24/1000} =$ **663**.

22.45. Under uniform inflation, all of the percentiles would increase by the same amount.
Since here the percentiles increase by very different amounts, the mathematics of uniform inflation are <u>not</u> a good model.
Since the higher percentiles have increased (much) faster than the lower percentiles, in fact the lower percentiles have decreased, the shape of the distribution is changing so as to have a (significantly) heavier righthand tail in 2012 than in 1973.
<u>Comment</u>: If for example the distribution of wages were Single Parameter Pareto, then the parameter α would have gotten smaller from 1973 to 2012.

22.46. E. In 2020 severity is Inverse Gamma with $\alpha = 6$ and $\theta = (1.2)(10) = 12$.

Fourth moment is: $\dfrac{\theta^4}{(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)} = \dfrac{12^4}{(5)(4)(3)(2)} =$ **172.8**.

Alternately, in 2014 the fourth moment is: $\dfrac{\theta^4}{(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)} = \dfrac{10^4}{(5)(4)(3)(2)} = 83.333$.

Under inflation, the fourth moment is multiplied by $(1+r)^4$.
$(1.2^4)(83.333) =$ **172.8**.

22.47. A. & **22.48. E.** For a Pareto Distribution:
$E[X] - E[X \wedge d] = \theta/(\alpha-1) - \{\theta/(\alpha-1)\}\{1 - (\theta/(d + \theta))^{\alpha-1}\} = \theta^\alpha / \{(\alpha-1)(d + \theta)^{\alpha-1}\}$.
The insurance premium in each year is: $1.3(E[X] - E[X \wedge 50]) = 1.3\theta^\alpha / \{(\alpha-1)(50 + \theta)^{\alpha-1}\}$.
The reinsurance covers the layer excess of 300 in ground up loss, and the reinsurance premium is: $1.1(E[X] - E[X \wedge 300]) = 1.1\theta^\alpha / \{(\alpha-1)(300 + \theta)^{\alpha-1}\}$.
Thus $R_i/P_i = (1.1/1.3)\ (50 + \theta)^{\alpha-1} / (300 + \theta)^{\alpha-1}$.
In 2015 this ratio is: $(1.1/1.3)\ (50 + 200)^2 / (300 + 200)^2 =$ **0.212**.
In 2020, the Pareto has $\alpha = 3$ and $\theta = (1.25)(200) = 250$.
Thus in 2020 this ratio is: $(1.1/1.3)\ (50 + 250)^2 / (300 + 250)^2 =$ **0.252**.
<u>Comment</u>: Similar to SOA M, 11/05, Q.28 (2009 Sample Q.209).

22.49. E. The losses in 2015 follow an Exponential with mean: (1.06)(3000) = 3180.
The payments excess of deductible d follow the same Exponential with mean 3180.
Thus we cannot determine d.
Alternately, deflate the deductible to 2014: d/1.06.
The payments excess of d/1.05 in 2014 follow the same Exponential with mean 3000.
Inflating to 2015, the payments excess of d in 2015 follow an Exponential with mean:
(1.06)(3000) = 3180. Thus we cannot determine d.
Alternately, the average payment per payment is:
$(1.06)\ (E[X] - E[X \wedge d/1.06])\ /\ S(d/1.06) = (1.06)\ \{3000 - (3000)(1 - e^{-d/1.06})\}\ /\ e^{-d/1.06} = 3180.$
Thus we cannot determine d.

22.50. B. Applying inflation, in 2020 the Pareto distribution has $\alpha = 4$ and $\theta = (1.3)(500) = 650$.
After truncating and shifting from below, one gets another Pareto Distribution with
$\alpha = 4$ and $\theta = 650 + 200 = 850$.
Thus the nonzero payments are Pareto with $\alpha = 4$ and $\theta = 850$.
This has mean: $\theta/(\alpha - 1) = 850/3 = 283.33$, second moment: $2\theta^2 / \{(\alpha - 1)(\alpha - 2)\} = 240{,}833$,
and variance: $240{,}833 - 283.33^2 = 160{,}557$.
The probability of a nonzero payment is the probability that a loss is greater than the deductible of 200; for the inflated Pareto, $S(200) = \{650/(650+200)\}^4 = 0.342$.
Thus the payments of the insurer can be thought of as an aggregate distribution,
with Bernoulli frequency with mean 0.342 and Pareto severity with $\alpha = 4$ and $\theta = 850$.
The variance of this aggregate distribution is:
(Mean Frequency)(Variance of Severity) + (Mean Severity)^2(Variance of Frequency) =
$(0.342)(160{,}557) + (283.33^2)\ \{(0.342)(1 - 0.342)\} =$ **72,975**.
One can also think of this as a two-point mixture between a severity that is always zero and a severity that is the truncated and shifted Pareto, with the former with weight 1 - 0 342 and the latter with weight 0.342. The mean of this mixture is: (0.658)(0) + (0.342)(283.33) = 96.90.
The second moment of this mixture is: (0.658)(0) + (0.342)(240,833) = 82,365.
The variance of this mixture is: $82{,}365 - 96.90^2 =$ **72,975**.

22.51. B. After severity increases by 20%:

Probability	Severity	Payment with 100 deductible
0.50	120	20
0.40	300	200
0.10	600	500

Average payment per loss: (0.5)(20) + (0.4)(200) + (0.1)(500) = 140.
Expected total payment = (70)(140) = **9800**.

22.52. E. The average payment per loss is: $(1+r)\ c\ (E[X\wedge L/(1+r)] - E[X\wedge d/(1+r)])$.
In 2016 this is: $E[X] - E[X\wedge 50{,}000]$. In 2020 this is: $(1.05^4)\ (E[X] - E[X\wedge 50{,}000/1.05^4])$.
For the Pareto with $\theta = 10{,}000$ and $\alpha = 1.5$: $E[X] = 10{,}000 / (1.5 - 1) = 20{,}000$.
$E[X \wedge x] = \{\theta/(\alpha-1)\}\{1 - (\theta/(x+\theta))^{\alpha-1}\} = (20{,}000)\ \{1 - \sqrt{10{,}000/(10{,}000+x)}\ \}$.

$E[X \wedge 50{,}000] = (20{,}000)\ \{1 - \sqrt{1/6}\ \} = 11{,}835$.

$E[X \wedge 50{,}000/1.05^4] = E[X \wedge 41{,}135] = (20{,}000)\ \{1 - \sqrt{10{,}000/51{,}135}\ \} = 11{,}156$.

Thus, $\dfrac{\text{expected losses paid by the reinsurer in 2020}}{\text{expected losses paid by the reinsurer in 2016}} = \dfrac{\text{average payment per loss in 2020}}{\text{average payment per loss in 2016}} =$

$\dfrac{(1.05^4)\ (20{,}000 - 11{,}156)}{20{,}000 - 11{,}835} = \mathbf{1.317}$.

Alternately, the Excess Ratio, $R(x) = 1 - E[X\wedge x]/E[X]$. For the Pareto, $E[X] = \theta/(\alpha-1)$ and $E[X\wedge x] = \{\theta/(\alpha-1)\}\{1 - (\theta/(x+\theta))^{\alpha-1}\}$. Therefore $R(x) = \{\theta/(x+\theta)\}^{\alpha-1}$.
In 2016, the Pareto has $\theta = 10{,}000$ and $\alpha = 1.5$, so $R(x) = \sqrt{10{,}000/(10{,}000+x)}$.

$R(50{,}000) = \sqrt{1/6} = 40.82\%$.

The excess ratio at 50,000 in 2020 is the same as the excess ratio at $50{,}000/1.05^4 = 41{,}135$ in 2016. In 2016, $R(41{,}135) = \sqrt{10{,}000/51{,}135} = 44.22\%$.

Thus, $\dfrac{\text{expected losses paid by the reinsurer in 2020}}{\text{expected losses paid by the reinsurer in 2016}} = 1.05^4\ (44.22\%/40.82\%) = \mathbf{1.317}$.

Alternately, in 2020 the Pareto has parameters $\theta = (1.05^4)(10{,}000) = 12{,}155$ and $\alpha = 1.5$.
Thus in 2020, the excess ratio at 50,000 is: $\sqrt{12{,}155/(12{,}155+50{,}000)} = 44.22\%$.

Proceed as before.
<u>Comment</u>: Similar to 3, 11/00, Q.42 (2009 Sample Q.120).
Over a fixed limit the excess losses increase more quickly than the overall inflation rate, which over 4 years would be $1.05^4 = 1.216$.

22.53. (a) The mean of the Pareto in 2023 is: $\frac{\theta}{\alpha - 1}$ = 2000/(3-1) = 1000.

The Pareto is a scale family. Therefore, the scale parameter θ is multiplied by the inflation factor of 1.09 and the shape parameter α remains the same.
Thus in 2026 we have a Pareto Distribution with parameters α = 3 and θ = (1.09)(2000) = 2180.
We want in 2026: 1000 = E[X] - E[X ∧ d].

$$\Rightarrow 1000 = \frac{\theta}{\alpha - 1} \left(\frac{\theta}{d + \theta}\right)^{\alpha-1} = \frac{2180}{3 - 1} \left(\frac{2180}{d + 2180}\right)^{3-1}. \Rightarrow 1.09 = (1 + d/2180)^2. \Rightarrow d = \mathbf{96.0}.$$

(b) We want in 2026: 1000 = E[X ∧ L] - E[X ∧ 50].

$$\Rightarrow 1000 = \frac{\theta}{\alpha - 1} \left\{\left(\frac{\theta}{d + \theta}\right)^{\alpha-1} - \left(\frac{\theta}{L + \theta}\right)^{\alpha-1}\right\} = \frac{2180}{3 - 1} \left\{\left(\frac{2180}{50 + 2180}\right)^{3-1} - \left(\frac{2180}{L + 2180}\right)^{3-1}\right\}.$$

$$\Rightarrow 0.91743 = 0.95566 - \left(\frac{2180}{L + 2180}\right)^2. \Rightarrow L/2180 + 1 = 5.1144. \Rightarrow L = \mathbf{8969}.$$

Comment: Similar to ASTAM, 11/23, Q.1 parts (c) and (d).

22.54. E. 1. False. $F_Z(Z) = F_X(Z / 1.1)$ so that $f_Z(Z) = f_X(Z / 1.1) / 1.1$. 2. True. 3. True.

22.55. C. For the Burr distribution **θ is transformed by inflation to θ (1+r)**.
This follows from the fact that θ is the scale parameter for the Burr distribution.
The shape parameters α and γ remain the same.
From first principles, one makes the change of variables Z = (1+r) X. For the Distribution Function one just sets $F_Z(z) = F_X(x)$; one substitutes for x = z / (1+r).
$F_Z(z) = F_X(x) = 1 - (1/\{1+(x/\theta)^\tau\})^\alpha = 1 - \{1/(1+\{z / (1+r)\theta)^\tau\})\}^\alpha$.
This is a Burr Distribution with parameters: α, θ (1+r), and γ.

22.56. A. The mean of a Pareto is θ/(α-1).
Therefore, θ = (α-1) (mean) = (3-1) (25,000) = 50,000.
Prior to the impact of inflation: 1 - F(100,000) = $\{50{,}000 / (50{,}000 + 100{,}000)\}^\alpha = 1/3^3$ = 0.0370.
Under uniform inflation for the Pareto, θ is multiplied by 1.2 and α is unchanged.
Thus the new θ is (50,000)(1.2) = 60,000.
Thus after inflation: 1 - F(100,000) = $\{\theta / (\theta + 100{,}000)\}^\alpha = (6/16)^3$ = 0.0527.
The increase is: 0.0527 - 0.0370 = **0.0157**.

22.57. A. Inflation has been 10% per year for 7 years. Thus the inflation factor is 1.1^7 = 1.949.
Under uniform inflation, the Pareto has θ increase to (1+r)θ, while α remains the same.
Thus in 1992 the Pareto Distribution has parameters: 2 and (12,500)(1.949) = 24,363.
For the Pareto $E[X \wedge x] = \{\theta/(\alpha-1)\}\{1 - (\theta/(\theta+x))^{\alpha-1}\}$. Thus in 1992,
$E[X \wedge 200{,}000] = \{24{,}363/(2-1)\}\{1-(24{,}363/(24{,}363+200{,}000))^{2-1}\}$ = **21,717**.
Alternately, the 200,000 limit in 1992 corresponds to 200,000 / 1.949 = 102,617 limit in 1985.
In 1985, $E[X \wedge 102{,}617] = \{12{,}500/(2-1)\}\{1-(12{,}500/(12{,}500+102{,}617))^{2-1}\}$ = 11,143.
In order to inflate to 1992, multiply by 1.949: (1.949)(11,143) = **21,718**.

22.58. C. 1. False. For the Inverse Gaussian, both μ and θ are multiplied by 1+r.
2. False. θ becomes θ(1+r). (The Generalized Pareto acts like the Pareto under inflation. The scale parameter is multiplied by the inflation factor.) 3. True.

22.59. B. Since b divides x everywhere that x appears in the density function, b is a scale parameter. Therefore, under uniform inflation we get a Erlang Distribution with b multiplied by (1+r). Alternately, one can substitute for x = z / (1+r).
For the density function $f_Z(z) = f_X(x) / (1+r)$.
Thus $f(z) = (z/(1+r)b)^{c-1}\ e^{-z/(1+r)b}\ / (1+r)\{\ b\ (c-1)!\ \}$, which is an Erlang Distribution with b multiplied by (1+r) and with c unchanged.
<u>Comment:</u> The Erlang Distribution is a special case of the Gamma Distribution, with c integral.
c ⇔ α, and b ⇔ θ.
Recall that that under change of variables applied to the density you need to divide by
$\frac{dz}{dx} = 1 + r$, since $\frac{dF}{dz} = \frac{dF}{dx} \Big/ \frac{dz}{dx}$.

22.60. D. The mean of a Burr distribution is: $\theta\Gamma(1 + 1/\gamma)\Gamma(\alpha - 1/\gamma) / \Gamma(\alpha)$ =
$\theta\Gamma(1+2)\Gamma(3 - 2)/\Gamma(3) = \theta\Gamma(1)\ \Gamma(3)/\Gamma(3) = \theta$. Under uniform inflation the mean increases from 10,000 to (10,000)(1.44) = 14,400. After inflation, the chance that a claim exceeds $40,000 is:
$S(40{,}000) = \{1 / (1 + (40{,}000/\theta)^\gamma)\}^\alpha = \{1 / (1 + (40{,}000/14{,}400)^{0.5})\}^3$ = **0.0527**.
Alternately, one can compute the chance of exceeding 40,000 / 1.44 = 27,778 prior to inflation:
$S(27{,}778) = \{1 / (1 + (27{,}778/10{,}000)^{0.5})\}^3$ = **0.0527**.

22.61. B. Under uniform inflation for the LogNormal we get another LogNormal,
but μ becomes μ + ln(1+r) while σ stays the same.
Thus in this case μ' = 17.953 + ln(1.1) = 18.048, while σ remains 1.6028.

22.62. C. For the Exponential Distribution $E[X \wedge x] = \theta\ (1 - e^{-x/\theta})$.
During 1992 the distribution is an Exponential Distribution with $\theta = 1$ and the average value of the capped losses is $E[X \wedge 1] = 1 - e^{-1} = 0.6321$.
During 1993 the distribution is an Exponential Distribution with $\theta = 1.1$.
Thus in 1993, $E[X \wedge 1] = 1.1\{1 - e^{-1/1.1}\} = 0.6568$.
The increase in capped losses between 1993 and 1992 is: 0.6568 / 0.6321 = **1.039**.
Comments: The rate of inflation of 3.9% for the capped losses with a fixed limit is less than the overall rate of inflation of 10%.

22.63. B. Prior to inflation in 1991, $F(x) = 1 - x^{-5}$, $x > 1$. After inflation in 1992,
$F(x) = 1 - (x/1.1)^{-5}$, $x > 1.1$. $f(y) = 5(1.1^5)x^{-6}$. LER(1.2) = $E[X \wedge 1.2]$ / E[X].

$$E[X \wedge 1.2] = \int_{1.1}^{1.2} x\ f(x)\ dx + (1.2)\{S(1.2)\} = (1.1^5)(5/4)\{(1.1^{-4}) - (1.2^{-4})\} + (1.2)\{(1.2/1.1)^{-5}\}$$

= 0.404 + 0.777 = 1.181

$$E[X] = \int_{1.1}^{\infty} x\ f(x)\ dx = (1.1^5) \int_{1.1}^{\infty} x^{-5}\ dx = (1.1^5)\ (5/4)(1.1^{-4}) = 1.375.$$

LER(1.2) = $E[X \wedge 1.2]$ / E[X] = 1.181 / 1.375 = **0.859**.
Comment: Remember that under uniform inflation the domain of the Distribution Function also changes; in 1992 x > 1.1. This is a Single Parameter Pareto with $\alpha = 5$ and $\theta = 1.1$.
$E[X \wedge x] = \theta\ [\{\alpha - (x/\theta)^{1-\alpha}\} / (\alpha - 1)]$. $E[X \wedge 1.2] = 1.1[\{5 - (1.2/1.1)^{1-5}\} / (5 - 1)] = 1.181$.
$E[X] = \theta\ \alpha / (\alpha - 1) = 1.1(5/4) = 1.375$.
$LER(x) = 1 - (1/\alpha)(x/\theta)^{1-\alpha}$. $LER(1.2) = 1 - (1/5)(1.2/1.1)^{-4} = 1 - (0.2)(0.7061) = 0.859$.
Note that one could instead deflate the 1.2 deductible in 1992 to a 1.2/1.1 = 1.0909 deductible in 1991 and then work with the 1991 distribution function.

22.64. B. The distribution for the 1993 losses is an exponential distribution $F(x) = 1 - e^{-x}$.
In order to convert into 1994 dollars, the parameter of 1 is multiplied by 1 plus the inflation rate of 5%; thus the revised parameter is 1.05. The capped losses which are given by the Limited Expected Value are for the exponential: $E[X \wedge x] = (1 - e^{-x/\theta})\theta$.
Thus in 1993 the losses capped to 1 ($million) is $E[X \wedge 1] = (1 - e^{-1}) / 1 = 0.6321$.
In 1994 with $\theta = 1.05$, $E[X \wedge 1] = (1 - e^{-0.9524})(1.05) = 0.6449$.
The increase in capped losses is: 0.6449 / 0.6321 = 1.019, or **1.9%** inflation.
Alternately rather than working with the 1994 distribution one can translate everything back to 1993 dollars and use the 1993 distribution. In 1993 dollars the 1994 limit of 1 is only 1/1.05 = 0.9524. Thus the capped losses in 1994 are in 1993 dollars $E[X \wedge 0.9524] = (1 - e^{-0.9524})$.
In 1994 dollars the 1994 capped losses are therefore $1.05E[X \wedge 0.9524] = 0.6449$.
The solution is therefore 0.6449 / 0.6321 = 1.019, or **1.9%** inflation.

22.65. A. Statement 1 is false. In fact the coefficient of variation as well as the skewness are dimensionless quantities which are unaffected by a change in scale and are therefore unchanged under uniform inflation. Specifically in this case the new mean is the prior mean times (1 + r), the new variance is the prior variance times $(1+r)^2$.
Therefore, the new coefficient of variation = new standard deviation / new mean =
(1+ r) prior standard deviation / (1 + r) prior mean = prior standard deviation / prior mean = prior coefficient of variation.
Statement 3 is false. In fact, $E[Z \wedge d(1+r)] = (1+r)\ E[X \wedge d]$. The left hand side is the Limited Expected Value in the later year, with a limit of d(1+r); we have adjusted d, the limit in the prior year, in order to keep up for inflation via the factor 1+r. This yields the Limited Expected Value in the prior year, except multiplied by the inflation factor to put it in terms of the subsequent year dollars, which is the right hand side. For example, if the expected value limited to $1 million is $300,000 in the prior year, then after uniform inflation of 10%, the expected value limited to $1.1 million is $330,000 in the later year.
In terms of the definition of the Limited Expected Value:

$$E[Z \wedge d(1+r)] = \int_0^{d(1+r)} z\ f_Z(z)\ dz\ + S_Z(d(1+r))d(1+r) = \int_0^{d} (1+r)\ x\ f_X(x)\ dx\ + S_X(d)d(1+r)$$

$= (1+r)E[X \wedge d]$.
Where we have applied the change of variables, z = (1+d) x and thus $F_Z(d(1+r)) = F_X(d)$, and $f_X(x)\ dx = f_Z(z)\ dz$.
Statement 2 is true. The mean residual life at d in the prior year is given by $e_X(d) =$ { mean of X - $E[X \wedge d]$ } / $\{1 - F_X(d)\}$. Similarly, the mean residual life at d(1+r) in the later year is given by $e_Z(d(1+r))$ = {mean of Z - E[Z; d(1+r)]} / $\{1 - F_Z(d(1+r))\}$ =
{ $(1+r)\ E[X] - (1+r)E[X \wedge d]$ } / $\{1 - F_X(d)\} = (1+r)e_X(d)$. Thus the mean residual life in the later year is multiplied by the inflation factor of (1+r), provided the limit has been adjusted to keep up with inflation. For example, if the mean residual life beyond $1 million is $3 million in the prior year, then after uniform inflation of 10%, the mean residual life beyond $1.1 million is $3.3 million in the subsequent year.

22.66. B. Losses uniform on [0, 10,000] in 1991 become
uniform on: $[0, 1.05^2(10,000)] = [0, 11,025]$ in 1993.

$$\text{LER}(500) = \left\{ \int_0^{500} x\, f(x)\, dx + S(500)(500) \right\} / \int_0^{11,025} x\, f(x)\, dx$$

We have f(x) = 1/11,025 for $0 \le x \le 11,025$. F(500) = 500 / 11,025 = 0.04535.
Thus, LER(500) = $\{(1/11,025)(500^2)/2 + (1-0.04535)(500)\}$ / (11,025 / 2) = **0.0886.**
Alternately, the LER(500) in 1993 is the LER(500/1.1025) = LER(453.51) in 1991.

$$\text{In 1991: } E[X \wedge 453.51] = \int_0^{453.51} x/10,000\, dx + S(453.51)(453.51) =$$

10.28 + 432.96 = 443.24. Mean in 1991 = 10,000 / 2 = 5000.
In 1991: LER(453.51) = $E[X \wedge 453.51]$ / mean = 443.24 / 5000 = 0.0886.

22.67. C. $F(x) = 1 - x^{-3}$, $x \ge 1$ in 1993 dollars.
A loss exceeding $2.2 million in 1994 dollars is equivalent to a loss exceeding
$2.2 million / 1.1 = $2 million in 1993 dollars.
The probability of the latter is: $1 - F(2) = 2^{-3} = 1/8 =$ **0.125**.
Alternately, the distribution function in 1994 dollars is: $G(x) = 1 - (x/1.1)^{-3}$, $x \ge 1.1$.
Therefore, $1 - G(2.2) = (2.2/1.1)^{-3} = 1/8 =$ **0.125**.
Comment: Single Parameter Pareto Distribution.

22.68. D.

Probability	1993 Amount Loss Amount	1993 Insurer Payment	1994 Amount Loss Amount	1994 Insurer Payment
0.1667	1000	0	1050	0
0.1667	2000	500	2100	600
0.1667	3000	1500	3150	1650
0.1667	4000	2500	4200	2700
0.1667	5000	3500	5250	3750
0.1667	6000	4500	6300	4800
Average	3500.00	2083.33	3675.00	2250

2250 / 2083 = 1.080, therefore the insurer's payments increased **8%**.
Comment: Inflation on the losses excess of the deductible is greater than that of the ground up losses.

22.69. E. The distribution for the 1993 losses is an exponential distribution $F(x) = 1 - e^{-0.1x}$.
In order to convert into 1994 dollars, the parameter of 1/0.1 is multiplied by 1 plus the inflation rate of 10%; thus the revised parameter is 1.1/0.1 = 1/0.0909. Thus the 1994 distribution function is $G(x) = 1 - e^{-0.0909x}$, where x is in 1994 dollars. The next step is to write down (in 1994 dollars) the Truncated and Shifted distribution function for a deductible of d:
$F_P(x) = \{G(x+d) - G(d)\} / \{1 - G(d)\} = \{e^{-0.0909d} - e^{-0.0909(x+d)}\} / e^{-0.0909d} = 1 - e^{-0.0909x}$.
$F_p(5) = 1 - e^{-(0.0909)(5)} =$ **0.3653**.
Alternately, \$5 in 1994 dollars corresponds to \$5 / 1.1 = \$4.545 in 1993 dollars.
In 1993, the Truncated and Shifted distribution function for a deductible of d:
$G(x) = \{F(x+d) - F(d)\} / \{1 - F(d)\} = \{e^{-0.1d} - e^{-0.1(x+d)}\} / e^{-0.1d} = 1 - e^{-0.1x}$.
$G(4.545) = 1 - e^{-0.1(4.545)} =$ **0.3653**.
Comment: Involves two separate questions: how to adjust for the effects of inflation and how to adjust for the effects of truncated and shifted data. Note that for the exponential distribution, after truncating and shifting the new distribution function does not depend on the deductible amount d.

22.70. B. Under uniform inflation, the parameters of a LogNormal become:
$\mu' = \mu + \ln(1.1) = 10 + 0.09531 = 10.09531$, $\sigma' = \sigma = \sqrt{5}$.
Using the formula for the limited expected value of the LogNormal: $E[X \wedge \$2{,}000{,}000] =$
$\exp(10.09531 + 5/2)\ \Phi[\ln(2{,}000{,}000) - 10.09531 - 5)/\sqrt{5}\ ] +$
$(2{,}000{,}000)\{1 - \Phi[\ln(2{,}000{,}000) - 10.09531)/\sqrt{5}\ ]\} = 295{,}171\ \Phi[-0.26] + (2{,}000{,}000)(1 - \Phi[1.97])$
= (295,171)(0.3974) + (2,000,000)(0.0244) = **\$166 thousand**.
Alternately, using the original LogNormal Distribution, the average payment per loss in 1994 is:
$1.1\ E_{1993}[X \wedge 2 \text{ million} / 1.1] = 1.1\ E_{1993}[X \wedge 1{,}818{,}182] =$
$1.1\ \{\ \exp(10 + 5/2)\ \Phi[\ln(1{,}818{,}182) - 10 - 5)/\sqrt{5}\ ] + (1{,}818{,}182)\{1 - \Phi[\ln(1{,}818{,}182) - 10)/\sqrt{5}\ ]\}\ \} =$
$1.1\{268{,}337\ \Phi[-0.26] + (1{,}818{,}182)(1 - \Phi[1.97]) =$
(1.1){(268,337)(0.3974) + (1,818,182)(0.0244)} = **\$166 thousand**.

22.71. E. One can put y = 1.05x, where y is the claim size in 1994 and x is the claim size in 1993. Then let g(y) be the p.d.f. for y, g(y)dy = f(x)dx = f(y/1.05) dy/1.05.

$$g(y) = \frac{\exp[-0.5(\{(y/1.05)-1000\}/100)^2]}{\sqrt{2\pi}\ (100)(1.05)} = \frac{\exp[-0.5\{(y-1050)/105\}^2]}{\sqrt{2\pi}\ (105)}.$$

This is again a Normal Distribution with both μ and σ multiplied by the inflation factor of 1.05.
Comment: As is true in general, under uniform inflation, both the mean and the standard deviation have been multiplied by the inflation factor of 1.05. Assuming you remember that a Normal Distribution is reproduced under uniform inflation, you can use this general result to arrive at the solution to this particular problem, since for the Normal, μ is the mean and σ is the standard deviation.

22.72. (a) $\text{LER}(1000) = \frac{26 + 115 + 387 + 449 + 609 + 774 + (4)(1000)}{30{,}307} =$

6360 / 30,307 = **21.0%**.

$\text{LER}(5000) = \frac{26 + 115 + 387 + 449 + 609 + 774 + 2131 + (3)(5000)}{30{,}307}$

= 19,491 / 30,307 = **64.3%**.

(b) $E[X] = 1/\lambda$. $E[X \wedge x] = (1 - e^{-\lambda x})/\lambda$. $\Rightarrow$ $\text{LER}[x] = 1 - e^{-\lambda x}$.

LER(1000) = 1 - exp[-(0.00033)(1000)] = **28.1%**.

LER(5000) = 1 - exp[-(0.00033)(5000)] = **80.8%**.

(c) Average payment per loss is: $E[X] - E[X \wedge 5000] = e^{-\lambda 5000} / \lambda$.

Prior to inflation the average payment per loss is: exp[-(0.00033)(5000)] / 0.00033 = 581.97.

After inflation, loss are Exponential with hazard rate = 0.00033/1.1 = 0.00030.

After inflation the average payment per loss is: exp[-(0.0003)(5000)] / 0.0003 = 743.77.

743.77 / 581.97 = 1.278. $\Leftrightarrow$ **27.8% inflation**.

22.73. B. $0.95 = p_0 = S(1) = e^{-1/\theta}$. $\Rightarrow$ $\theta = -1/\ln(0.95) = 19.5$.

Y is also Exponential with twice the mean of X: (2)(19.5) = 39. $f_Y(1) = e^{-1/39}/39 =$ **0.0250**.

22.74. C. If the losses are uniformly distributed on [0,2500] in 1994 then they are uniform on [0,2575] in 1995. (Each boundary is multiplied by the inflation factor of 1.03.)

$$\text{LER}(100) = \left\{ \int_0^{100} x\, f(x)\, dx + S(100)(100) \right\} / \int_0^{2575} x\, f(x)\, dx =$$

$$\left\{ \int_0^{100} x\, (1/2575)\, dx + \{1 - (100/2575)\}\, (100) \right\} / \int_0^{2575} x\, (1/2575)\, dx =$$

$\{(1/2575)(100^2)/2 + 100 - (1/2575)(100^2)\} / (2575/2) =$ **7.62%**.

Alternately, \$100 in 1995 is equivalent to 100/1.03 = \$97.09 in 1994. In 1994:

$$\text{LER}(97.09) = \left\{ \int_0^{97.09} x\, f(x)\, dx + (1 - F(97.09))(97.09) \right\} / \int_0^{2500} x\, f(x)\, dx =$$

$$\left\{ \int_0^{97.09} x\, (1/2575)\, dx + (1 - (97.09/2500))(97.09) \right\} / \int_0^{2500} x\, (1/2575)\, dx =$$

$\{(1/2500)(97.09^2)/2 + 97.09 - (1/2500)(97.09^2)\} / (2500/2) =$ **7.62%**.

22.75. D. Under uniform inflation the parameters of the Pareto become 2 and 1000(1.1) = 1100.
The expected number of insurer payments is 10 losses per year times the percent of losses greater than 100: $10S(100) = 10\,\{1100/(1100+100)\}^2 =$ **8.40**.

Alternately, after inflation the \$100 deductible is equivalent to 100/1.1 = 90.91.

For the original Pareto with $\alpha = 2$ and $\theta = 1000$, $10\{1-F(90.91)\} = 10\,\{1000/1090.91\}^2 = 8.40$.

22.76. C. Under uniform inflation the scale parameter θ of the Pareto is multiplied by the inflation factor, while the shape parameter α remains the same. Therefore the size of loss distribution in 1995 has parameters: $\theta = (500)(1.05) = 525$, $\alpha = 1.5$.
$F(x) = 1 - \{525/(525+x)\}^{1.5}$. The distribution function of the data truncated from below at 200 is:
$G(x) = \{F(m) - F(200)\} / \{1-F(200)\} =$
$\{(525/(525+200))^{1.5} - (525/(525+x))^{1.5}\} / (525/(525+200))^{1.5} = 1 - (725/(525+x))^{1.5}$.
At the median m of the distribution truncated from below G(m) = 0.5.
Therefore, $1 - (725/(525+m))^{1.5} = 0.5. \Rightarrow (725/(525+m))^{1.5} = 0.5$.
Thus $\{(525+m)/725\} = 2^{1/1.5} = 1.587$. Solving, m = (725)(1.587) - 525 = **626**.

22.77. B. 1. Above the franchise deductible, the insured is paid the whole loss amount.
Thus there is no incentive for the insured to reduce the magnitude of a loss.
In fact, if the loss is close to the franchise deductible, the insured has an incentive to make sure the loss exceeds the franchise deductible.
2. Should have multiplied by one minus the loss elimination ratio.
3. True.

22.78. a) Pure premium is: (mean frequency) $E[X \wedge 1 \text{ million}]$ = (0.15) (75,845) = **$11,377**.
b) Pure premium is:
(mean frequency) ($E[X \wedge 1 \text{ million}] - E[X \wedge 1000]$) = (0.15)(75,845 - 962) = **11,232**.
The frequency of (non-zero) claims is: (0.15)(1 - 0.0734) = **0.13899**.
Thus the average severity of (non-zero) claims is: 11,232 / 0.13899 = **$80,812**.
c) After inflation, the pure premium is: (0.15)(1.1) ($E[X \wedge 1 \text{ million} / 1.1] - E[X \wedge 1000 /1.1]$) =
(0.15)(1.1)(73,493 - 878) = 11,981.
The increase in the pure premium due to inflation is: 11,981 / 11,232 - 1 = **6.7%**.
Comment: In taking the ratio in part (c), the claim frequency drops out.
The rate of inflation for loses excess of a fixed deductible is greater than the overall rate of inflation, while the rate of inflation for losses capped by a fixed limit is less than the overall rate of inflation. Thus the rate of inflation in part (c) could turn out to be either greater than or less than 10%.

22.79. B. After 20% uniform inflation, the parameters of the LogNormal are:
$\mu' = \mu + \ln(1+r) = 7 + \ln(1.2) = 7.18$, while σ is unchanged at 2.
$F(2000) = \Phi[\{\ln(2000) - 7.18\} / 2] = \Phi[0.21] = 0.5832$.
Thus the expected number of claims per year greater than 2000 is:
10{1 - F(2000)} = (10)(1 - 0.5832) = **4.17**.
Alternately, one can deflate the deductible amount of 2000, which is then 2000/1.2 = 1667, and use the original LogNormal Distribution.
The expected number of claims per year greater than 1667 in the original year is:
$10(1 - F(1667)) = (10)(1 - \Phi[\{\ln(1667) - 7\}/ 2]) = (10)(1 - \Phi[0.21]) = (10)(1 - 0.5832) = 4.17$.
Comment: Prior to inflation, the expected number of claims per year greater than 2000 is:
$10(1 - F(2000)) = (10)(1 - \Phi[\{\ln(2000) - 7\} / 2]) = (10)(1 - \Phi[0.30]) = 3.82$.

22.80. E. (10 million / 0.8) + (9 million / 0.9) = **22.5 million**.
Comment: Prior to working with observed losses, they are commonly brought to one common level of inflation.

22.81. D. For the Pareto Distribution, LER(x) = E[X∧x] / E[X] = 1 - $(\theta/(\theta+x))^{\alpha-1}$.
In the later year, losses have doubled, so the scale parameter of the Pareto has doubled, so θ = 2k, rather than k.
For θ = 2k and α = 2: LER(x) = 1 - {2k/(2k+x)} = x / (2k + x). Thus LER(2k) = 2k / (4k) = **1/2**.

22.82. C. The behavior of the LogNormal under uniform inflation is explained by noting that multiplying each claim by a factor of 1.1 is the same as adding a constant amount ln(1.1) to the log of each claim. (For the LogNormal, the log of the sizes of the claims follow a Normal distribution.) Adding a constant amount to a normal distribution, gives another normal distribution, with the same variance but with the mean shifted. Thus under uniform inflation for the LogNormal, μ becomes μ + ln(1.1). The parameter σ remains the same.

22.83. C. An inflation factor of 0.25 applied to a Weibull Distribution, gives another Weibull with scale parameter: (0.25)θ = (0.25)(625) = 156.25, while the shape parameter τ is unaffected.
Thus Z is a Weibull with parameters θ = 156.25 and τ = 0.5.

22.84. C. For the Exponential Distribution, under uniform inflation θ is multiplied by the inflation factor. In this case, the inflation factor is 2, so the new theta is (1000)(2) = 2000.
Prior to inflation the percent of losses that exceed the deductible of 500 is:
$e^{-500/1000} = e^{-0.5}$ = 0.6065.
After inflation the percent of losses that exceed the deductible of 500 is: $e^{-500/2000} = e^{-0.25}$ = 0.7788. Thus the number of losses that exceed the deductible increased by a factor of 0.7788/0.6065 = 1.284. Since there were 10 losses expected prior to inflation, there are (10)(1.284) = **12.8** claims expected to exceed the 500 deductible after inflation.
Comment: One can also do this question by deflating the 500 deductible to 250. Prior to inflation, S(250) = $e^{-250/1000} = e^{-0.25}$ and S(500) = $e^{-500/1000} = e^{-0.5}$. Thus if 10 claims are expected to exceed 500, then there a total of $10/e^{-0.5}$ claims. Thus the number of clams expected to exceed 250 is: $(10e^{0.5})(e^{-0.25}) = 10e^{0.25}$ = 12.8.

22.85. D. Under uniform inflation, for the Burr, theta is multiplied by (1+r), thus theta becomes:
1.1 $\sqrt{1000}$ = **34.8**.
Comment: For a mixed distribution, under uniform inflation each of the individual distributions is transformed just as it would be if an individual distribution. In this case, the Pareto has new parameters α = 1 and θ = (1000)(1.1) = 1100, while the Burr has new parameters α = 1, θ = (1.1) $\sqrt{1000}$ = 1210, and γ = 2. The weights applied to the distributions remain the same.

22.86. B. The mean of the 1997 LogNormal is: $\exp((\mu + \ln k) + \sigma^2/2)$.
$F_{96}[\text{Mean}_{97}] = \Phi[(\ln(\text{Mean}_{97}) - \mu)/\sigma] = \Phi[(\mu + \ln k + \sigma^2/2 - \mu)/\sigma] = \Phi[(\ln k + \sigma^2/2)/\sigma]$.
Since we are given that in 1996 100p% of the losses exceed the mean of the losses in 1997,
$1 - F_{96}[\text{Mean}_{97}] = p$. Thus $F_{96}[\text{Mean}_{97}] = 1 - p$.
Thus $\Phi[(\ln k + \sigma^2/2) / \sigma] = 1 - p$. Since the Normal Distribution is symmetric,
$\Phi[-(\ln k + \sigma^2/2) / \sigma] = p$. Thus by the definition of z_p, $-(\ln k + \sigma^2/2)/\sigma = z_p$.
Therefore, $\sigma^2/2 + z_p\sigma + \ln k = 0$. $\Rightarrow \sigma = \mathbf{-z_p \pm \sqrt{z_p^2 - 2\ln k}}$.

22.87. D. At 10% per year for three years, the inflation factor is $1.1^3 = 1.331$. Thus greater than 500 in 1998 corresponds to greater than 500/1.331 = 376 in 1995. At least 45 and at most 50 claims are <u>less</u> than 376 in 1995. Therefore, between 50% and 55% of the total of 100 claims are <u>greater</u> than 376 in 1995. Therefore, **between 50% and 55%** of the total of 100 claims are greater than 500 in 1998.
<u>Comment</u>: One could linearly interpolate that about 52% or 53% of the claims are greater than 500 in 1998.

22.88. D. Deflate the 15,000 deductible in the later year back to the prior year:
15,000/1.5 = 10,000. In the prior year, the average non-zero payment is:
$(E[X] - E[X \wedge 10{,}000]) / S(10{,}000) = (20{,}000 - 6000) / (1 - 0.6) = 14{,}000 / 0.4 = 35{,}000$.
Inflating to the subsequent year: (1.5)(35,000) = **52,500**.
<u>Comment</u>: If the limit keeps up with inflation, so does the mean residual life.

22.89. C. In 1999, one has a Pareto with parameters 2 and 1.06θ.
$S_{1999}(d) = \{1.06\theta / (1.06\theta + d)\}^2$. $S_{1998}(d) = \{\theta / (\theta + d)\}^2$.
$r = S_{1999}(d) / S_{1998}(d) = 1.06^2 \{(\theta + d) / (1.06\theta + d)\}^2 = 1.1236 \{(1 + \theta/d) / (1 + 1.06\theta/d)\}^2$
As d goes to infinity, r goes to **1.1236**.
<u>Comment</u>: Alternately, $S_{1999}(d) = S_{1998}(d/1.06) = \{\theta / (\theta + d/1.06)\}^2$.

22.90. C. After 10% inflation, the survival function at 1000 is what it was originally at 1000/1.1 = 909.09. $S(1000) = 1 - F(1000) = 1 - \Phi[\{\ln(1000)-\mu\} / \sigma] = 1 - \Phi[0] = 0.5$.
$S(909.09) = 1 - \Phi[\{\ln(909.09)-\mu\}/\sigma] = 1 - \Phi[-0.06] = 0.5239$.
S(909.09) / S(1000) = 0.5239 / 0.5 = 1.048. An increase of **4.8%**.
<u>Comment</u>: After inflation one has a LogNormal with $\mu = 6.9078 + \ln(1.1)$, $\sigma = 1.5174$.

22.91. Assume for simplicity that the expected frequency is 5. ⇔ One loss of each size.

Loss	Contribution to Layer 0-50	Contribution to Layer 50-100	Contribution to Layer 100-200	Contribution to Layer 200-∞	Total
40	40	0	0	0	40
80	50	30	0	0	80
120	50	50	20	0	120
160	50	50	60	0	160
200	50	50	100	0	200
Total	240	180	180	0	600

For the next year, increase each size of loss by 10%:

Loss	Contribution to Layer 0-50	Contribution to Layer 50-100	Contribution to Layer 100-200	Contribution to Layer 200-∞	Total
44	44	0	0	0	44
88	50	38	0	0	88
132	50	50	32	0	132
176	50	50	76	0	176
220	50	50	100	20	220
Total	244	188	208	20	660

Trend for layer from 0 to 50 is: 244/240 - 1 = **1.7%.**
Trend for layer from 50 to 100 is: 188/180 - 1 = **4.4%.**
Trend for layer from 100 to 200 is: 208/180 - 1 = **15.6%.**
Comment: The limited losses in the layer from 0 to 50 increase slower than the overall rate of inflation 10%, while the excess losses in the layer from 200 to ∞ increase faster. The losses in middle layers, such as 50 to 100 and 100 to 200, can increase either slower or faster than the overall rate of inflation, depending on the particulars of the situation.

22.92. A. Y follows a Pareto Distribution with parameters: $\alpha = 2$ and $\theta = (1.10)(100) = 110$.
Thus $e_Y(k) = (k+\theta)/(\alpha-1) = k + 110$. $e_X(k) = k + 100$.
$e_Y(k) / e_X(k) = (k+110) / (k+100) = 1 + 10/(k+110)$.
Therefore, as k goes from zero to infinity, $e_Y(k) / e_X(k)$ goes from **1.1 to 1**.

22.93. a. The frequency of claims exceeding the franchise deductible is:
S(1000) 0.05 = (1 - 0.09) (0.05) = **4.55%**.
The average payment per payment for an ordinary deductible of 1000 is:
$(E[X] - E[X \wedge 1000]) / \{1 - F(1000)\} = (10,000 - 945) / (1 - 0.09) = 9950.55$.
The average payments per payment for a franchise deductible of 1000 is 1000 more:
$10,950.55.
b. After inflation, the average payment per payment for an ordinary deductible of 1000 is:
$(1.1)(E[X] - E[X \wedge 909]) / S(909)\} = (1.1)(10,000 - 870) / (1 - 0.075) = 10,857.30$.
The average payments per payment for a franchise deductible of 1000 is 1000 more:
$11,857.30.
Subsequent to inflation, the frequency of payments is: (1 - 0.075) (0.05) = 4.625%,
and the pure premium is: (4.625%)($11,857.30) = **$548.40**.
Comment: Prior to inflation, the pure premium is: (4.55%)($10,950.55) = $498.25.

22.94. B. During 1994, there is a 100/1000 = 10% chance that nothing is paid.
If there is a non-zero payment, it is uniformly distributed on (0, 900).
Thus the mean amount paid is: 90%(450) = 405.
The second moment of the amount paid is: $(90\%)(900^2)/3 = 243{,}000$.
Thus in 1994, the standard deviation of the amount paid is: $\sqrt{243{,}000 - 405^2} = 281.02$.
In 1995, the losses are uniformly distributed from (0, 1050). During 1995, there is a 100/1050 chance that nothing is paid. If there is a non-zero payment it is uniformly distributed on (0, 950).
Thus the mean amount paid is: (950/1050)(950/2) = 429.76.
The second moment of the amount paid is: $(950/1050)(950^2)/3 = 272{,}182.5$.
Thus in 1994, the standard deviation of the amount paid is: $\sqrt{272{,}182.5 - 429.76^2} = 295.78$.
% increase in the standard deviation of amount paid is: 295.78/281.02 - 1 = **5.25%**.
Alternately, the variance of the average payment per loss under a maximum covered loss of u and a deductible of d is: $E[(X \wedge u)^2] - E[(X \wedge d)^2] - 2d\{E[X \wedge u] - E[X \wedge d]\} - \{E[X \wedge u] - E[X \wedge d]\}^2$.
With no maximum covered loss (u = ∞), this is:
$E[X^2] - E[(X \wedge d)^2] - 2d\{E[X] - E[X \wedge d]\} - \{E[X] - E[X \wedge d]\}^2$.
For the uniform distribution on (a, b), the limited moments are for a ≤ x ≤ b :

$$E[(X \wedge d)^n] = \int_a^x y^n / (b-a)\ dy + x^n S(x) = (x^{n+1} - a^{n+1})/\{(n+1)(b-a)\} + x^n(b-x)/(b-a) =$$

$$\{(n+1)x^n b - a^{n+1} - n\, x^{n+1}\} / \{(n+1)(b-a)\}.$$

In 1994, a = 0, b = 1000, and d =100. $E[X \wedge 100] = \{(2)(100)(1000) - 100^2\}/2000 = 95$.
$E[(X \wedge 100)^2] = ((3)(100^2)(1000) - (2)100^3)/3000 = 9333.33$. The variance of the average payment per loss is: $1000^2/3 - 9333.33 - (2)(100)(500 - 95) - (500 - 95)^2 = 78{,}975$.
Similarly, in 1995, $E[X \wedge 100] = ((2)(100)(1050) - 100^2)/\{(2)(1050)\} = 95.238$.
$E[(X \wedge 100)^2] = ((3)(100^2)(1050) - (2)100^3)/((3)(1050)) = 9365.08$.
In 1995, the variance of the average payment per loss is:
$1050^2/3 - 9365.08 - (2)(100)(525 - 95.238) - (525 - 95.238)^2 = 87{,}487$.
% increase in the standard deviation of amount paid is: $\sqrt{87{,}487/78{,}975} - 1 =$ **5.25%**.

<u>Comment</u>: The second moment of the uniform distribution on (a, b) is: $(b^3 - a^3) / \{3(a-b)\}$.
When a = 0, this is $b^2/3$. The amount paid is a mixture of two distributions, one always zero and the other uniform. For example, in 1994, the amount paid is a 10%-90% mixture of a distribution that is always zero and a uniform distribution on (0, 900). The second moments of these distributions are zero and $900^2/3 = 270{,}000$. Thus the second moment of the amount paid is: (10%)(0) + (90%)(270000) = 243,000.

22.95. C. Prior to inflation, $0.250 = F(1000) = 1 - e^{-1000/\theta}$. ⇒ θ = 3476.
After inflation, θ = (2)(3476) = 6952. $F(1000) = 1 - e^{-1000/6952} =$ **0.134**.

22.96. D. $E[X \wedge 10] = -0.025(10^2) + (1.475)(10) - 2.25 = 10$.
$E[X \wedge 11] = -0.025(11^2) + (1.475)(11) - 2.25 = 10.95$.
$E[X \wedge 20] = -0.025(20^2) + (1.475)(20) - 2.25 = 17.25$
$E[X \wedge 22] = -0.025(22^2) + (1.475)(22) - 2.25 = 18.10$.
In 2000, there is a deductible of 11 and a maximum covered loss of 22, so the expected payments are: $E[X \wedge 22] - E[X \wedge 11] = 18.10 - 10.95 = 7.15$.
A deductible of 11 and maximum covered loss of 22 in the year 2001, when deflated back to the year 2000 correspond to a deductible of 11/1.1 = 10 and a maximum covered loss of 22/1.1 = 20.
Therefore, reinflating back to the year 2001, the expected payments in the year 2001 are:
$(1.1)(E[X \wedge 20] - E[X \wedge 10]) = (1.1)(17.25 - 10) = 7.975$.
The ratio of expected payments in 2001 over the expected payments in the year 2000 is:
7.975/ 7.15 = **1.115**.
Alternately, the insurer's average payment per loss is: $(1+r)\ c\ (E[X \wedge L/(1+r)] - E[X \wedge d/(1+r)])$.
c = 100%, L = 22, d = 11. r = 0.1 for the year 2001 and r = 0 for the year 2000.
Then proceed as previously.

22.97. C. The expected frequency is: 10,000,000 / 2000 = 5000.
$E[X \wedge x] = \{\theta/(\alpha-1)\}\{1 - (\theta/(x+\theta))^{\alpha-1}\} = (2000)\{1 - (2000/(x + 2000))\} = 2000\ x/(x + 2000)$.
$E[X] - E[X \wedge 3000] = 2000 - 1200 = 800$.
⇒ Expected losses paid by reinsurer: (5000)(800) = 4 million.
The ceded premium is: (1.1)(4 million) = **4.4 million**.
Alternately, the Excess Ratio, $R(x) = 1 - E[X \wedge x]/E[X]$. For the Pareto, $E[X] = \theta/(\alpha-1)$ and
$E[X \wedge x] = \{\theta/(\alpha-1)\}\{1- (\theta/(x+\theta))^{\alpha-1}\}$. Therefore $R(x) = (\theta/(x+\theta))^{\alpha-1}$.
In this case, $\theta = 2000$ and $\alpha = 2$, so R(x) = 2000/(x+2000). R(3000) = 40%.
The expected excess losses are: (40%)(10,000,000) = 4 million.
The ceded premium is: (1.1)(4 million) = **4.4 million**.

22.98. E. In 2001 the Pareto has $\theta = 2000$ and $\alpha = 2$, so in 2002 the Pareto has parameters $\theta = (1.05)(2000) = 2100$ and $\alpha = 2$. In 2002, R(3000) = 2100/(3000+2100) = 41.18%.
The expected losses in 2002 are: (1.05)(10 million) = 10.5 million.
The expected excess losses in 2002 are: (41.18%)(10.5 million) = 4.324 million.
The ceded premium in 2002 is: (1.1)(4.324 million) = 4.756 million.
C_{2002} / C_{2001} = 4.756/4.4 = **1.08**.
Alternately, the excess ratio at 3000 in 2002 is the same as the excess ratio at 3000/1.05 = 2857 in 2001. In 2001, R(2857) = 2000/(2857+2000) = 41.18%. Proceed as before.
Alternately, the average payment per loss is: $(1+r)\ c\ (E[X \wedge L/(1+r)] - E[X \wedge d/(1+r)])$.
In 2001 this is: $E[X] - E[X \wedge 3000]$. In 2002 this is: $(1.05)(E[X] - E[X \wedge 3000/1.05])$.
C_{2002} / C_{2001} = (avg. payment per loss in 2002)/(average payment per loss in 2001) =
$(1.05)(E[X] - E[X \wedge 2857])/(E[X] - E[X \wedge 3000]) = (1.05)(2000 - 1176)/(2000 - 1200) =$ **1.08**.
Comment: Over a fixed limit the excess losses increase more quickly than the overall inflation rate of 5%.

22.99. D. After severity increases by 50%:

Probability	Severity	Payment with 100 deductible
0.25	60	0
0.25	120	20
0.25	180	80
0.25	300	200

Average payment per loss: (0 + 20 + 80 + 200)/4 = 75.
Expected total payment = (300)(75) = **22,500**
Comment: Expect 300 payments: 75@0, 75@ 20, 75@80, and 75@ 200, for a total of 22,500.

22.100. B. Prior to inflation, $S(100) = e^{-100/200} = 0.6065$.
After inflation, severity is Exponential with $\theta = (1.2)(200) = 240$.
$S(100) = e^{-100/240} = 0.6592$.
Percentage increase in the number of reported claims next year:
(1.1)(0.6592/0.6065) - 1 = **19.6%**.

22.101. D. After inflation, severity is Pareto with $\theta = (1.2)(1500) = 1800$, and $\alpha = 4$.
Expected payment per loss: $E[X] - E[X \wedge 100] = \{\theta/(\alpha-1)\} - \{\theta/(\alpha-1)\}\{1 - (\theta/(\theta+100))^{\alpha-1}\}$
$= \{1800/(4 - 1)\}(1800/(1800 + 100))^{4-1}$ = **510.16**.
Alternately, the average payment per loss in the later year is:
$(1+r)\ c\ (E[X \wedge u/(1+r)] - E[X \wedge d/(1+r)]) = (1.2)(1)(E[X] - E[X \wedge 100/1.2]) =$
$1.2\{500 - 500(1 - (1500/1583.33)^3)\}$ = **510.16**.

22.102. A. 1. True.
2. False. Should be α and $(1+i)\theta$. In any case, α is a shape not a scale parameter.
3. False. Should be $(1+i)\theta$; the scale parameter is multiplied by one plus the rate of inflation.

22.103. B. First year: $E[X \wedge 5000] = (2000/(2-1))\ \{1 - (2000/(2000 + 5000))^{2-1}\} = 1429$.
Next year, d = 100, u = 5000, c = 80% and r = 4%, and the average payment per loss is:
$(1.04)(80\%)\{E[X \wedge 5000/1.04] - E[X \wedge 100/1.04]\} = 0.832\{E[X \wedge 4807.7] - E[X \wedge 96.15]\} =$
$(0.832)(2000)\{2000/(2000 + 96.15) - 2000/(2000 + 4807.7)\} = 1099$.
Reduction is: 1 - 1099/1429 = **23.1%**.
Alternately, in the next year we have a Pareto with $\alpha = 2$, and $\theta = (1.04)(2000) = 2080$.
Thus in the next year, $E[X \wedge 5000] = (2080/(2-1))\ \{1 - (2080/(2080 + 5000))^{2-1}\} = 1468.9$.
$E[X \wedge 100] = (2080/(2-1))\ \{1 - (2080/(2080 + 100))^{2-1}\} = 95.4$.
Thus the average payment per loss is: (80%)(1468.9 - 95.4) = 1098.8. Proceed as before.

22.104. B. In 2004 losses follow a Pareto with $\alpha = 2$ and $\theta = (1.2)(5) = 6$.
$E[X] = 6/(2 - 1) = 6$. $E[X \wedge 10] = \{\theta/(\alpha-1)\}\ \{1 - (\theta/(\theta+x))^{\alpha-1}\} = (6)\{1 - 6/(6 + 10)\} = 3.75$.
$LER(10) = E[X \wedge 10]/E[X] = 3.75/6 = 0.625 =$ **5/8**.

22.105. D. $S_{thisyear}(750{,}000) = \{400{,}000/(400{,}000 + 750{,}000)\}^2 = 0.12098$.
After inflation, the losses follow a Pareto distribution with $\alpha = 2$ and $\theta = (1.1)(400{,}000) =$ 440,000. $S_{nextyear}(750{,}000) = \{440{,}000/(440{,}000 + 750{,}000)\}^2 = 0.13671$.
$S_{nextyear}(750{,}000) / S_{thisyear}(750{,}000) = 0.13671/0.12098 =$ **1.130**.
Alternately, one can calculate the survival function at 750,000 next year, by deflating the 750,000 to this year. 750,000/1.1 = 681,818. $S_{nextyear}(7500{,}00) = S_{thisyear}(681{,}818) =$
$\{400{,}000/(400{,}000 + 681{,}818)\}^2 = 0.13671$. Proceed as before.

22.106. B. This is Pareto Distribution with $\alpha = 3$ and $\theta = 800$.
In 2006, losses follow a Pareto Distribution with $\alpha = 3$ and $\theta = (1.08)(800) = 864$.
With a franchise deductible, the total amount of those losses of size 300 or less are eliminated, while the full amount of all losses of size greater than 300 is paid.

$$\text{Losses eliminated} = \int_0^{300} x\, f(x) dx = E[X \wedge 300] - 300S(300) =$$

$(864/2)\{1 - (864/(300 + 864))^2\} - (300)\{864/(300 + 864)\}^3 = 71.30$.
Loss elimination ratio = Losses eliminated / mean = 71.30/(864/2) = **16.5%.**
Alternately, the expected payment per loss with an ordinary deductible would be:
$E[X] - E[X \wedge 300] = (864/2) - (864/2)\{1 - (864/(300 + 864))^2\} = 238.02$.
With the franchise deductible one pays 300 more on each loss of size exceeding 300 than under the ordinary deductible: $238.02 + 300S(300) = 238.02 + (300)\{864/(300 + 864)\}^3 = 360.71$.
$E[X] = 864/2 = 432$. Loss Elimination Ratio is: $1 - 360.71/432 =$ **16.5%.**

22.107. D. For a Pareto Distribution, $E[X] - E[X \wedge d] = \theta/(\alpha-1) - \{\theta/(\alpha-1)\}\{1 - (\theta/(d + \theta))^{\alpha-1}\} =$
$\theta^\alpha / \{(\alpha-1)(d + \theta)^{\alpha-1}\}$.
The premium in each year is: $1.2(E[X] - E[X \wedge 600]) = 1.2\theta^\alpha / \{(\alpha-1)(600 + \theta)^{\alpha-1}\}$.
If the reinsurance covers the layer excess of d in ground up loss, then the reinsurance premium is: $1.1(E[X] - E[X \wedge d]) = 1.1\theta^\alpha / \{(\alpha-1)(d + \theta)^{\alpha-1}\}$.
In 2005, the Pareto has $\alpha = 2$ and $\theta = 3000$.
$R_{2005} = 0.55P_{2005}$. $\Rightarrow (1.1)\ 3000^2/(d + 3000) = (0.55)(1.2)\,3000^2/3600$.
$\Rightarrow (0.66)(d + 3000) = (1.1)(3600)$. $\Rightarrow d = 3000$.
In 2006, the losses follow a Pareto Distribution with $\alpha = 2$ and $\theta = (3000)(1.2) = 3600$.
$P_{2006} = (1.2)3600^2/4200$. $R_{2006} = (1.1)3600^2/6600$.
$R_{2006}/P_{2006} = (1.1)(4200)/\{(1.2)(6600)\} =$ **0.583**.
<u>Comment</u>: The higher layer increases more due to inflation, and therefore the ratio of R/P has to increase, thereby eliminating choices A, B, and C.
One could have instead described the reinsurance as covering the layer excess of 600 + d in ground up loss, in which case d = 2400 and one obtains the same final answer.
$P_{2005} = 3000$. $R_{2005} = 1650$. $P_{2006} = 3703$. $R_{2006} = 2160$.

22.108. D. The losses in dollars are all 30% bigger.
For example, 10,000 euros ⇔ 13,000 dollars.
This is mathematically the same as 30% uniform inflation.
We get another Lognormal with σ = **2** the same, and $\mu' = \mu + \ln(1+r) = 8 + \ln(1.3) =$ **8.26**.
Comment: The mean in dollars should be 1.3 times the mean in euros. The mean in euros is: $\exp[8 + 2^2/2] = 22{,}026$. For the choices we get means of: $\exp[6.15 + 2.26^2/2] = 6026$, $\exp[7.74 + 2^2/2] = 16{,}984$, $\exp[8 + 2.6^2/2] = 87{,}553$, $\exp[8.26 + 2^2/2] = 28{,}567$, and $\exp[10.4 + 2.6^2/2] = 965{,}113$. Eliminating all but choice D.

22.109. A. In 2005, $S(250) = (1000/1250)^3 = 0.512$.
In 2006, the Pareto distribution has $\alpha = 3$ and $\theta = (1.1)(1000) = 1100$.
In 2006, $S(250) = (1100/1350)^3 = 0.54097$.
Increase in expected number of claims: (14)(0.54097 - 0.512) = **0.406** claims.
Alternately, deflate 250 in 2006 back to 2005: 250/1.1 = 227.27.
In 2005, $S(227.27) = (1000/1227.27)^3 = 0.54097$. Proceed as before.
Comment: We make no use of the fact that frequency is Poisson.

22.110. D. In 4 years, severity is Pareto with parameters $\alpha = 4$ and $\theta = (1.06^4)3000 = 3787$.
Under Policy R, the expected cost per loss is the mean: 3787/(4 - 1) = 1262.
Under Policy S, the expected cost per loss is: $E[X \wedge 3000] - E[X \wedge 500] =$
$\{\theta/(\alpha-1)\}\{(\theta/(\theta+500))^{\alpha-1} - (\theta/(\theta+3000))^{\alpha-1}\} = (1262)\{(3787/4287)^3 - (3787/6787)^3\} = 651$.
Difference is: 1262 - 651 = **611**.
Alternately, under Policy S, the expected cost per loss, using the Pareto for year t, is:
$(1.06^4)\{E[X \wedge 3000/1.06^4] - E[X \wedge 500/1.06^4]\} =$
$(1.2625)\{E[X \wedge 2376] - E[X \wedge 396]\} =$
$(1.2625)\{\{3000/(4\text{-}1)\}\{1 - (3000/5376)^{4-1}\} - \{3000/(4\text{-}1)\}\{1 - (3000/3396)^{4-1}\} =$
$(1.2625)\{826 - 311\} = 650$.
Difference is: 1262 - 650 = **612**.
Comment: This exam question should have said: "Policy S has a deductible of $500 and a maximum covered loss of $3,000."

22.111.
Compare the contributions to the layer from 100,000 to 200,000 before and after inflation:

Claim	Loss	Contribution to Layer	Inflated Loss	Contribution to Layer
A	35,000	0	37,800	0
B	125,000	25,000	135,000	35,000
C	180,000	80,000	194,400	94,400
D	206,000	100,000	222,480	100,000
E	97,000	0	104,760	4,760
Total		205,000		234,160

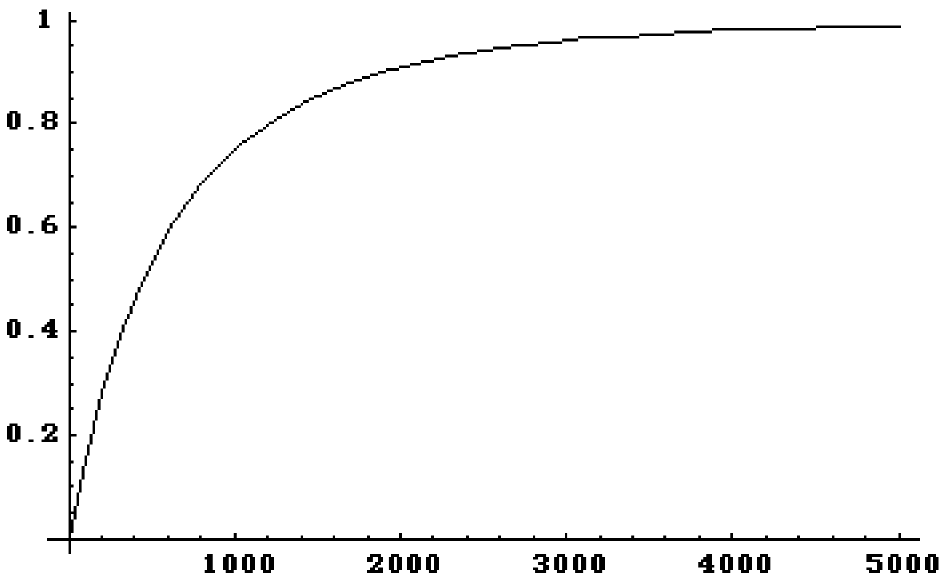
234,160/205,000 = 1.142. **14.2%** effective trend on this layer.

Section 23, Lee Diagrams

A number of important loss distribution concepts can be displayed graphically.
Important concepts will be illustrated using this graphical approach of Lee.[158]

While this material is not on your exam, graphically oriented students may benefit from looking at this material. You may find that it helps you to remember formulas that are used on your exam.

Below is shown a conventional graph of a Pareto Distribution with $\alpha = 4$ and $\theta = 2400$:



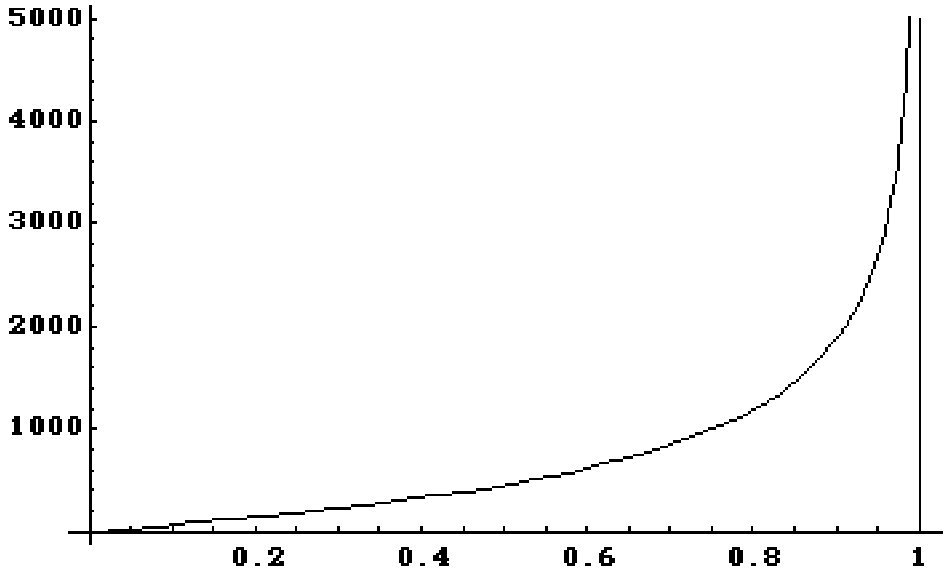
Exercise: For a Pareto Distribution with $\alpha = 4$ and $\theta = 2400$, what is F(1000)?
[Solution: $F(x) = 1 - \left(\frac{\theta}{\theta + x}\right)^{\alpha}$. $F(1000) = 1 - (2400/3400)^4 = 0.752$.]

In the conventional graph, the x-axis correspond to size of loss, while the y-axis corresponds to probability. Thus for example, the above graph of a Pareto includes the point (1000, 0.752).

In contrast, **"Lee Diagrams" have the x-axis correspond to probability, while the y-axis corresponds to size of loss.**

[158] "The Mathematics of Excess of Loss Coverage and Retrospective Rating --- A Graphical Approach", by Y.S. Lee, PCAS LXXV, 1988. *Lee cites "A Practical Guide to the Single Parameter Pareto Distribution", by Steven Philbrick, PCAS 1985. Philbrick in turn points out the similarity to the treatment of Insurance Charges and Savings in "Fundamentals of Individual Risk Rating and Related Topics", by Richard Snader.*

Here is the Lee Diagram of a Pareto Distribution with $\alpha = 4$ and $\theta = 2400$:



For example, since F(1000) = 0.752, the point (0.752, 1000) is on the curve. Note the way that the probability approaches a vertical asymptote of unity as the claim size increases towards infinity.[159]

Advantages of this representation of Loss Distributions include the intuitively appealing features:[160]

1. The mean is the area under the curve.[161]

2. A loss limit is represented by a horizontal line, and excess losses lie above the line.

3. Losses eliminated by a deductible lie below the horizontal line represented by the deductible amount.

4. After the application of a trend factor, the new loss distribution function lies above the prior distribution.

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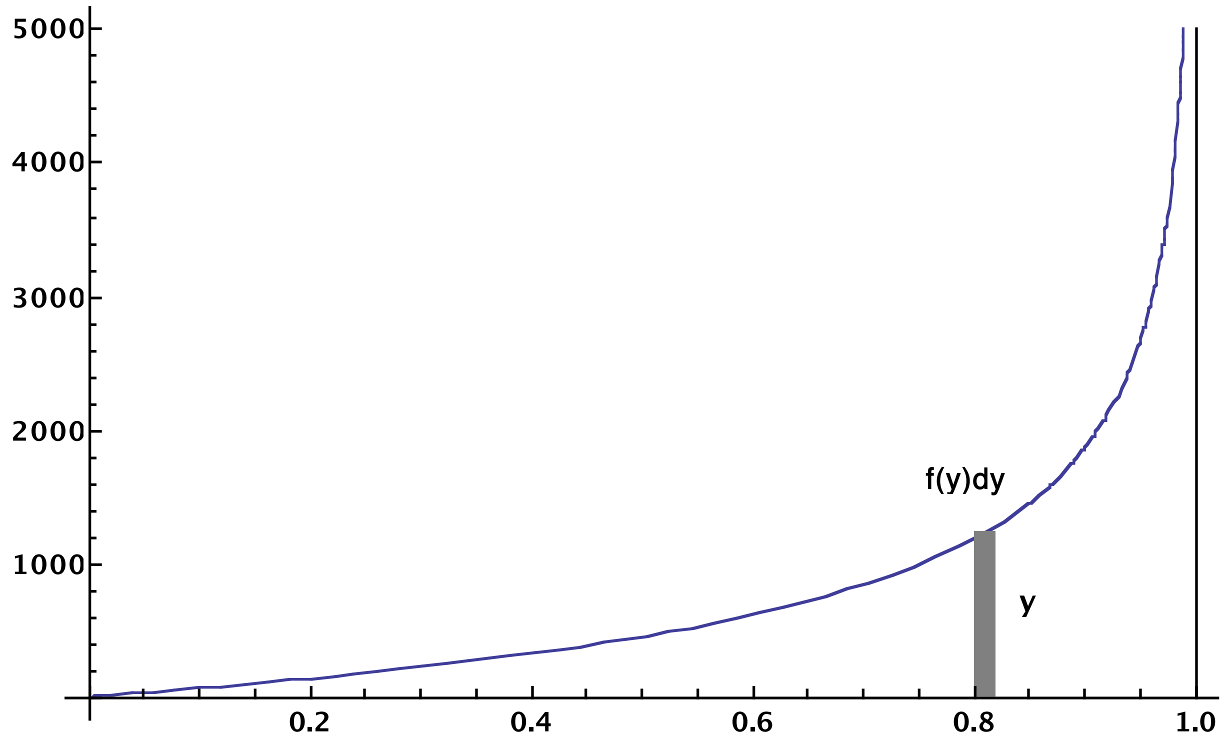
[159] $F(x) \to 1$, as $x \to \infty$.

[160] "A Practical Guide to the Single Parameter Pareto Distribution", by Steven Philbrick, PCAS 1985.

[161] As discussed below. In a conventional graph, the area under a distribution function is infinite, the area under the survival function is the mean, and the area under the density is one.

Means:

One can compute the mean of this size of loss distribution as the integral from 0 to ∞ of y f(y)dy. As shown below, this is represented by summing narrow vertical strips, each of width f(y)dy and each of height y, the size of loss.[162]

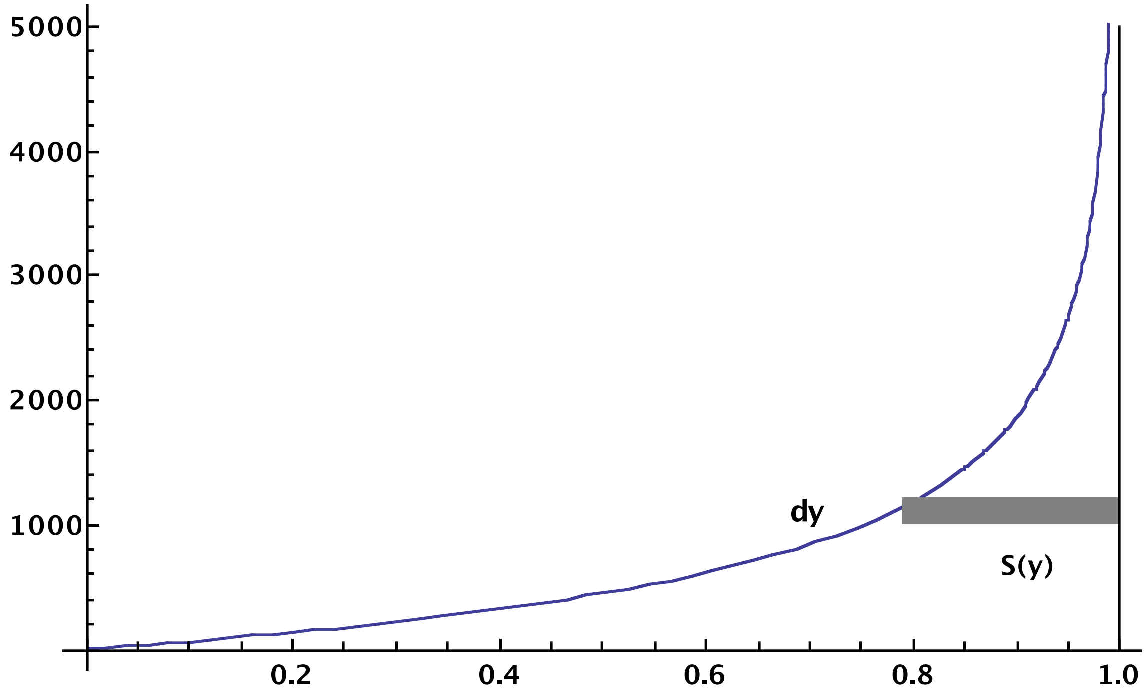


Summing over all y, would give the area under the curve. Thus **the mean is the area under the curve**.[163]

[162] y = size of loss. x = F(y) = Prob[Y ≤ y]. f(y) = dF/dy = dx/dy. dx = (dx/dy)dy = f(y)dy.
The width of each vertical strip is dx = f(y)dy.

[163] In this case, the mean = $\theta/(\alpha-1)$ = 2400/3 = 800.

Alternately as shown below, one can compute the area under the curve by summing narrow horizontal strips, each of height dy and each of width 1 - F(y) = S(y), the survival function.[164] Summing over all y, would give the area under the curve.



Two integrals can be gotten from each other via integration by parts:

$$\int_0^\infty S(y)\ dy = \int_0^\infty y\ f(y)\ dy = \text{mean claim size.}$$

Thus the area under the curve can be computed in either of two ways. The mean is either the integral of the survival function, via summing horizontal strips, or the integral of y f(y), via summing vertical strips .

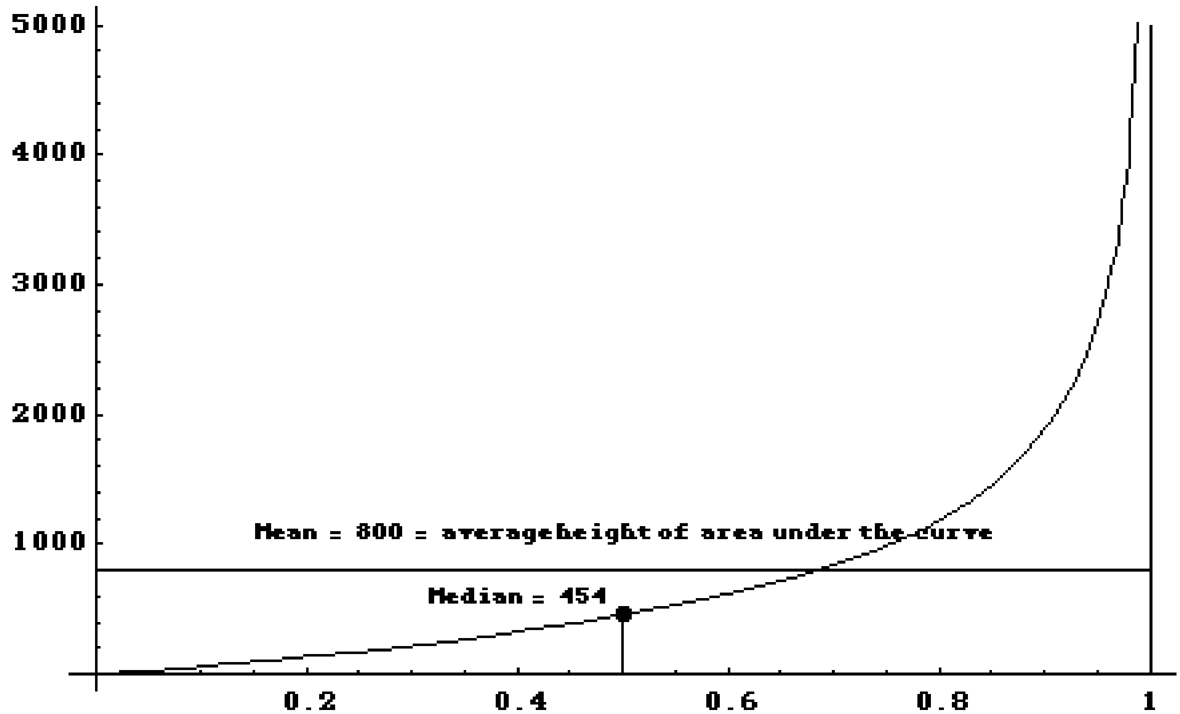
In the Pareto example:

$$\int_0^\infty \left(\frac{2400}{y+2400}\right)^4 dy = 800 = \int_0^\infty y\ 4\ \frac{2400^4}{(y+2400)^5}\ dy = \text{mean claim size.}$$

[164] Each horizontal strip goes from x = F(y) to x = 1, and thus has width 1 - F(x) = S(x).

Mean vs. Median:

The Lee Diagram below, illustrates why for loss distributions skewed to the right, i.e., with positive coefficient of skewness, the mean is usually greater than the median.[165]

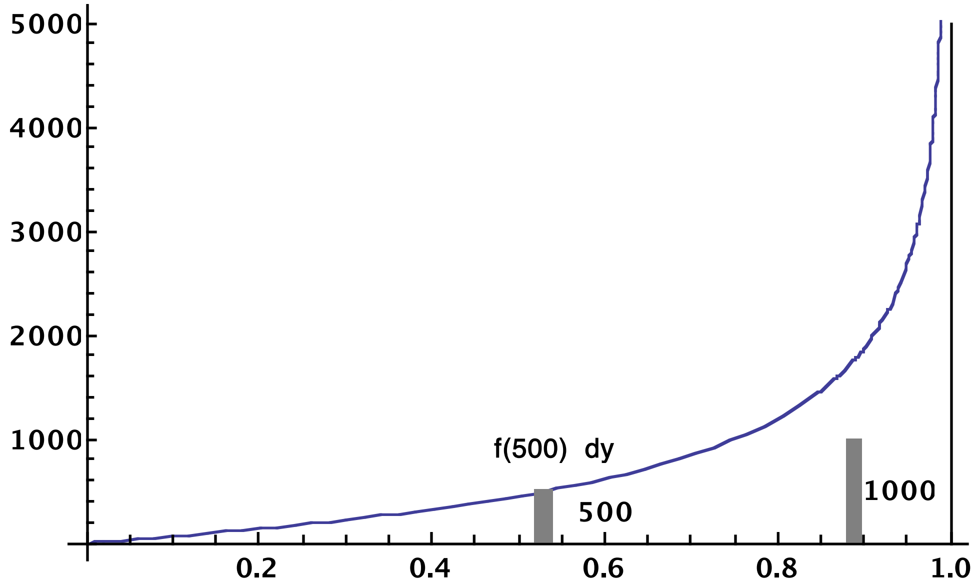


The area under the curve is the mean; since the width is one, the average height of the curve is the mean. On the other hand, the median is the height at which the curve reaches a probability of one half, which in this case is less than the average height of the curve. The diagram would be similar for most distributions significantly skewed to the right, and the median is less than the mean.

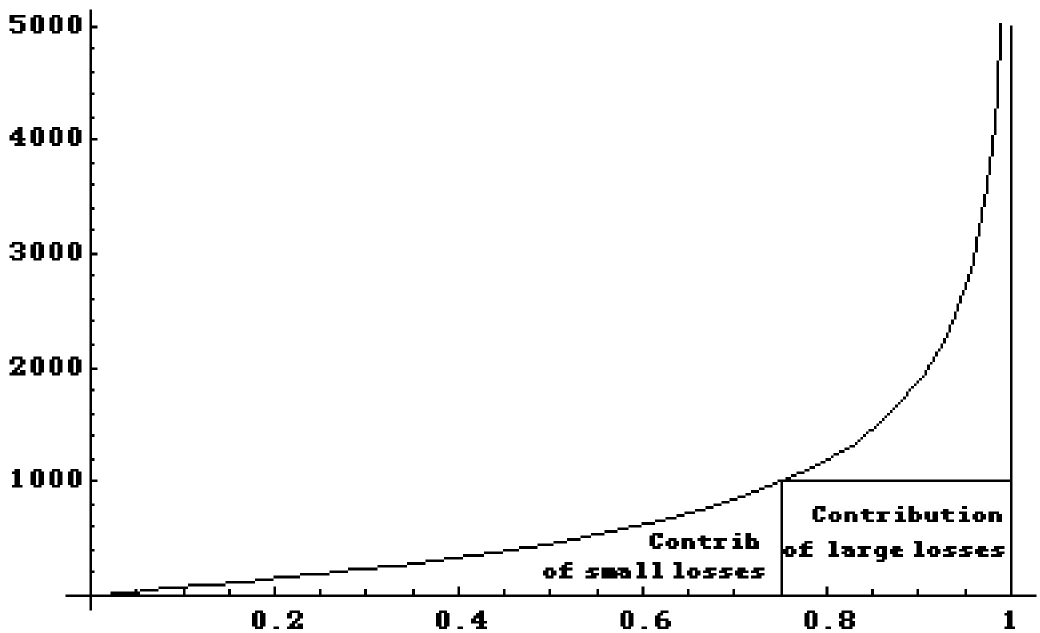
[165] For distributions with skewness close to zero, the mean is usually close to the median. (For symmetric distributions, the mean equals the median.) Almost all loss distributions encountered in practical applications by casualty actuaries have substantial positive skewness, with the median significantly less than the mean.

<u>Limited Expected Value</u>:

A loss of size less than 1000 contributes its size to the Limited Expected Value at 1000, E[X∧1000]. A loss of size greater than or equal to 1000 contributes 1000 to E[X∧1000]. These contributions to E[X∧1000] correspond to vertical strips:

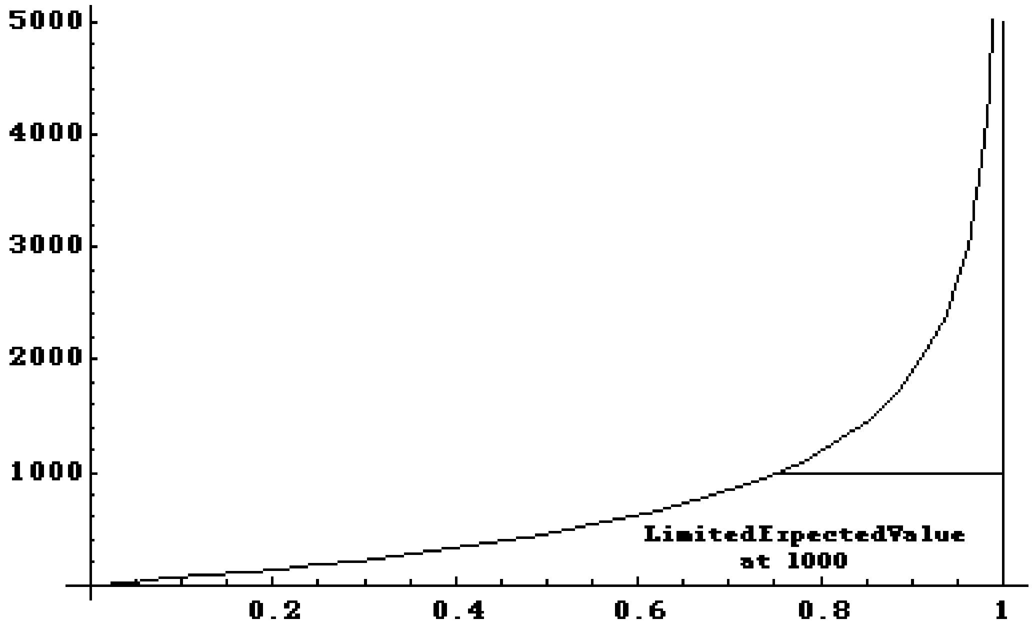


The contribution to E[X∧1000] of the small losses, as a sum of vertical strips, is the integral from 0 to 1000 of y f(y) dy. The contribution to E[X∧1000] of the large losses, is the area of the rectangle of height 1000 and width S(1000): 1000 S(1000).

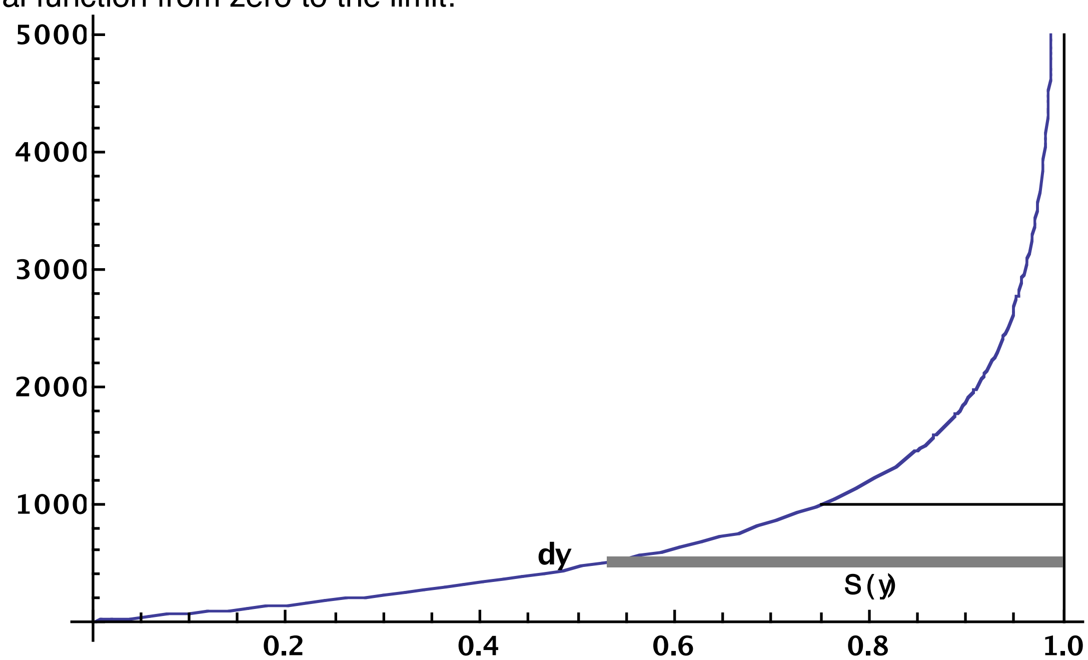


These 2 pieces correspond to the 2 terms: $E[X \wedge 1000] = \int_0^{1000} y\, f(y)\, dy + 1000 S(1000)$.

Adding up these two types of contributions, E[X∧1000] corresponds to the area under both the distribution curve and the horizontal line y = 1000:



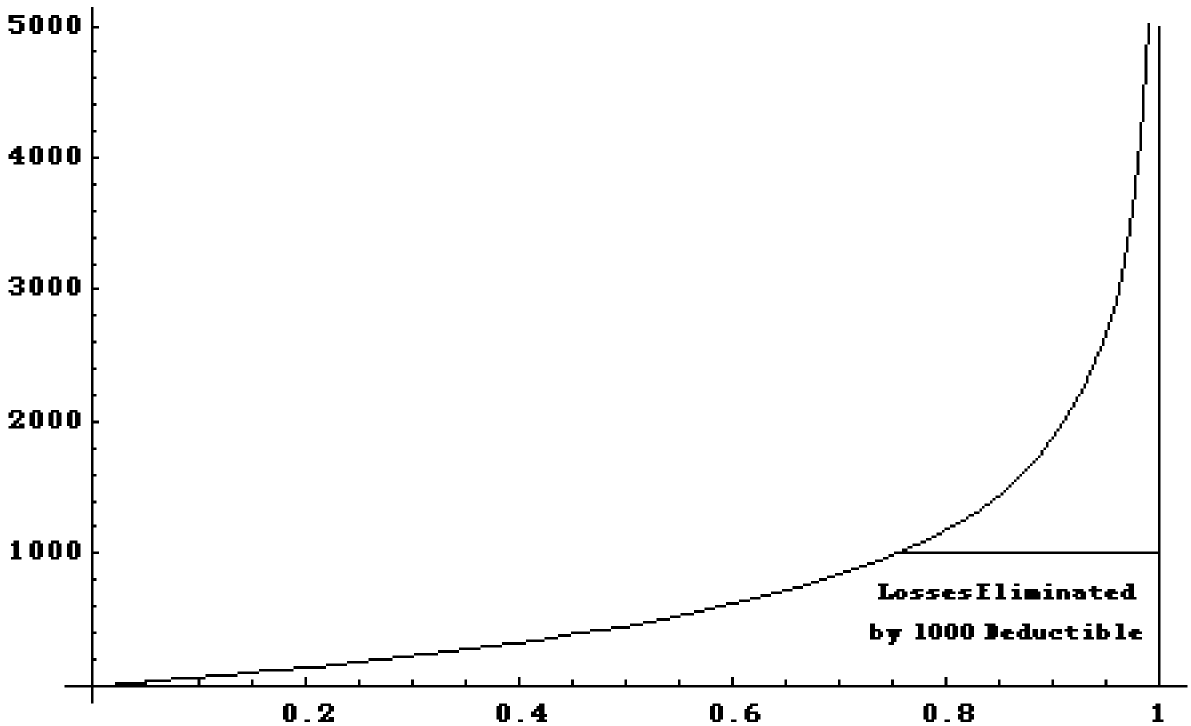
In general, **E[X∧L] ⇔ the area below the curve and also below the horizontal line at L.**
Summing horizontal strips, the Limited Expected Value also is equal to the integral of the survival function from zero to the limit:



$$E[X \wedge 1000] = \int_0^{1000} S(t)\ dt .$$

Losses Eliminated:

A loss of size less than 1000 is totally eliminated by a deductible of 1000. For a loss of size greater than or equal to 1000, 1000 is eliminated by a deductible of size 1000. Summing these contributions as vertical strips, as shown below the losses eliminated by a 1000 deductible correspond to the area under both the curve and y = 1000.



Losses Eliminated by 1000 Deductible ⇔ area under both the curve and y = 1000
⇔ $E[X \wedge 1000]$.

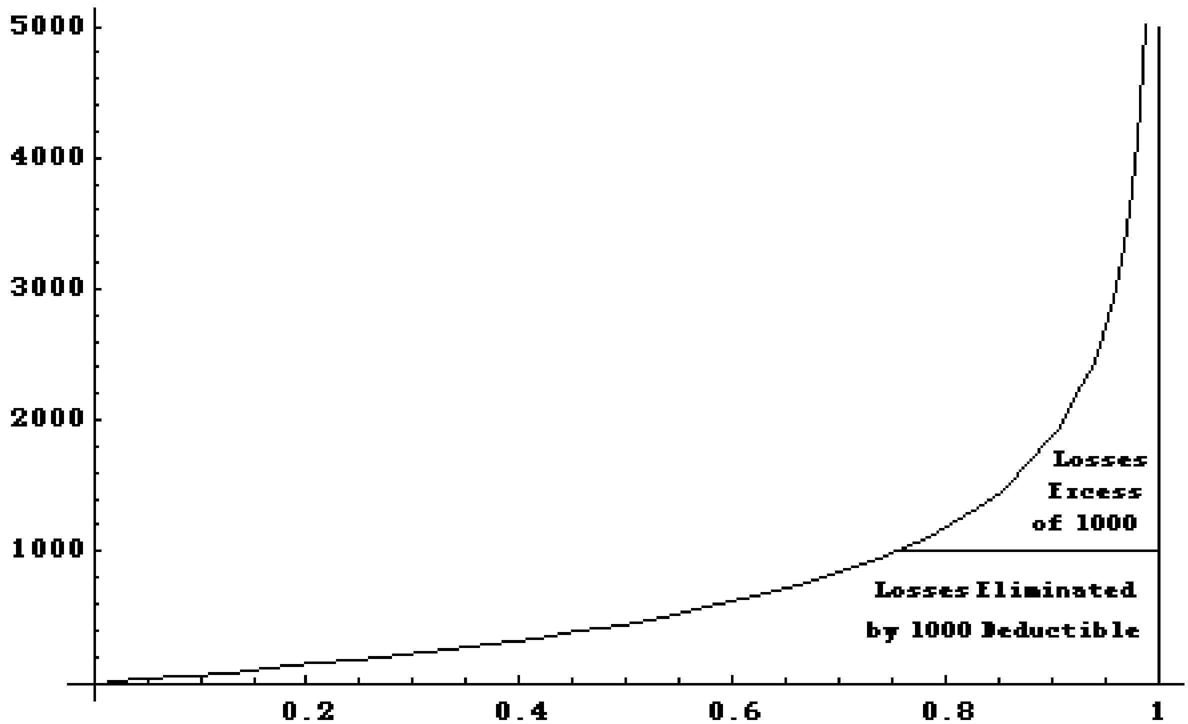
In general, **the losses eliminated by a deductible d ⇔ the area below the curve and also below the horizontal line at d.**

Loss Elimination Ratio (LER) = losses eliminated / total losses.
The Loss Elimination Ratio is represented by the ratio of the area under both the curve and y = 1000 to the total area under the curve.[166] One can either calculate the area corresponding to the losses eliminated by summing of horizontal strips of width S(t) or vertical strips of height t limited to x. Therefore: $LER(x) = \int_0^x S(t)\, dt \,/\, m = \{\int_0^x t\, f(t)\, dt + xS(x)\} \,/\, m = E[X \wedge x] \,/\, E[X]$.

[166] In the case of a Pareto with $\alpha = 4$ and $\theta = 2400$, the Loss Elimination Ratio at 1000 is 518.62 / 800 = 64.8% .

Excess Losses:

A loss of size y > 1000, contributes y - 1000 to the losses excess of a 1000 deductible. Losses of size less than or equal to 1000 contribute nothing. Summing these contributions, as shown below the losses excess of a 1000 deductible corresponds to the area under the distribution curve but above y = 1000.



This area under the distribution curve but above y = 1000, are the losses excess of 1000, $E[(X - 1000)_+]$, the numerator of the Excess Ratio.[167] The denominator of the Excess Ratio is the mean, the whole area under the distribution curve. Thus the Excess Ratio, R(1000) = $E[(X - 1000)_+] / E[X]$, corresponds to the ratio of the area under the curve but above y = 1000, to the total area under the curve.[168]

Since the Losses Eliminated by a 1000 deductible and the Losses Excess of 1000 sum to the total losses, $E[(X - 1000)_+]$ = Losses Excess of 1000 = total losses - losses limited to 1000 ⇔ $E[X] - E[X \wedge 1000]$. ⇒ LER(1000) + R(1000) = 1.

167 In the case of a Pareto with $\alpha = 4$ and $\theta = 2400$, the area under the curve and above the line y=1000 is: $\theta^a (\theta+y)^{1-\alpha} / (\alpha-1) = 2400^4 / \{(3400^3)3\} = 281.38$.

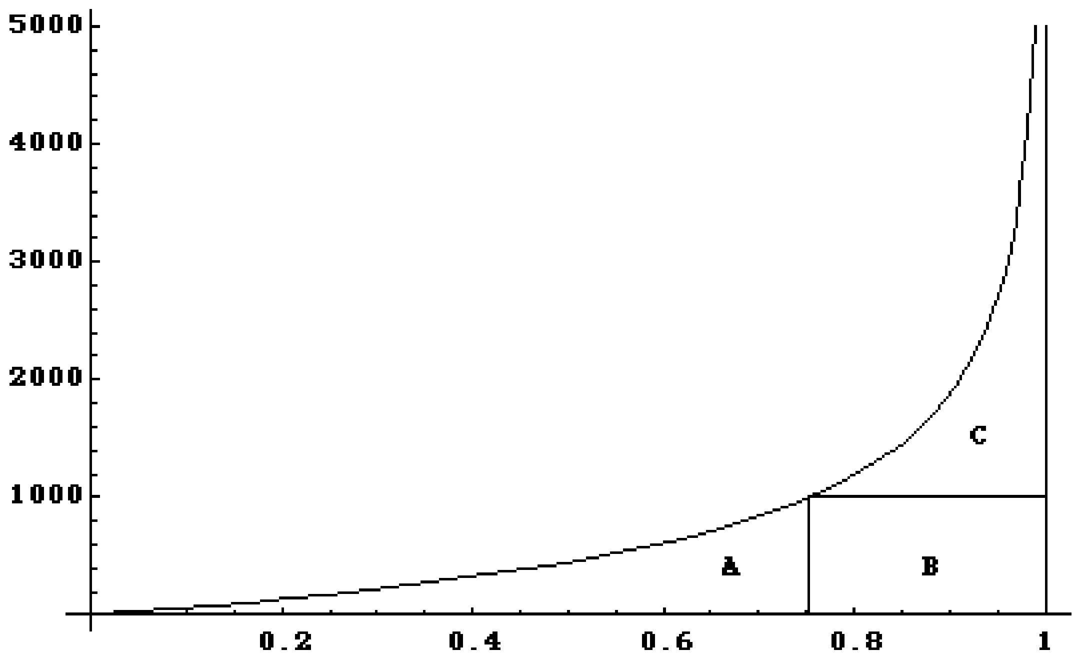
168 In the case of a Pareto with $\alpha = 4$ and $\theta = 2400$, the Excess Ratio at 1000 is: 281.38 / 800 = 35.2% = $(2400 / 3400)^3 = \{\theta/(\theta+x)\}^{\alpha-1}$. R(1000) = 1 - LER(1000) = 1 - 0.648 = 0.352.

$$E[(X - 1000)_+] = \int_{1000}^{\infty} S(t)\ dt = \int_{1000}^{\infty} (t - 1000)\ f(t)\ dt.$$

The first integral ⇔ summing of horizontal strips.
The second integral ⇔ summing of vertical strips.
$E[(X - 1000)_+]$ ⇔ area under the distribution curve but above y = 1000.

In general, **the losses excess of u ⇔ the area below the curve and also above the horizontal line at u.**

As was done before, one can divide the limited losses into two Area A and B.



$$\text{Area A} = \int_0^{1000} t\ f(t)\ dt = 271.$$

Area B = (1000)S(1000) = (1000)(.248) = 248.
Area A + Area B = E[X∧1000] = 519.
Area C = E[X] - E[X∧1000] = 800 - 519 = 281.

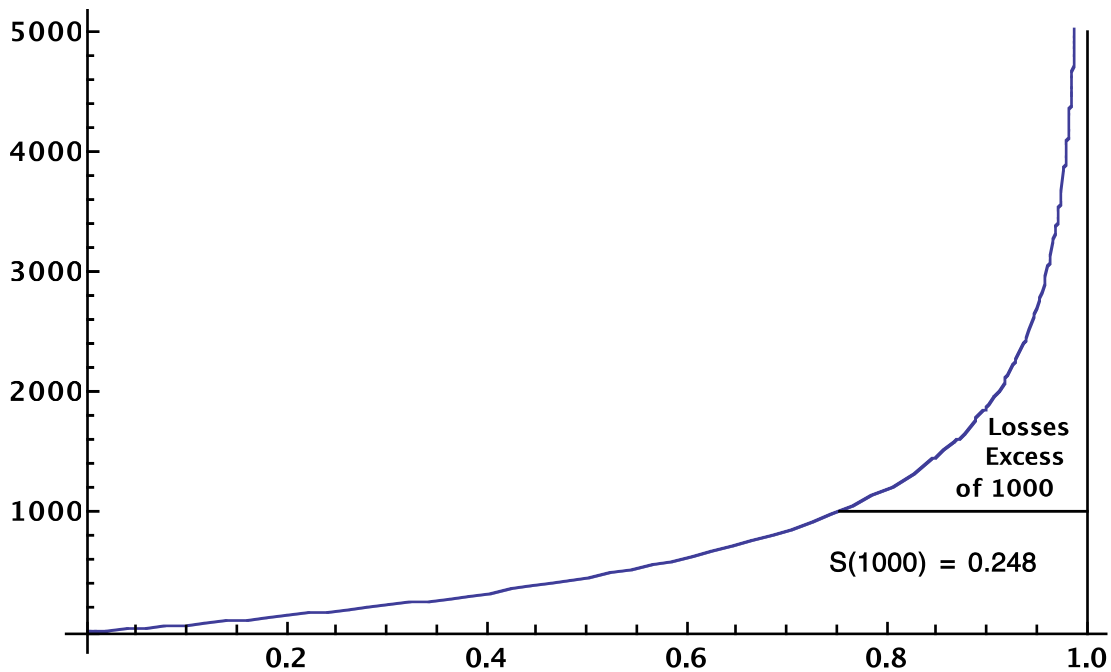
Excess Ratio at 1000 = C / (A + B + C) = 281/ 800 = 35%.

<u>*Mean Excess Loss (Mean Residual Life)*</u>:

The Mean Excess Loss can be written as the excess losses divided by the survival function:

$$e(x) = \int_x^\infty S(t)\ dt \,/\, S(x) = \int_x^\infty (t - x)\ f(t)\ dt \,/\, S(x)\,.$$

For example, the losses excess of a 1000 deductible corresponds to the area under the distribution curve but above y = 1000:



This area under the distribution curve but above y = 1000 is the numerator of the Mean Excess Loss.[169] The first integral above corresponds to the summing of horizontal strips. The second integral above corresponds to the summing of vertical strips. The denominator of the Excess Ratio is the survival function S(1000) = 0.248, which is the width along the horizontal axis of the area corresponding to the numerator.

Thus the Mean Excess Loss, e(1000), corresponds to the average height of the area under the curve but above y = 1000.[170] For example, in this case that average height is e(1000) = 1133. However, since the curve extends vertically to infinity, it is difficult to use this type of diagram to distinguish the mean excess loss, particularly for heavy-tailed distributions such as the Pareto.

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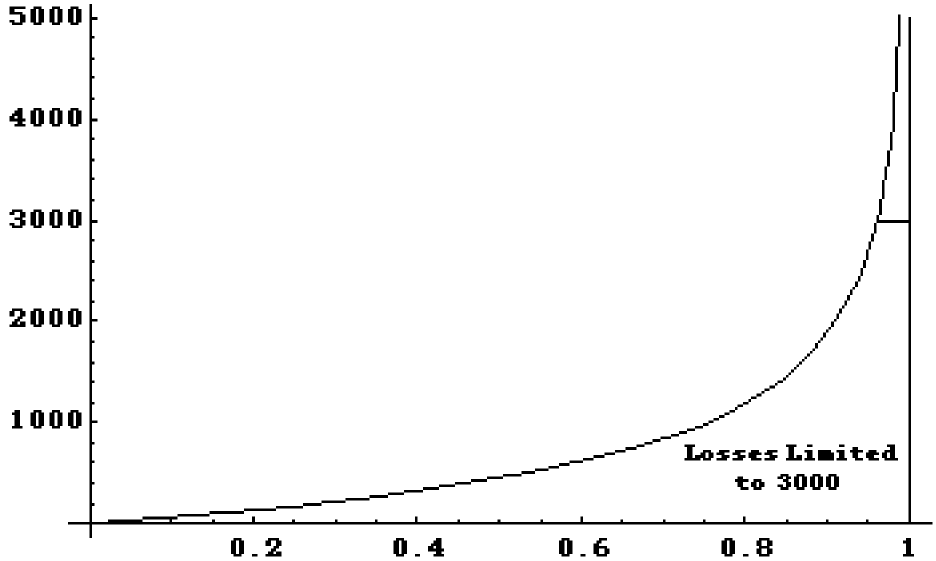
[169] The numerator of the Mean Excess Loss is the same as the numerator of the Excess Ratio. In the case of a Pareto with α = 4 and θ = 2400, the area under the curve and above the line y = 1000 is:
$\theta^\alpha (\theta+y)^{1-\alpha} / (\alpha-1) = 2400^4 / \{(3400^3)3\} = 281.38$.

[170] In the case of a Pareto with α = 4 and θ = 2400, the Mean Excess Loss at 1000 is 281.38 / 0.2483 = 1133, since $S(x) = \{\theta/(\theta+x)\}^\alpha = (2400 / 3400)^4 = 0.2483$. Alternately, for the Pareto $e(x) = (\theta+x)/(\alpha-1) = 3400/3 = 1133$.

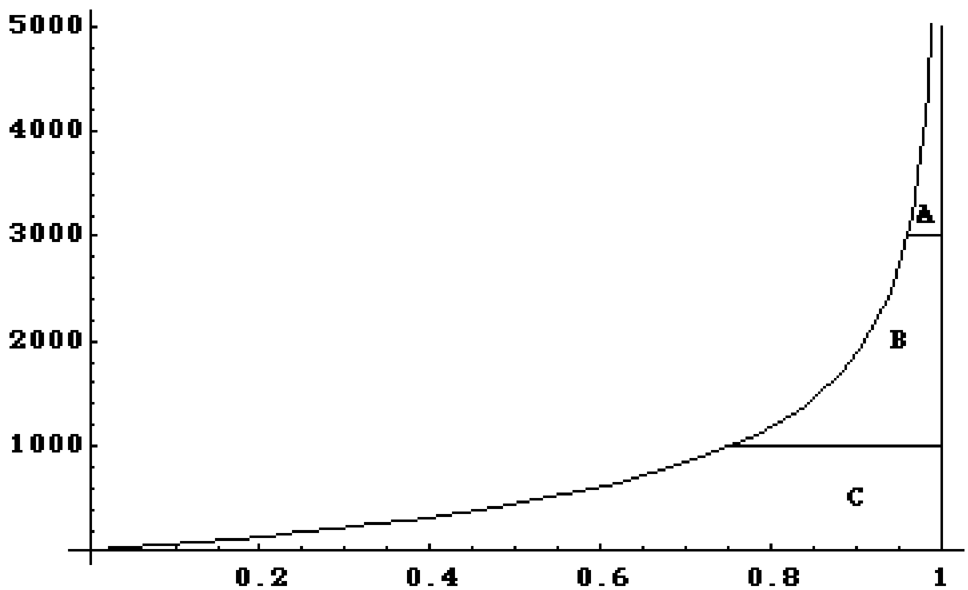
Layers of Loss:

Below is shown the effect of imposing a 3000 maximum covered loss:



Amount paid with a 3000 maximum covered loss ⇔ the losses censored from above at 3000 ⇔ $E[X \wedge 3000]$ ⇔ the area under both the curve and the horizontal line y = 3000.

The Layer of Loss between 1000 and 3000 would be those dollars paid in the presence of both a 1000 deductible and a 3000 maximum covered loss.
As shown below, the layer of losses from 1000 to 3000 is the area under the curve but between the horizontal lines y = 1000 and y = 3000 ⇔ Area B.



Area A ⇔ Losses excess of 3000 ⇔ Losses not paid due to 3000 maximum covered loss.
Area C ⇔ Losses limited to 1000 ⇔ Losses not paid due to 1000 deductible.
Area B ⇔ Layer from 1000 to 3000
⇔ Losses paid with 1000 deductible and 3000 maximum covered loss.

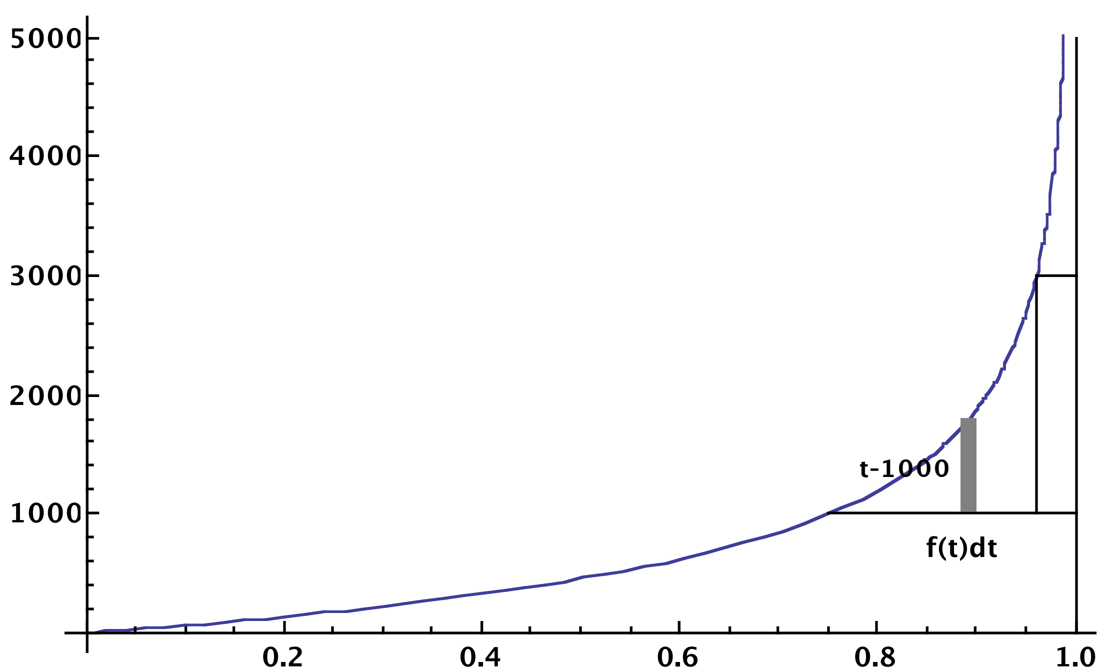
Summing horizontal strips, Area B can be thought of as the integral of the survival function from 1000 to 3000:

$$\text{Layer of losses from 1000 to 3000} = \int_{1000}^{3000} S(t)\ dt = E[X \wedge 3000] - E[X \wedge 1000].$$

This area is also equal to the difference of two limited expected values: the area below the curve and y = 3000, E[X∧3000], minus the area below the curve and y = 1000, E[X∧1000].

In general, **the layer from d to u ⇔ the area under the curve but also between the horizontal line at d and the horizontal line at u.**

Summing vertical strips this same area can also be expressed as the sum of an integral and the area of a rectangle:



Layer of losses from 1000 to 3000 = $\int_{1000}^{3000} (t - 1000)\, f(t)\, dt + (3000 - 1000)\, S(3000)$.

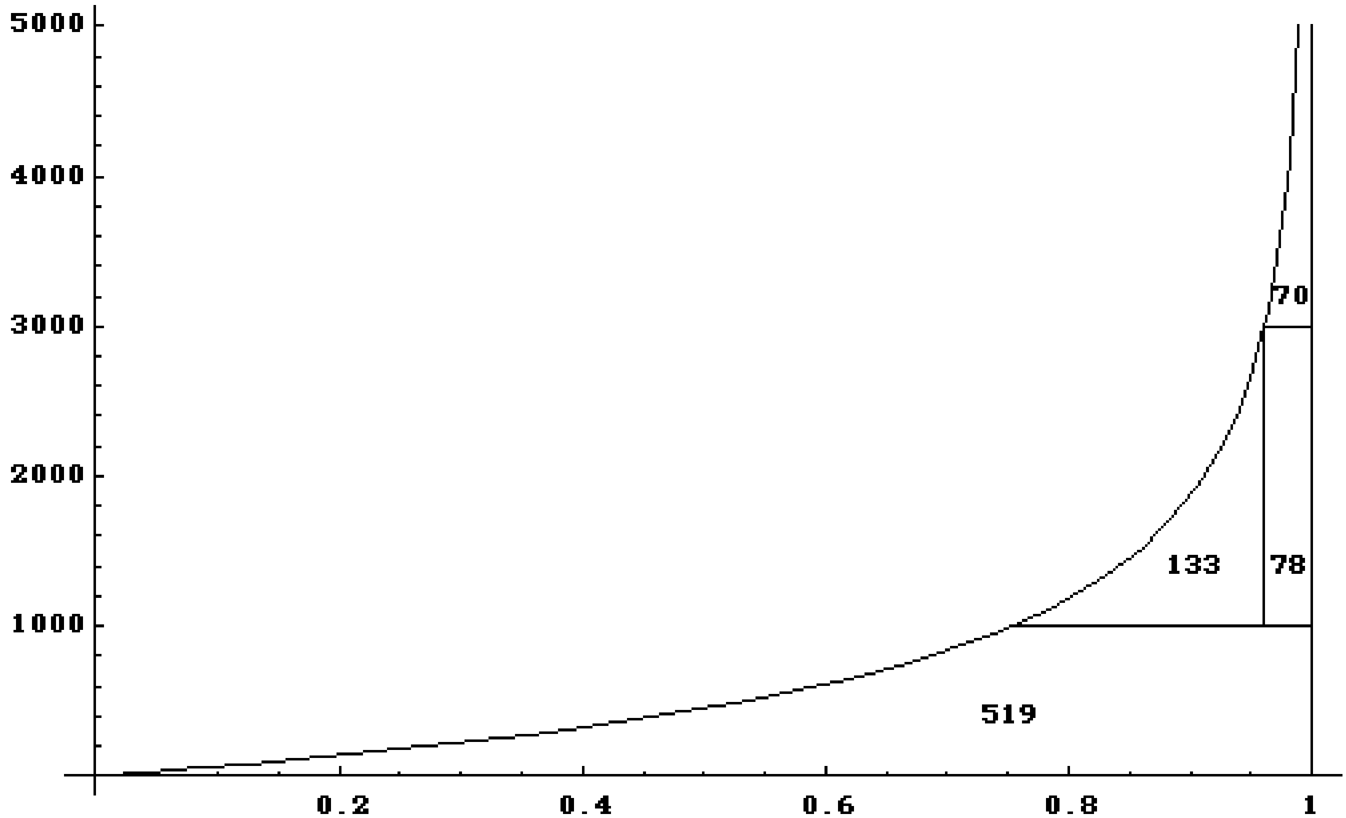
For the Pareto example, losses excess of 3000 = E[X] - E[X∧3000] =
2400/3 - (2400/3){1 - (2400/(2400 + 3000))^3} = 800 - 729.767 = 70.233.
Losses limited to 1000 = E[X∧1000] = (2400/3){1 - (2400/(2400 + 1000))^3) = 518.624.
Layer from 1000 to 3000 = E[X∧3000] - E[X∧1000] = 729.767 - 518.624 = 211.143.
(Losses limited to 1000) + (Layer from 1000 to 3000) + (Losses excess of 3000) =
518.624 + 211.143 + 70.233 = 800 = Mean ⇔ total losses.

Alternately, the layer from 1000 to 3000 = $\int_{1000}^{3000} (t - 1000)\, f(t)\, dt + (3000 - 1000)\, S(3000)$ =

$$\int_{1000}^{3000} (t - 1000)\, (4)\, \frac{2400^4}{(t + 2400)^5}\, dt + (2000)\{2400/(3000+2400)\}^4$$

= 133.106 + 78.037 = 211.143.

The area below the Pareto distribution curve, which equals the mean of 800, can be divided into four pieces, where the layer from 1000 to 3000 has been divided into the two terms calculated above:



Exercise: For a Pareto Distribution with $\alpha = 4$ and $\theta = 2400$, determine the expected losses in the following layers: 0 to 500, 500 to 1000, 1000 to 1500, and 1500 to 2000.

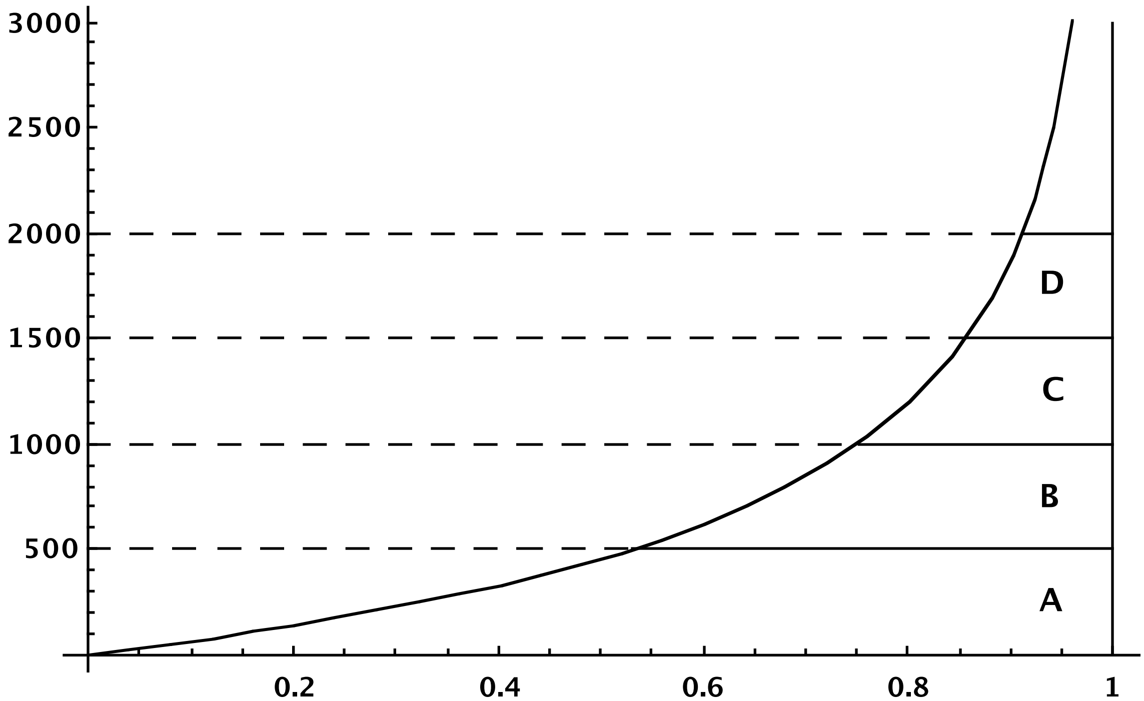
[Solution: $E[X \wedge x] = \frac{\theta}{\alpha - 1} \left\{1 - \left(\frac{\theta}{\theta + x}\right)^{\alpha - 1}\right\} = 800 \left\{1 - \left(\frac{2400}{2400 + x}\right)^{3}\right\}$.

$E[X \wedge 500] = 347$. $E[X \wedge 1000] = 519$. $E[X \wedge 1500] = 614$. $E[X \wedge 2000] = 670$.
Expected Losses in layer from 0 to 500: $E[X \wedge 500] = 347$.
Expected Losses in layer from 500 to 10000: $E[X \wedge 1000] - E[X \wedge 500] = 519 - 347 = 172$.
Expected Losses in layer from 1000 to 1500: $E[X \wedge 1500] - E[X \wedge 1000] = 614 - 519 = 95$.
Expected Losses in layer from 1500 to 2000: $E[X \wedge 2000] - E[X \wedge 1500] = 670 - 614 = 56$.]

For a given width, lower layers of loss are larger than higher layers of loss.[171]

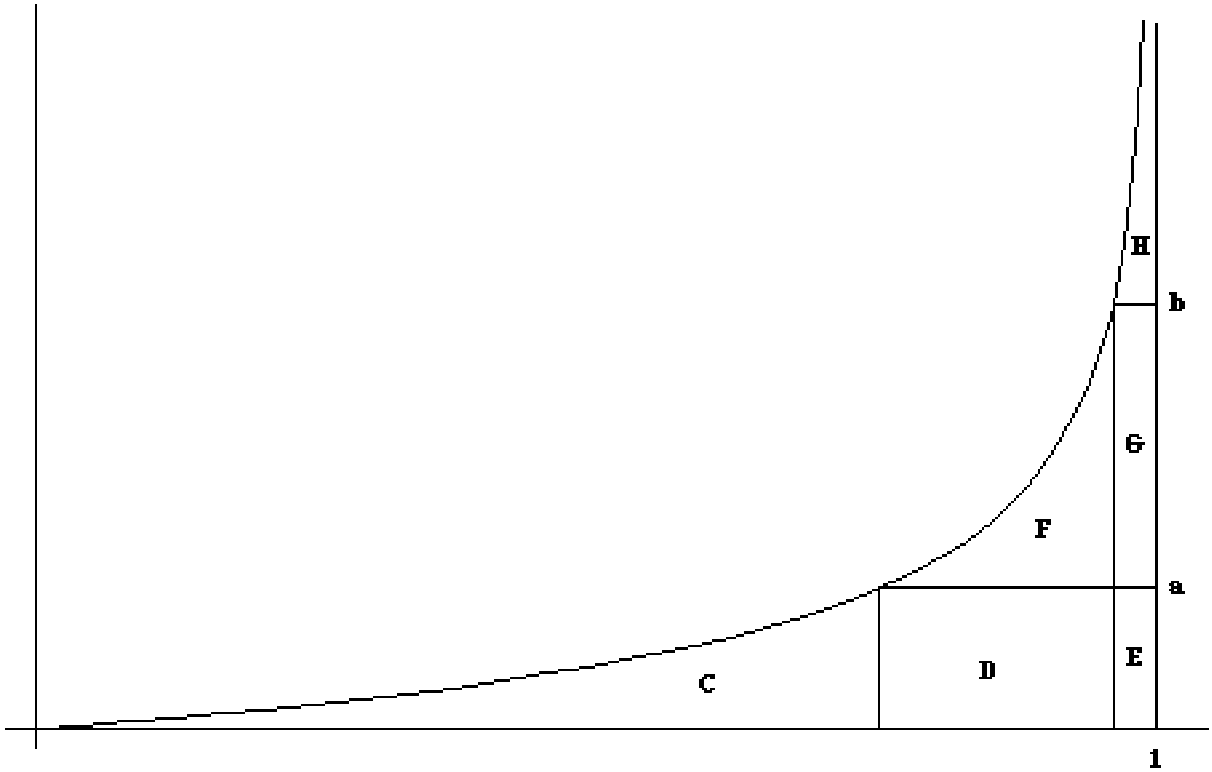
[171] *Therefore, incremental increased limits factors decrease as the limit increases.*
These ideas are discussed in "On the Theory of Increased Limits and Excess of Loss Pricing," by Robert Miccolis, with discussion by Sheldon Rosenberg, PCAS 1997.

This is illustrated in the following Lee Diagram:



Area A > Area B > Area C > Area D.

Various Formulas for a Layer of Loss:[172]



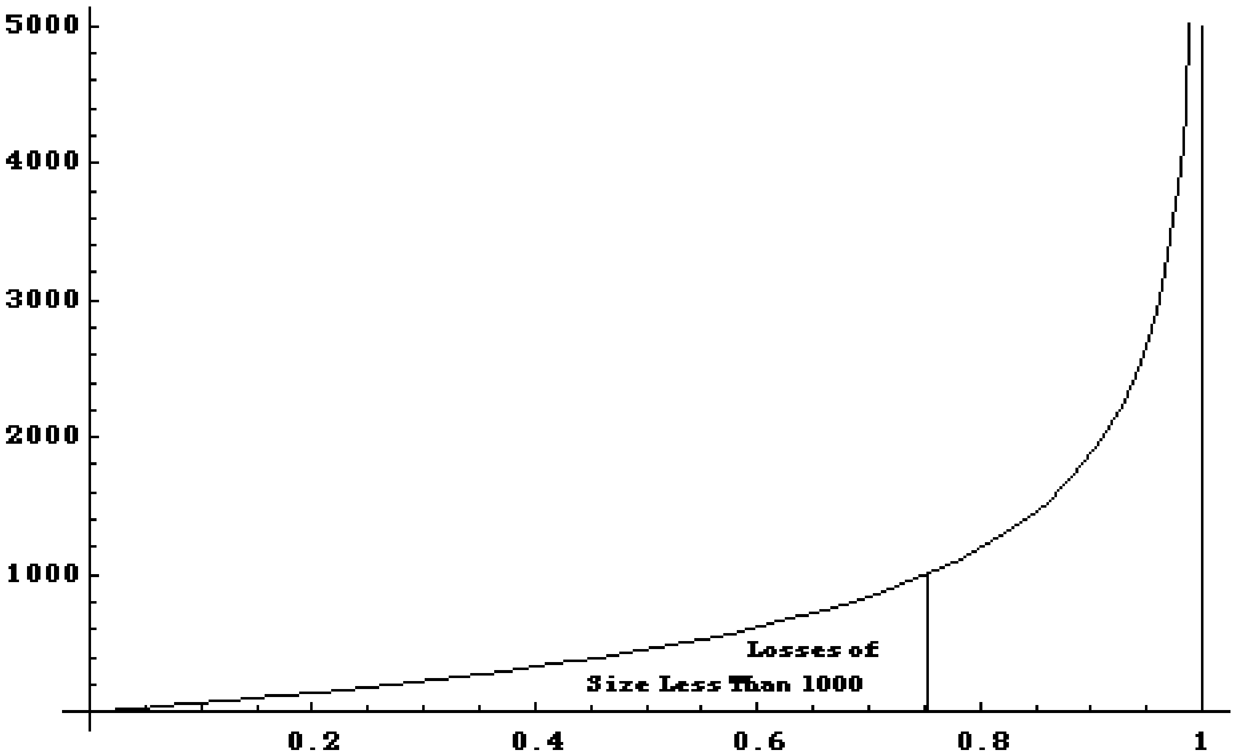
In the above diagram, the layer from a to b is: Area F + Area G.
There are various different formulas for the layer from a to b.

Algebraic Expression for Layer from a to b	Corresponding Areas on the Lee Diagram
$E[X \wedge b] - E[X \wedge a]$	(C + D + E + F + G) - (C + D + E)
$E[(X - a)_+] - E[(X - b)_+]$	(F + G + H) - H
$\int_a^b (y - a)\ f(y)\ dy + (b-a)S(b)$	F + G
$\int_a^b y\ f(y)\ dy + (b-a)S(b) - a\{F(b)-F(a)\}$	(D + F) + G - D
$\int_a^b y\ f(y)\ dy + bS(b) - aS(a)$	(D + F) + (E + G) - (D + E)

172 See pages 58-59 of "The Mathematics of Excess of Loss Coverage and Retrospective Rating --- A Graphical Approach" by Y.S. Lee, PCAS LXXV, 1988.

Average Sizes of Losses in an Interval:

As shown below, the dollars of loss on losses of size less than 1000 correspond to the area under the curve and to the left of the vertical line x = F(1000) = 0.7517:



Summing up vertical strips, this left hand area corresponds to the integral of y f(y) from 0 to 1000. As discussed previously, this area is the contribution of the small losses, one of the two pieces making up E[X∧1000]. The other piece of E[X∧1000] was the contribution of the large losses, 1000S(1000). Thus the dollars of loss on losses of size less than 1000 can also be calculated as E[X∧1000] - 1000S(1000) or as the difference of the corresponding areas.

In this case, the area below the curve and to the left of x = F(1000) is:

$$\int_0^{1000} y\, f(y)\, dy = E[X \wedge 1000] - 1000S(1000) = 518.62 - 248.27 = 270.35.$$ [173]

Therefore, the average size of the losses of size less than 1000 is:
270.35 / F(1000) = 270.35 / 0.75173 = 359.64.

In general, **the losses of size a to b ⇔ the area below the curve and also between the vertical line at a and the vertical line at b**.

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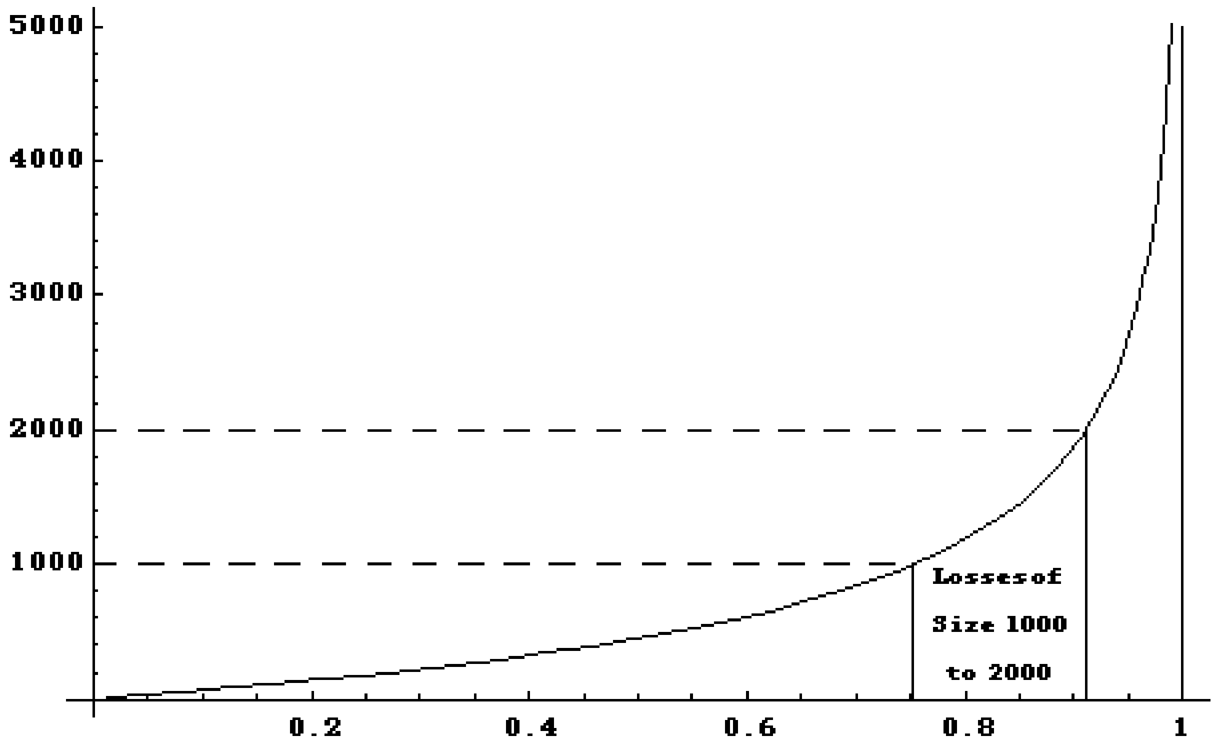
[173] In the case of a Pareto with $\alpha = 4$ and $\theta = 2400$, $E[X \wedge 1000] = (2400/3)\{1 - (2400/3400)^3\} = 518.62$, and $S(1000) = (2400/3400)^4 = 0.24827$.

As shown below, the dollars of loss on losses of size between 1000 and 2000 correspond to the area under the curve and between the vertical lines the vertical line x = F(1000) = 0.7517 and x = F(2000) = 0.9115:



Summing up vertical strips, this area corresponds to the integral of y f(y) from 1000 to 2000. This area between the two vertical lines can also be thought of as the difference between the area to the left of x = F(2000) and that to the left of x = F(1000). In other words the dollars of loss on losses between 1000 and 2000 are the difference between the dollars of loss on losses of size less than 2000 and dollars of loss on losses of size less than 1000.

In this case, the area below the curve and to the left of x = F(2000) is:
$E[X \wedge 2000] - 2000S(2000) = 670.17 - (2000)(0.088519) = 493.13$.
The area below the curve and to the left of x = F(1000) is:
$E[X \wedge 1000] - 1000S(1000) = 518.62 - 248.27 = 270.35$.

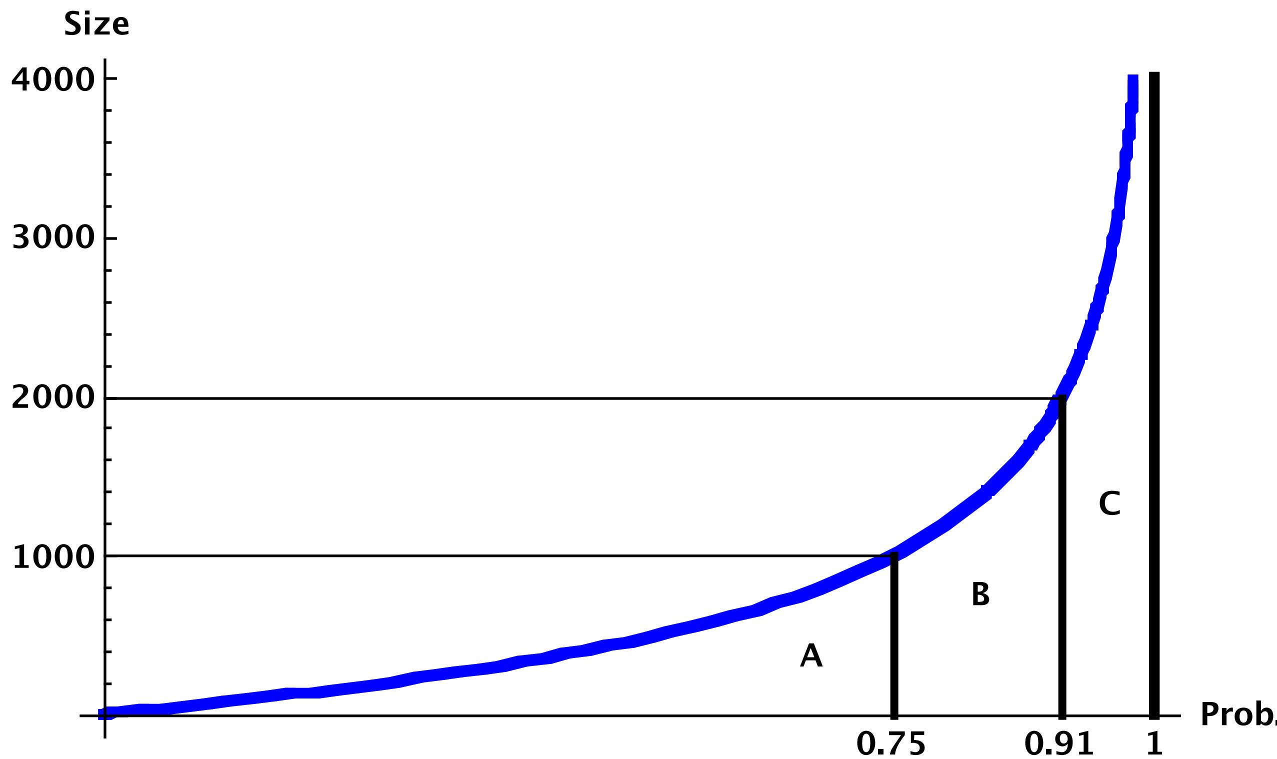
The area between the vertical lines is the difference:

$$\int_{1000}^{2000} y\, f(y)\, dy = \int_0^{2000} y\, f(y)\, dy - \int_0^{1000} y\, f(y)\, dy = 493.13 - 270.35 = 222.78.$$

The average size of the losses of size between 1000 and 2000 is:
222.78 / {F(2000) -F(1000)} = 222.78 / (0.9115 - 0.7517) = 1394.

The numerator is the area between the vertical lines at F(2000) and F(1000) and below the curve, while the denominator is the width of this area. The ratio is the average height of this area, the average size of the losses of size between 1000 and 2000.

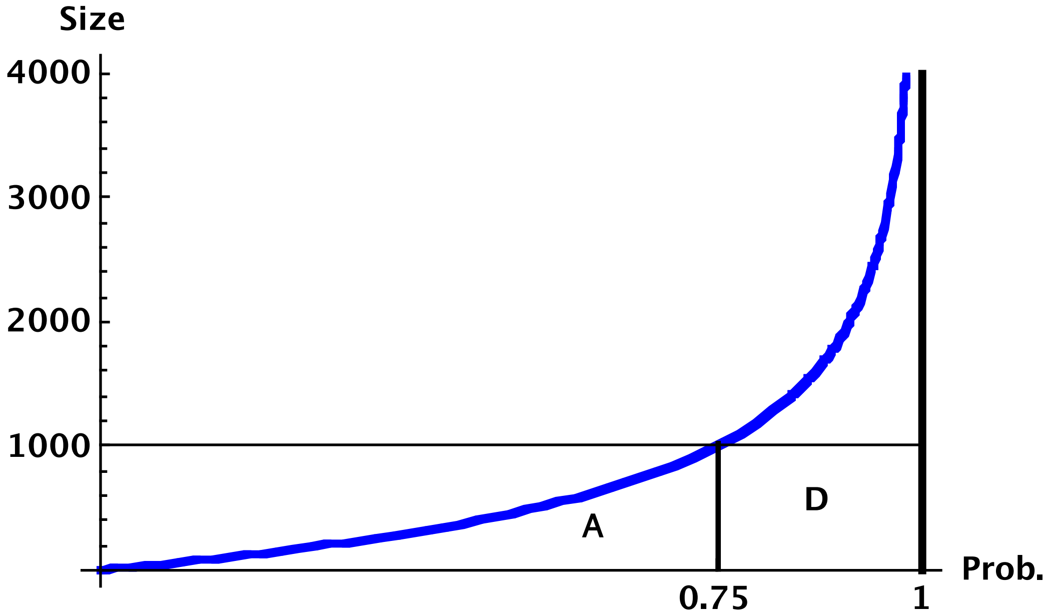
This is all summarized In the following Lee Diagram.
Area A = Dollars of loss on losses of size < 1000.
Area C = Dollars of loss on losses of size > 2000.
Area B = Dollars of loss on losses of size between 1000 and 2000.



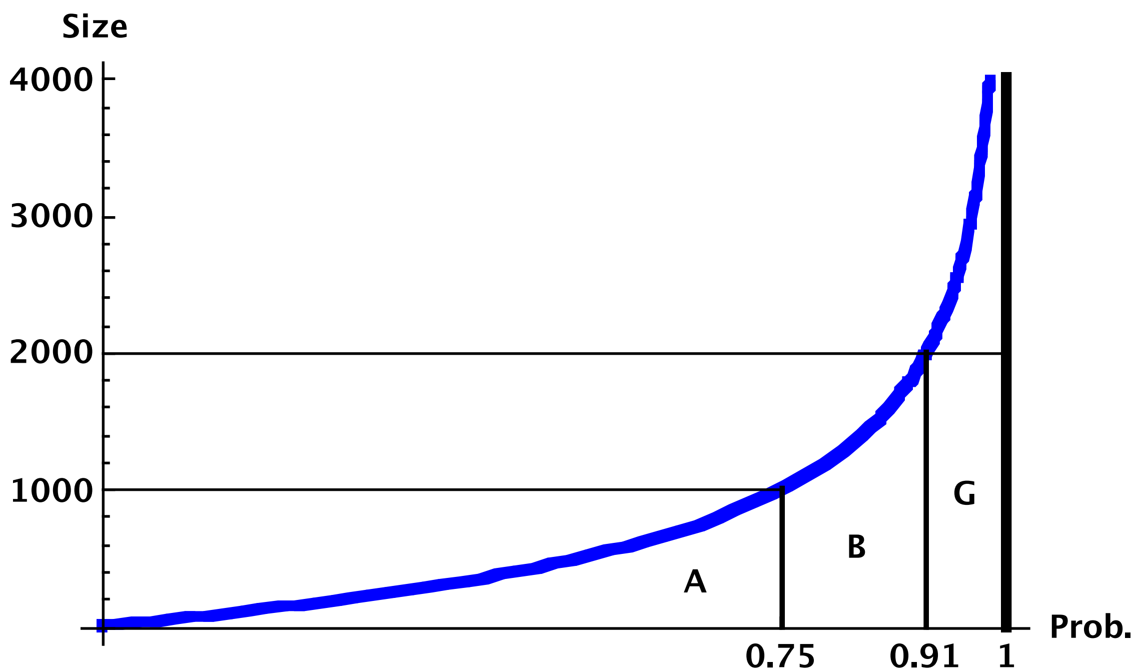
The average size of the losses of size between 1000 and 2000 is:

$$\frac{\text{Area B}}{F(2000) - F(1000)} = \frac{222.78}{0.9115 - 0.7517} = 1394.$$

E[X∧1000] = A + D = A + 1000S(1000). ⇒ Area A = E[X∧1000] - 1000S(1000).



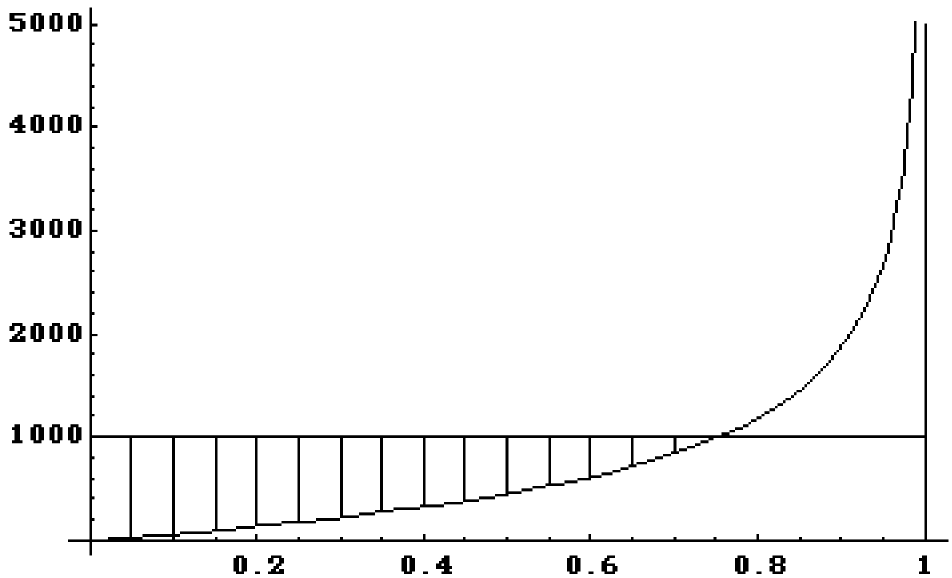
E[X∧2000] = A + B + G = A + B + 2000S(2000).
⇒ Area A + Area B = E[X∧2000] - 2000S(2000).



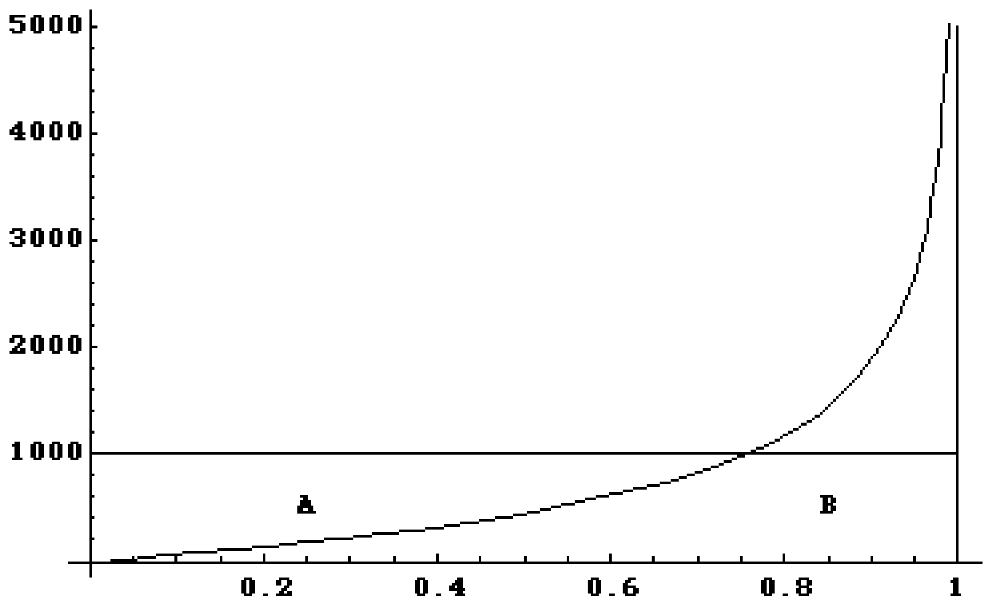
⇒ Dollars of loss on losses of size between 1000 and 2000 = Area B =
(A+B) - A = {E[X∧2000] - 2000S(2000)} - {E[X∧1000] - 1000S(1000)}.

Expected Amount by Which Losses are Less than a Given Value:

Assume we take 1000 - x for x ≤ 1000 and 0 for x > 1000. This is the amount by which X is less than 1000, or $(1000 - X)_+$. As shown below for various x < 1000, $(1000 - X)_+$ is the height of a vertical line from the curve to 1000, or from x up to 1000:



The expected amount by which X is less than 1000, $E[(1000 - X)_+]$, is Area A below:



Area B = $E[X \wedge 1000]$. Area A + Area B = area of a rectangle of width 1 and height 1000 = 1000.
$E[(1000 - X)_+]$ = the expected amount by which losses are less than 1000

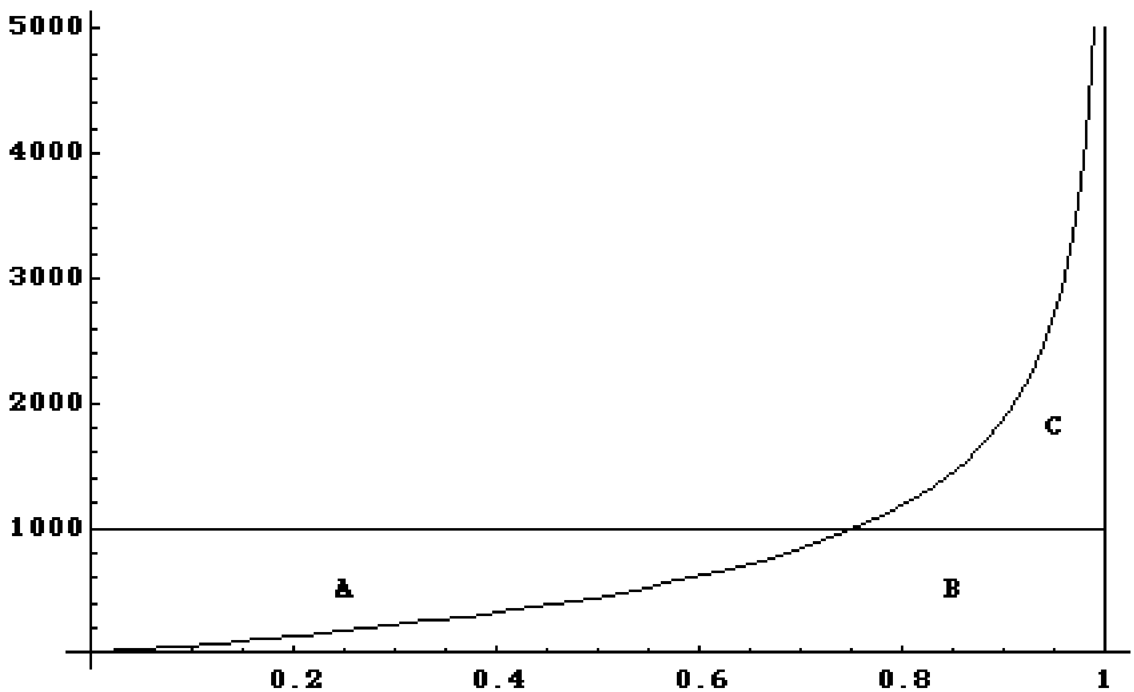
= Area A = 1000 - Area B = 1000 - $E[X \wedge 1000]$.

We can also write area A as a sum of horizontal strips: $\int_0^{1000} F(x)\ dx$.

In general, $E[(b - X)_+] = \int_0^b F(x)\ dx = \int_0^b \{1 - S(x)\}\ dx = b - E[X \wedge b]$.

<u>$E[(X - d)_+]$ versus $E[(d - X)_+]$:</u>

In the following Lee Diagram, Area A = $E[(1000 - X)_+]$ and Area C = $E[(X - 1000)_+]$.

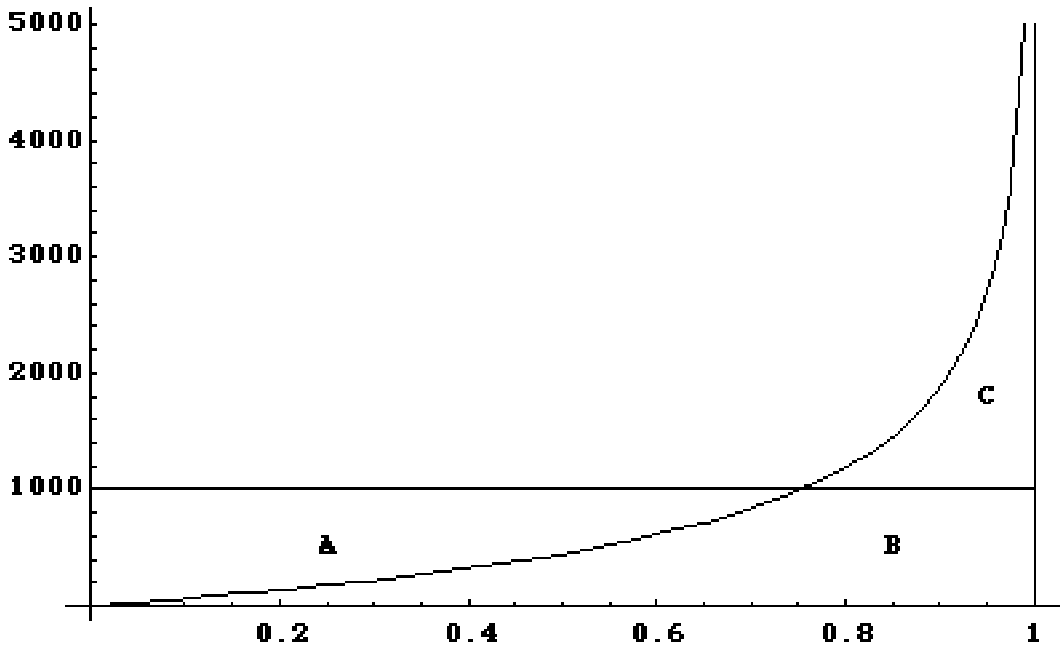


Area A + Area B = a rectangle of height 1000 and width 1.
Therefore, A + B = 1000.
B = 1000 - A = 1000 - $E[(1000 - X)_+]$.

Area B + Area C = area under the curve.
Therefore, B + C = E[X].
B = E[X] - C = E[X] - $E[(X - 1000)_+]$.

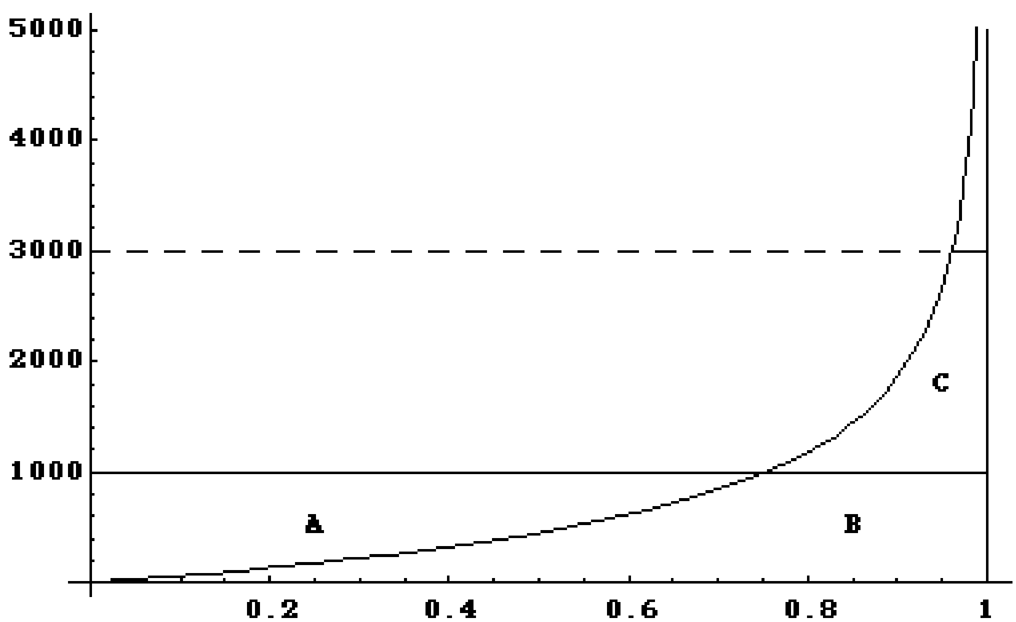
Therefore, $1000 - E[(1000 - X)_+] = E[X] - E[(X - 1000)_+]$.
Therefore, $E[(X - 1000)_+] - E[(1000 - X)_+] = E[X] - 1000 = E[X - 1000]$.
In general, $E[(X-d)_+] - E[(d-X)_+] = E[X] - d = E[X - d]$.

Payments Subject to a Minimum:



Max[X, 1000] = A + B + C = (A + B) + C = 1000 + (E[X] - E[X ∧ 1000]).
Max[X, 1000] = A + B + C = A + (B + C) = (1000 - E[X ∧ 1000]) + E[X].

Payments Subject to both a Minimum and a Maximum:

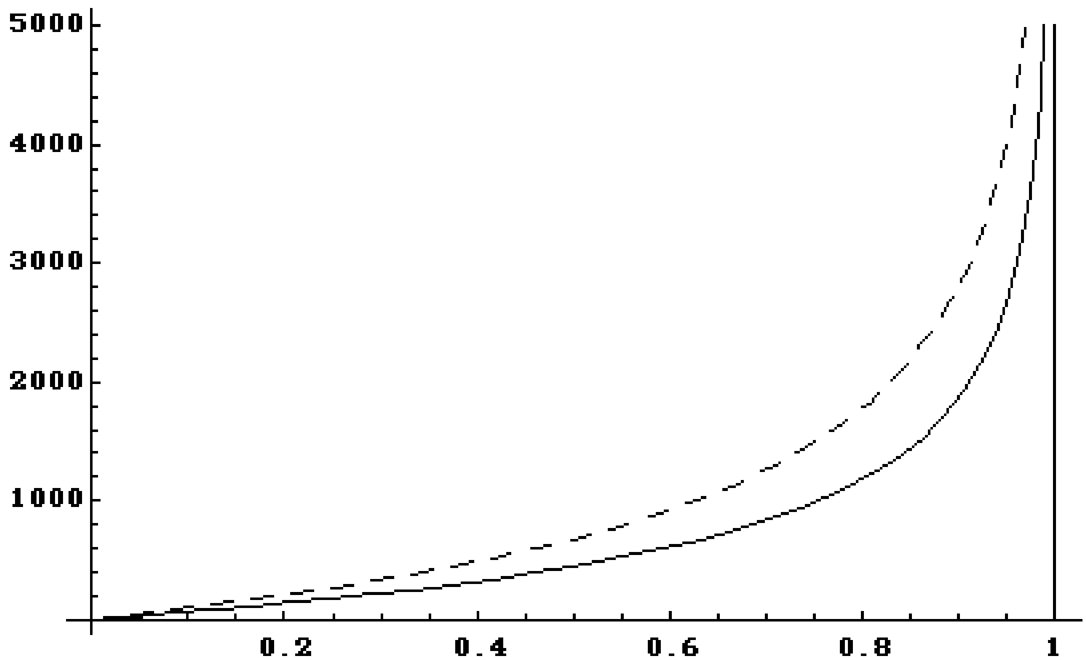


Min[Max[X, 1000], 3000] = A + B + C = (A + B) + C = 1000 + (E[X ∧ 3000] - E[X ∧ 1000]).
Min[Max[X, 1000], 3000] = A + B + C = A + (B + C) = (1000 - E[X ∧ 1000]) + E[X ∧ 3000].

Inflation:

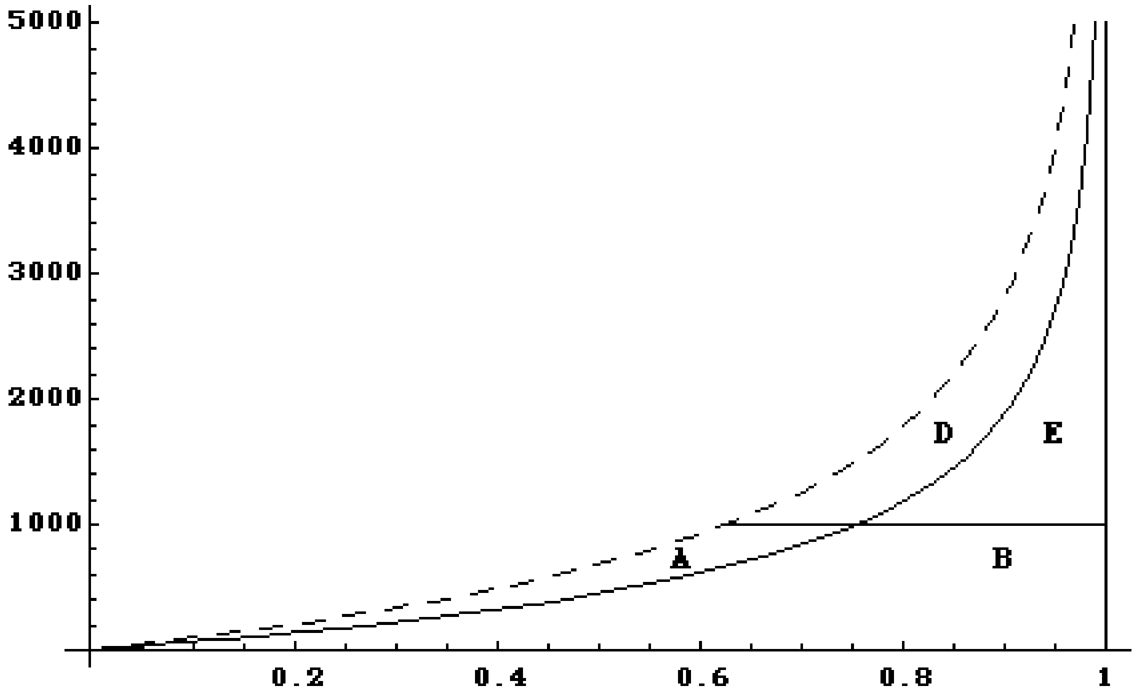
After 50% uniform inflation is applied to the original Pareto distribution, with $\alpha = 4$ and $\theta = 2400$, the revised distribution is also a Pareto with $\alpha = 4$ but with $\theta = (1.5)(2400) = 3600$.

Here are the original Pareto (solid) and the Pareto after inflation (dashed):



The increase in the losses due to inflation corresponds to the area between the distribution curves. The total area under the new curve is: (1.5)(800) = 3600 / (4-1) = 1200. The area under the old curve is 800. The increase in losses is the difference = 1200 - 800 = (0.5)(800) = 400. The increase in losses is 50% from 800 to 1200.

The losses excess of 1000 are above the horizontal line at 1000. Prior to inflation the excess losses are below the original Pareto (solid), Area E. After inflation the excess losses are below the revised Pareto (dashed), Areas D + E:



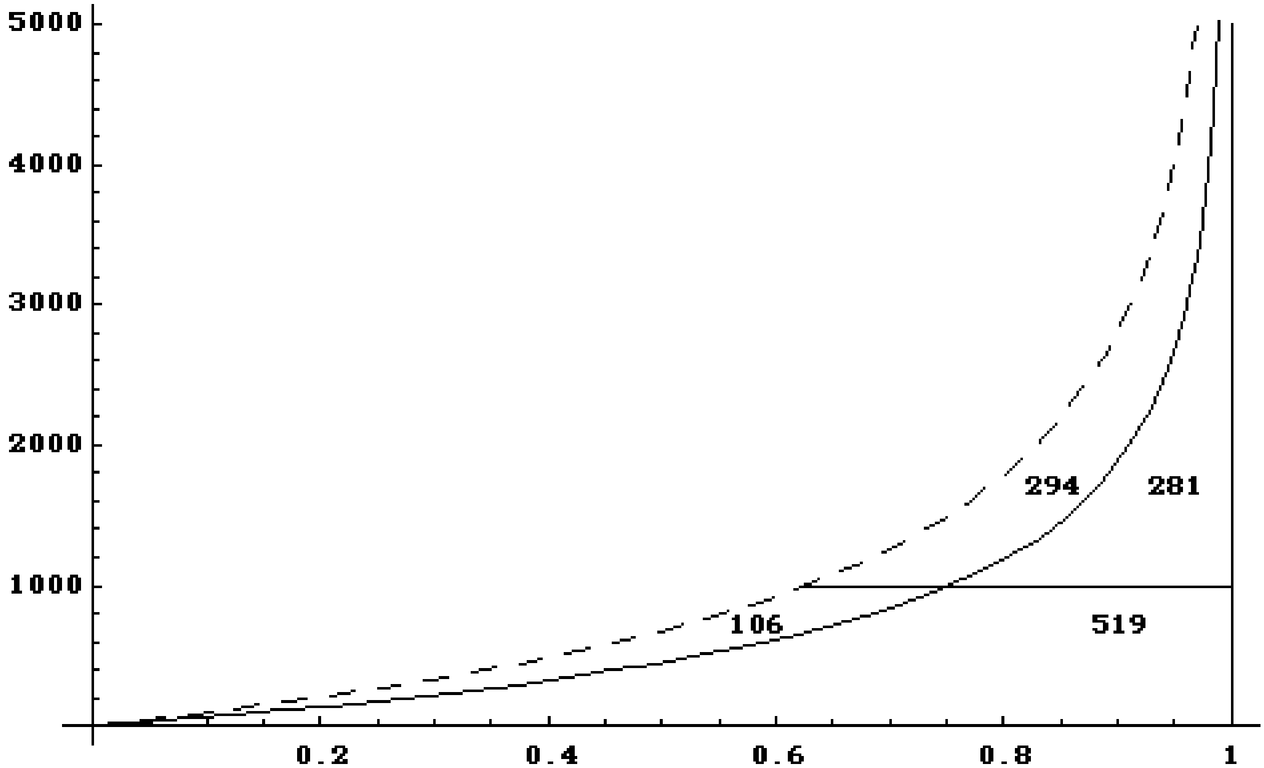
Area D represents the increase in excess losses due to inflation, while Area A represents the increase in limited losses due to inflation. Note that the excess losses have increased more quickly (as a percent) than the total losses, while the losses limited to 1000 have increased less quickly (as a percent) than the total losses.

The loss excess of 1000 for a Pareto with $\alpha = 4$ and $\theta = 3600$ is: 1200 - E[X∧1000] = 1200 - 624.80 = 575.20, Areas D + E above. The loss excess of 1000 for a Pareto with $\alpha = 4$ and $\theta = 2400$ is: 800 - E[X∧1000] = 800 - 518.62 = 281.38, Area E above. Thus under uniform inflation of 50%, in this case the losses excess of 1000 have increased by 104.4%, from 281.38 to 575.20. The increase in excess losses is Area D above, 575.20 - 281.38 = 293.82.

The loss limited to 1000 for a Pareto with $\alpha = 4$ and $\theta = 3600$ is: E[X∧1000] = 624.80. The loss limited to 1000 for a Pareto with $\alpha = 4$ and $\theta = 2400$ is: E[X∧1000] = 518.62. Thus under uniform inflation of 50%, in this case the losses limited to 1000 have increased by only 20.5%, from 518.62 to 624.80. The increase in limited losses is Area A above, 624.80 - 518.62 = 106.18.

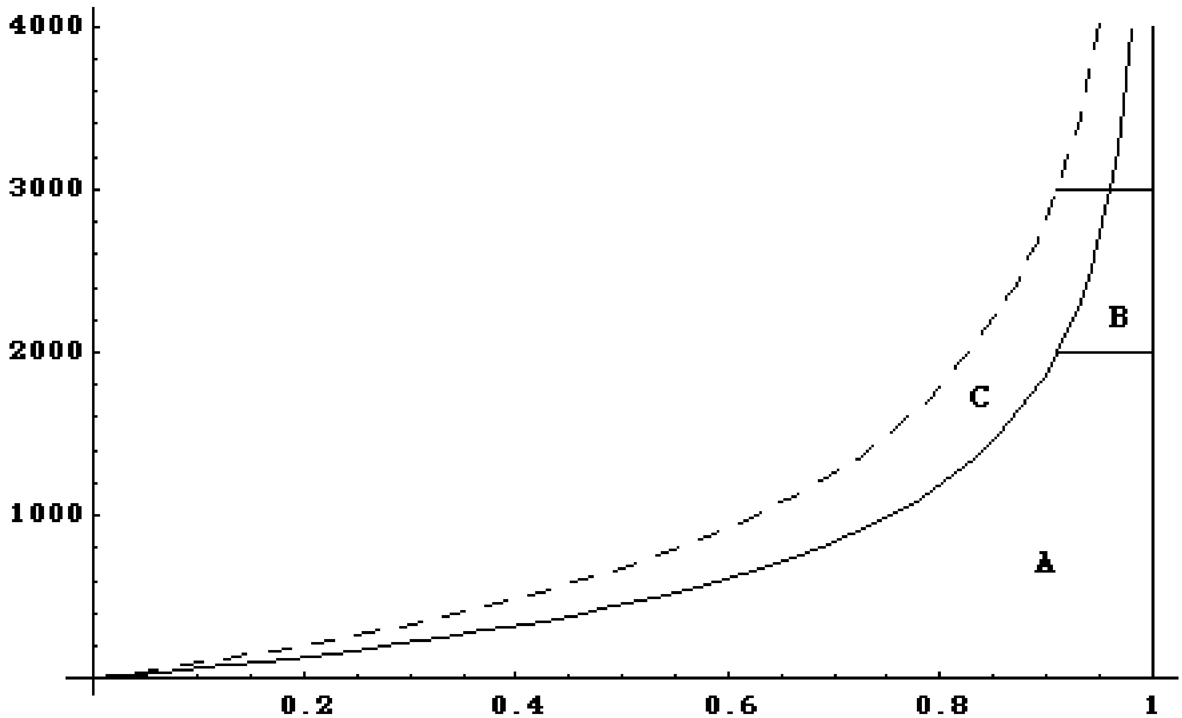
The total losses increase from 800 to 1200; Area A + Area D = 293.82 + 106.18 = 400.

Another version of this same Lee Diagram, showing the numerical areas:



% Increase in Limited Losses is: 106/519 = 20% < 50%.
% Increase in Excess Losses is: 294/281 = 105% > 50%.
% Increase in Total Losses is: (106 + 294)/ (519 + 281) = 400/800 = 50%.

In the earlier year, the losses limited to 2000 are below the horizontal line at 2000, and below the solid Pareto, Area A in the Lee Diagram below.
In the later year, the losses limited to 3000 are below the horizontal line at 3000, and below the dotted Pareto, Areas A + B + C in the Lee Diagram below.
Every loss in combined Areas A + B + C is exactly 1.5 times the height of a corresponding loss in Area A.



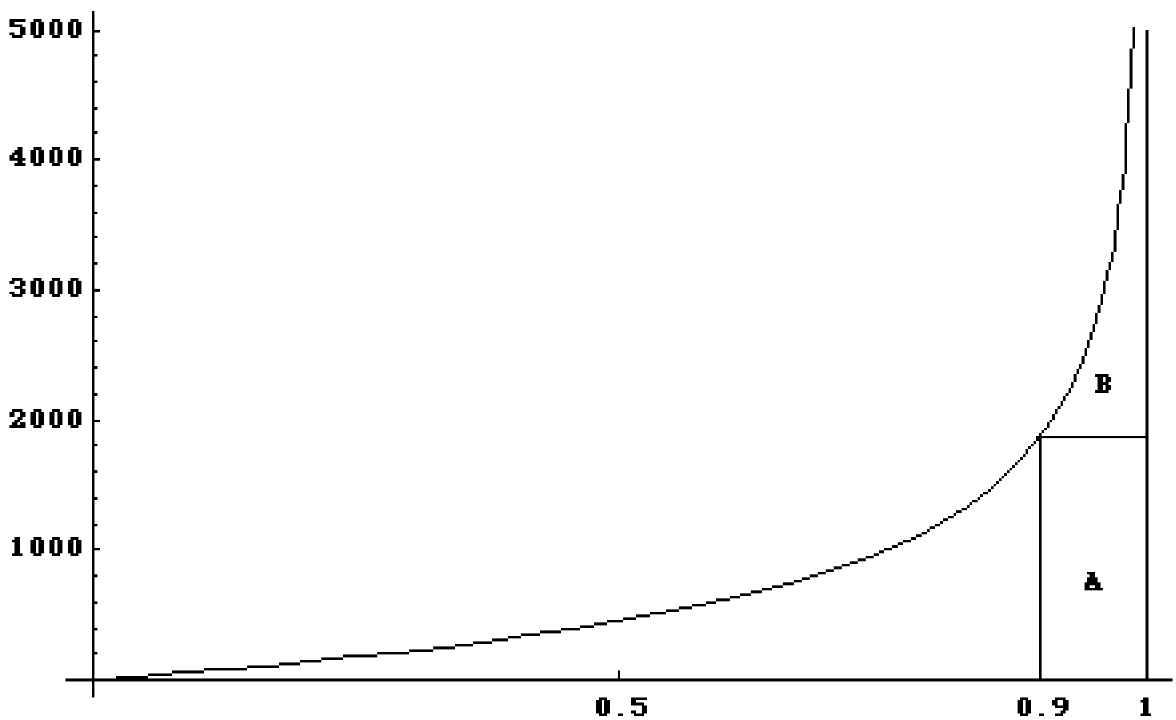
Showing that $E_{\text{later year}}[X \wedge 3000] = 1.5\ E_{\text{earlier year}}[X \wedge 3000/1.5] = 1.5\ E_{\text{earlier year}}[X \wedge 2000]$.

Excess Shortfall (ES):[174]

The Excess Shortfall of a loss distribution is defined as: $ES_p \cong E[X \mid X > \pi_p]$,
where the percentile π_p is such that $F(\pi_p) = p$.

Exercise: For a Pareto Distribution with $\alpha = 4$ and $\theta = 2400$, determine $\pi_{0.90}$.
[Solution: $0.90 = 1 - \{2400/(2400 + x)\}^4$. $\Rightarrow$ x = 1868.]

$ES_{0.90}$ is the average size of those losses of size greater than $\pi_{0.90} = 1868$.
The denominator of $ES_{0.90}$ is: 1 - 0.90 = 0.10.
The numerator of $ES_{0.90}$ is Area A + Area B in the following Lee Diagram:



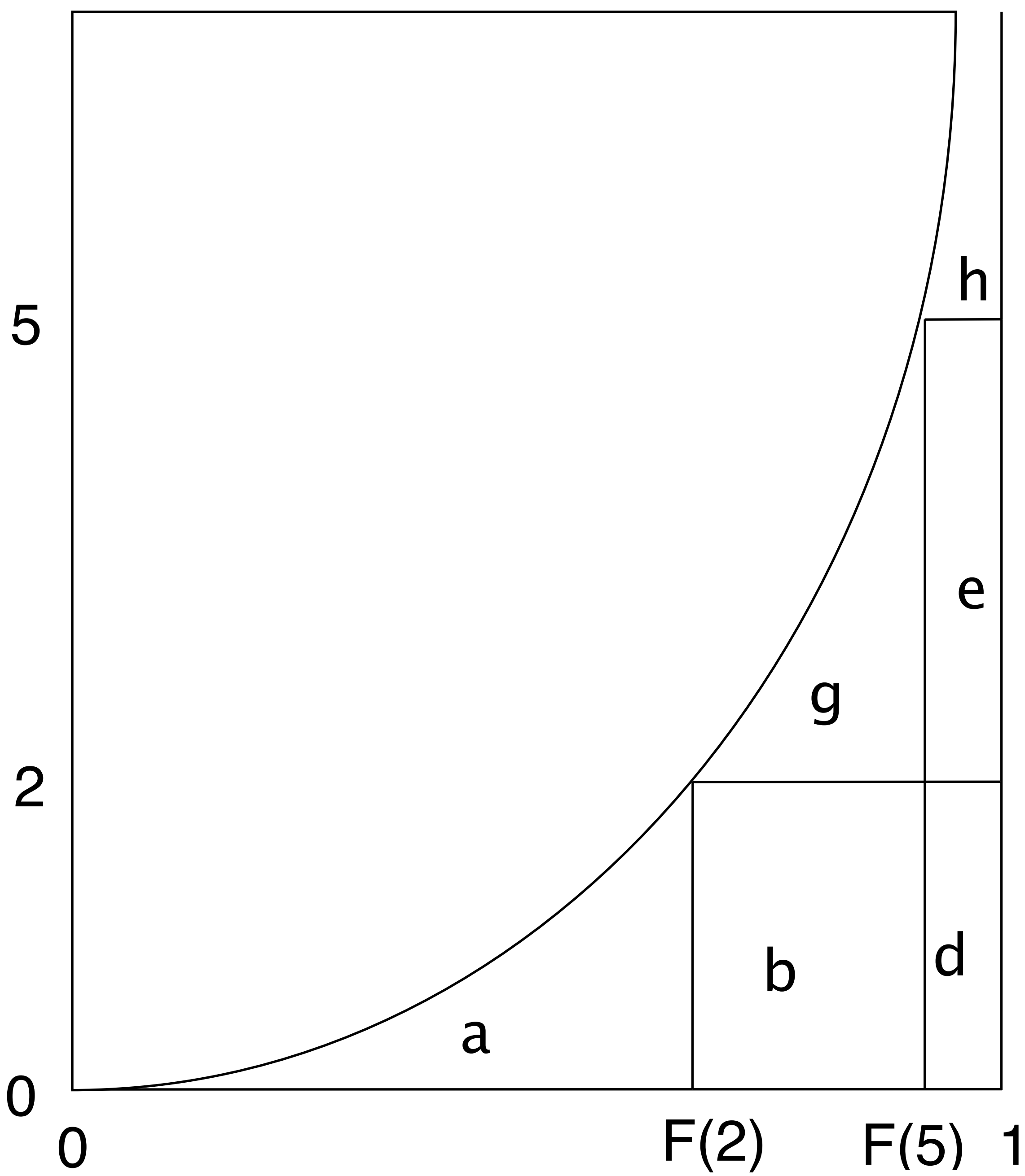
Therefore, $ES_{0.90}$ is the average height of Areas A + B.
Area A has height $\pi_{0.90} = 1868$.
Area B is the expected losses excess of $\pi_{0.90} = 1868$.
The average height of Area B is the mean excess loss, $e(1868) = e(\pi_{0.90})$.

Therefore, $ES_{0.90} = \pi_{0.90} + e(\pi_{0.90})$. In general, $ES_p = \pi_p + e(\pi_p)$.

[174] See "Mahler's Guide to Extreme Value Theory".

Problems:

Use the following information for the next 26 questions:
The size of loss distribution F(x), with corresponding survival function S(x) and density f(x), is shown in the following diagram, with probability along the horizontal axis and size of loss along the vertical axis. Express each of the stated quantities algebraically in terms of the six labeled areas in the diagram: α, β, γ, δ, ε, η.



<u>23.1</u> (1 point) E[X].

23.2 (1 point) Losses from claims of size less than 2.

23.3 (1 point) Portion of total losses in the layer from 0 to 2.

23.4 (1 point) $\int_0^2 x \, dF(x) + 2\{1 - F(2)\}$.

23.5 (1 point) Portion of total losses from claims of size more than 2.

23.6 (1 point) Portion of total losses in the layer from 0 to 5.

<u>23.7</u> (1 point) $E[X \wedge 5]$.

23.8 (1 point) Portion of total losses from claims of size less than 5.

23.9 (1 point) Portion of total losses in the layer from 2 to 5.

23.10 (1 point) R(2) = excess ratio at 2 = 1 - LER(2).

23.11 (1 point) $\int_5^\infty x \, dF(x)$.

23.12 (1 point) Portion of total losses in the layer from 2 to ∞.

23.13 (1 point) LER(5) = loss elimination ratio at 5.

23.14 (1 point) 2(F(5)-F(2)).

23.15 (1 point) Portion of total losses from claims of size between 2 and 5.

23.16 (1 point) $\int_5^\infty S(t) \, dt$.

23.17 (1 point) LER(2) = loss elimination ratio at 2.

23.18 (1 point) $\int_5^{\infty} (t-5)\ f(t)\ dt$.

23.19 (1 point) 2S(5).

23.20 (1 point) R(5) = excess ratio at 5 = 1 - LER(5).

23.21 (1 point) e(5) = mean excess loss at 5.

23.22 (1 point) $\int_2^{5} (t-2)\ f(t)\ dt$.

23.23 (1 point) Losses in the layer from 5 to ∞.

23.24 (1 point) $\int_2^{5} (1 - F(t))\ dt$.

23.25 (1 point) 3S(5).

23.26 (1 point) e(2) = mean excess loss at 2.

23.27 (2 points) Using Lee's "The Mathematics of Excess of Loss Coverages and Retrospective Rating -- A Graphical Approach," show graphically why the limited expected value increases at a decreasing rate as the limit is increased.
Label all axes and explain your reasoning in a brief paragraph.

23.28 (2 points) Losses follow an Exponential Distribution with mean 500.
Using Lee's "The Mathematics of Excess of Loss Coverages and Retrospective Rating -- A Graphical Approach," draw a graph to show the expected losses from those losses of size 400 to 800. Label all axes.

Use the following information for the next five questions:
Prior to the effects of any maximum covered loss or deductible, losses follow a Weibull Distribution with $\theta = 300$ and $\tau = 1/2$.
Using Lee's "The Mathematics of Excess of Loss Coverages and Retrospective Rating -- A Graphical Approach," draw a graph to show the expected payments. Label all axes.

23.29 (1 point) With no deductible and no maximum covered loss.

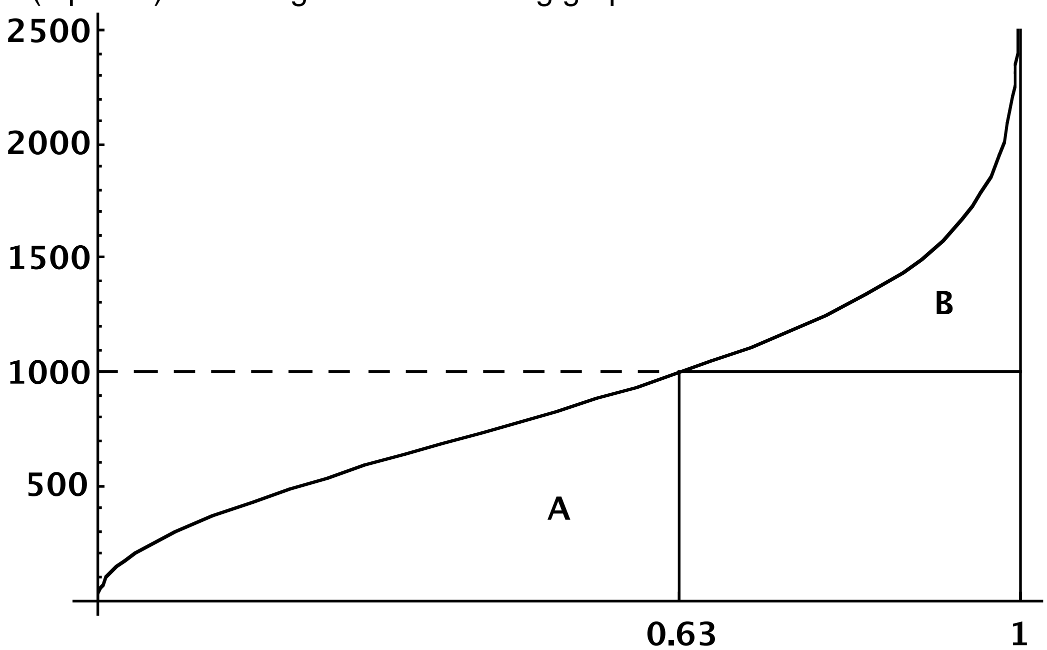
23.30 (1 point) With a 500 deductible and no maximum covered loss.

23.31 (1 point) With no deductible and a 1500 maximum covered loss.

23.32 (1 point) With a 500 deductible and a 1500 maximum covered loss.

23.33 (1 point) With a 500 franchise deductible and no maximum covered loss.

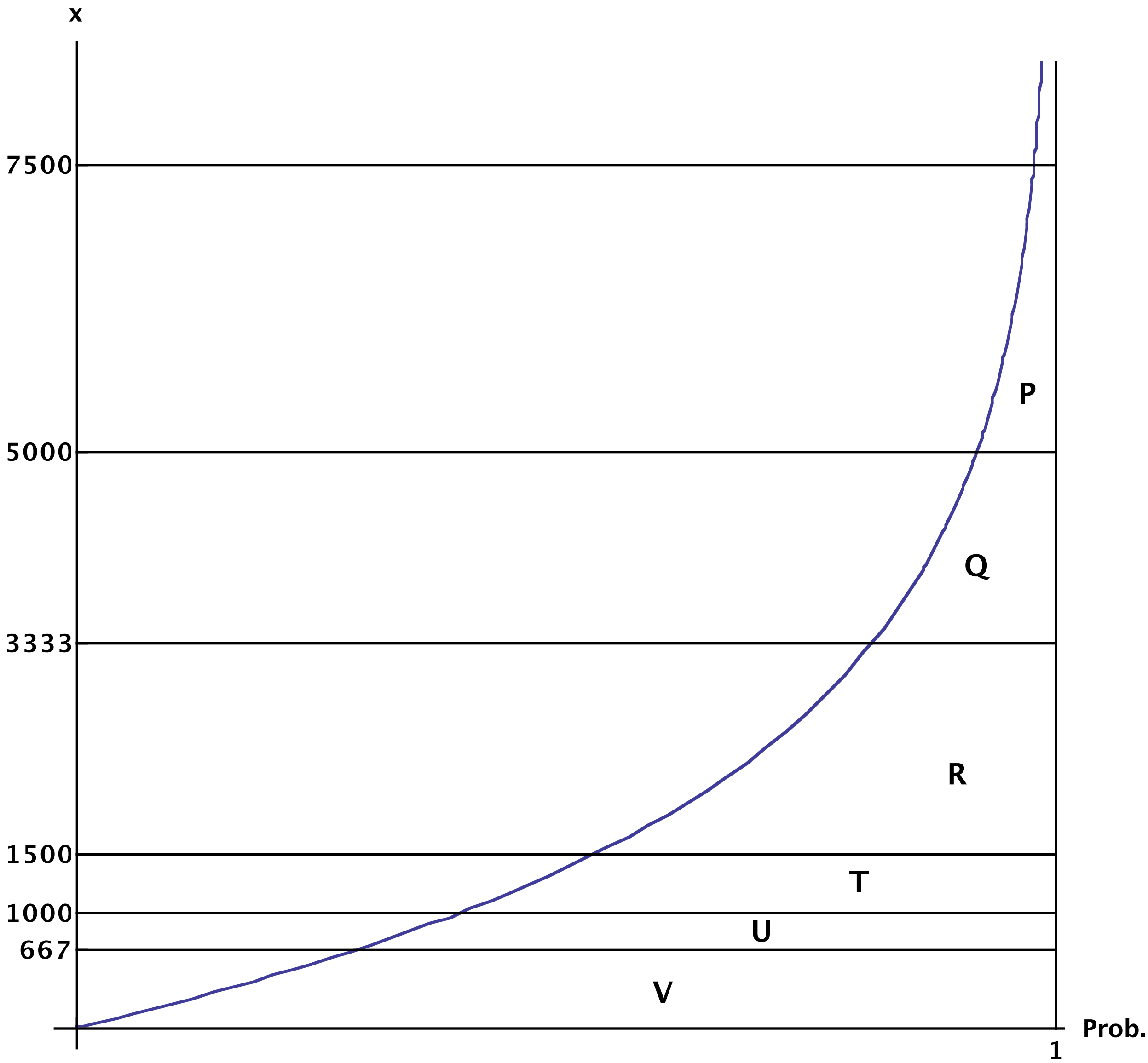
23.34 (2 points) You are given the following graph of the cumulative loss distribution:



- Size of the area labeled A = 377.
- Size of the area labeled B = 139.

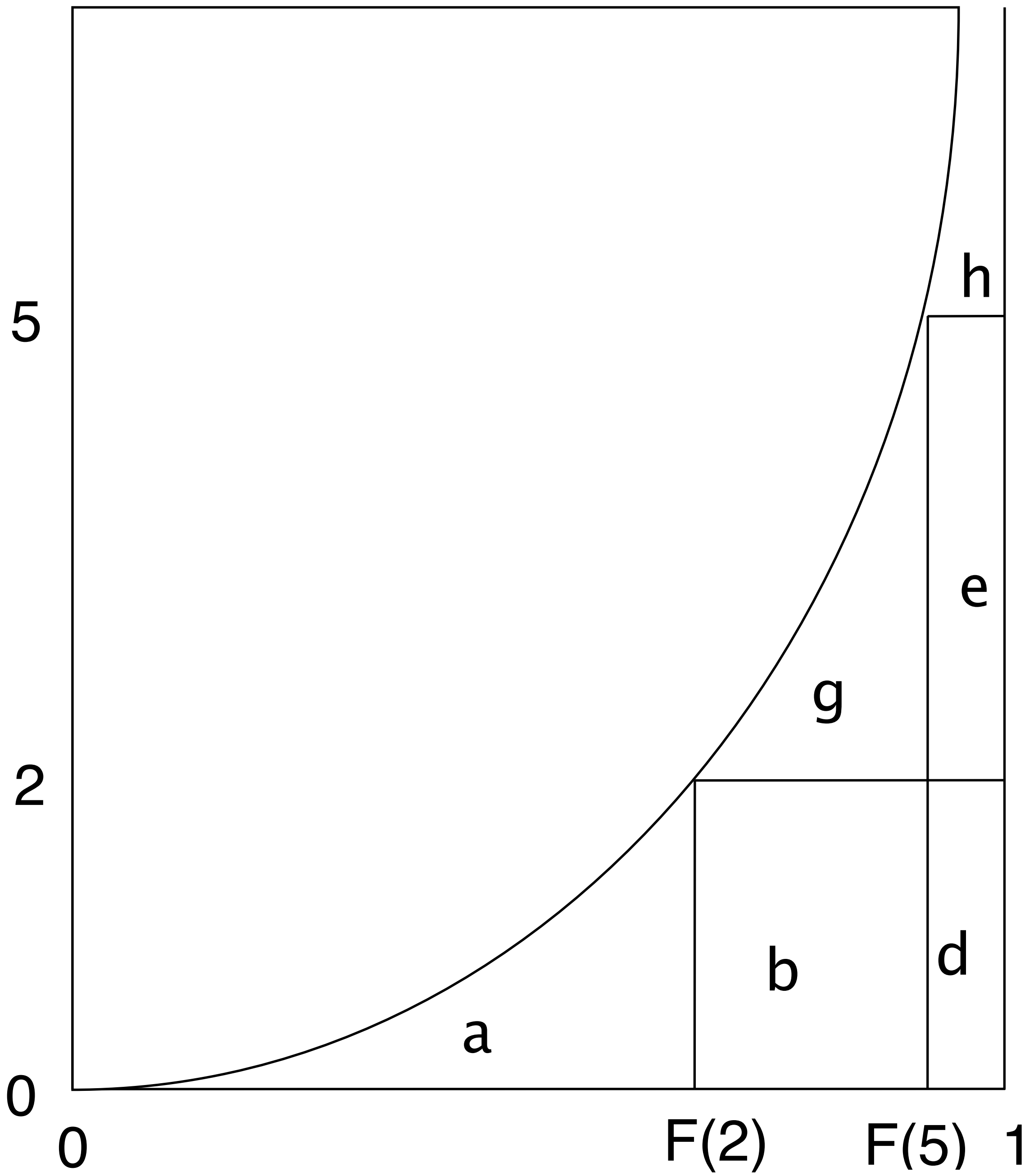
Calculate the loss elimination ratio at 1000.
A. Less than 0.6
B. At least 0.6, but less than 0.7
C. At least 0.7, but less than 0.8
D. At least 0.8, but less than 0.9
E. At least 0.9

23.35 (3 points) The following graph is of the cumulative loss distribution in 2001:

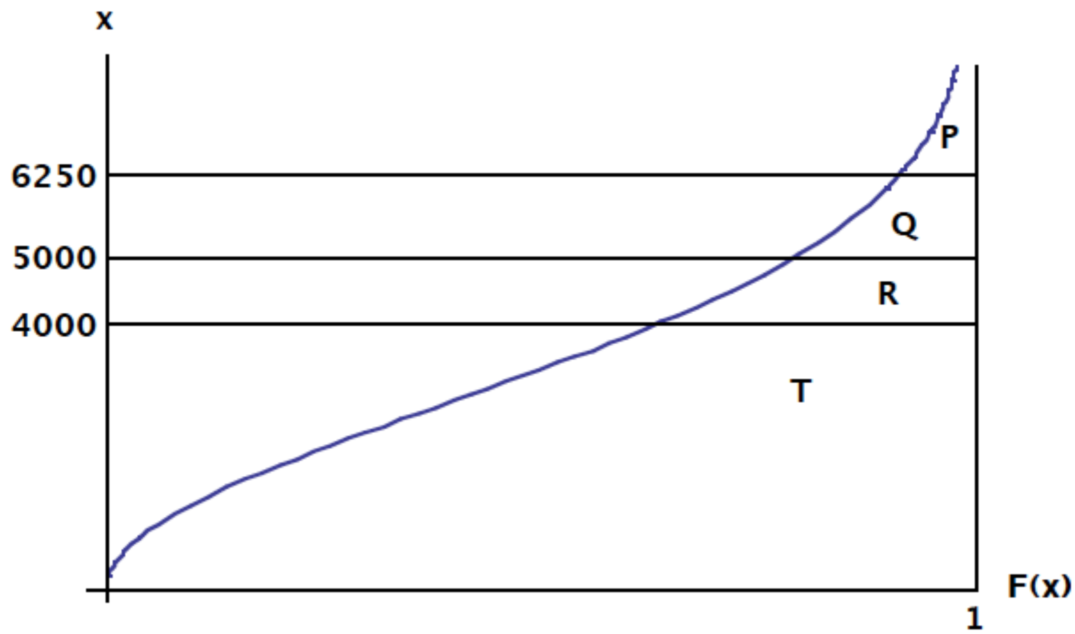


There is a total of 50% inflation between 2001 and 2008.
A policy in 2008 has a 1000 deductible and a 5000 maximum covered loss.
Which of the following represents the expected size of loss under this policy?
A. 1.5(Q + R + T)
B. 1.5(R + T + U)
C. (P + Q + R)/1.5
D. (Q + R + T)/1.5
E. None of A, B, C, or D.

23.36 (2 points) The size of loss distribution is shown in the following diagram, with probability along the horizontal axis and size of loss along the vertical axis.
Which of the following represents the expected losses under a policy with a franchise deductible of 2 and a maximum covered loss of 5?
A. g + e B. a + b + g C. d + e + h D. b + g + d + e E. b + g + d + e + h



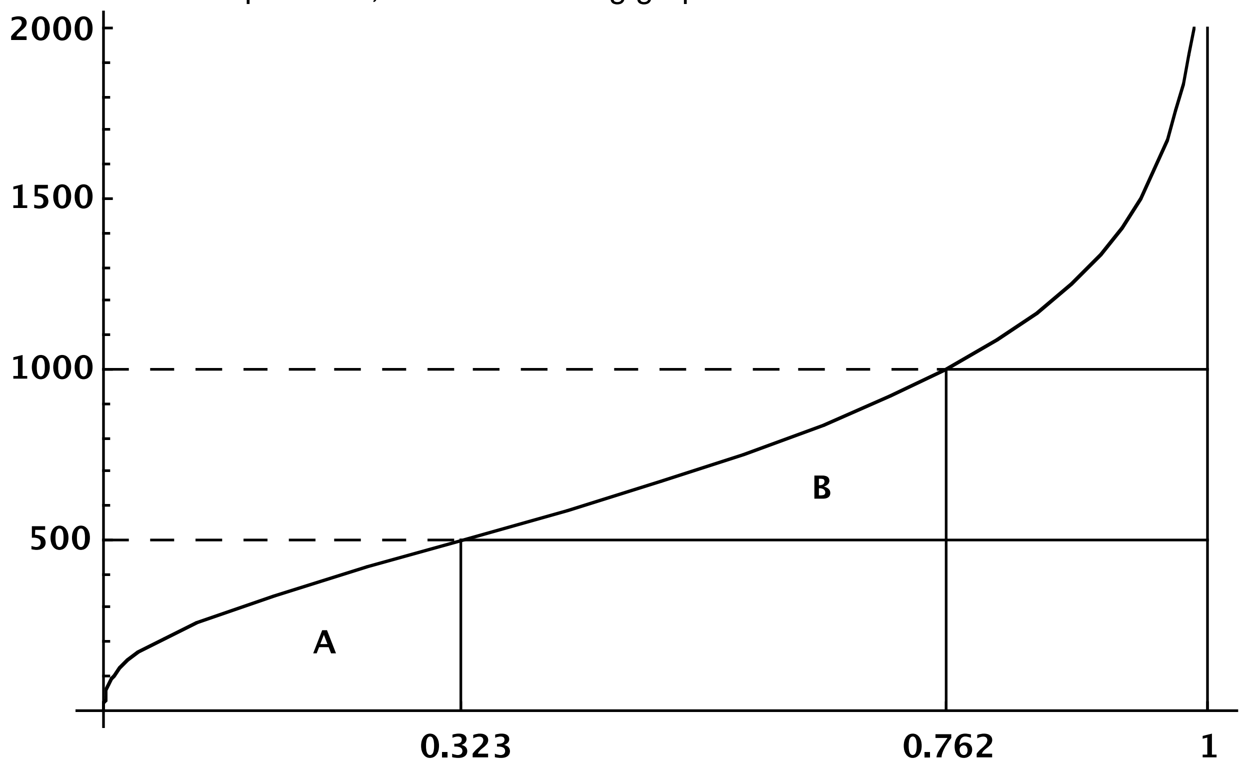
23.37 (2 points) The following graph shows the distribution function, F(x), of loss severities.



A policy has a 5000 maximum covered loss and an 80% coinsurance.
Which of the following represents the expected size of loss under this policy?
A. Q + R
B. R + T
C. Q + R + T
D. P + Q + R + T
E. None of A, B, C, or D.

23.38 (3 points) For a Pareto Distribution with $\alpha = 4$ and $\theta = 3$, draw a Lee Diagram, showing the curtate expectation of life, e_0.

For the next nine questions, use the following graph of the cumulative loss distribution:



Size of the area labeled A = 107. Size of the area labeled B = 98. The mean size of loss is 750.
Calculate the following items:

23.39 (1 point) The average size of loss for those losses of size less than 500.

23.40 (1 point) The loss elimination ratio at 500.

23.41 (1 point) The average payment per loss with a deductible of 500 and a maximum covered loss of 1000.

23.42 (1 point) The mean excess loss at 500.

23.43 (1 point) The average size of loss for those losses of size between 500 and 1000.

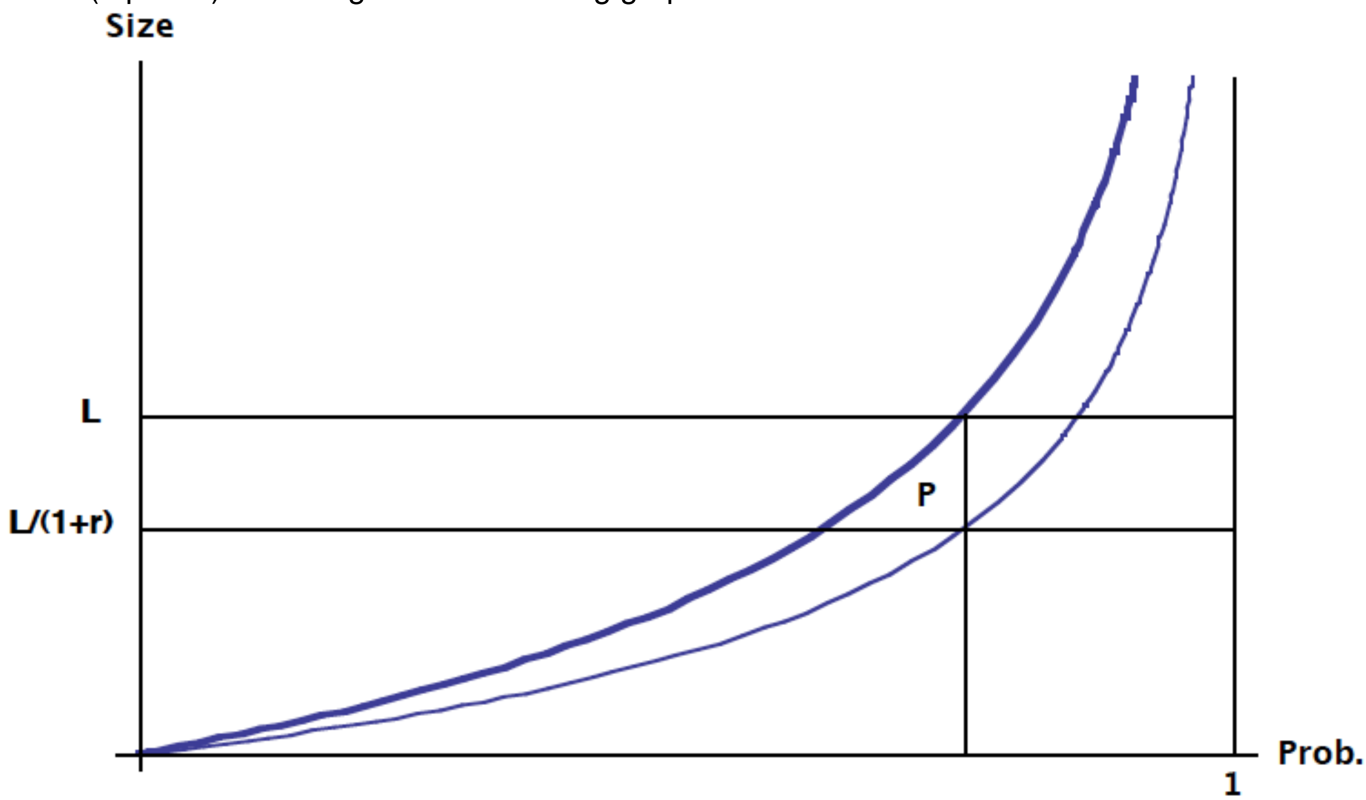
23.44 (1 point) The loss elimination ratio at 1000.

23.45 (1 point) The average payment per payment with a deductible of 500 and a maximum covered loss of 1000.

23.46 (1 point) The mean excess loss at 1000.

23.47 (1 point) The average size of loss for those losses of size greater than 1000.

23.48 (3 points) You are given the following graph of cumulative distribution functions.



The thicker curve is the cumulative distribution function for the size of loss in a later year, F(x), with corresponding Survival Function S(x) and density f(x).
There is total inflation of r between this later year and an earlier year.
The thinner curve is the cumulative distribution function for the size of loss in this earlier year.
Which of the following is an expression for Area P?

A. $\int_{L/(1+r)}^{L} S(x)\ dx\ - S(L)Lr$.

B. $\int_{L/(1+r)}^{L} x\ f(x)\ dx\ - L\{S(L/(1+r)) - S(L)\}$.

C. $(1+r) \int_{L/(1+r)}^{L} S(x)\ dx\ - S(L)Lr/(1+r)$.

D. $\int_{L/(1+r)}^{L} x\ f(x)\ dx\ - L\{F(L) - F(L/(1+r))\}/(1+r)$.

E. None of the above.

23.49 (1 point) Let a size of loss distribution be given by F(x) with density f(x).
Let S(x) = 1 - F(x).
For the layer of losses from d to u, give two different forms each involving integrals.
Briefly describe how these two forms relate to the graphical Lee diagrams.

23.50 (1 point) Draw a graph of a loss distribution F(x) in the manner described by Lee.
What geometrical quantity corresponds to P(X ≤ L)?

23.51 (3.5 points) For Workers Compensation Insurance in Massachusetts, let
R = workers weekly wage divided by the state average weekly wage.
You are given the following information:

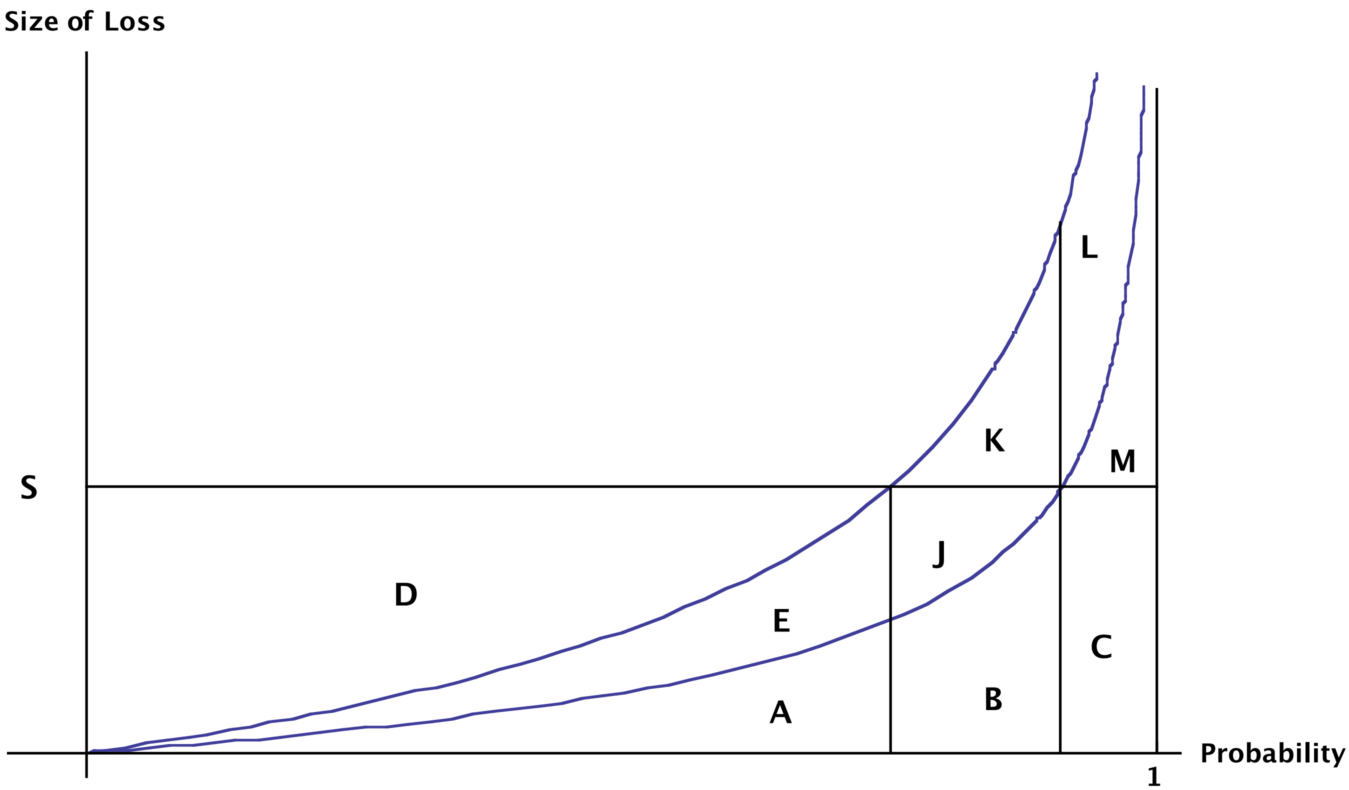
R	Percentage of Workers with Wages At Most R times the State Average Weekly Wage
0.25	2.2%
0.50	11.3%
0.75	35.0%
1.00	57.5%
1.25	74.0%
1.50	85.3%
1.75	92.9%
2.00	96.9%
2.50	99.3%
3.00	99.7%

(a) (2.5 points) Draw a Lee Diagram. Label the axes.
(b) (0.5 point) Label the area corresponding to the percentage of wages earned by those making at most 150% of the state average weekly wage.
(c) (0.5 point) Assume injured workers are paid a benefit equal to their average weekly wage, subject to a maximum benefit of the state average weekly wage.
Label the area corresponding to the average benefit paid.

23.52 (8 points) Losses on a policy have the following distribution:
- 50% probability of a loss between \$0 and \$10,000
- 30% probability of a loss between \$10,000 and \$25,000
- 20% probability of a loss between \$25,000 and \$100,000

Losses are uniformly distributed within each range.
Assume a 40% trend is applied uniformly to all losses.
a. (3 points) Draw a Lee diagram depicting the cumulative loss distribution described above before and after the 40% trend. Label all relevant features of the diagram.
b. (5 points) Use Lee diagrams to calculate the amounts in the layer from 20,000 to 70,000 both prior and subsequent to trend, and thus to calculate implied trend for this layer.

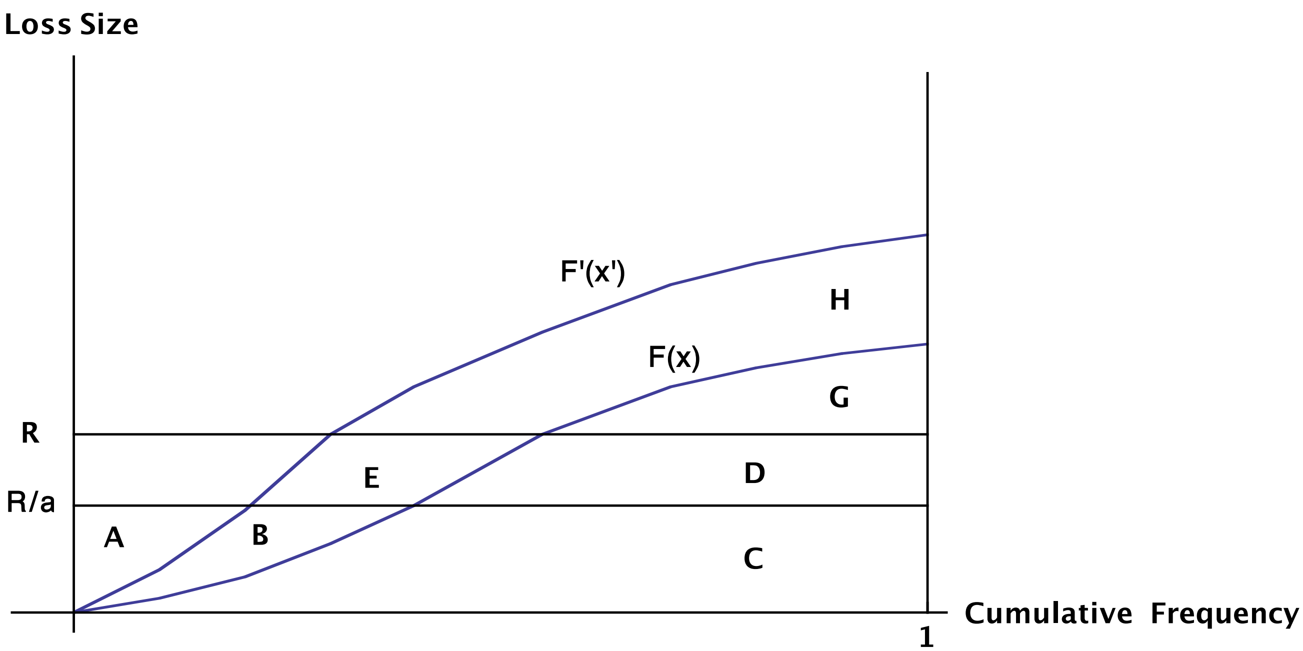
23.53 (4 points) As a result of benefit reforms, claims in a given line of insurance have been sharply reduced in size. The curves in the diagram below represent the severity distribution before and after reform. An insurance company writes coverage in this line of insurance, excess of a self-insured retention S.



Define small claims as those that were of size ≤ S prior to the benefit reform.
Define medium claims as those that were of size > S prior to the benefit reform, and are of size ≤ S after the benefit reform.
Define large claims as those that are of size > S after the benefit reform.
Put each of the verbal descriptions in terms of the labeled areas and S.

1. Reduction due to benefit reform in the insured's expected retained losses from small claims.
2. After benefit reform, contribution to the insured's expected retained losses from small claims.
3. Reduction due to benefit reform in the insured's expected retained losses from medium claims.
4. After benefit reform, contribution to the insured's expected retained losses from medium claims.
5. Reduction due to benefit reform in the insured's expected retained losses from large claims.
6. After benefit reform, contribution to the insured's expected retained losses from large claims.
7. Reduction due to benefit reform in the insurer's expected losses from small claims.
8. After benefit reform, contribution to the insurer's expected losses from small claims.
9. Reduction due to benefit reform in the insurer's expected losses from medium claims.
10. After benefit reform, contribution to the insurer's expected losses from medium claims.
11. Reduction due to benefit reform in the insurer's expected losses from large claims.
12. After benefit reform, contribution to the insurer's expected losses from large claims.
13. Prior to benefit reform, the insurer's losses eliminated by the retention S. Use Area D.

23.54 (CAS9, 11/92, Q.8) (1 point)
According to Lee in "The Mathematics of Excess of Loss Coverages and Retrospective Rating - A Graphical Approach," which of the following statements are true?
Assume that claims of all sizes inflate by a constant factor a (x' = ax). The cumulative distribution function is F(x) before inflation and F'(x') after inflation. R is the retention.



1. (G + H) / G > a.
2. (B + C + D + E) / (D + C) > a.
3. (B + C + D + E) = a C.

A. 1 only B. 2 only C. 3 only D. 1 and 2 E. 1 and 3

23.55 (CAS9, 11/94, Q.40) (2 points) Answer the following based on "The Mathematics of Excess of Loss Coverage and Retrospective Rating - A Graphical Approach" by Lee.
For a random variable, X, that can have a value in the interval (0, b], the limited expected value function of X with limit equal to t is given by the function:
$E[X \wedge t] = t - (0.5)(t^2 / b)$; for $0 < t \leq b$.
Using this limited expected value function, calculate the probability density function f(x) for x.
Show all of your work.

23.56 (CAS9, 11/94, Q.41) (3 points) Answer the questions below using graphs in the style of Lee in his paper "The Mathematics of Excess of Loss Coverage and Retrospective Rating - A Graphical Approach."
A company writes two lines of business, A and B. Each line has identical loss characteristics, except that their severity distributions (which have the same mean of $10,000 and the same approximately normal form) have different standard deviations. The standard deviation of A is large, although there is insignificant probability of claims near zero. The standard deviation of B is small. Assume that policy limits do not apply to coverage under A or B.
a. (1.5 points) For a deductible of $8,000, how would you expect the loss elimination ratios (LERs) for the two lines to compare?
b. (1.5 points) Suppose that the annual claim cost trend factor is α. Assume positive trend, with trend factor $\alpha > 1.000$, and assume that α is uniform by size of loss.
Compare the expected effect of this trend on the LERs for these two lines.

23.57 (CAS9, 11/96, Q.39) (2 points) Using Lee's "The Mathematics of Excess of Loss Coverages and Retrospective Rating - A Graphical Approach," draw a graph and use letters to show the impact of inflation on each of the following types of loss.
Assume that claims of all sizes inflate by a constant factor of 1 + r.
1. total losses
2. basic limit losses
3. excess limit losses
For each type of loss, describe the impact as the ratio of losses after inflation to losses before inflation, and compare this impact to the overall inflation rate of 1 + r.

23.58 (CAS9, 11/99, Q.34) (2 points) Using Lee's "The Mathematics of Excess of Loss Coverages and Retrospective Rating - A Graphical Approach," answer the following:
a. (0.5 point) If the total limits inflation rate is 6%, describe why the inflation rate for the basic limits coverage is lower than 6%.
b. (1 point) Use Lee to graphically justify your answer.
c. (0.5 point) What are the two major reasons why the inflation rate in the excess layer is greater than the total limits inflation rate?

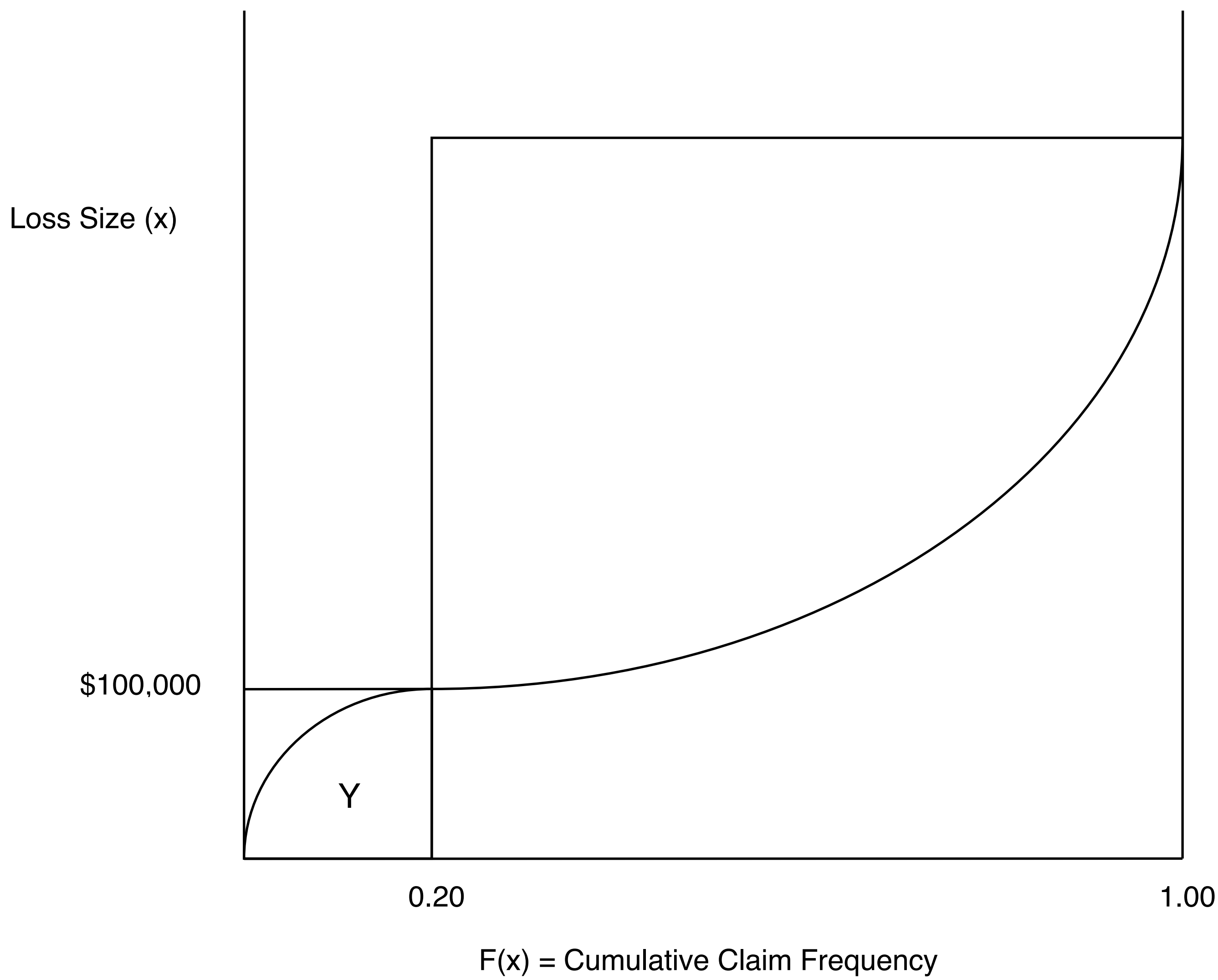
23.59 (CAS9, 11/02, Q.43) (3 points) a. (1.5 points) Using Lee's "The Mathematics of Excess of Loss Coverages and Retrospective Rating - A Graphical Approach," draw a graph to show what the expected losses would be for an excess of loss contract, covering losses in excess of retention (R), subject to a maximum limit (L). Include all appropriate labels on the graph.
b. (1.5 points) Assuming retention (R) but no maximum limit (L), describe how inflation will affect the expected losses for the excess cover relative to unlimited ground up losses. Explain your answer graphically or in words.

23.60 (5, 5/03, Q.14) (1 point)

Given $E[x] = \int_0^{\infty} x\, f(x)\, dx = \$152{,}500$

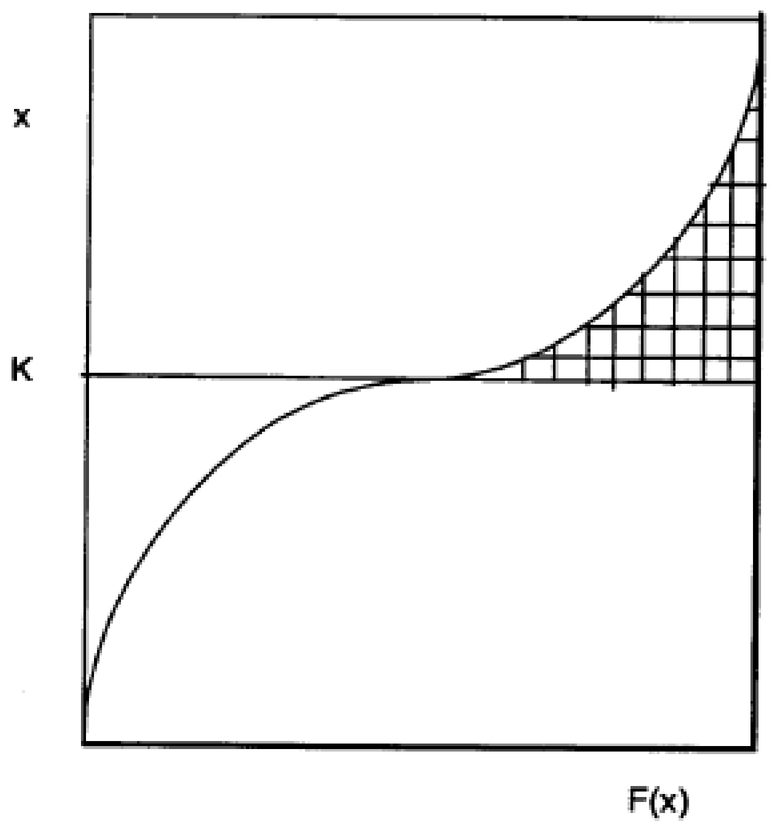
and the following graph of the cumulative loss distribution, F(x), as a function of the size of loss, x, calculate the excess ratio at $100,000.

- Size of the area labeled Y = $12,500



A. Less than 0.3
B. At least 0.3, but less than 0.5
C. At least 0.5, but less than 0.7
D. At least 0.7, but less than 0.9
E. At least 0.9

23.61 (CAS3, 11/03, Q.20) (2.5 points) Let X be the size-of-loss random variable with cumulative distribution function F(x) as shown below:



Which expression(s) below equal(s) the expected loss in the shaded region?

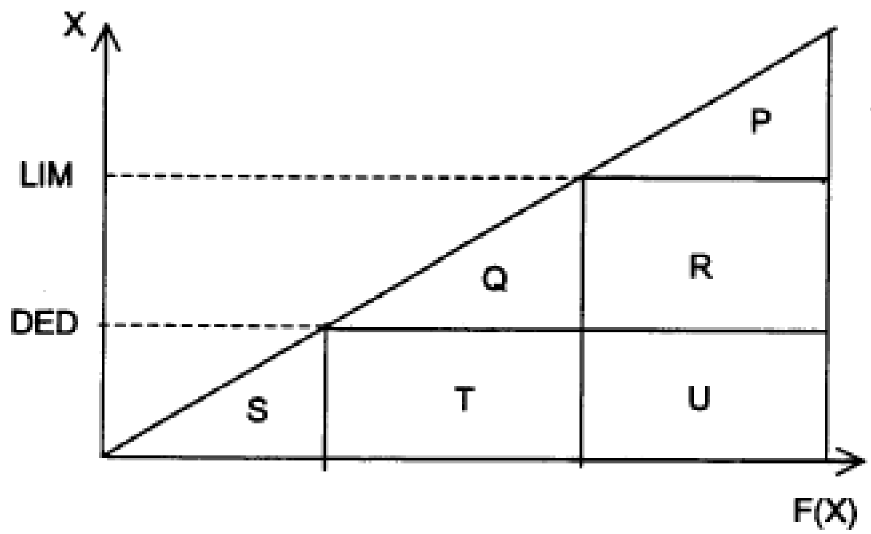
I. $\int_K^{\infty} x \, dF(x)$

II. $E(x) - \int_0^K x \, dF(x) - K[1-F(K)]$

III. $\int_K^{\infty} [1 - F(x)] \, dx$

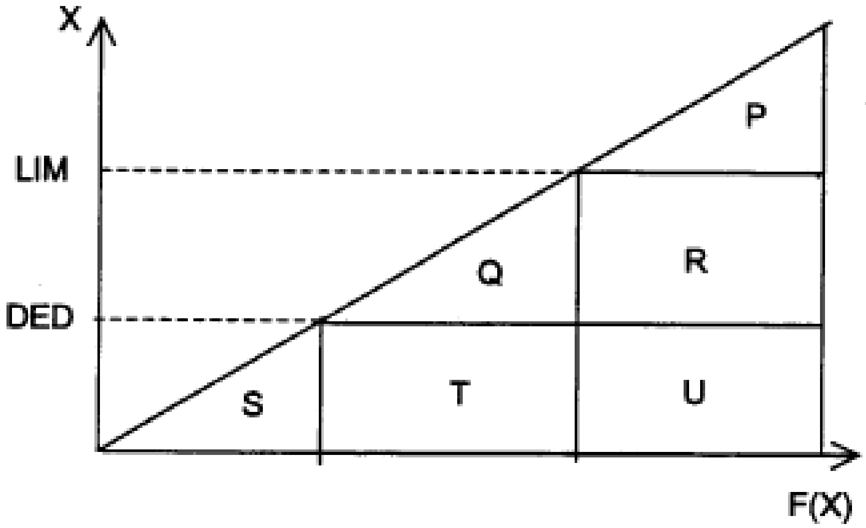
A. I only
B. II only
C. III only
D. I and III only
E. II and III only

23.62 (CAS3, 11/03, Q.23) (2.5 points)
F(x) is the cumulative distribution function for the size-of-loss variable, X.
P, Q, R, S, T, and U represent the areas of the respective regions.
What is the expected value of the insurance payment on a policy with a deductible of "DED" and a limit of "LIM"? (For clarity, that is a policy that pays its first dollar of loss for a loss of DED + 1 and its last dollar of loss for a loss of LIM.)



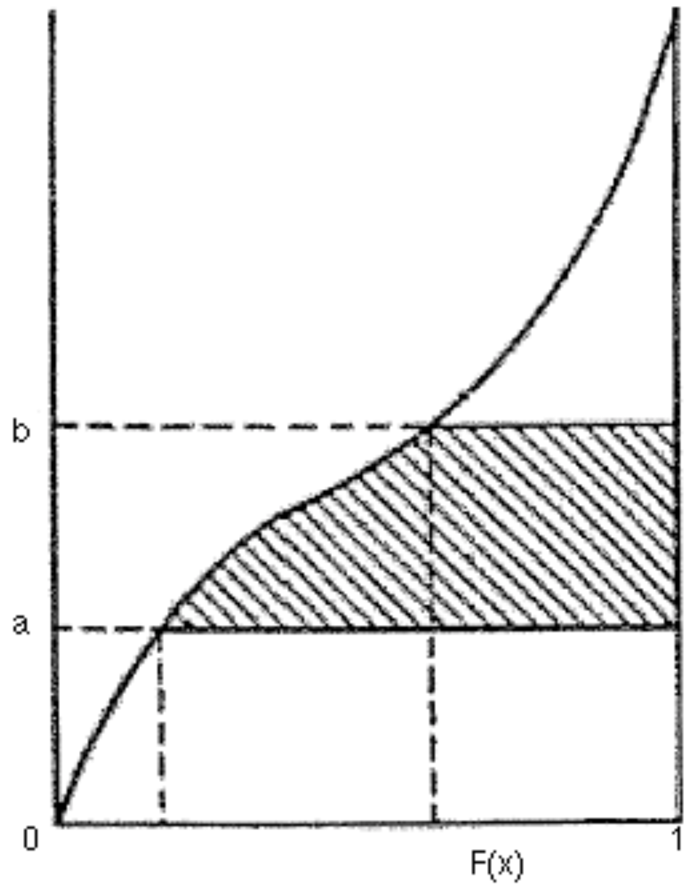
A. Q B. Q+R C. Q+T D. Q+R+T+U E. S+T+U

23.63 (CAS3, 5/04, Q.33) (2.5 points)
F(x) is the cumulative distribution function for the size-of-loss variable, X.
P, Q, R, S, T, and U represent the areas of the respective regions.
What is the expected value of the savings to the insurance company of implementing a franchise deductible of “DED" and a limit of “LIM" to a policy that previously had no deductible and no limit? (For clarity, that is a policy that pays its first dollar of loss for a loss of DED + 1 and its last dollar of loss for a loss of LIM.)



A. S B. S+P C. S+Q+P D. S+P+R+U E. S+T+U+P

23.64 (CAS3, 11/04, Q.30) (2.5 points) Let X be a random variable representing an amount of loss. Define the cumulative distribution function F(x) as F(x) = Pr(X ≤ x).



Determine which of the following formulas represents the shaded area.

A. $\int_a^b x \, dF(x) + a - b + aF(b) - bF(a)$

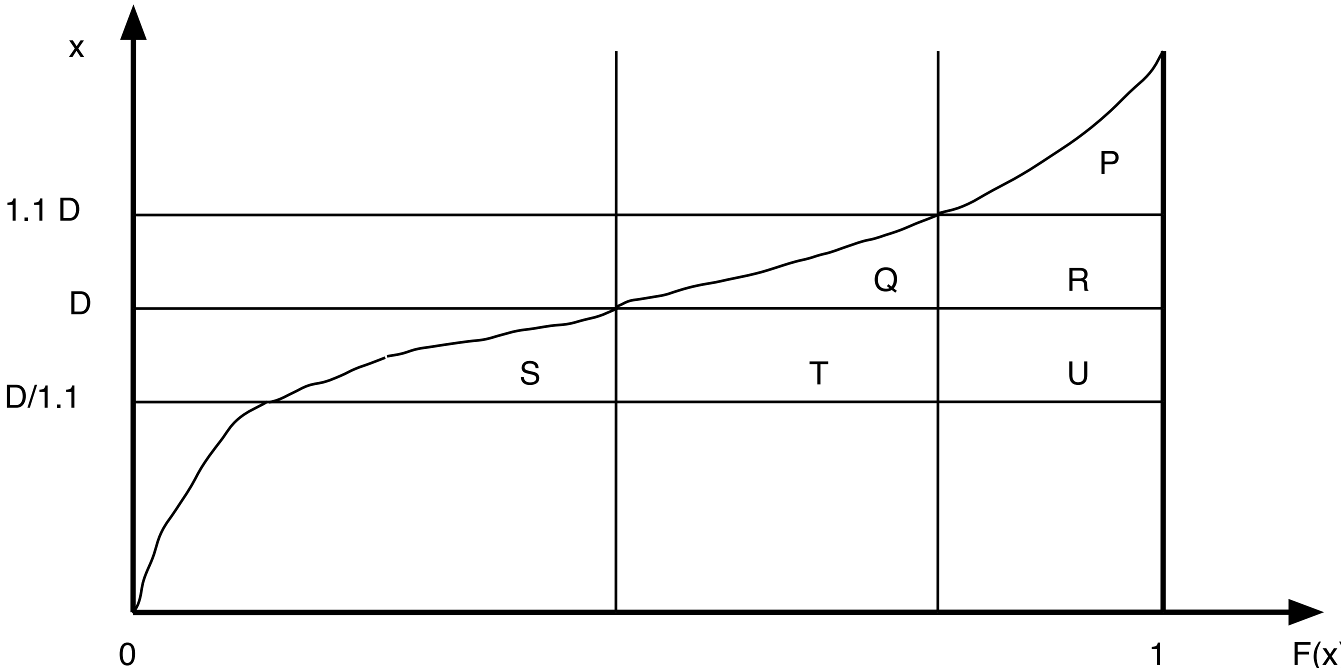
B. $\int_a^b x \, dF(x) + a - b + aF(a) - bF(b)$

C. $\int_a^b x \, dF(x) - a + b + aF(b) - bF(a)$

D. $\int_a^b x \, dF(x) - a + b + aF(a) - bF(b)$

E. $\int_a^b x \, dF(x) - a + b - aF(a) + bF(b)$

23.65 (CAS3, 5/06, Q.28) (2.5 points)
The following graph shows the distribution function, F(x), of loss severities in 2005.



Loss severities are expected to increase 10% in 2006 due to inflation.
A deductible, D, applies to each claim in 2005 and 2006.
Which of the following represents the expected size of loss in 2006?
A. P
B. 1.1P
C. 1.1(P+Q+R)
D. P+Q+R+S+T+U
E. 1.1(P+Q+R+S+T+U)

23.66 (CAS8, 11/11, Q.11) (3 points) Losses follow a uniform distribution between $0 and $100.
Assume a 10% trend is applied uniformly to all losses.
Use a Lee diagram to calculate the implied trend for the layer from $25 to $75
($50 excess of $25.)
Label all relevant features of the diagram.

23.67 (CAS8, 11/12, Q.22) (5 points)
The current deductible pricing for an auto insurer is based on the following claim distribution:

Size of Loss	Number of Claims
$100	21
$250	50
$500	42
$1,000	37
$5,000	22

An actuary wants to review the effect of loss trend on the insurer's loss elimination ratios.
a. (1.5 point) Calculate the loss elimination ratio for a straight $500 deductible assuming no trend adjustment.
b. (2.5 points) Assuming no frequency trend, calculate the percentage change in the loss elimination ratio for a straight $500 deductible assuming a ground-up loss severity trend of 10%.
c. (1 point) Explain why the loss cost for a given straight deductible policy can increase by more than the ground-up severity trend.

23.68 (CAS8, 11/14, Q.6) (9 points) Losses on a policy have the following distribution:
● 60% probability of a loss between $0 and $250,000
● 30% probability of a loss between $250,000 and $500,000
● 10% probability of a loss between $500,000 and $1 million
Losses are uniformly distributed within each range.
Assume a 20% trend is applied uniformly to all losses.
a. (4 points) Draw a diagram depicting the cumulative loss distribution described above before and after the 20% trend. Label all relevant features of the diagram.
b. (5 points) Calculate the implied trend for the layer $500,000 excess of $500,000. (The layer from $500,000 to $1 million.)

Solutions to Problems:

23.1. $\alpha+\beta+\gamma+\delta+\varepsilon+\eta$, the mean is the area under the distribution curve.

23.2. α, the result of summing vertical strips under the curve from zero to F(2).

23.3. $(\alpha+\beta+\delta)/(\alpha+\beta+\gamma+\delta+\varepsilon+\eta)$ = E[X ∧ 2] / E[X].

23.4. E[X ∧ 2] is: $\alpha+\beta+\delta$, the area under the curve and the horizontal line at 2.

23.5. $(\beta+\gamma+\delta+\varepsilon+\eta) / (\alpha+\beta+\gamma+\delta+\varepsilon+\eta) = 1 - \alpha / (\alpha+\beta+\gamma+\delta+\varepsilon+\eta)$.

23.6. $(\alpha+\beta+\gamma+\delta+\varepsilon) / (\alpha+\beta+\gamma+\delta+\varepsilon+\eta) = 1 - \eta / (\alpha+\beta+\gamma+\delta + \varepsilon+\eta)$ = E[X∧ 2]) / E[X].

23.7. $\alpha+\beta+\delta+\gamma+\varepsilon$, the area under the curve and the horizontal line at 5.

23.8. $(\alpha+\beta+\gamma) / (\alpha+\beta+\gamma+\delta+\varepsilon+\eta)$, the numerator is the result of summing vertical strips under the curve from zero to F(5), while the denominator is the total area under the curve.

23.9. $(\gamma+\varepsilon) / (\alpha+\beta+\gamma+\delta+\varepsilon+\eta)$ = (E[X∧ 5] - E[X∧ 2]) / E[X].

23.10. $(\gamma+\varepsilon+\eta) / (\alpha+\beta+\gamma+\delta+\varepsilon+\eta)$ = (E[X] - E[X∧ 2])) / E[X].

23.11. Losses from claims of size more than 5: $\delta+\varepsilon+\eta$.

23.12. $(\gamma+\varepsilon+\eta) / (\alpha+\beta+\gamma+\delta+\varepsilon+\eta)$ = (E[X] - E[X∧ 2])) / E[X].

23.13. $(\alpha+\beta+\gamma+\delta+\varepsilon) / (\alpha+\beta+\gamma +\delta+\varepsilon+\eta) = 1 - \eta / (\alpha+\beta+\gamma+\delta+\varepsilon+\eta)$ = E[X∧ 5] / E[X].

23.14. β, a rectangle of height 2 and width F(5) - F(2).

23.15. $(\beta+\gamma) / (\alpha+\beta+\gamma+\delta+\varepsilon+\eta)$, the numerator is the result of summing vertical strips under the curve from F(2) to F(5), while the denominator is the total area under the curve.

23.16. η = E[X] - E[X∧ 5], the sum of horizontal strips of length S(t) = 1-F(t) between the horizontal lines at 5 and ∞.

23.17. $(\alpha+\beta+\delta) / (\alpha+\beta+\gamma+\delta+\varepsilon+\eta)$ = E[X∧ 2]) / E[X].

23.18. η, the sum of vertical strips of height t-5 between the vertical lines at F(5) and 1.

23.19. δ, a rectangle of height 2 and width S(5).

23.20. $\eta / (\alpha+\beta+\gamma+\delta+\varepsilon+\eta) = (E[X] - E[X \wedge 5]) / E[X]$.

23.21. $e(5) = \eta/S(5)$. But, $\delta = 2S(5)$ and $\varepsilon = 3S(5)$. Thus $e(5) = 3\eta/\varepsilon$ or $2\eta/\delta$.

23.22. γ, the sum of vertical strips of height t-2 between the vertical lines at F(2) and F(5).

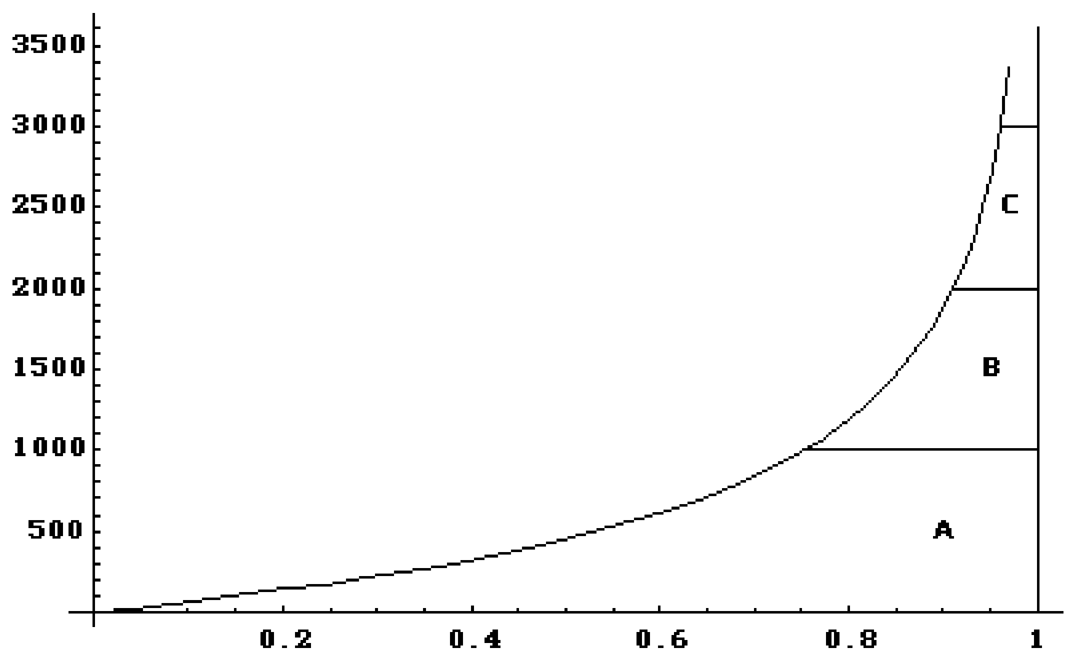
23.23. $\eta = E[(X - 5)_+] = E[X] - E[X \wedge 5]$.

23.24. $\gamma+\varepsilon = E[X \wedge 5] - E[X \wedge 2]$, the sum of horizontal strips of length 1-F(t) between the horizontal lines at 2 and 5.

23.25. ε, a rectangle of height 5 -2 and width 1-F(5) = S(5).

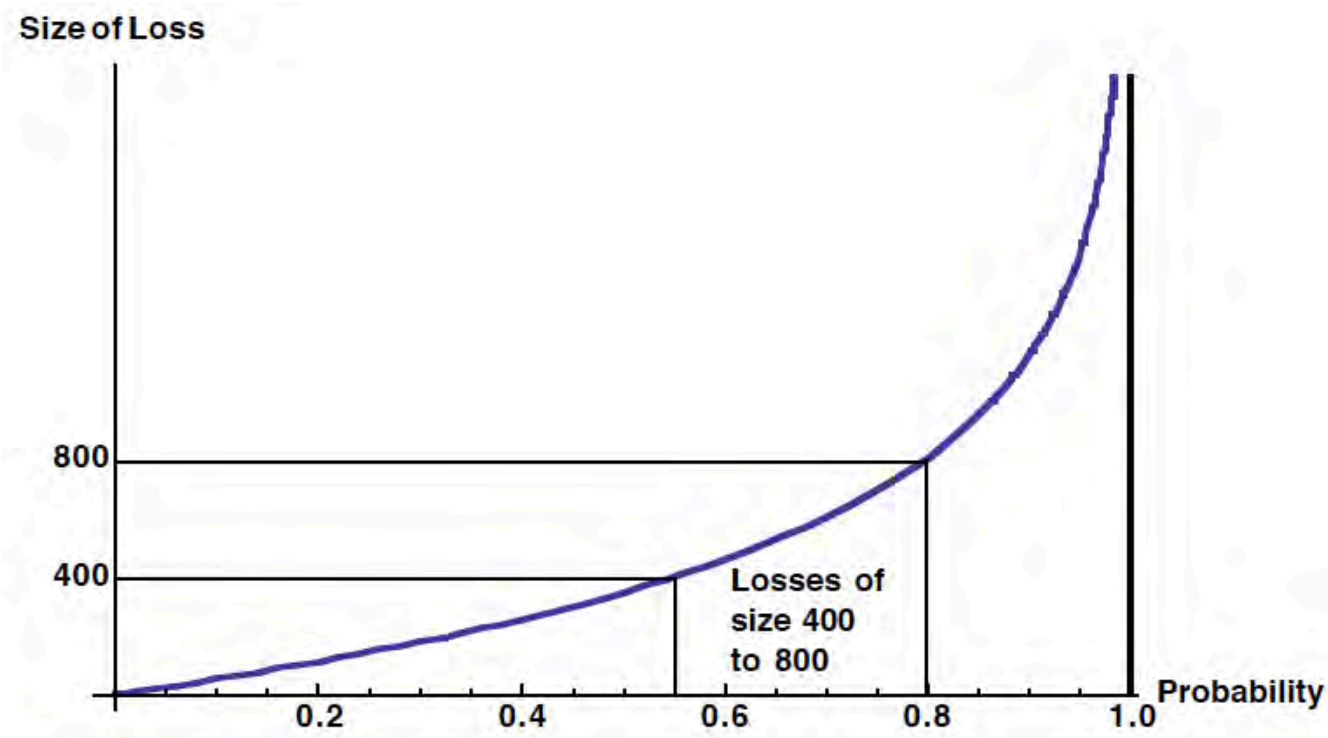
23.26. e(2) = losses excess of 2 / S(2) = $(\gamma+\varepsilon+\eta)/S(2)$.
But, $\beta+\delta = 2S(2)$. $\Rightarrow e(2) = 2(\gamma+\varepsilon+\eta) / (\beta+\delta)$.

23.27. In the Lee diagram below, $E[X \wedge 1000]$ = Area A.

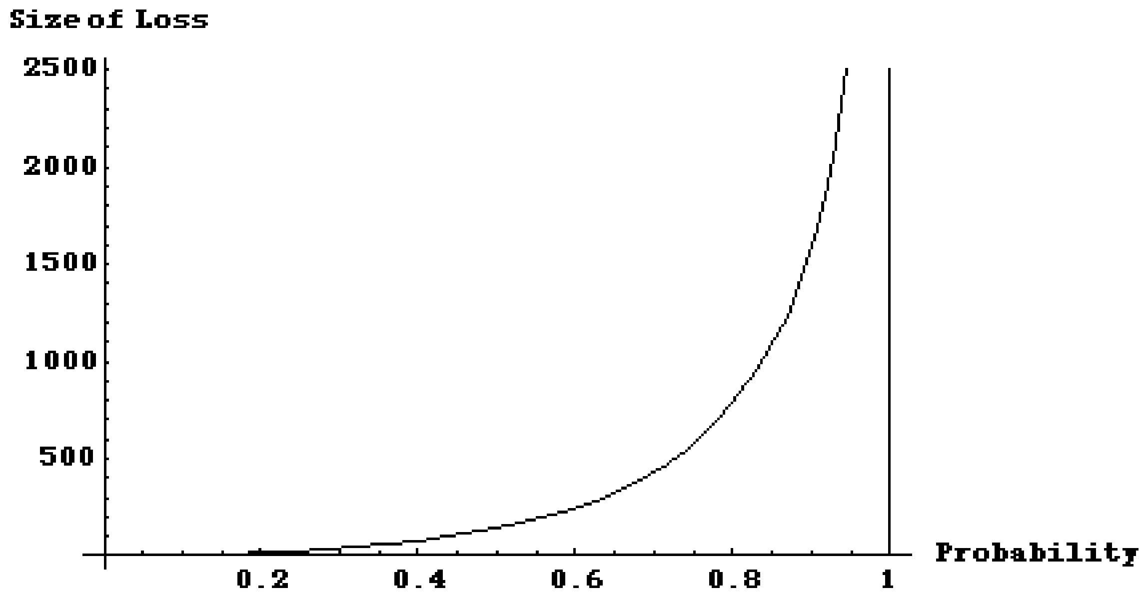


$E[X \wedge 2000]$ = Area A + Area B. Therefore, Area B = $E[X \wedge 2000] - E[X \wedge 1000]$.
Area B is the increase in the limited expected value due to increasing the limit from 1000 to 2000. Similarly, Area C is the increase in the limited expected value due to increasing the limit by another 1000. Area C < Area B, and therefore the increase in the limited expected value is less. In general the areas of a given height get smaller as one moves up the diagram, as the curve moves closer to the righthand asymptote. Therefore, the rate of increase of the limited expected value decreases.
<u>Comment</u>: The diagram was based on a Pareto Distribution with $\alpha = 4$ and $\theta = 2400$.

23.28. If y = size of loss and x = probability, then for an Exponential with θ = 500,
x = 1 - exp[-y/500]. Therefore, y = -500 ln[1 - x].
loss of size 400 ⇔ probability = 1 - $e^{-0.8}$ = 0.551.
loss of size 800 ⇔ probability = 1 - $e^{-1.6}$ = 0.798.
Losses of size 400 to 800 corresponds to the area below the curve and between vertical lines at 0.551 and 0.798:

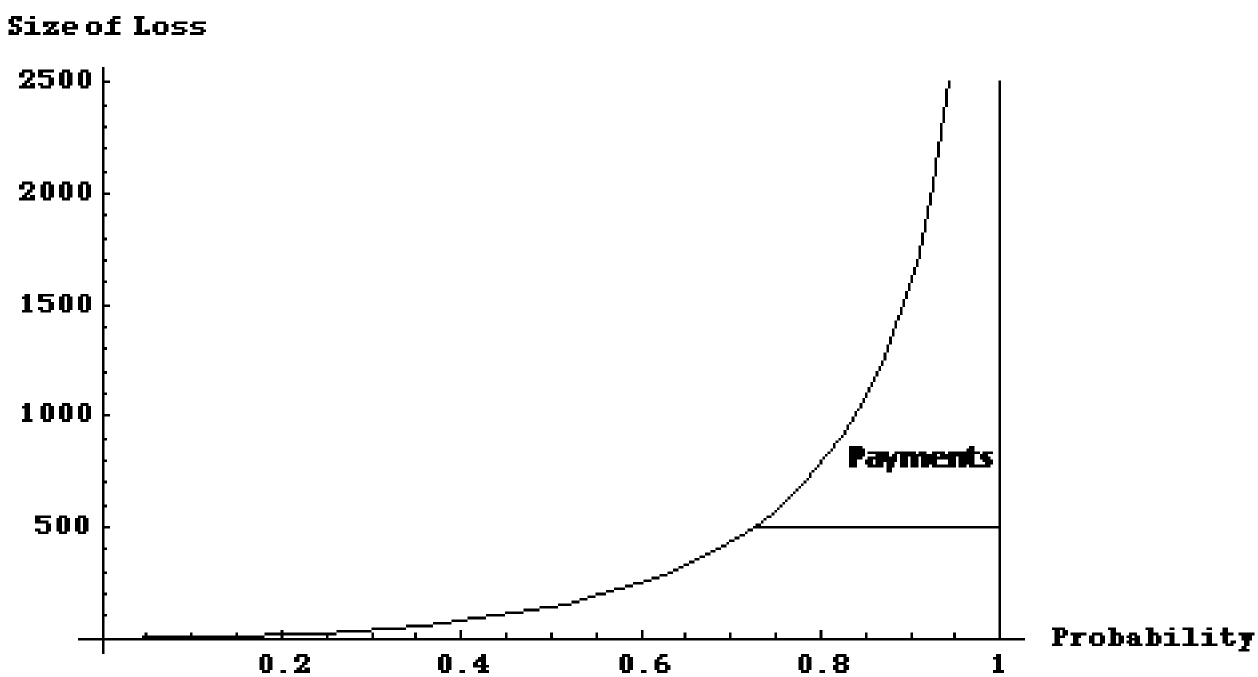


23.29. If y = size of loss and x = probability, then for a Weibull with θ = 300 and τ = 1/2, $x = 1 - \exp[-(y/300)^{0.5}]$. Therefore, $y = 300 \ln[1 - x]^2$. Lee Diagram:

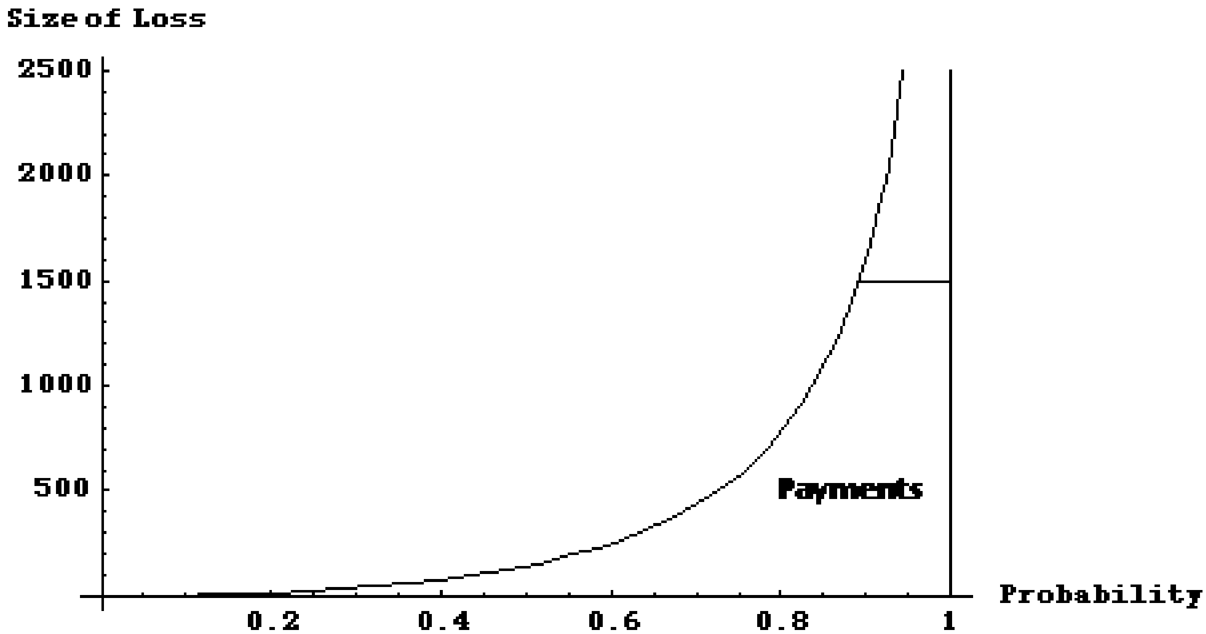


Comment: One has to stop graphing at some size of loss, unless one has infinite graph paper! In this case, I only graphed up to 2500.

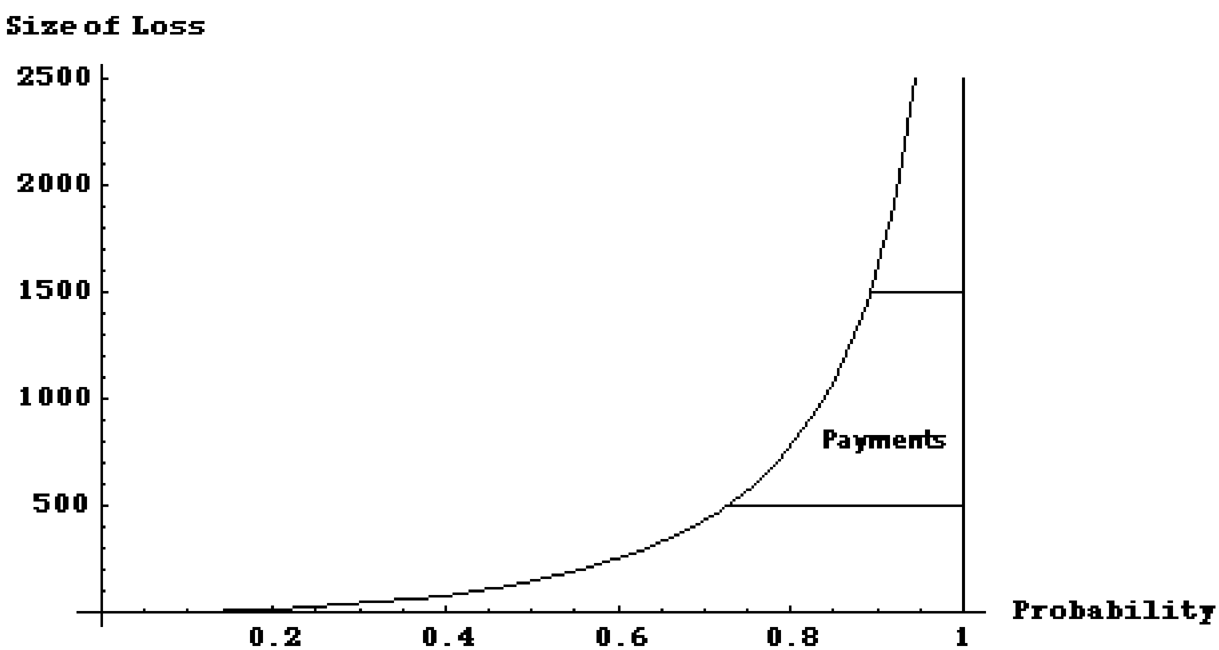
23.30. The payments with a 500 deductible and no maximum covered loss are represented by the area above the line at height 500 and to the right of the curve:



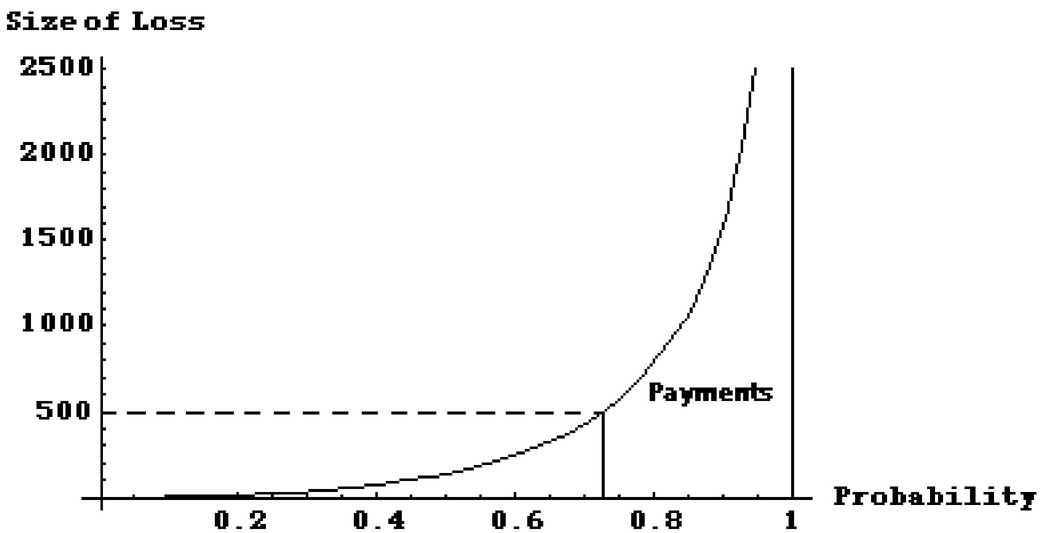
23.31. The payments with no deductible and a 1500 maximum covered loss are represented by the area below the line at height 1500, and to the right of the curve:



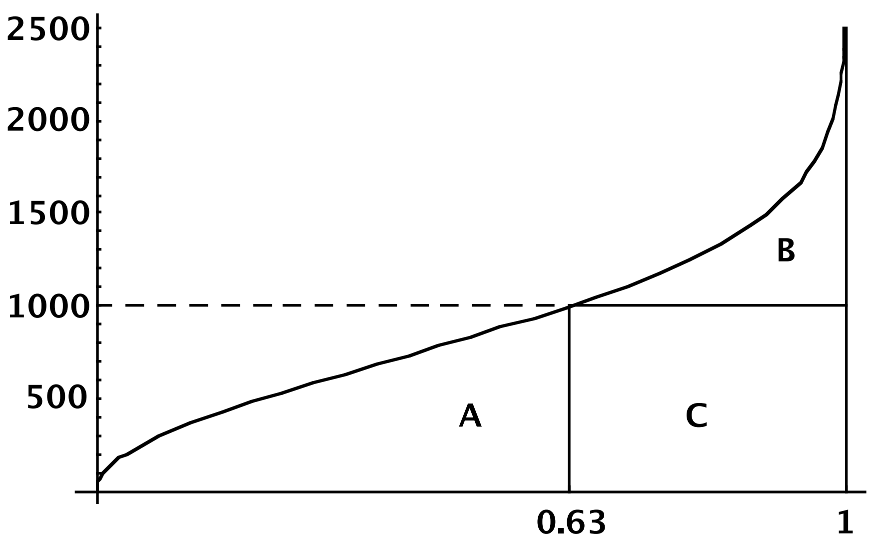
23.32. The payments with a 500 deductible and a 1500 maximum covered loss are represented by the area above the line at height 500, below the line at height 1500, and to the right of the curve:



23.33. Under a 500 franchise deductible, nothing is paid on a loss of size 500 or less, and the whole loss is paid for a loss of size greater than 500. The payments with a 500 franchise deductible and no maximum covered loss are the losses of size greater than 500, represented by the area to the right of the vertical line at F(500) and below the curve:



23.34. D. Area C is a rectangle with height 1000 and width (1 - 0.63) = 0.37, with area 370.



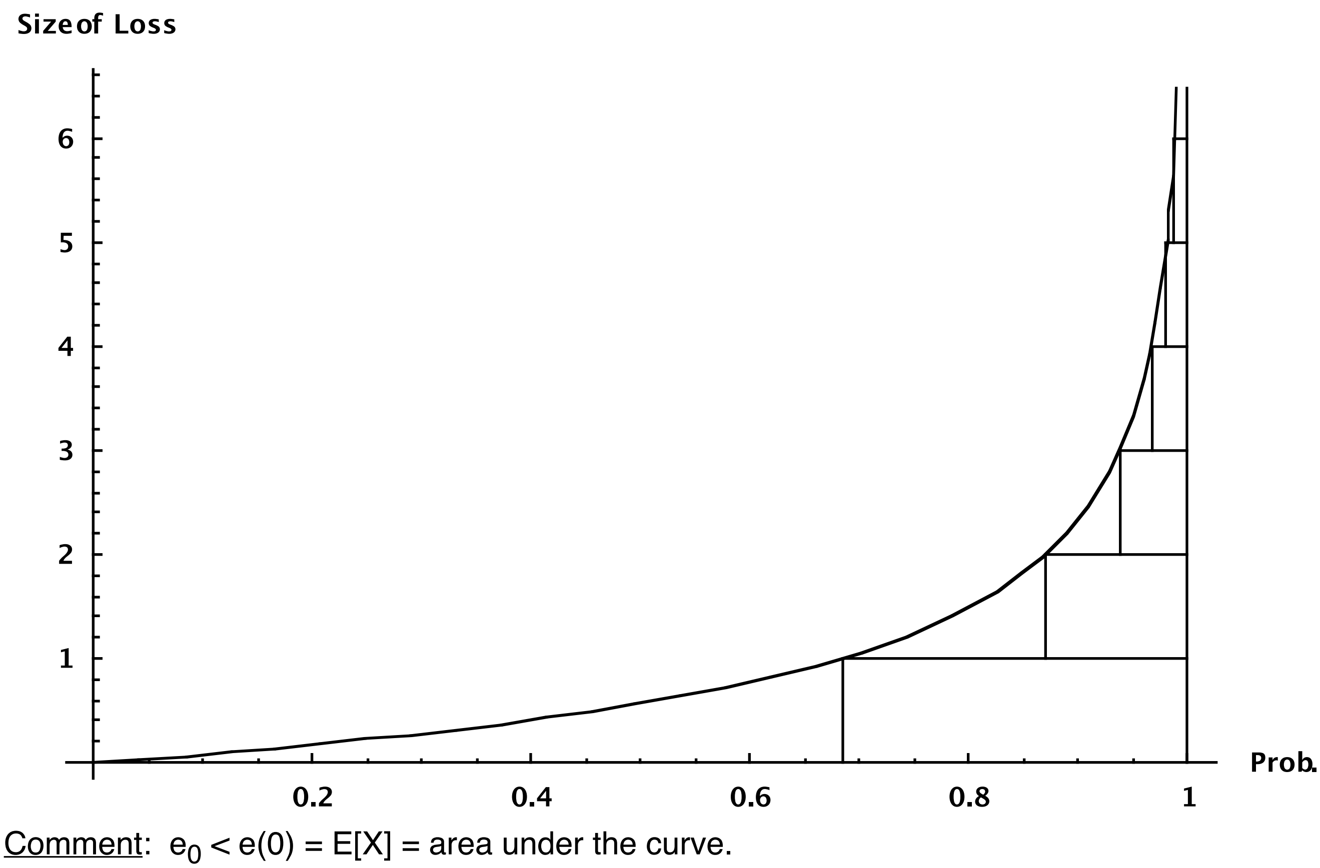
Expected losses limited to 1000 = Area A + Area C = 377 + 370 = 747.
E[X] = Area A + Area B + Area C = 377 + 139 + 370 = 886.
Loss elimination ratio at 1000 = $E[X \wedge 1000]/E[X]$ = 747/886 = **84.3%**.
<u>Comment</u>: Similar to 5, 5/03, Q.14.
The Lee Diagram was based on a Weibull Distribution with θ = 1000 and τ = 2.

23.35. B. Deflate 5000 from 2008 to 2001, where it is equivalent to: 5000/1.5 = 3333.
Deflate 1000 from 2008 to 2001, where it is equivalent to: 1000/1.5 = 667. Then the average expected loss in 2001 is the area between horizontal lines at 667 and 3333, and under the curve: R+T+U. In order to get the expected size of loss in 2008, reinflate back up to the 2008 level, by multiplying by 1.5: **1.5(R + T + U)**.
Comment: Similar to CAS3, 5/06, Q.28.

23.36. D. A policy with a franchise deductible of 2 pays the full amount of all losses of size greater than 2. This is the area under the curve and to the right of the vertical line at 2: b + g + d + e + h. However, there is also a maximum covered loss of 5, which means the policy does not pay for the portion of any loss greater than 5, which eliminates area h, above the horizontal line at 5. Therefore the expected payments are: b + g + d + e.
Comment: Similar to CAS3, 5/04, Q.33.

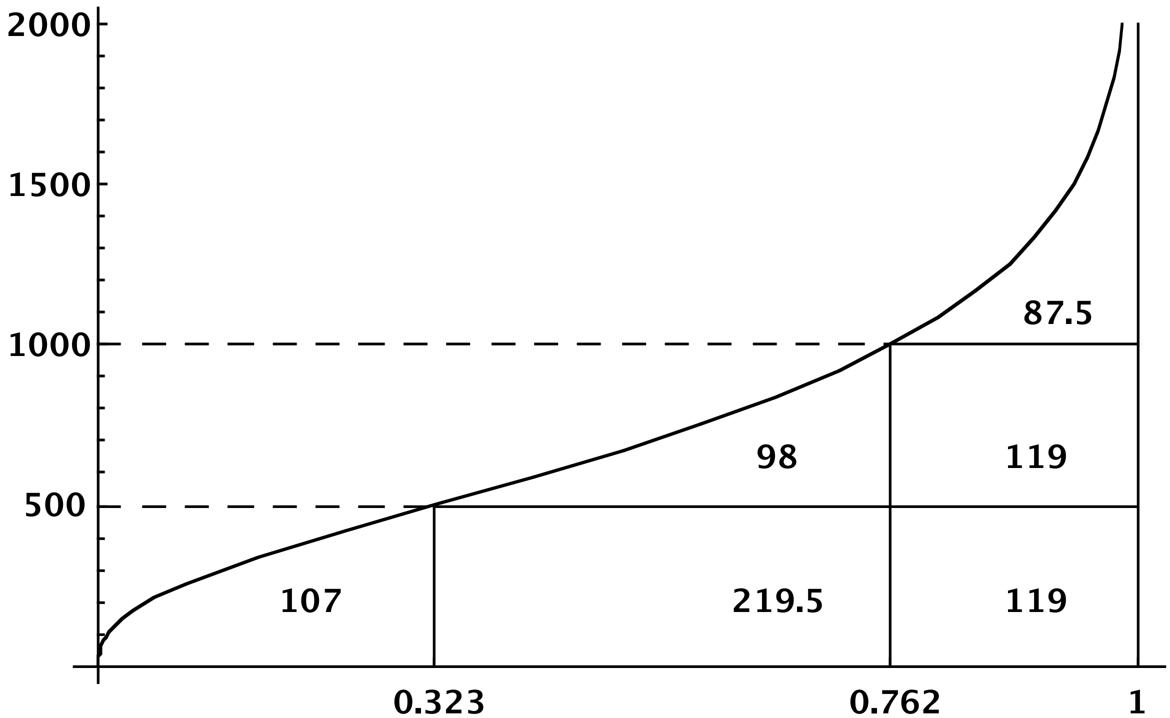
23.37. E. Prior to the effect of the coinsurance, the expected size of loss is below the line at 5000 and below the curve: R + T. We multiply by 80% before paying under this policy.
The expected size of loss is: **0.8(R + T)**.

23.38. The curtate expectation of life, e_0, is the sum of a series of rectangles, each with height 1, with areas: $S(1) = (3/4)^4$, $S(2) = (3/5)^4$, $S(3) = (3/6)^4$, etc.
The first six of these rectangles are shown below:



Comment: $e_0 < e(0) = E[X]$ = area under the curve.
The curtate expectation of life is discussed in Actuarial Mathematics.

23.39. We are given Area A = 107 and Area B is 98. We can get the areas of three rectangles.
One rectangle has width: 0.762 - 0.323 = 0.439, height 500, and area: (0.439)(500) = 219.5.
Two rectangles have width: 1 - 0.762 = 0.238, height 500, and area: (0.238)(500) = 119.
The total area under the curve is equal to the mean, given as 750.
Therefore, the area under the curve and above the horizontal line at 1000 is:
750 - (107 + 219.5 + 98 + 119 + 119) = 87.5.



The average size of loss for those losses of size less than 500 is:
(dollars from losses of size less than 500)/F(500) = 107/0.323 = **331**.
Comment: The Lee Diagram was based on a Gamma Distribution with $\alpha = 3$ and $\theta = 250$.

23.40. LER(500) = E[X∧500]/E[X] = (107 + 219.5 + 119)/750 = **59.4%**.

23.41. The average payment per loss with a deductible of 500 and maximum covered loss of 1000 is: layer from 500 to 1000 ⇔
the area under the curve and between the horizontal lines at 500 and 1000 ⇔
98 + 119 = **217**.

23.42. e(500) = (losses excess of 500)/S(500) = (98 + 119 + 87.5) / (1 - 0.323) = **450**.

23.43. The average size of loss for those losses of size between 500 and 1000 is:
(dollars from losses of size between 500 and 1000) / {F(1000) - F(500)} =
(219.5 + 98) / (0.762 - 0.323) = **723**.

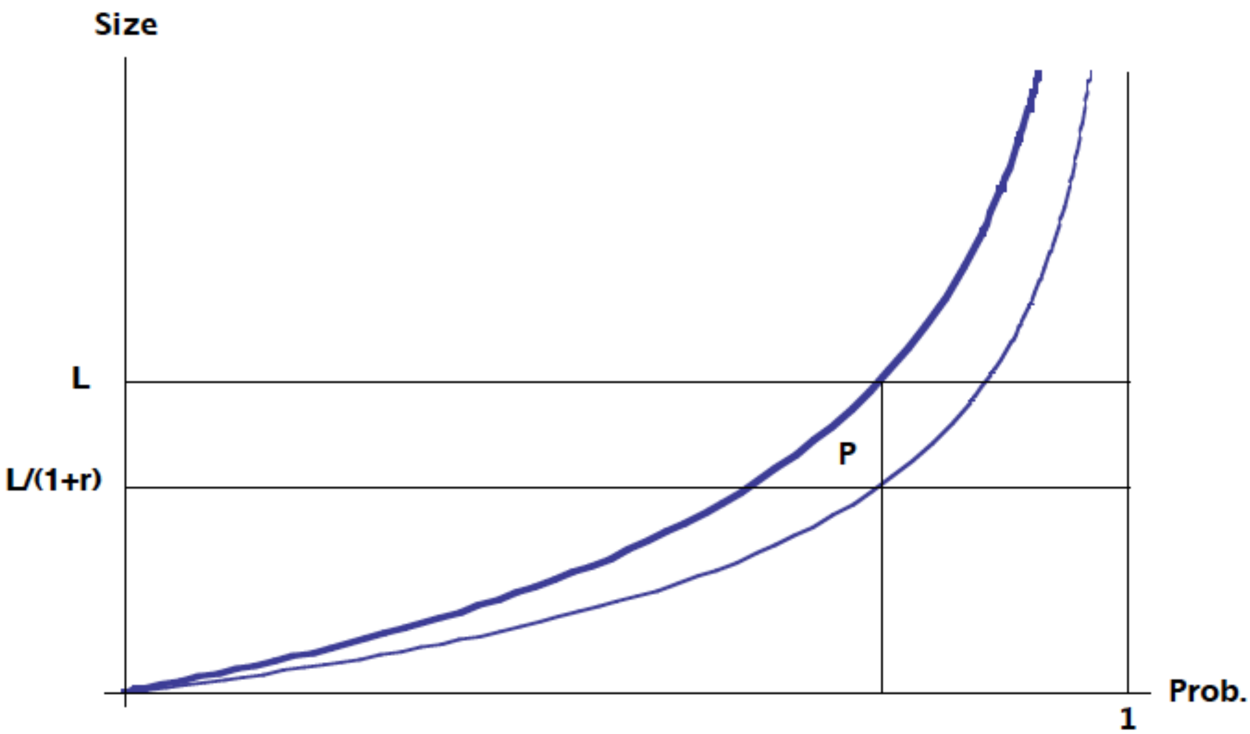
23.44. The loss elimination ratio at 1000 = E[X∧1000] / E[X] =
(107 + 219.5 + 119 + 98 + 119) / 750 = **88.3%**.
Alternately, excess ratio at 1000 is: 87.5/750 = 11.7%. ⇒ LER(1000) = 1 - 11.7% = 88.3%.

23.45. The average payment per (non-zero) payment with a deductible of 500 and maximum covered loss of 1000 is: (average payment per loss)/S(500) = 217 / (1 - 0.323) = **321**.

23.46. e(1000) = (losses excess of 1000) / S(1000) = 87.5 / (1 - 0.762) = **368**.

23.47. The average size of loss for those losses of size greater than 1000 is:
(dollars from losses of size > 1000)/S(1000) = (87.5 + 119 + 119) / (1 - 0.762) = **1368**.

23.48. D. The rectangle below Area P plus Area P, represent those losses of size between L/(1+r) and L.

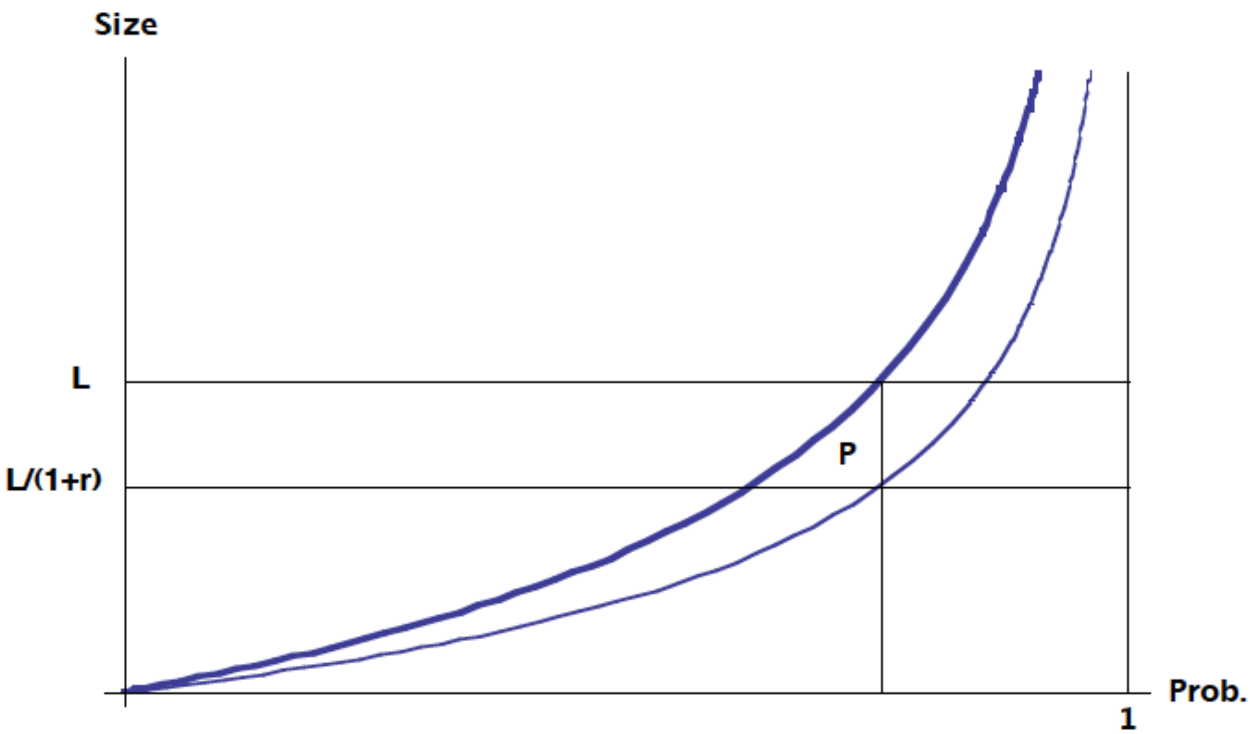


⇒ Area P + Rectangle = $\int_{L/(1+r)}^{L} x\, f(x)\, dx$.

The rectangle has height: L/(1+r), and width: F(L) - F(L/(1+r)).

⇒ Area P = $\int_{L/(1+r)}^{L} x\, f(x)\, dx$ - {F(L) - F(L/(1+r))} L/(1+r).

Comment: Area P can be written in other ways as shown below:



Area P plus the rectangle to the left of Area P, represents the layer of loss from L/(1+r) to L.

$\Rightarrow$ Area P + Rectangle = $\int_{L/(1+r)}^{L} S(x)\ dx$.

The rectangle has height: L - L/(1+r), and width: S(L).

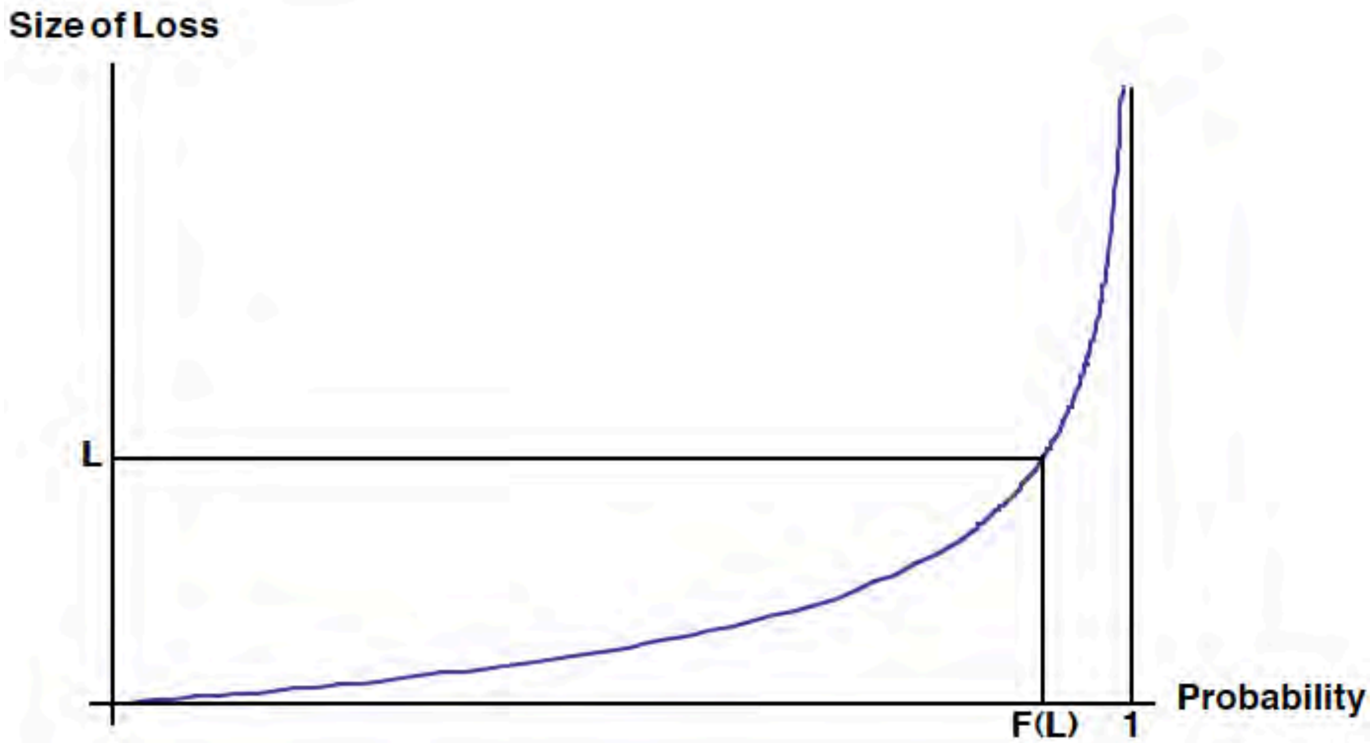
$\Rightarrow$ Area P = $\int_{L/(1+r)}^{L} S(x)\ dx$ - S(L){L - L/(1+r)} = $\int_{L/(1+r)}^{L} S(x)\ dx$ - S(L) L r/(1+r).

23.49. The layer can be gotten by adding horizontal strips: $\int_d^u S(x)\ dx$.

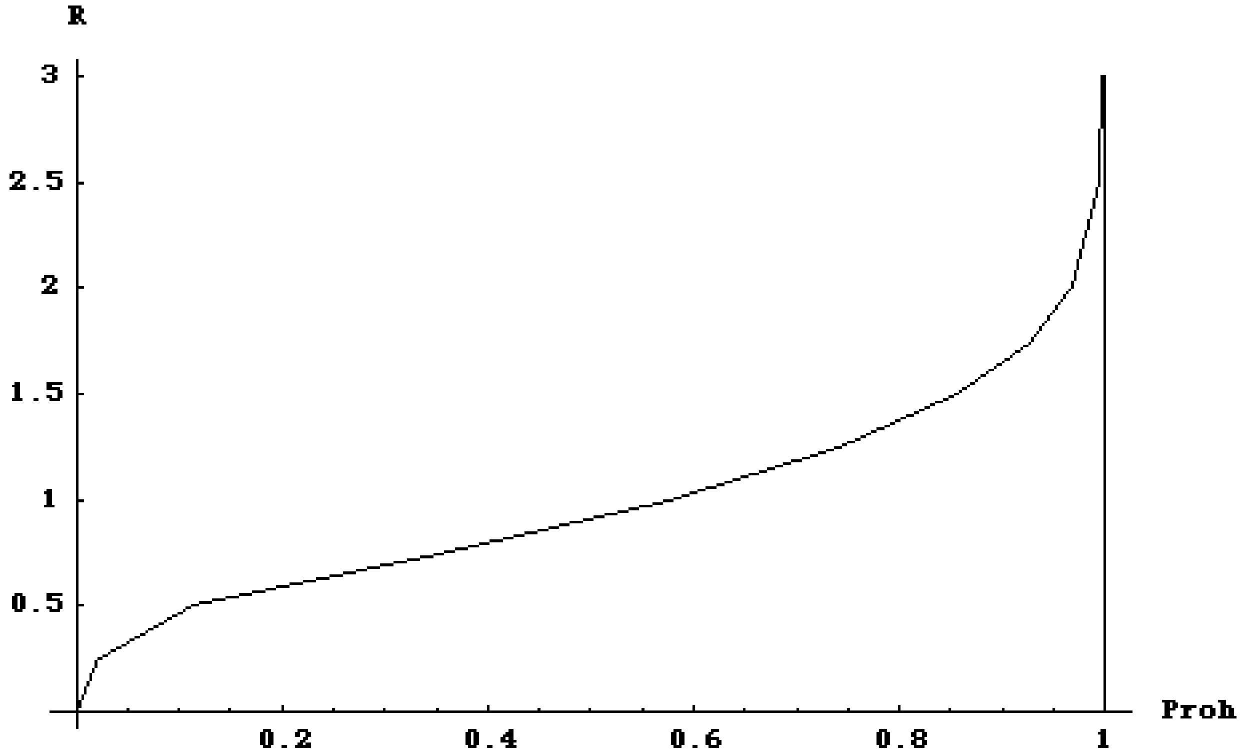
Also, the layer can be gotten by adding vertical strips: $\int_d^u (x - d)\ f(x)\ dx\ + (u-d)S(u)$.

Comment: In the second version, the first term is the contribution of medium losses, while the second term is the contribution of large losses.

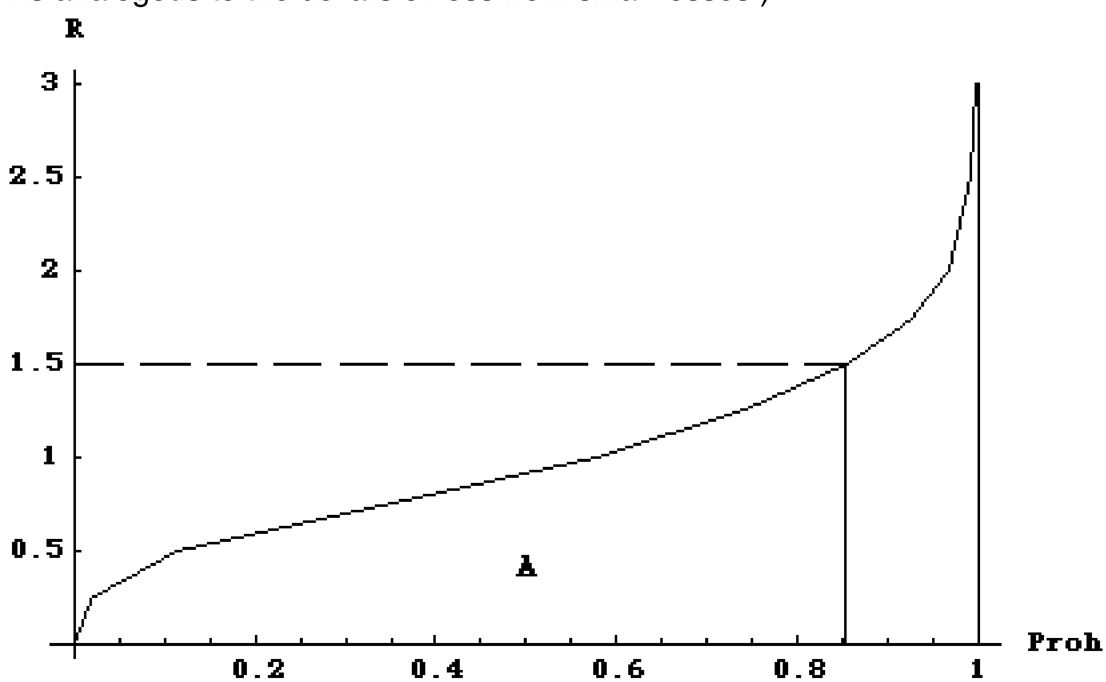
23.50. The vertical axis is size of loss. Where the horizontal line at height L hits the curve is F(L).
Dropping a vertical line, where we hit the horizontal-axis has distance of F(L) from the origin.



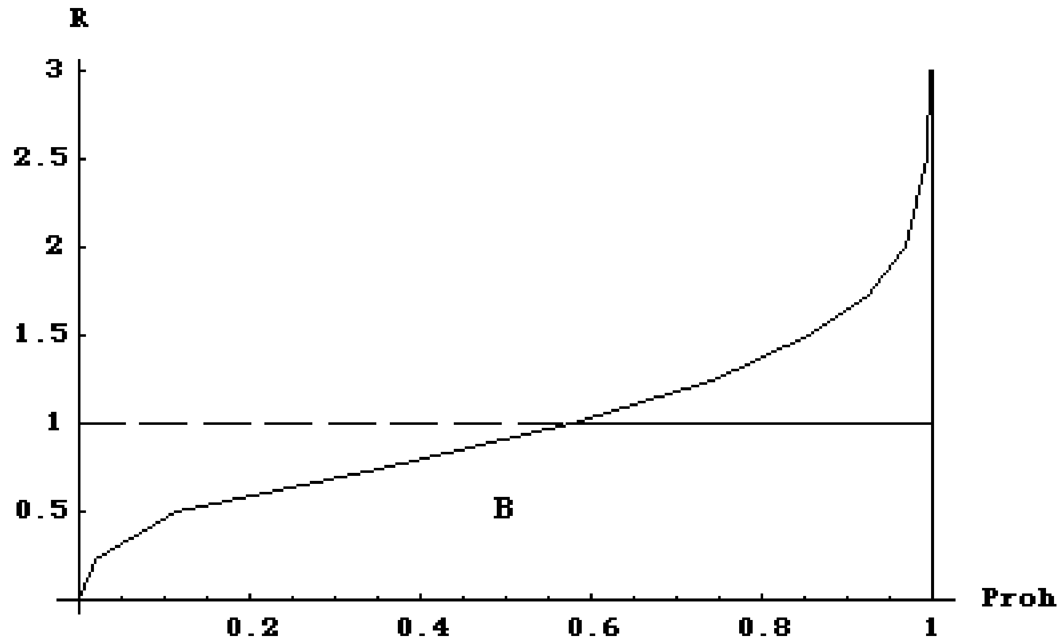
23.51. (a) As an approximation, we connect by straight lines the points:
(0, 0), (0.022, 0.25), (0.113, 0.5), (0.350, 0.75), (0.575, 1), etc.



(b) 85.3% of workers earn at most 150% of the state average weekly wage.
The area corresponding to the percentage of wages earned by those making at most 150% of the state average weekly wage is below the curve and to the left of the vertical line at 0.853, Area A.
(Area A is analogous to the dollars of loss from small losses.)

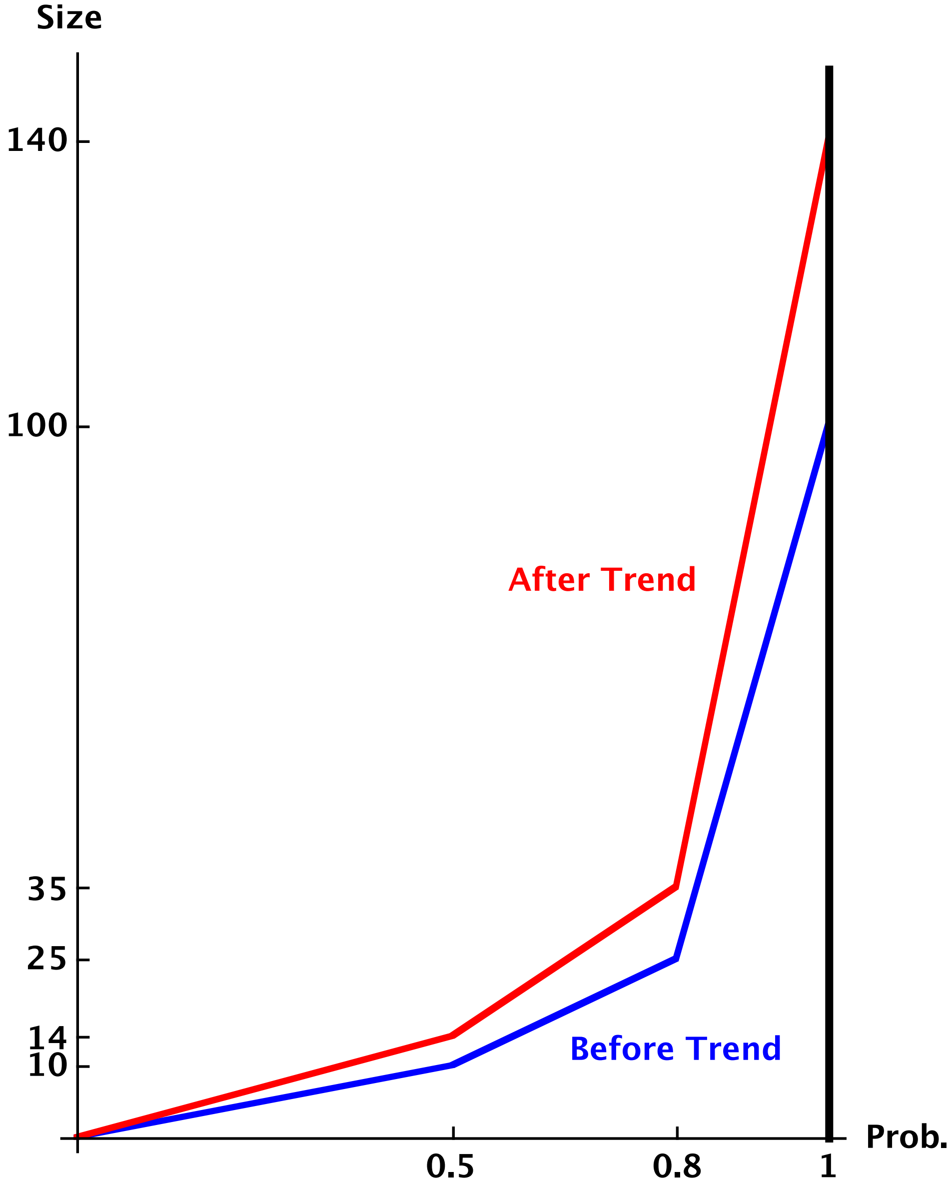


(c) The benefits are capped at the State Average Weekly Wage, corresponding to R = 1. Therefore, the area corresponding to the average benefit paid is below the curve and also below a horizontal line at 1, Area B.
(Area B is analogous to the percentage of loss dollars in the layer from 0 to 1.)

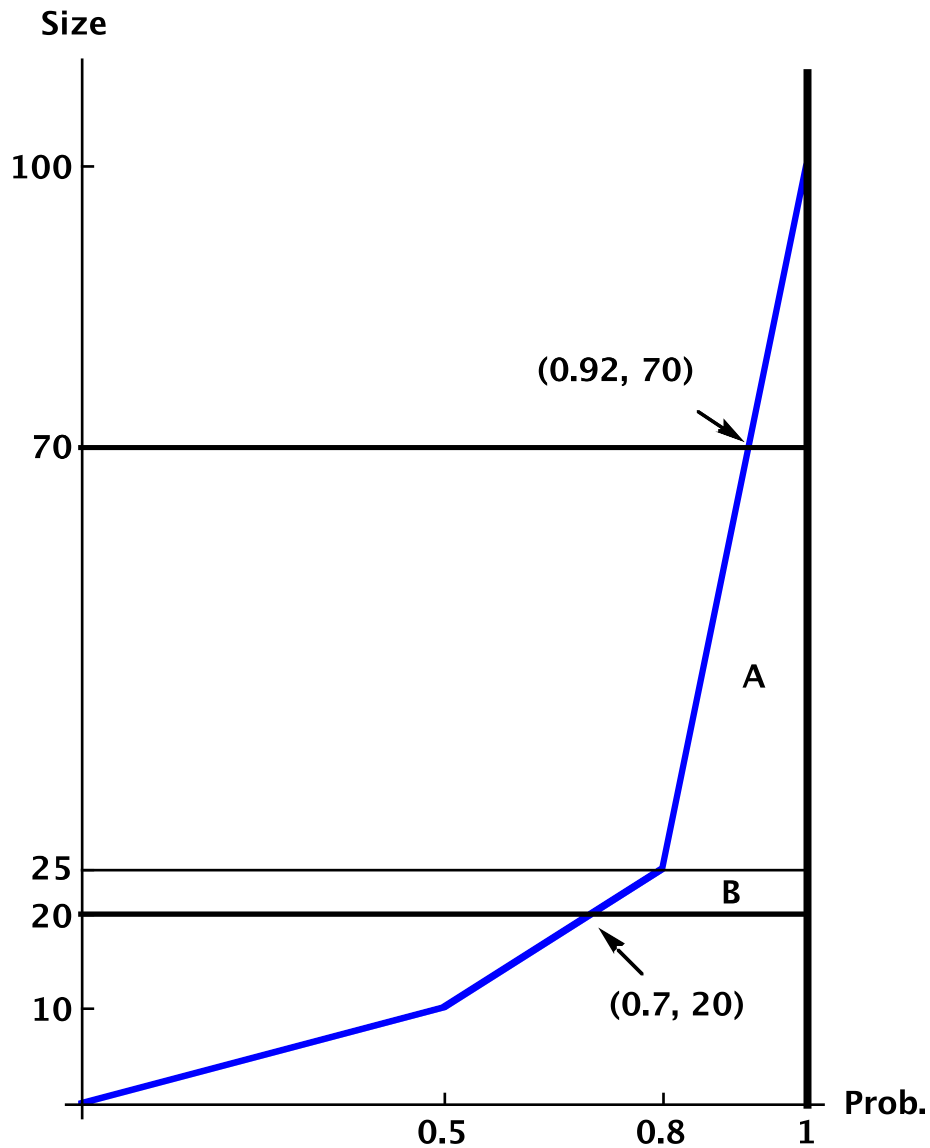


Comment: Since R is with respect to the State Average Weekly Wage, the area under the curve should be 1.

23.52. (a) Subsequent to trend, we have three uniform distributions:
from 0 to 14,000; from 14,000 to 35,000; and from 35,000 to 140,000.
A Lee Diagram, with size in thousands:

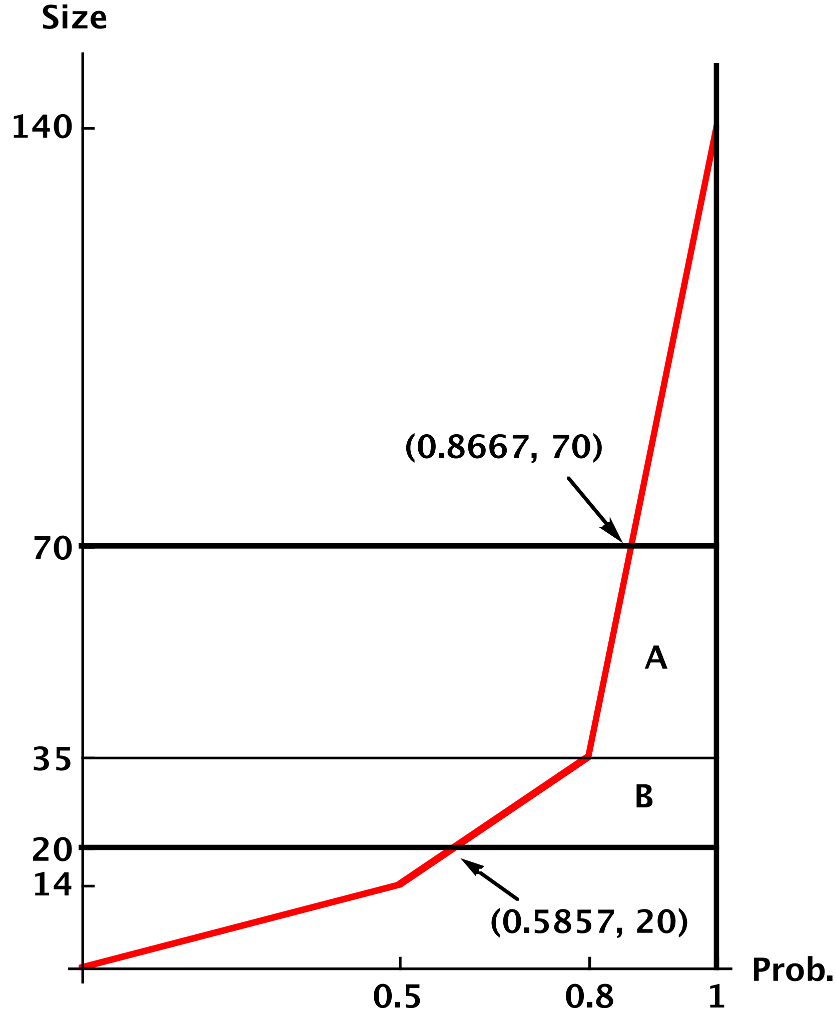


(b) Prior to trend the distribution function at 20 is: (1/3)(0.5) + (2/3)(0.8) = 0.7.
Prior to trend the distribution function at 70 is: (30/75)(0.8) + (45/75)(1) = 0.92.
Prior to trend, the layer from 20,000 to 70,000 is the area below the distribution and between horizontal lines at heights 20,000 and 70,000.
This is Areas A plus B in the following Lee Diagram.



Area A is a trapezoid, with area: (70 - 25)(0.2 + 0.08)/2 = 6.3 (thousand).
Area B is a trapezoid, with area: (25 - 20)(0.3 + 0.2)/2 = 1.25 (thousand).
Prior to trend, the losses in the layer are: 6.3 + 1.25 = 7.55 (thousand).

After trend the distribution function at 20 is: (15/21)(0.5) + (6/21)(0.8) = 0.5857.
After trend the distribution function at 70 is: (2/3)(0.8) + (1/3)(1) = 0.8667.
After trend the Lee Diagram is as follows, with the excess layer being the sum of Areas A and B:



Area A is a trapezoid with area: (70 - 35)(0.2 + 0.1333)/2 = 5.833 (thousand).
Area B is a trapezoid with area: (35 - 20)(0.4143 + 0.2)/2 = 4.607 (thousand).
Thus after trend the excess layer is: 5.833 + 4.607 = 10.440 (thousand).
The implied trend for the layer \$50,000 excess of \$20,000 is: 10.440/7.55 - 1 = **38.3%**.
Comment: Similar to 8, 11/14, Q. 6.
Prior to trend:
$E[X \wedge 70{,}000] = (0.5)(5K) + (0.3)(17.5K) + (0.2)(45/75)(47.5K) + (0.2)(30/75)(70K) = 18.05K.$
$E[X \wedge 20{,}000] = (0.5)(5K) + (0.3)(2/3)(15K) + (0.3)(1/3)(20K) + (0.2)(20K) = 11.5K.$
$E[X \wedge 70{,}000] - E[X \wedge 20{,}000] = 18.45K - 11.5K = 7550.$
Subsequent to trend:
$E[X \wedge 70{,}000] = (0.5)(7K) + (0.3)(24.5K) + (0.2)(1/3)(52.5K) + (0.2)(2/3)(70K) = 23{,}683.$
$E[X \wedge 20K] = (0.5)(7K) + (0.3)(6/21)(17K) + (0.3)(15/21)(20K) + (0.2)(20K) = 13{,}243.$
$E[X \wedge 70{,}000] - E[X \wedge 20{,}000] = 23{,}683 - 13{,}243 = 10{,}440.$

23.53. The higher curve is prior to benefit reform, while the lower curve is after benefit reform. (Unlike inflation, here the loss sizes went down.)
The losses prior to benefit reform are below the higher curve.
The losses after benefit reform are below the lower curve.
The reduction in losses due to benefit reform are between the curves.
Retained losses are below the horizontal line at S.
Insurer's losses are above the horizontal line at S.
1. E.
2. A.
3. J.
4. B.
5. Zero. The contribution from large claims to the insured's retained losses are the same before and after benefit reform, C.
6. C.
7. Zero. There is no contribution from small claims to the insurer's losses both before and after benefit reform.
8. Zero.
9. K.
10. Zero.
11. L
12. M.
13. Prior to benefit reform, the losses eliminated by the retention S are: A + B + C + E + J.
However, S = D + A + B + C + E + J.
Therefore, prior to benefit reform, the losses eliminated = S - Area D.
Comment: After benefit reform, the losses eliminated by the retention S are:
A + B + C = S - Area D - Area E - Area J.

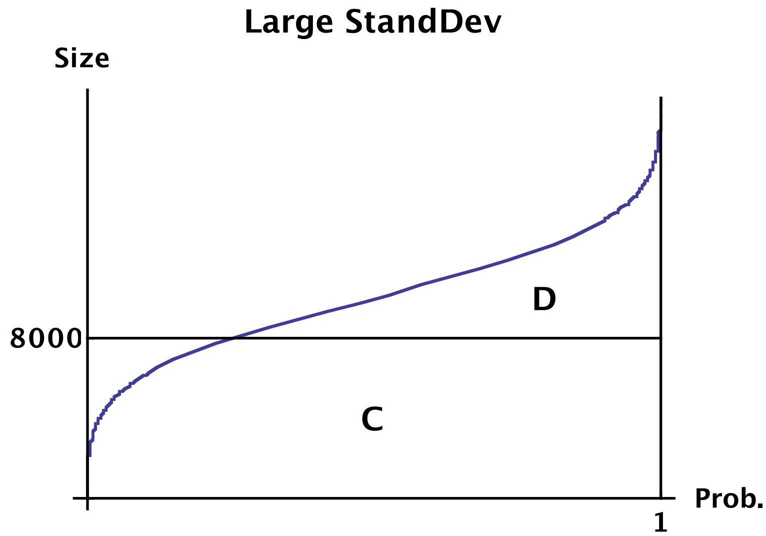
23.54. E. 1. True. (G + H) / G is the ratio of excess losses after inflation to excess losses prior to inflation. Excess losses increase at a rate greater than the overall rate of inflation.
2. False. (B + C + D + E) / (D + C) is the ratio of retained losses after inflation to retained losses prior to inflation. Retained losses increase at a rate less than the overall rate of inflation.
3. True. B + C + D + E is the retained losses after inflation.
Area C is what the retained losses would be prior to inflation at a retention of R/a.
One way to get the retained losses after inflation is to take the losses below the deflated retention (limit) and multiply by the inflation factor, in other words to take: a C.
(This is the idea behind the formula for the average payment per loss.)

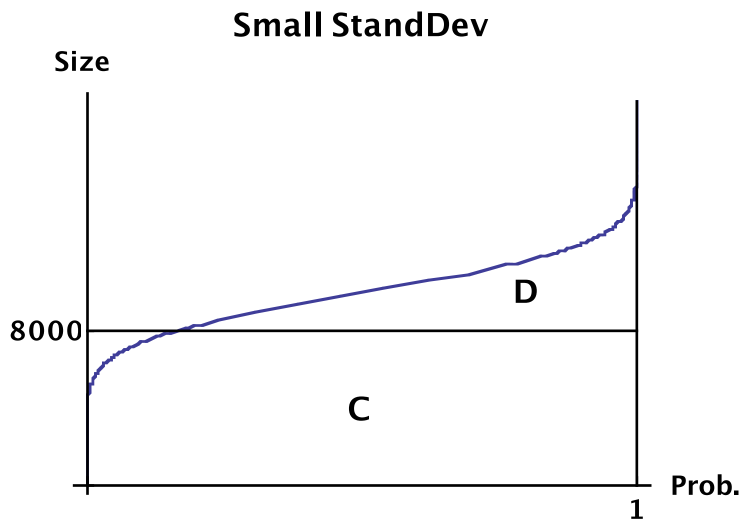
23.55. If S(x) = 1 - F(x), the survival function, then the limited expected value is:

$$E[X \wedge t] = \int_0^t S(x)\, dx.$$

Therefore, S(t) is the derivative with respect to t of the limited expected value.
Thus, S(t) = 1 - t/b.
f(t) = -S'(t) = 1/b.
Comment: A uniform distribution.

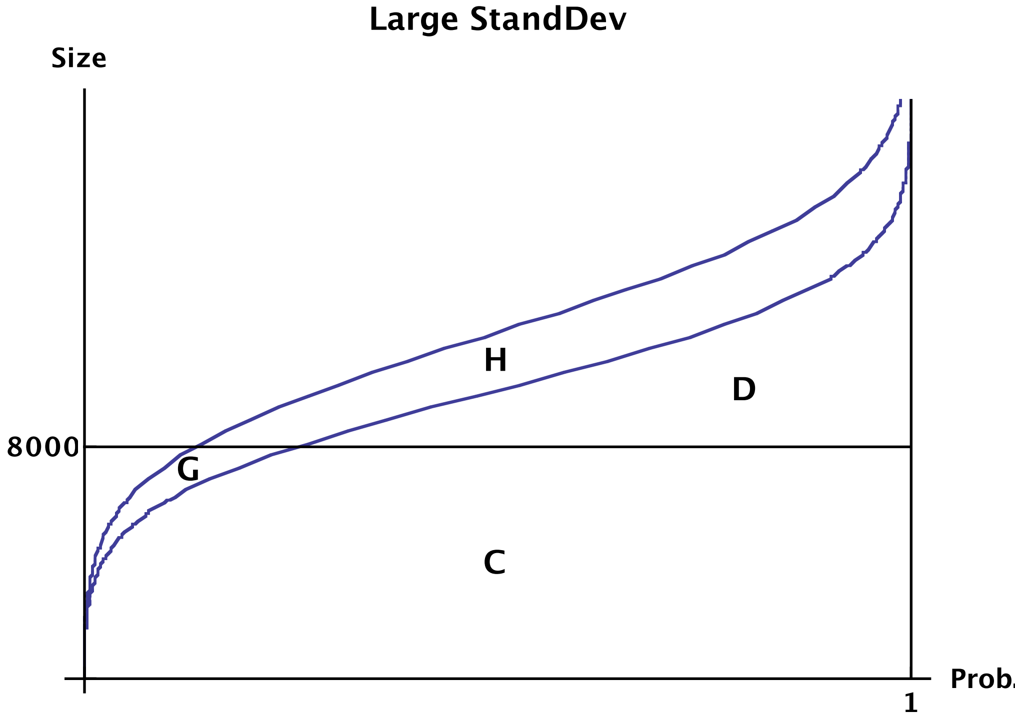
23.56. a. The large standard deviation case has more probability in both tails.
The LER is in each case: C/(C + D).

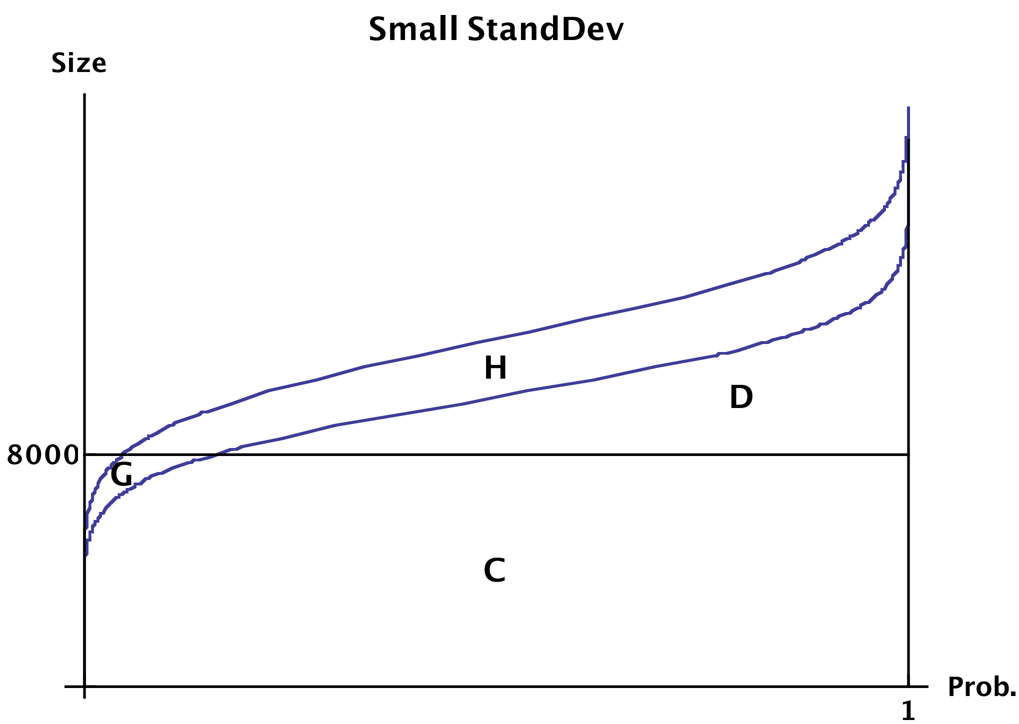




Since the distributions have the same mean, the denominators of the two LERs are equal.
For the larger standard deviation, Area C is smaller.
For the larger standard deviation, the LER is smaller.

b. Under uniform inflation, excess losses increase faster than limited losses; the LER declines.
In each case, the increase due to inflation in the losses eliminated is area G.

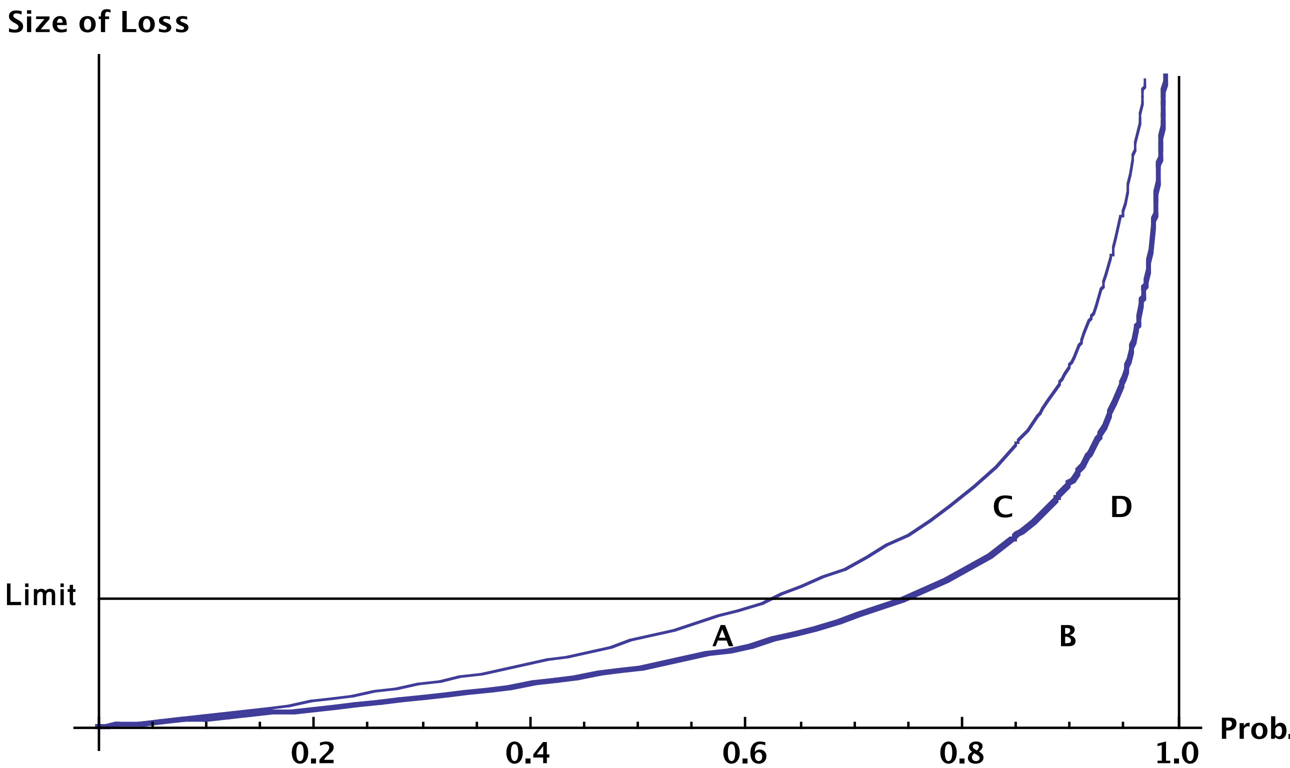




Since the distributions have the same mean, and the same inflation factor is applied to both, we can concentrate on the changes in the losses eliminated.
For the smaller standard deviation, G is smaller, than for the larger standard deviation.
With a smaller standard deviation, the LER declines more due to inflation.

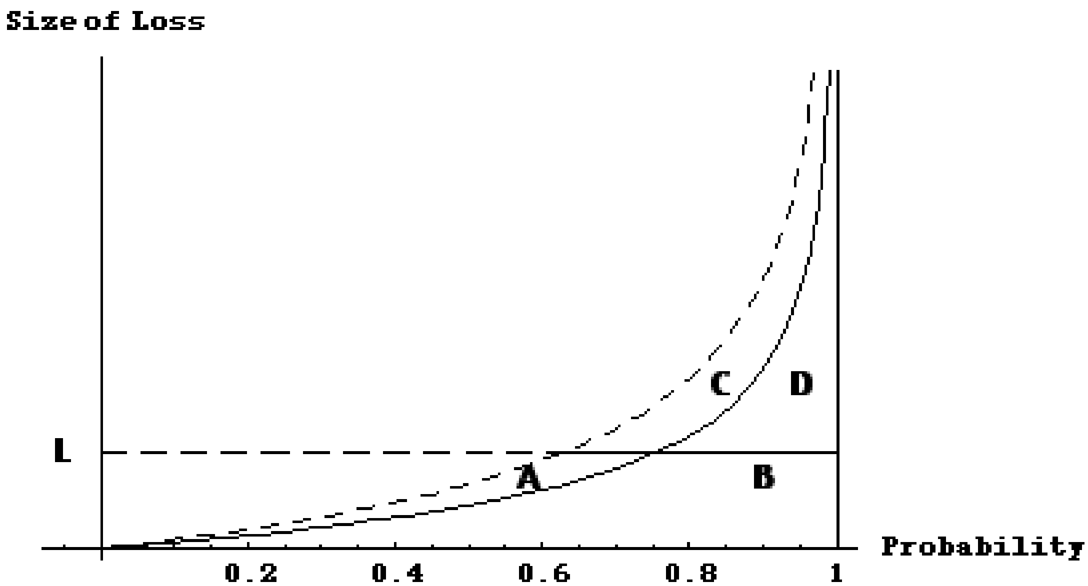
Comment: For the intuition on theoretical problems, it often helps to substitute simple numbers.
We are told that both lines have means of $10,000, but they have different standard deviations.
Let's make up some simple numbers that fit this scenario:
The line with the large standard deviation also has three claims: $5,000, $10,000, and $15,000.
The line with the small standard deviation has three claims: $9,000, $10,000, and $11,000.
With a deductible of $8,000, the LER for the large standard deviation is:
($5,000 + $8,000 + $8,000) / $30,000 = 70%.
With a deductible of $8,000, the LER for the small standard deviation is:
($8,000 + $8,000 + $8,000) / $30,000 = 80%.
With a larger standard deviation, the LER is less.
For Part (b) of the examination question, we continue with our numerical example. Suppose that inflation is 10%.
For the large standard deviation with claims of $5,000, $10,000, $15,000, the losses inflate by 10%, and the amount eliminated increases. Instead of $5,000 + $8,000 + $8,000, the amount eliminated becomes $5,500 + $8,000 + $8,000.
For the large standard deviation, the LER is now: 21,500/(5500 + 11,000 + 16,500) = 65.2%.
For the small standard deviation with claims of $9,000, $10,000, $11,000, the losses inflate by 10%, but the amount eliminated doesn't change; it is still $8,000 for each claim.
For the small standard deviation, the LER is now: 24,000/(9900 + 11,000 + 12,100) = 72.7%.
Both LERs decrease due to inflation.
For the large standard deviation, the LER went from 70% to 65.2%.
For the small standard deviation, the LER went from 80% to 72.7%.
With a smaller standard deviation, the LER is more affected by inflation.
In my graphs, the small standard deviation line is a Normal Distribution with $\mu = 10{,}000$ and $\sigma = 2000$, while the large standard deviation line has $\mu = 10{,}000$ and $\sigma = 3000$.
I took the inflation factor $\alpha = 1.2$.

23.57. The basic limits losses are below the horizontal line at height equal to the limit.
Prior to inflation the basic limits losses are below the original distribution (thick), Area B.
After inflation the basic limits losses are below the inflated distribution (thinner), Areas A + B.
The excess limits losses are above the horizontal line.
Prior to inflation the excess limits losses are below the original distribution (thick), Area D.
After inflation the excess losses are below the inflated distribution (thinner, Areas C + D.



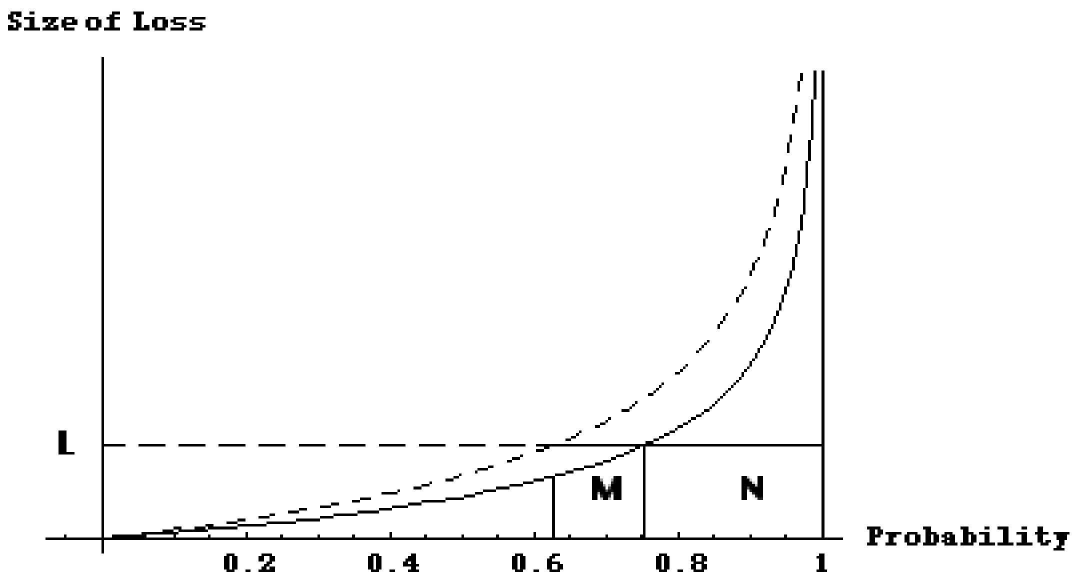
Area C represents the increase in excess losses due to inflation, while Area A represents the increase in limited losses due to inflation.
(A+B)/B = 1 + A/B = ratio of basic limit losses after inflation and before inflation < 1+ r.
(C+ D)/D = 1 + C/D = ratio of excess limit losses after inflation and before inflation > 1 + r.
(A + B + C + D) / (B + D) = 1 + (A + C) / (B + D)
= ratio of total losses after inflation and before inflation = 1 + r.

23.58. a. Some losses will hit the basic limit before they get the total increase from inflation, while some were already at the basic limit so inflation won't increase them.
For example, if the basic limit is $100,000, then a loss of $125,000 will still contribute $100,000 to the basic limit after inflation. A loss of $98,000 will increase to $103,880 after inflation, and would then contribute $100,000 to the basic limit, an increase of only 100/98 - 1 = 2.04% in that contribution, less than 6%.
b. Let L be the basic limit. The solid curve refers to the losses prior to inflation, while the dashed curve refers to the losses after inflation:



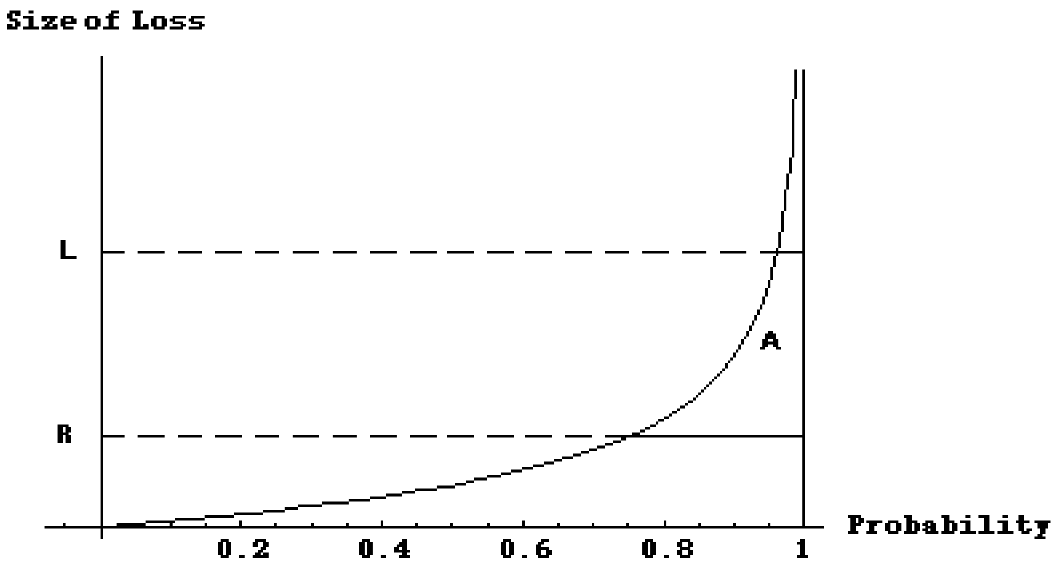
The expected excess losses prior to inflation are Area D.
The increase in the expected excess losses due to inflation is Area C.
The expected basic limit losses prior to inflation are Area B.
The increase in the expected basic limit losses due to inflation is Area A.
Area C is larger compared to Area D, than is Area A compared to Area B.
Therefore, the basic limit losses increase slower due to inflation than do the excess losses.
Since the unlimited ground up losses are the sum of the excess and basic limit losses,
the basic limit losses increase slower due to inflation than the unlimited ground up losses.

Looked at from a somewhat different point of view, those losses in Area M will have their contributions to the basic limit increase at a rate less than 6%, while those losses in Area N, will not have their contribution to the basic limit increases at all due to inflation.

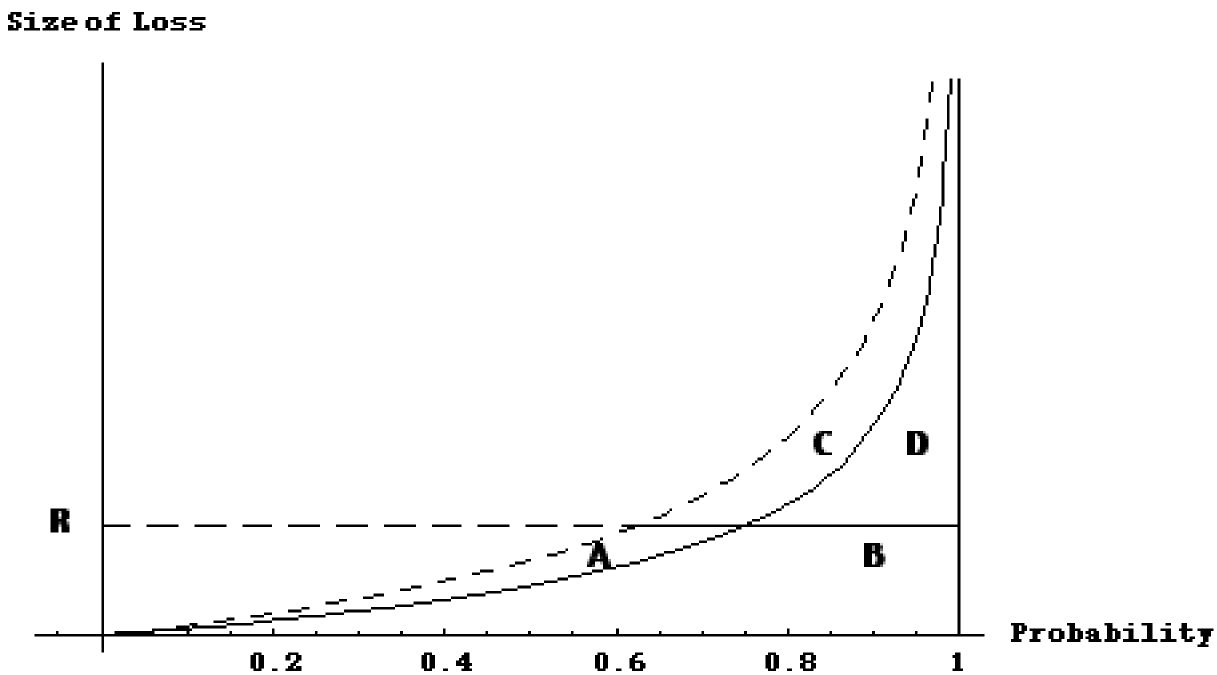


c. All losses that were already in the excess layer receive the full increase from inflation. Some losses that did not contribute to the excess layer will, after inflation, contribute something to the excess layer.
<u>Comment</u>: In order to allow one to see what is going on, the Lee Diagrams in my solution are based on much more than 6% inflation.

23.59. a. The expected losses for an excess of loss contract, covering losses in excess of retention (R), subject to a maximum limit (L) is represented by Area A, the area above the horizontal line at R, below the horizontal line at L, and below the curve for the distribution of sizes of loss.

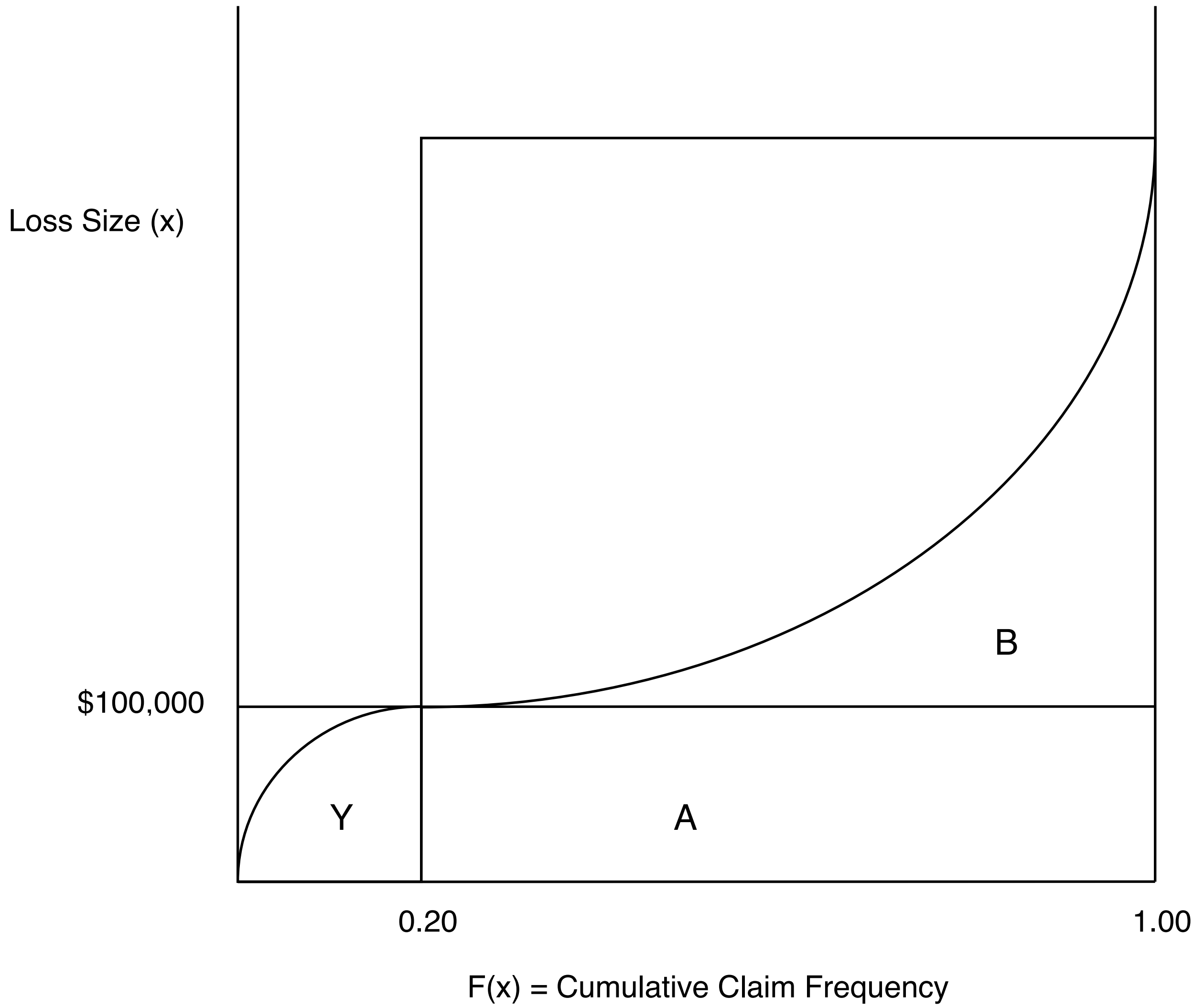


b. The solid curve is the loss distribution prior to inflation, while the dashed curve is the loss distribution after inflation.



The expected excess losses prior to inflation are Area D.
The increase in the expected excess losses due to inflation is Area C.
The expected retained losses prior to inflation are Area B.
The increase in the expected retained losses due to inflation is Area A.
Area C is larger compared to Area D, than is Area A compared to Area B.
Therefore, the excess losses increase faster due to inflation than do the retained losses.
Since the unlimited ground up losses are the sum of the excess and retained losses, the excess losses increase faster due to inflation than the unlimited ground up losses.
<u>Comment</u>: Excess losses increase faster than the overall rate of inflation.
Limited losses increase slower than the overall rate of inflation.

23.60. B. Area Y + Area A + Area B = E[x] = $152,500.
Area A is a rectangle with height $100,000 and width 0.8, with area $80,000.
Area Y is given as $12,500.
Expected losses excess of $100,000 = Area B = $152,500 - $12,500 - $80,000 = $60,000.
Excess ratio at $100,000 = (Expected losses excess of $100,000)/E[X] =
$60,000 / $152,500 = **0.393**.



Comment: Loss Elimination Ratio at $100,000 is: 1 - 0.393 = 0.607.
Not one of the usual size of loss distributions encountered in casualty actuarial work.

23.61. E. The shaded area represents the losses excess of K = $E[(X-K)_+] = E[X] - E[X \wedge K]$ =

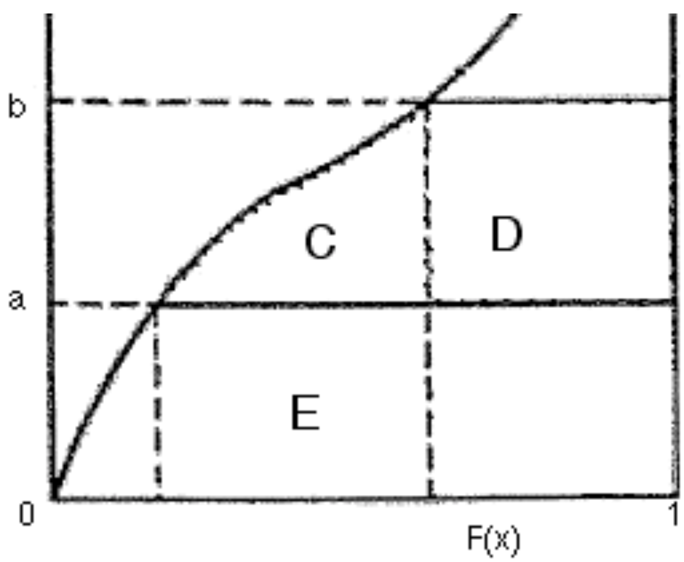
$$E[X] - \left\{ \int_0^K x\, f(x)\, dx + K S(K) \right\} = \int_K^\infty x\, f(x)\, dx - K\, S(K) = \int_K^\infty (x-K)\, f(x)\, dx = \int_K^\infty S(x)\, dx .$$

Since S(x) = 1 - F(x) and f(x) dx = dF(x), statements II and III are true.
Statement I is false; it would be true if the integrand were x-K rather than x.

23.62. B. The layer of loss from DED to LIM is the area under the curve and between the horizontal lines at DED and LIM: **Q + R**.

23.63. B. Under a franchise deductible one does not pay any losses of size less than DED, but pays the whole of any loss of size greater than DED. Due to the franchise deductible, one saves S, the area corresponding to the losses of size less than or equal to DED. Due to the limit, one pays at most LIM for any loss, so one saves P, the area representing the expected amount excess of LIM. The expected savings are: **S + P**.
Comment: Similar to CAS3, 11/03, Q.23, except here there is a franchise deductible rather than an ordinary deductible. The effect of the franchise deducible is to left truncate the data at DED, which removes area S.

23.64. D. Label some of the areas in the Lee Diagram as follows:



D = (b-a)S(b).
E = a{F(b) - F(a)}

C + E = losses on losses of size between a and b = $\int_a^b x \, dF(x)$.

Shaded Area = C + D = (C + E) + D - E =

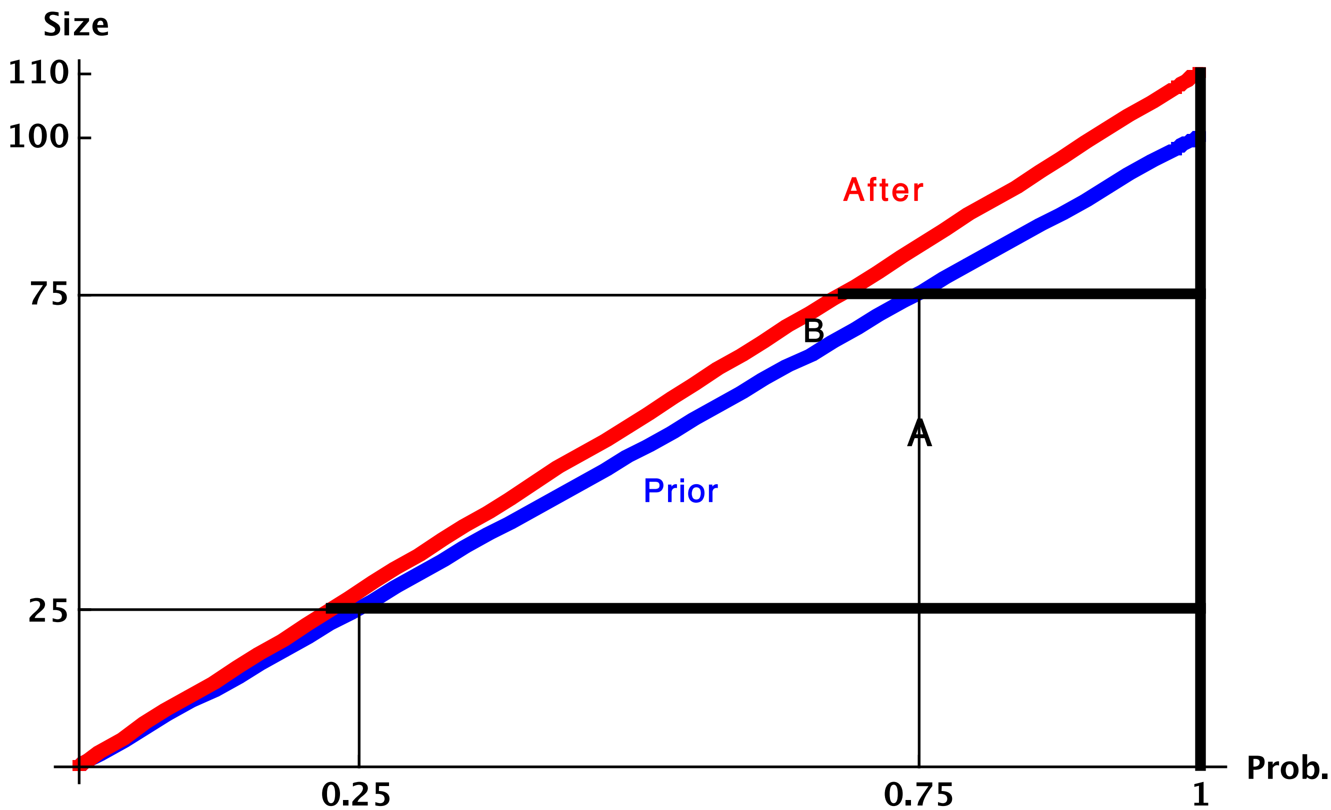
$$\int_a^b x \, dF(x) + (b-a)S(b) - a\{F(b) - F(a)\} = \int_a^b x \, dF(x) - a + b + aF(a) - bF(b).$$

Alternately, the shaded area is the layer from a to b:

$$E[X \wedge b] - E[X \wedge a] = \int_0^b x \, dF(x) + bS(b) - \int_0^a x \, dF(x) - aS(a) = \int_a^b x \, dF(x) - a + b + aF(a) - bF(b).$$

23.65. E. Deflate D from 2006 to 2005, where it is equivalent to a deductible of: D/1.1.
Then the average expected loss in 2005 is the area above the deductible, D/1.1, and under the curve: P+Q+R+S+T+U. In order to get the expected size of loss in 2006, reinflate back up to the 2006 level, by multiplying by 1.1: **1.1(P+Q+R+S+T+U)**.

23.66. Prior to inflation, the losses are uniform from 0 to 100.
On the following Lee Diagram, this is represented by the straight line from (0, 0) to (1, 100).
After inflation the losses are uniform from 0 to 110.
On the following Lee Diagram, this is represented by the straight line from (0, 0) to (1, 110).



Area A = losses in the layer prior to inflation = (height) (average width) =

$(75 - 50)\ \frac{(1 - 0.25) + (1 - 0.75)}{2} = 25.$

Area A + Area B = losses in the layer after inflation = (height) (average width) =

$(75 - 50)\ \frac{(1 - 25/110) + (1 - 75/110)}{2} = 27.27.$

⇒ Area B = 2.27. ⇒ Inflation for the layer = B / A = 2.27/25 = **9.1%**.
Alternately, one can use limited expected values. Geometrically a limited expected value is the area below the curve and also below a horizontal line at the limit.

$$E[X \wedge 75] = \int_0^{75} (x/100)\ dx + (0.25)(75) = 46.875.$$

$$E[X \wedge 25] = \int_0^{25} (x/100)\, dx + (0.75)(25) = 21.875.$$

$$E[X \wedge 75/1.1] = \int_0^{75/1.1} (x/100)\, dx + (1 - 0.75/1.1)(75/1.1) = 44.938.$$

$$E[X \wedge 25/1.1] = \int_0^{25/1.1} (x/100)\, dx + (1 - 0.25/1.1)(25/1.1) = 20.145.$$

The inflation factor for the layer is:

$$\frac{(1.1)\,(E[X \wedge 75/1.1] - E[X \wedge 25/1.1])}{E[X \wedge 75] - E[X \wedge 25]} = \frac{(1.1)\,(44.938 - 20.145)}{46.875 - 21.875} = 1.091.$$

The implied trend for the layer \$50 excess of \$25 is **9.1%**.

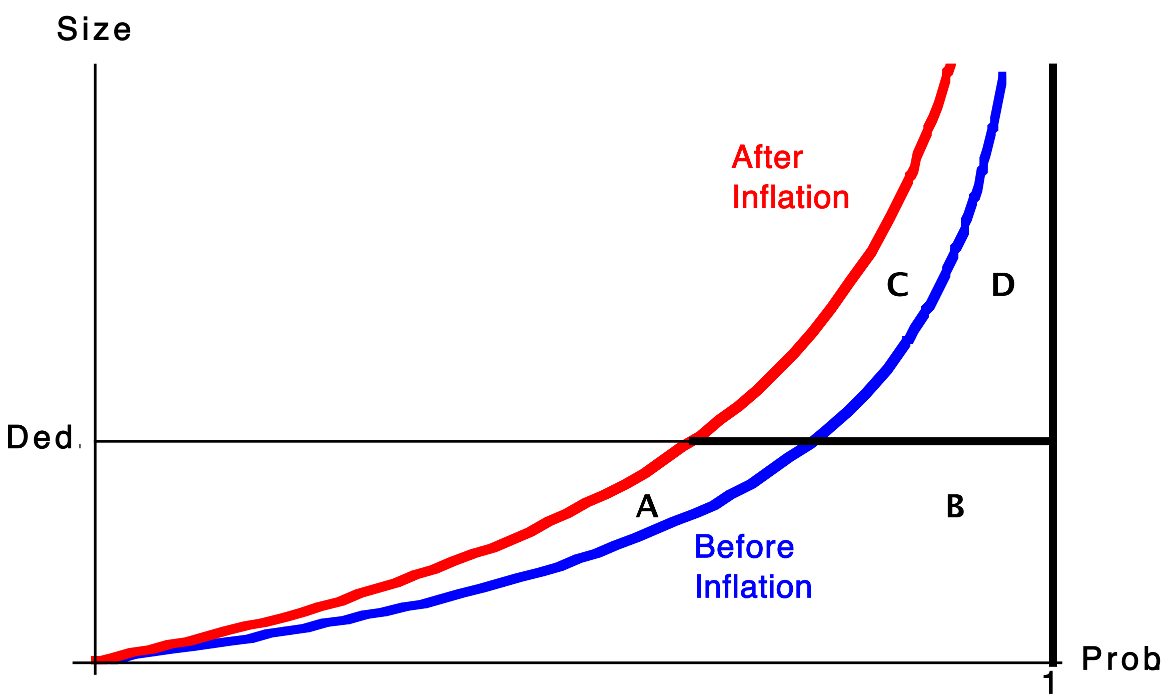
Comment: Area A is a trapezoid. Area A plus Area B is another trapezoid.
A layer like this can inflate either slower or faster than the overall rate of inflation.

Losses in layer prior to inflation: $\int_{25}^{75} (x - 25)/100\, dx + (1 - 75/100)\,(75 - 25) = 25.$

Losses in layer after to inflation: $\int_{25}^{75} (x - 25)/110\, dx + (1 - 75/110)\,(75 - 25) = 27.27.$

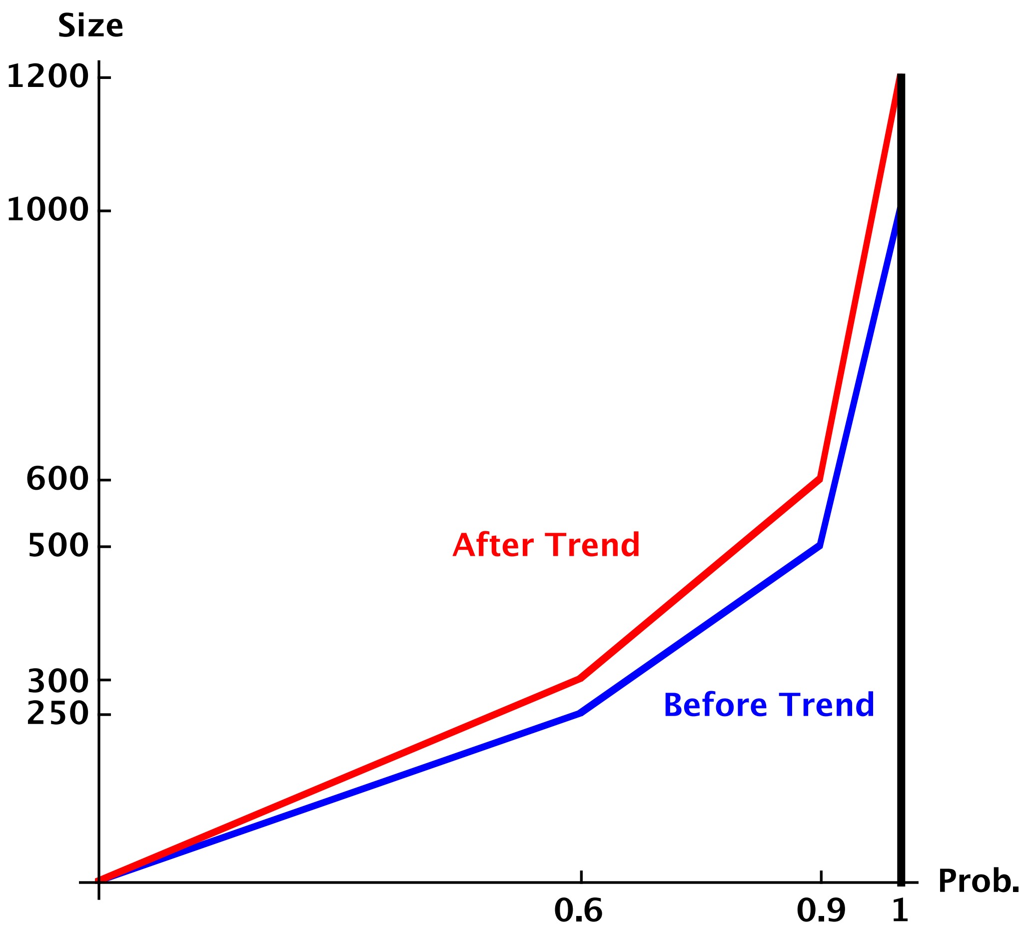
The alternate solution takes the ratio of average payments per loss, after inflation and prior to inflation.

23.67. a) Losses eliminated: (21)(100) + (50)(250) + (42 + 37 + 22)(500) = 65,100.
Total losses: (21)(100) + (50)(250) + (42)(500) + (37)(1000) + (22)(5000)= 182,600.
loss elimination ratio: 65,100 / 182,600 = **35.65%**.
Alternately, the excess losses are: (37)(1000 - 500) + (22)(5000 - 500) = 117,500.
Loss elimination ratio: 1 - 117,500 / 182,600 = **35.65%**.
b) The total losses increase by 10%: (1.1)(182,600) = 200,860.
Losses eliminated: (21)(110) + (50)(275) + (42 + 37 + 22)(500) = 66,560.
After inflation, loss elimination ratio: 66,560/200,860 = 33.14%.
Percentage change in LER is: 33.14% / 35.65% - 1 = **-7.0%**.
c) The total losses increase at the overall rate of inflation. The loss elimination ratio declines under uniform inflation with a fixed limit (as occurred in this example.) Therefore the loss cost for a given straight deductible policy increases by more than the ground-up severity trend.
The reason why the loss elimination ratio declines, is because some large claims have already had the whole deductible amount eliminated prior to inflation and no more will be eliminated after inflation.
Alternately, under uniform inflation, some claims for which the insurer paid nothing prior to inflation it will pay something for after inflation. In this example, a claim for 500 prior to inflation will pierce the covered layer after inflation. Also for large claims the payments by the insurer will increase faster than the overall rate of inflation. For example, for a $1000 claim prior to inflation the insurer paid $500, but after inflation will pay 1100 - 500 = 600, an increase of 20%. Thus in combination the loss cost for a given straight deductible policy will increase by more than the ground-up severity trend. Losses excess of a fixed limit increase faster than the overall rate of inflation!
Alternately, look at the following Lee diagram:

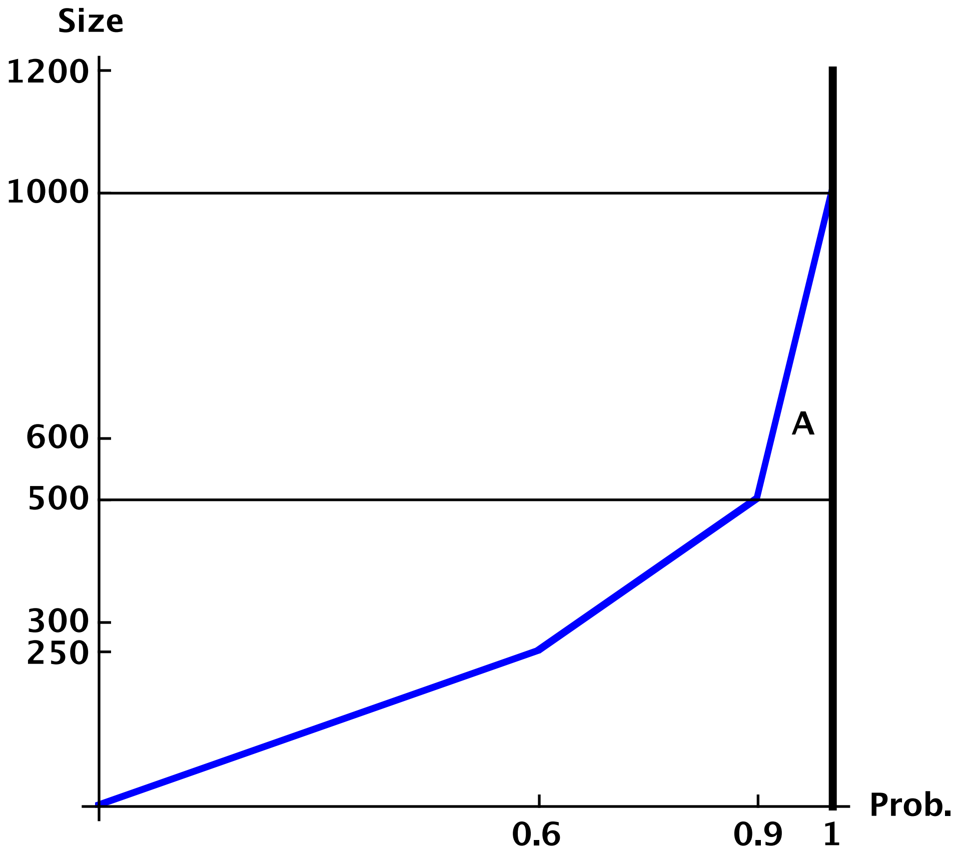


C / D is the increase in the losses paid by the insurer due to inflation.
(C + A) / (D + B) is the increase in the total losses due to inflation. C / D > (C + A) / (D + B).

23.68. (a) Subsequent to trend, we have three uniform distributions:
from 0 to 300,000; from 300,000 to 600,000; and from 600,000 to 1,200,000.
Here is a Lee Diagram, with size in thousands:

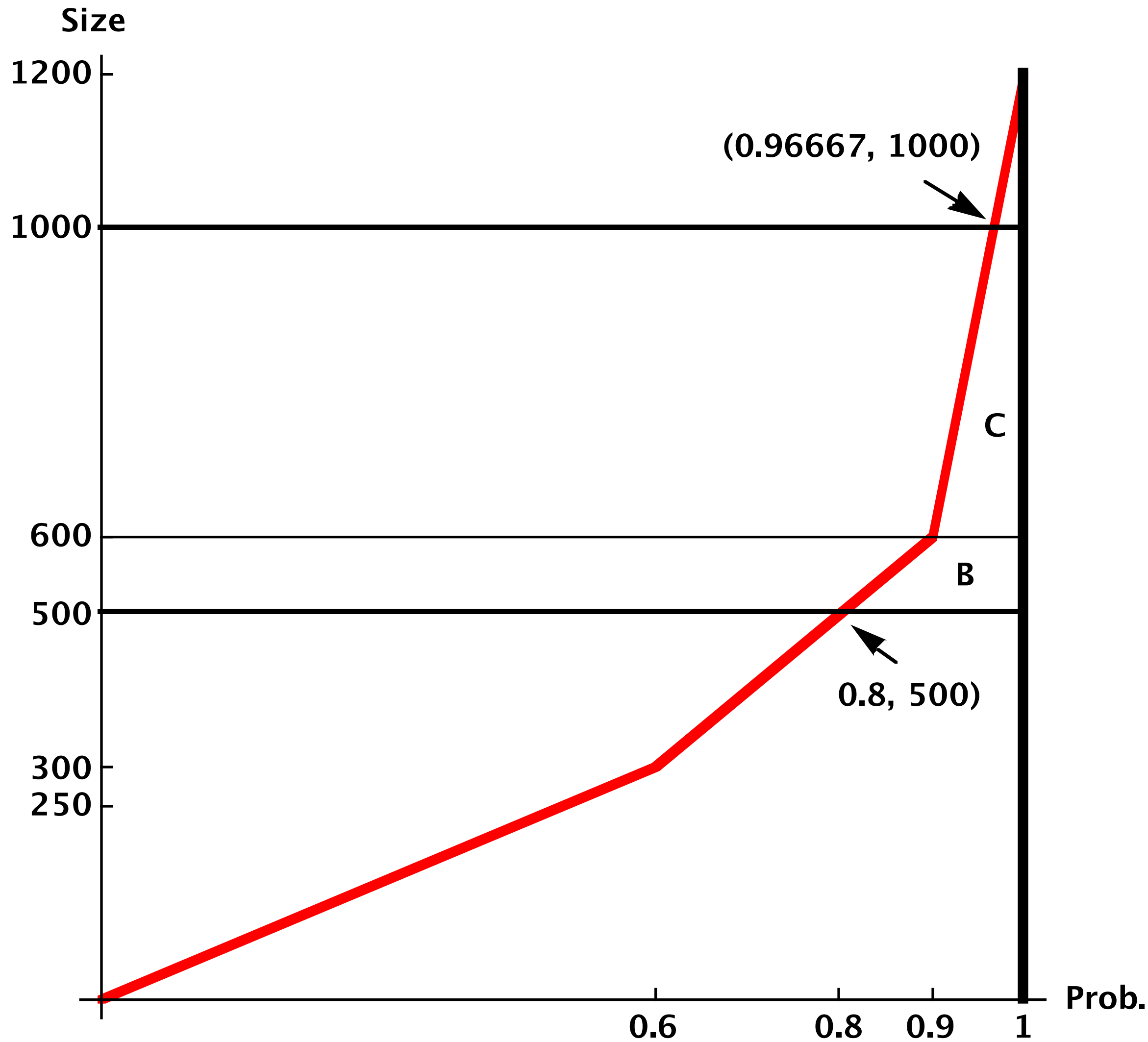


(b) Prior to trend, the layer from 500,000 to 1,000,000 is the area below the distribution and between horizontal lines at heights 500,000 and 1,000,000.
This is Area A in the following Lee Diagram.



Area A is a triangle, with base 0.1 and height 500, and thus area: (0.1)(500)/2 = 25 (thousand).

After trend the distribution function at 500 is: (1/3)(0.6) + (2/3)(0.9) = 0.8.
After trend the distribution function at 1000 is: (1/3)(0.9) + (2/3)(1) = 0.96667.
After trend the Lee Diagram is as follows, with the excess layer being the sum of Areas B and C:



Area B is a trapezoid with height 100 and widths 0.1 and 0.2, and thus area:
(100)(0.1 + 0.2)/2 = 15 (thousand).
Area C is a trapezoid with height 400 and widths 0.1 and 0.03333, and thus area:
(400)(0.1 + 0.03333)/2 = 26.667 (thousand).
Thus after trend the excess layer is: Area B + Area C = 15 + 26.667 = 41.667 (thousand).
The implied trend for the layer $500,000 excess of $500,000 is: 41.667/25 - 1 = **66.7%**.

Alternately, both prior and posterior to trend, the first interval contributes nothing to the excess layer.
Prior to trend, the second interval contributes nothing to the excess layer.
After trend, the second interval is uniform from 300,000 to 600,000.
After trend, the second interval contributes to the excess layer:

$$(1/300{,}000) \int_{500{,}000}^{600{,}000} (x - 500{,}000)\ dx = (1/300{,}000)\ (100{,}000^2 / 2) = 16{,}667.$$

Prior to trend, the third interval contributes to the excess layer:

$$(1/500{,}000) \int_{500{,}000}^{1{,}000{,}000} (x - 500{,}000)\ dx = (1/500{,}000)\ (500{,}000^2 / 2) = 250{,}000.$$

After trend, the third interval is uniform from 600,000 to 1,200,000,
and contributes to the excess layer:

$$(1/600{,}000) \int_{600{,}000}^{1{,}000{,}000} (x - 500{,}000)\ dx + (200{,}000/600{,}000)(500{,}000) =$$

$(1/600{,}000)\ (500{,}000^2 / 2 - 100{,}000^2/2) + 166{,}667 = 366{,}667.$
Thus prior to trend the expected losses in the excess layer are:
(60%)(0) + (30%)(0) + (10%)(250,000) = $25,000.
After to trend the expected losses in the excess layer are:
(60%)(0) + (30%)(16,667) + (10%)(366,667) = $41,667.
The implied trend for the layer $500,000 excess of $500,000 is: 41,667/25,000 - 1 = **66.7%**.
Alternately, prior to trend: $E[X \wedge 1 \text{ million}]$ = (0.6)(125K) + (0.3)(375K) + (0.1)(750K) = 262.5K.
$E[X \wedge 500K]$ = (0.6)(125K) + (0.3)(375K) + (0.1)(500K) = 237.5K.
$E[X \wedge 1 \text{ million}] - E[X \wedge 500K]$ = 262.5K - 237.5K = 25,000.
Subsequent to trend, we have three uniform distributions:
from 0 to 300,000; from 300,000 to 600,000; and from 600,000 to 1,200,000.
Break the last interval into 600K to 1000K and 1000K to 1200K.
$E[X \wedge 1 \text{ million}]$ = (0.6)(150K) + (0.3)(450K) + (0.1)(2/3)(800K) + (0.1)(1/3)(1000K) = 311,667.
Break the middle interval into 300K to 500K and 500K to 600K.
$E[X \wedge 500K]$ = (0.6)(150K) + (0.3)(2/3)(400K) + (0.3)(1/3)(500K) + (0.1)(500K) = 270,000.
$E[X \wedge 1 \text{ million}] - E[X \wedge 500K]$ = 311,667 - 270,000 = 41,667.
The implied trend for the layer $500,000 excess of $500,000 is: 41,667/25,000 - 1 = **66.7%**.
Comment: In part (a) it would have been helpful if the question specified whether or not they wanted a Lee Diagram.

Section 24, Continuous Mixtures of Models[175]

Mixing models is a technique that provides a greater variety of loss distributions.[176]

Discrete mixtures can be extended to a continuous case such as the Inverse Gamma - Exponential situation, to be discussed below. Instead of an n-point mixture, one can take a continuous mixture of severity distributions.

Mixture Distribution ⇔ Continuous Mixture of Models.

Review of 2-point Mixtures:[177]

For example, let A be a Pareto Distribution with parameters $\alpha = 2.5$ and $\theta = 10$,
while B is a LogNormal Distribution with parameters $\mu = 0.5$ and $\sigma = 0.8$.
Let p = 0.10, the weight for the Pareto Distribution.

If we let G(x) = pA(x) + (1-p)B(x) = 0.1 A(x) + 0.9 B(x),
then G is a Distribution Function since G(0) = 0 and G(∞) = 1.
G is a mixed “Pareto-LogNormal” Distribution.

For example, for the Pareto Distribution F(10) = 0.8232, while for the LogNormal Distribution F(10) = 0.9879. Thus for the mixed distribution, F(10) = (0.1)(0.8232) + (0.9)(0.9879) = 0.9714.

In general, one can take a weighted average of any two Distribution Functions:
G(x) = p A(x) + (1-p)B(x). Such a Distribution Function H, called a 2-point mixture of models, will generally have properties that are a mixture of those of A and B.[178]

One can create a very large number of possible combinations by choosing various types of distributions for A and B. The mixture will have a number of parameters equal to the sum of the number of parameters of the two distributions A and B, plus one more for p, the weighting parameter. The mixed Pareto-LogNormal Distribution discussed above has 2 + 2 + 1 = 5 parameters: α, θ, μ, σ, and p.

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[175] See Section 5.2.4 of Loss Models.
See also “Mahler’s Guide to Conjugate Priors”.
[176] Mixtures can be performed of either frequency distributions or loss distributions.
[177] See Loss Models Section 4.2.3, on the syllabus of Exam FAM.
[178] One could also mix together three or more distributions,

An Example of Continuous Mixture:

For example assume that each individual's future lifetime is exponentially distributed with mean $1/\lambda$, and over the population, λ is uniformly distributed over (0.05, 0.15).
$u(\lambda) = 1/0.1 = 10$, $0.05 \le \lambda \le 0.15$.

Then the probability that a person picked at random lives more than 20 years is:

$$S(20) = \int S(20; \lambda)\, u(\lambda)\, d\lambda = \int_{0.05}^{0.15} e^{-20\lambda}\ (1/0.10)\ d\lambda = (10/20)(e^{-1} - e^{-3}) = 15.9\%.$$

The density at 20 of this mixture distribution is:

$$\int f(20; \lambda)\, u(\lambda)\, d\lambda = \int_{0.05}^{0.15} \lambda e^{-20\lambda}\ 10\ d\lambda = (10)(-\lambda e^{-20\lambda}/20 - e^{-20\lambda}/400)\Big]_{\lambda = 0.05}^{\lambda=0.15} = 0.0134.$$

In general, one takes a mixture of the density functions for specific values of the parameter ζ:

$$g(x) = \int f(x; \zeta)\, \pi(\zeta)\, d\zeta.$$

via some mixing distribution $\pi(\zeta)$.

For example, in the case where the severity is Exponential and the mixing distribution of their means is Inverse Gamma, we get the Inverse Gamma - Exponential process.

Inverse Gamma-Exponential:[179]

The sizes of loss for a particular policyholder are assumed to be Exponential with mean δ.
Given δ, the distribution function of the size of loss is $1 - e^{-x/\delta}$, while the density of the size of loss distribution is $(1/\delta)e^{-x/\delta}$. The mean of this Exponential is δ and its variance is δ^2.
Note that I have used δ rather than θ, so as to not confuse the scale parameter of the Exponential with that of the Inverse Gamma which is θ.

So for example, the density of a loss being of size 8 is $(1/\delta)e^{-8/\delta}$. If $\delta = 2$ this density is: $(1/2)e^{-4} = 0.009$, while if $\delta = 20$ this density is: $(1/20)e^{-0.4} = 0.034$.

Assume that the values of δ across a portfolio of policyholders are given by an Inverse Gamma distribution with $\alpha = 6$ and $\theta = 15$, with probability density function:

$$\pi(\delta) = \frac{\theta^\alpha \ e^{-\theta/\delta}}{\delta^{\alpha+1} \ \Gamma[\alpha]} = \frac{94921.875 \ e^{-15/\delta}}{\delta^7}, \ 0 < \delta < \infty.$$[180]

Note that this distribution has a mean of: $\theta / (\alpha-1) = 15 / (6-1) = 3$.

If we have a policyholder and do not know its expected mean severity, in order to get the density of the next loss being of size 8, one would weight together the densities of having a loss of size 8 given δ, using the a priori probabilities of δ: $\pi(\delta) = 94{,}921.875 \ e^{-15/\delta} / \delta^7$, and integrating from zero to infinity:

$$g(8) = \int_0^\infty \frac{e^{-8/\delta}}{\delta} \ \pi(\delta) \ d\delta = \int_0^\infty \frac{e^{-8/\delta}}{\delta} \ \frac{94921.875 \ e^{-15/\delta}}{\delta^7} \ d\delta = 94{,}921.875 \int_0^\infty \frac{e^{-23/\delta}}{\delta^8} \ d\delta$$

$= 94{,}921.875 \ (6!) / (23^7) = 0.0201$.

Where we have used the fact that the density of the Inverse Gamma Distribution integrates to unity over its support and therefore: $\int_0^\infty \frac{e^{-\theta/x}}{x^{\alpha+1}} \ dx = \frac{\Gamma[\alpha]}{\theta^\alpha} = \frac{(\alpha-1)!}{\theta^\alpha}$.

[179] This is mathematically equivalent to the Gamma-Exponential Conjugate Prior.
See "Mahler's Guide to Conjugate Priors".

[180] The Inverse Gamma Distribution has density: $f(x) = \theta^\alpha \ e^{-\theta/x} / \{\Gamma(\alpha) \ x^{\alpha+1}\}$.
In this case, the constant in front is: $\theta^\alpha / \Gamma(\alpha) = 15^6 / \Gamma(6) = 11{,}390{,}625 / 120 = 94{,}921.875$.

More generally, if the distribution of Exponential means δ is given by an Inverse Gamma distribution $\pi(\delta) = \frac{\theta^\alpha \ e^{-\theta/\delta}}{\delta^{\alpha+1} \ \Gamma[\alpha]}$, and we compute the density of having a claim of size x by integrating from zero to infinity: $g(x) = \int_0^\infty \frac{e^{-8/\delta}}{\delta} \pi(\delta) \ d\delta = \int_0^\infty \frac{e^{-x/\delta}}{\delta} \frac{\theta^\alpha \ e^{-\theta/\delta}}{\delta^{\alpha+1} \ \Gamma[\alpha]} \ d\delta =$

$$\frac{\theta^\alpha}{\Gamma[\alpha]} \int_0^\infty \frac{e^{-(\theta + x)/\delta}}{\delta^{\alpha+2}} \ d\delta = \frac{\theta^\alpha}{\Gamma[\alpha]} \frac{\Gamma[\alpha+1]}{(\theta+x)^{\alpha+1}} = \frac{\alpha\theta^\alpha}{(\theta+x)^{\alpha+1}}.$$[181]

Thus the (prior) mixed distribution is in the form of the Pareto distribution. Note that the shape parameter and scale parameter of the mixed Pareto distribution are the same as those of the Inverse Gamma distribution. For the specific example: α = 6 and θ = 15. Thus the mixed Pareto has $g(x) = 6(15^6)(15+x)^{-7}$. $g(8) = 6(15^6)(23)^{-7} = 0.0201$, matching the previous result.

For the Inverse Gamma-Exponential the (prior) mixed distribution is always a Pareto, with α = shape parameter of the (prior) Inverse Gamma
and θ = scale parameter of the (prior) Inverse Gamma.[182]

Note that for the particular case we get a mixed Pareto distribution with parameters of α = 6 and θ = 15, which has a mean of 15/(6-1) = 3, which matches the result obtained above. Note that the formula for the mean of an Inverse Gamma and a Pareto are both θ/(α−1).

Exercise: Each insured has an Exponential severity with mean δ. The values of δ are distributed via an Inverse Gamma with parameters α = 2.3 and θ = 1200. An insured is picked at random. What is the probability that its next claim will be greater than 1000?
[Solution: The mixed distribution is a Pareto with parameters α = 2.3 and θ = 1200.
$S(1000) = \left(\frac{\theta}{\theta+x}\right)^\alpha = \left(\frac{1200}{1000+1200}\right)^{2.3} = 24.8\%$.]

Hazard Rates of Exponentials Distributed via a Gamma:[183]

If the hazard rate of the Exponential, λ, is distributed via a Gamma(α, θ), then the mean 1/λ is distributed via an Inverse Gamma(α, 1/θ), and therefore the mixed distribution is Pareto.
If the Gamma has parameters α and θ, then the mixed Pareto has parameters α and 1/θ.

[181] Both the Exponential and the Inverse Gamma have terms involving powers of $e^{-1/\delta}$ and 1/δ.
[182] See Example 5.4 in Loss Models. See also 4B, 11/93, Q.26.
[183] See for example, SOA M, 11/05, Q.17.

Moments of Mixed Distributions:

The nth moment of a mixed distribution is the mixture of the nth moments for specific values of the parameter ζ:

$E[X^n] = E_\varsigma[E[X^n \mid \varsigma]]$.

Exercise: What is the mean for a mixture of Exponentials, mixed on the mean δ?
[Solution: For a given value of δ, the mean of a Exponential Distribution is δ. We need to weight these first moments together via the density of delta, $\pi(\delta)$:
$\int \delta\ \pi(\delta)\ d\delta$ = mean of $\pi(\delta)$, the distribution of δ.]

Thus the mean of a mixture of Exponentials is the mean of the mixing distribution. This result will hold whenever the parameter being mixed is the mean, as it was in the case of the Exponential.

For the case of a mixture of Exponentials via an Inverse Gamma Distribution with parameters α and θ, the mean of the mixed distribution is that of the Inverse Gamma, $\theta/(\alpha-1)$.

Exercise: What is the Second Moment of Exponentials, mixed on the mean δ?
[Solution: For a given value of δ, the second moment of an Exponential Distribution is $2\delta^2$.
We need to weight these second moments together via the density of delta, $\pi(\delta)$:
$\int 2\ \delta^2\ \pi(\delta)\ d\delta$ = 2(second moment of $\pi(\delta)$, the distribution of δ).]

Exercise: What is the variance of Exponentials mixed on the mean δ via an Inverse Gamma Distribution, as per Loss Models, with parameters α and θ?
[Solution: The second moment of the mixed distribution is:
2(second moment of the Inverse Gamma) = $2 \frac{\theta^2}{(\alpha-1)\ (\alpha-2)}$.
The mean of the mixed distribution is the mean of the Inverse Gamma: $\theta/(\alpha-1)$.
Thus the variance of the mixed distribution is: $2 \frac{\theta^2}{(\alpha-1)\ (\alpha-2)} - \left(\frac{\theta}{\alpha-1}\right)^2 = \frac{\alpha\theta^2}{(\alpha-1)^2(\alpha-2)}$.
Comment: The mixed distribution is a Pareto and this is indeed its variance.]

Normal-Normal:

The sizes of claims a particular policyholder makes is assumed to be Normal with mean m and known fixed variance s^2.[184]

Given m, the distribution function of the size of loss is: $\Phi[\frac{x-m}{s}]$,

while the density of the size of loss distribution is: $\phi[\frac{x-m}{s}] = \frac{\exp[-\frac{(x-m)^2}{2s^2}]}{s\sqrt{2\pi}}$.

So for example if s = 3, then the probability density of a claim being of size 8 is: $\frac{\exp[-\frac{(8-m)^2}{18}]}{3\sqrt{2\pi}}$.

If m = 2 this density is: $\frac{\exp[-2]}{3\sqrt{2\pi}}$ = 0.018, while if m = 20 this density is: $\frac{\exp[-8]}{3\sqrt{2\pi}}$ = 0.000045.

Assume that the values of m are given by another Normal Distribution with mean 7 and standard deviation of 2, with probability density function:[185]

$$\pi(m) = \frac{\exp[-\frac{(m-7)^2}{8}]}{2\sqrt{2\pi}}, \quad -\infty < m < \infty.$$

Note that 7, the mean of this distribution, is the a priori mean claim severity.

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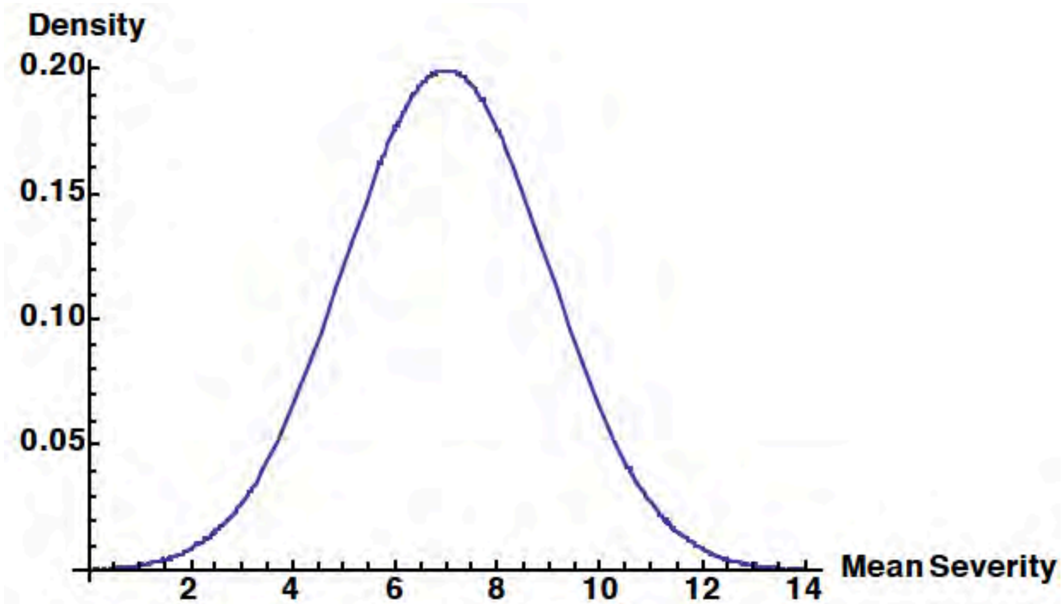
[184] Note I've used roman letter for parameters of the Normal likelihood, in order to distinguish from those of the Normal distribution of parameters discussed below.

[185] There is a very small but positive chance that the mean severity will be negative.

Below is displayed this distribution of hypothetical mean severities:[186]



If we have a risk and do not know what type it is, in order to get the chance of the next claim being of size 8, one would weight together the chances of having a claim of size 8 given m:

$\frac{\exp[-\frac{(8-m)^2}{18}]}{3\sqrt{2\pi}}$, using the a priori probabilities of m:

$\pi(m) = \frac{\exp[-\frac{(m-7)^2}{8}]}{2\sqrt{2\pi}}$, and integrating from minus infinity to infinity:

$$\int_{-\infty}^{\infty} \frac{\exp[-\frac{(8-m)^2}{18}]}{3\sqrt{2\pi}} \pi(m)\ dm = \int_{-\infty}^{\infty} \frac{\exp[-\frac{(8-m)^2}{18}]}{3\sqrt{2\pi}} \frac{\exp[-\frac{(m-7)^2}{8}]}{2\sqrt{2\pi}}\ dm =$$

$$\frac{1}{6\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\exp[-\{\frac{(8-m)^2}{18} + \frac{(m-7)^2}{8}\}]}{\sqrt{2\pi}}\ dm = \frac{1}{6\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\exp[-\{13m^2 - 190m + 697\} / 72]}{\sqrt{2\pi}}\ dm =$$

$$\frac{1}{6\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\exp[-\{m^2 - (190/13)m + (95/13)^2 + 697/13 - (95/13)^2\} / (72/13)]}{\sqrt{2\pi}}\ dm =$$

[186] Note that there is a small probability that a hypothetical mean is negative. When this situation is discussed further in "Mahler's Guide to Conjugate Priors," this will be called the prior distribution of hypothetical mean severities.

$$\frac{\exp[(-36/13^2)/(72/13)]}{6\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\exp[-(m - 95/13)^2 / \{(2)(6/\sqrt{13})^2\}]}{\sqrt{2\pi}} \, dm =$$

$$\frac{\exp[-1/26]}{6\sqrt{2\pi}} (6/\sqrt{13}) = \frac{\exp[-1/26]}{\sqrt{13}\sqrt{2\pi}} = 0.1065.$$

Where we have used the fact that a Normal Density integrates to unity:[187]

$$\int_{-\infty}^{\infty} \frac{\exp[-(m - 95/13)^2 / \{(2)(6/\sqrt{13})^2\}]}{\sqrt{2\pi}} \, dm = 6/\sqrt{13}.$$

More generally, **for the Normal-Normal, the mixed distribution is another Normal, with mean equal to that of the Normal distribution of parameters, and variance equal to the sum of the variances of the Normal distribution of parameters and the Normal likelihood**.[188]

For the specific case dealt with previously: $s = 3$, $\mu = 7$, and $\sigma = 2$, the mixed distribution has a Normal Distribution with a mean of 7 and variance of: $3^2 + 2^2 = 13$.

Thus the chance of having a claim of size x is: $\dfrac{\exp[-\frac{(x-7)^2}{26}]}{\sqrt{13}\sqrt{2\pi}}$.

For x = 8 this chance is: $\dfrac{\exp[-1/26]}{\sqrt{13}\sqrt{2\pi}} = 0.1065$.

This is the same result as calculated above.

[187] With mean of 95/13 and standard deviation of $6/\sqrt{13}$.

[188] The Expected Value of the Process Variance is the variance of the Normal Likelihood, the Variance of the Hypothetical Means is the variance of the Normal distribution of parameters, and the total variance is the variance of the mixed distribution. Thus this relationship follows from the general fact that the total variance is the sum of the EPV and VHM. See “Mahler’s Guide to Buhlmann Credibility.”

Other Mixtures:

There are many other examples of continuous mixtures of severity distributions. Here are some examples.[189] In each case the scale parameter is being mixed, with the other parameters in the severity distribution held fixed.

Severity	Mixing Distribution	Mixed Distribution
Exponential	Inverse Gamma: α, θ	Pareto: α, θ
Inverse Exponential	Gamma: α, θ	Inverse Pareto: $\tau = \alpha, \theta$
Weibull, $\tau = t$	Inverse Transformed Gamma: α, θ, $\tau = t$	Burr: α, θ, $\gamma = t$
Inverse Weibull, $\tau = t$	Transformed Gamma: α, θ, $\tau = t$	Inverse Burr: $\tau = \alpha$, θ, $\gamma = t$
Gamma, $\alpha = a$	Inverse Gamma: α, θ	Generalized Pareto: α, θ, $\tau = a$
Inverse Gamma, $\alpha = a$	Exponential: θ	Pareto: $\alpha = a$, θ
Inverse Gamma, $\alpha = a$	Gamma: α, θ	Generalized Pareto: $\alpha = a$, θ, $\tau = \alpha$
Transformed Gamma, $\alpha = a$, $\tau = t$	Inverse Transformed α, θ, $\tau = t$	Transformed Beta: α, θ, $\gamma = t$, $\tau = a$
Inverse Transformed Gamma, $\alpha = a$, $\tau = t$	Transformed Gamma: α, θ, $\tau = t$	Transformed Beta: $\alpha = a$, θ, $\gamma = t$, $\tau = \alpha$

[189] See the problems for illustrations of some of these additional examples.
Example 5.6 in Loss Models shows that mixing an Inverse Weibull via a Transformed Gamma gives an Inverse Burr.

For example, assume that the amount of an individual claim has an Inverse Gamma distribution with shape parameter α fixed and scale parameter q (rather than θ to avoid later confusion.) The parameter q is distributed via an Exponential Distribution with mean μ.
For the Inverse Gamma, $f(x \mid q) = q^\alpha e^{-q/x} / \{\Gamma[\alpha]\, x^{\alpha+1}\}$. For the Exponential, $\pi(q) = e^{-q/\mu} / \mu$.

$$f(x) = \int f(x \mid q)\, \pi(q)\, dq = \int_0^\infty \frac{q^\alpha e^{-q/x}}{\Gamma[\alpha]\, x^{\alpha+1}} \frac{e^{-q/\mu}}{\mu} dq = \int_0^\infty q^\alpha e^{-q(1/x + 1/\mu)}\, dq \;/ \{\mu\, \Gamma[a]\, x^{\alpha+1}\}$$

$= \{\Gamma[\alpha+1]/(1/x + 1/\mu)^{\alpha+1}\} / (\{\mu\, \Gamma[\alpha]\, x^{\alpha+1}\}) = \alpha \mu^\alpha/(x + \mu)^{\alpha+1}$.
This is the density of a Pareto Distribution with parameters α and $\theta = \mu$.
This is an example of an Exponential-Inverse Gamma, an Inverse Gamma Severity with shape parameter α, with its scale parameter mixed via an Exponential.[190]
The mixture is a Pareto Distribution, with shape parameter equal to that of the Inverse Gamma severity, and scale parameter equal to the mean of the Exponential mixing distribution.

<u>Exercise</u>: The severity for each insured is an Inverse Gamma Distribution with parameters $\alpha = 3$ and q. Over the portfolio, q varies via an Exponential Distribution with mean 500. What is the severity distribution for the portfolio as a whole?
[Solution: The mixed distribution is a Pareto Distribution with parameters $\alpha = 3, \theta = 500$.]

<u>Exercise</u>: In the previous exercise, what is the probability that a claim picked at random will be greater than 400?
[Solution: $S(400) = \{500/(400 + 500)\}^3 = 17.1\%$.]

<u>Exercise</u>: In the previous exercise, what is the expected size of a claim picked at random?
[Solution: Mean of a Pareto with $\alpha = 3$ and $\theta = 500$ is: 500/(3 - 1) = 250.
Alternately, the mean of each Inverse Gamma is: $E[X \mid q] = q/(3 - 1) = q/2$.
$E[X] = E_q[E[X \mid q]] = E_q[q/2] = E_q[q]/2$ = (mean of the Exponential Dist.)/2 = 500/ 2 = 250.]

<u>Exercise</u>: The severity for each insured is a Transformed Gamma Distribution with parameters $\alpha = 3.9$, q, and $\tau = 5$. Over the portfolio, q varies via an Inverse Transformed Gamma Distribution with parameters $\alpha = 2.4$, $\theta = 17$, and $\tau = 5$.
What is the severity distribution for the portfolio as a whole?
[Solution: Using the above chart, the mixed distribution is a Transformed Beta Distribution with parameters $\alpha = 2.4, \theta = 17$, $\gamma = 5$, and $\tau = 3.9$.]

[190] This differs from the more common Inverse Gamma-Exponential discussed previously, in which we have an Exponential severity, whose mean is mixed via the Inverse Gamma.

Frailty Models:[191]

Frailty models are most commonly applied in the context of survival models, but mathematically they are just examples of continuous mixtures. They involve a particular form of the hazard rate.

Recall that the hazard rate, h(x) = f(x) / S(x). Also S(x) = exp[-H(x)], where $H(x) = \int_0^x h(t)\, dt$.

Assume $h(x \mid \lambda) = \lambda\, a(x)$, where λ is a parameter which varies across the portfolio.[192]
a(x) is some function of x, and let $A(x) = \int a(x)\, dx$.

Then $H(x \mid \lambda) = \lambda A(x)$.
$S(x \mid \lambda) = \exp[-\lambda A(x)]$.
$S(x) = E_\lambda[S(x \mid \lambda)] = E_\lambda[\exp[-\lambda A(x)]] = M_\lambda[-A(x)]$,
where M_λ is the moment generating function of the distribution of λ.[193] [194]

For an Exponential Distribution, the hazard rate is constant and equal to one over the mean.
Thus if each individual has an Exponential Distribution, a(x) = 1, and λ = 1/θ.
A(x) = x, and $S(x) = M_\lambda[-x]$.

We have already discussed mixtures like this. For example, λ could be distributed uniformly from 0 to 2.[195] In that case, the general mathematical structure does not help very much.

However, let us assume each individual is Exponential and that λ is Gamma Distributed with parameters α and β. The Gamma has moment generating function $M(t) = (1 - \beta t)^{-\alpha}$.[196]
Therefore, $S(x) = (1 + \beta x)^{-\alpha}$. This is a Pareto Distribution with parameters α and θ = 1/β.[197]

This is mathematically equivalent to the Gamma-Exponential discussed previously.
If λ is Gamma Distributed with parameters α and β, then the means of the Exponentials, δ = 1/λ are distributed via an Inverse Gamma with parameters α and 1/β.
The mixed distribution is Pareto with parameters α and θ = 1/β.

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191 Section 5.2.5 in Loss Models.
192 An individual with lambda larger than average would have a higher than average hazard rate. Lambda is sometimes called the frailty random variable, while the distribution of lambda is called the frailty distribution.
193 The definition of the moment generating function is $M_y(t) = E_y[\exp[yt]]$.
See "Mahler's Guide to Aggregate Distributions."
194 The survival function of the mixture is the mixture of the survival functions.
195 See 3, 5/01, Q.28.
196 See Appendix A in the tables attached to the exam.
197 For the Pareto, $S(x) = (1 + x/\theta)^{-\alpha}$.

Exercise: What is the hazard rate for a Weibull Distribution?
[Solution: $h(x) = f(x)/S(x) = \{\tau(x/\theta)^\tau \exp(-(x/\theta)^\tau) / x\} / \exp(-(x/\theta)^\tau) = \tau x^{\tau-1} \theta^{-\tau}$.]

Therefore, we can put the Weibull for fixed τ into the form of a frailty model; $h(x) = \lambda\, a(x)$, by taking $a(x) = \tau x^{\tau-1}$ and $\lambda = \theta^{-\tau}$. Then, $A(x) = x^\tau$.

Therefore, if each insured is Weibull with fixed τ, with $\lambda = \theta^{-\tau}$, then $S(x) = M_\lambda[-A(x)] = M_\lambda[-x^\tau]$.

Exercise: Each insured has a Weibull Distribution with τ fixed. $\lambda = \theta^{-\tau}$ is Gamma distributed with parameters α and β. What is the form of the mixed distribution?[198]
[Solution: The Gamma has moment generating function $M(t) = (1 - \beta t)^{-\alpha}$.[199]
Therefore, $S(x) = (1 + \beta x^\tau)^{-\alpha}$. This is a Burr Distribution with parameters α, $\theta = 1/\beta^{1/\tau}$, and $\gamma = \tau$.
Comment: The Burr Distribution has $S(x) = (1 + (x/\theta)^\gamma)^{-\alpha}$.]

If in this case $\alpha = 1$, then λ has an Exponential Distribution, and the mixed distribution is a Loglogistic, a special case of the Burr for $\alpha = 1$.[200] If instead $\tau = 1$, then as discussed previously the mixed distribution is a Pareto, a special case of the Burr.

In general for a frailty model, $f(x \mid \lambda) = -\frac{dS(x \mid \lambda)}{dx} = \lambda\, a(x) \exp[-\lambda A(x)]$.

Therefore, $f(x) = E_\lambda[\lambda\, a(x) \exp[-\lambda A(x)]] = a(x)\, E_\lambda[\lambda \exp[-\lambda A(x)]] = a(x)\, M_\lambda'[-A(x)]$, *where* M_λ' *is the derivative of the moment generating function of the distribution of* λ.[201] [202]

For example, in the previous exercise, $M(t) = (1 - \beta t)^{-\alpha}$, *and* $M'(t) = \alpha\beta\, (1 - \beta t)^{-(\alpha+1)}$.
$f(x) = \alpha(x)\, M_\lambda'[-A(x)] = \tau\, x^{\tau-1}\, \alpha\beta\, (1 - \beta x^t)^{-(\alpha+1)}$.
The density of a Burr Distribution is: $(\alpha\gamma\, \theta^{-\gamma} x^{\gamma-1})(1 + (x/\theta)^\gamma)^{-(\alpha+1)}$.
This is indeed the density of a Burr Distribution with parameters α, $\theta = 1/\beta^{1/\tau}$, *and* $\gamma = \tau$.

For a frailty model, $h(x) = f(x)/S(x) = a(x)\, M_\lambda'[-A(x)] / M_\lambda[-A(x)] = a(x)\, \frac{d\, \ln(M_\lambda[-A(x)])}{d\lambda}$.[203]

Defining the cumulant generating function as $\psi_X(t) = \ln M_X(t) = \ln E[e^{\lambda x}]$, *then*
$h(x) = a(x)\, \psi_l'(-A(x))$, *where* ψ' *is the derivative of the cumulant generating function.*

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198 See Example 5.7 in Loss Models. This result is mathematically equivalent to mixing the scale parameter of a Weibull via an Inverse Transformed Gamma, resulting in a Burr, one of the examples listed previously.
199 See Appendix A in the tables attached to the exam.
200 See Exercise 5.13 in Loss Models.
201 $E_y[y \exp[yt]] = M_y'(t)$. See "Mahler's Guide to Aggregate Distributions."
202 The density function of the mixture is the mixture of the density functions.
203 The hazard rate of the mixture is not the mixture of the hazard rates.

For example in the previous exercise, $M(t) = (1 - \beta t)^{-\alpha}$*, and* $\psi(t) = \ln M(t) = -\alpha \ln(1 - \beta t)$.
$\psi'(t) = \alpha\beta/(1 - \beta t)$. $h(x) = a(x)\,\psi_I'(-A(x)) = \tau x^{\tau-1}\alpha\beta/(1 + \beta x^{t})$.

Exercise: What is the hazard rate for a Burr Distribution?
[Solution: $h(x) = f(x)/S(x) = \{(\alpha\gamma\theta^{-\gamma}x^{\gamma-1})(1 + (x/\theta)^{\gamma})^{-(\alpha+1)}\}/\{1 + (x/\theta)^{\gamma}\}^{-\alpha} =$
$(\alpha\gamma\theta^{-\gamma}x^{\gamma-1})/(1 + (x/\theta)^{\gamma})$.*]*

Thus the above h(x) is indeed the hazard rate of a Burr Distribution with parameters α,
$\theta = 1/\beta^{1/\tau}$*, and* $\gamma = \tau$.

<u>Problems</u>:

Use the following information for the next 3 questions:
Assume that the size of claims for an individual insured is given by an Exponential Distribution: $f(x) = e^{-x/\delta}/\delta$, with mean δ and variance δ^2.
Also assume that the parameter δ varies for the different insureds, with δ following an Inverse Gamma distribution: $g(\delta) = \dfrac{\theta^\alpha \ e^{-\theta/\delta}}{\delta^{\alpha+1} \ \Gamma[\alpha]}$, for $0 < \delta < \infty$.

24.1 (2 points) An insured is picked at random and is observed until it has a loss.
What is the probability density function at 400 for the size of this loss?

A. $\dfrac{1}{(\theta + 400)^\alpha}$ B. $\dfrac{\theta^\alpha}{(\theta + 400)^\alpha}$ C. $\dfrac{\alpha\ \theta^\alpha}{(\theta + 400)^\alpha}$ D. $\dfrac{\theta^\alpha}{(\theta + 400)^{\alpha+1}}$ E. $\dfrac{\alpha\ \theta^\alpha}{(\theta + 400)^{\alpha+1}}$

24.2 (2 points) What is the unconditional mean severity?

A. $\dfrac{\theta}{\alpha - 1}$ B. $\dfrac{\theta}{\alpha}$ C. $\dfrac{\alpha - 1}{\theta}$ D. $\dfrac{\alpha}{\theta}$ E. None of A, B, C, or D

24.3 (3 points) What is the unconditional variance?

A. $\dfrac{\theta^2}{\alpha - 1}$ B. $\dfrac{2\ \alpha\ \theta^2}{\alpha - 1}$ C. $\dfrac{\alpha\ \theta^2}{(\alpha - 1)\ (\alpha - 2)}$ D. $\dfrac{\alpha\ \theta^2}{(\alpha - 1)^2\ (\alpha - 2)}$ E. $\dfrac{2\ \alpha\ \theta^2}{(\alpha - 1)^2\ (\alpha - 2)}$

24.4 (3 points) The severity distribution of each risk in a portfolio is given by a Weibull Distribution, with parameters $\tau = 1/3$ and θ, with θ varying over the portfolio via an Inverse Transformed Gamma Distribution: $g(\theta) = \dfrac{(7^{2.5}/3)\ \exp[-7/\theta^{1/3}]}{\theta^{11/6}\ \Gamma(2.5)}$. What is the mixed distribution?

A. Burr B. Generalized Pareto C. Inverse Burr
D. LogLogistic E. ParaLogistic

<u>**24.5**</u> (3 points) You are given the following:
- The amount of an individual claim, X, follows an exponential distribution function with probability density function: $f(x \mid \lambda) = \lambda\ e^{-\lambda x}$, x, $\lambda > 0$.
- The parameter λ, follows a Gamma distribution with probability density function $\pi(\lambda) = (4/3)\ \lambda^4\ e^{-2\lambda}$, $\lambda > 0$.

Determine the unconditional probability that x > 7.
A. 0.00038 B. 0.00042 C. 0.00046 D. 0.00050 E. 0.00054

<u>24.6</u> (2 points) Consider the following frailty model:
- $h(x \mid \lambda) = \lambda\, a(x)$.
- $a(x) = 4\, x^3$.
- Λ follows an Exponential Distribution with mean 0.007.

Determine S(6).
A. 7% B. 8% C. 9% D. 10% E. 11%

<u>24.7</u> (3 points)
The future lifetimes of a certain population consisting of 1000 people is modeled as follows:
(i) Each individual's future lifetime is exponentially distributed with constant hazard rate λ.
(ii) Over the population, λ is uniformly distributed over (0.01, 0.11).
For this population, all of whom are alive at time 0, calculate the number of deaths expected between times 3 and 5.
(A) 75 (B) 80 (C) 85 (D) 90 (E) 95

24.8 (2 points) You are given the following:
- The number of miles that an individual car is driven during a year is given by an Exponential Distribution with mean μ.
- μ differs between cars.
- μ is distributed via an Inverse Gamma Distribution with parameters $\alpha = 3$ and $\theta = 25{,}000$.

What is the probability that a car chosen at random will be driven more than 20,000 miles during the next year?
(A) 9% (B) 11% (C) 13% (D) 15% (E) 17%

24.9 (2 points) You are given the following:
- The IQs of actuaries are normally distributed with mean 135 and standard deviation 10.
- Each actuary's score on an IQ test is normally distributed around his true IQ, with standard deviation of 15.

What is the probability that Abbie the actuary scores between 145 and 155 on his IQ test?
A. 10% B. 12% C. 14% D. 16% E. 18%

<u>24.10</u> (2 points) Consider the following frailty model:
- $S_{X|\Lambda}(x \mid \lambda) = e^{-\lambda x}$.
- Λ follows a Gamma Distribution with $\alpha = 3$ and $\theta = 0.01$.

Determine S(250).
A. Less than 3%
B. At least 3%, but less than 4%
C. At least 4%, but less than 5%
D. At least 5%, but less than 6%
E. At least 6%

24.11 (3 points) The severity distribution of each risk in a portfolio is given by an Inverse Weibull Distribution, $F(x) = \exp[-(q/x)^4]$, with q varying over the portfolio via a Transformed Gamma Distribution with parameters $\alpha = 1.3$, $\theta = 11$, and $\tau = 4$.
What is the probability that the next loss will be of size less than 10?

Hint: For a Transformed Gamma Distribution, $f(x) = \dfrac{\tau x^{\tau\alpha-1} \exp\left[-\left(\frac{x}{\theta}\right)^{\tau}\right]}{\theta^{\tau\alpha}\,\Gamma(\alpha)}$.

A. Less than 32%
B. At least 32%, but less than 34%
C. At least 34%, but less than 36%
D. At least 36%, but less than 38%
E. At least 38%

24.12 (3 points) You are given the following:
- The amount of an individual loss in 2002, follows an exponential distribution with mean $3000.
- Between 2002 and 2007, losses will be multiplied by an inflation factor.
- You are uncertain of what the inflation factor between 2002 and 2007 will be, but you estimate that it will be a random draw from an Inverse Gamma Distribution with parameters $\alpha = 4$ and $\theta = 3.5$.

Estimate the probability that a loss in 2007 exceeds $5500.
A. Less than 18%
B. At least 18%, but less than 19%
C. At least 19%, but less than 20%
D. At least 20%, but less than 21%
E. At least 21%

Use the information on the following frailty model for the next two questions:
- Each insured has a survival function that is Exponential with hazard rate λ.
- The hazard rate varies across the portfolio via an Inverse Gaussian Distribution with $\mu = 0.015$ and $\theta = 0.005$.

24.13 (3 points) Determine S(65).
A. 56% B. 58% C. 60% D. 62% E. 64%

24.14 (2 points) For the mixture what is the hazard rate at 40?
A. 0.0060 B. 0.0070 C. 0.0080 D. 0.0090 E. 0.0100

24.15 (4 points) You are given the following:
- The amount of an individual claim has an Inverse Gamma distribution with shape parameter $\alpha = 4$ and scale parameter q (rather than θ to avoid later confusion.)
- The parameter q is distributed via an Exponential Distribution with mean 100.

What is the probability that a claim picked at random will be of size greater than 15?
A. Less than 50%
B. At least 50%, but less than 55%
C. At least 55%, but less than 60%
D. At least 60%, but less than 65%
E. At least 65%

24.16 (3 points) You are given the following:
- X is a Normal Distribution with mean zero and variance v.
- v is distributed via an Inverse Gamma Distribution with $\alpha = 10$ and $\theta = 10$.

Determine the form of the mixed distribution.

24.17 (2 points) You are given the following:
- The amount of an individual claim has an exponential distribution given by:
  $p(y) = (1/\delta)\, e^{-y/\delta}$, $y > 0$, $\delta > 0$
- The parameter δ has a probability density function given by:
  $f(\delta) = (4000/\delta^4)\, e^{-20/\delta}$, $\delta > 0$

Determine the variance of the claim severity distribution.
A. 150 B. 200 C. 250 D. 300 E. 350

24.18 (3 points) Consider the following frailty model:
- $h(x \mid \lambda) = \lambda\, a(x)$.
- $a(x) = \dfrac{0.004}{\sqrt{1 + 0.008x}}$.
- Λ follows a Gamma Distribution with $\alpha = 6$ and $\theta = 1$.

Determine S(11).
A. 70% B. 72% C. 74% D. 76% E. 78%

24.19 (3 points) You are given the following:
- The amount of an individual loss this year, follows an Exponential Distribution with mean $8000.
- Between this year and next year, losses will be multiplied by an inflation factor.
- The inflation factor follows an Inverse Gamma Distribution with parameters $\alpha = 2.5$ and $\theta = 1.6$.

Estimate the probability that a loss next year exceeds $10,000.
A. Less than 21%
B. At least 21%, but less than 22%
C. At least 22%, but less than 23%
D. At least 23%, but less than 24%
E. At least 24%

24.20 (2 points) Severity is LogNormal with parameters μ and 0.3.
μ varies across the portfolio via a Normal Distribution with parameters 5 and 0.4.
What is probability that a loss chosen at random exceeds 200?
(A) 25% (B) 27% (C) 29% (D) 31% (E) 33%

Use the following information for the next two questions:
For each class, sizes of loss are Exponential with mean μ.
Across a group of classes μ varies via an Inverse Gamma Distribution with parameters $\alpha = 3$ and $\theta = 1000$.

24.21 (2 points) For a class picked at random, what is the expected value of the loss elimination ratio at 500?
A. 50% B. 55% C. 60% D. 65% E. 70%

24.22 (6 points) What is the correlation across classes of the loss elimination ratio at 500 and the loss elimination ratio at 200?
A. Less than 96%
B. At least 96%, but less than 97%
C. At least 97%, but less than 98%
D. At least 98%, but less than 99%
E. At least 99%

24.23 (3 points) Severity is uniformly distributed from 0 to $\sqrt{\beta}$.
β in turn is uniformly distributed from 10 to 50.
Determine the variance of the mixed distribution.
A. 2.2 B. 2.4 C. 2.6 D. 2.8 E. 3.0

24.24 (3 points) Consider the following frailty model:
- $h(x \mid \lambda) = \lambda a(x)$.
- $a(x) = 1.09^x$.
- Λ follows a Gamma Distribution with $\alpha = 4$ and $\theta = 1/50{,}000$.

Determine S(90).
A. 14% B. 16% C. 18% D. 20% E. 22%

24.25 (2 points) A randomly chosen transformer has a lifetime in hours that is normally distributed with a mean of α and a standard deviation of β.
While β is fixed, α is normally distributed with a mean of 6000 hours and a standard deviation of σ.
You are given that the 90th percentile of the lifetime of a randomly chosen transformer is 9200 hours.
Find the probability that a randomly selected transformer has a lifetime of at most 5000 hours.
A. 29% B. 32% C. 35% D. 38% E. 41%

24.26 (5 points) Severity is LogNormal with parameters m and v.
m varies across the portfolio via a Normal Distribution with parameters μ and σ.
Determine the mean severity of this mixture.
Hint: Since the density of the Normal Distribution integrates to one

$$\int_{-\infty}^{\infty} \exp[-\frac{(x-a)^2}{2b^2}]\ dx = \sqrt{2\pi}\ b. \Rightarrow \int_{-\infty}^{\infty} \exp[-\frac{(x^2-2ax)}{2b^2}]\ dx = \sqrt{2\pi}\ b \exp[\frac{a^2}{2b^2}].$$

24.27 (4B, 5/93, Q.19) (2 points) You are given the following:

- The amount of an individual claim has an exponential distribution given by:
  $p(y) = (1/\mu)\ e^{-y/\mu}$, $y > 0$, $\mu > 0$.
- The parameter m has a probability density function given by: $f(\mu) = (400/\mu^3)e^{-20/\mu}$, $\mu > 0$.

Determine the mean of the claim severity distribution.
A. 10 B. 20 C. 200 D. 2000 E. 4000

24.28 (4B, 11/93, Q.26) (3 points) You are given the following:

- The amount of an individual claim, Y, follows an exponential distribution function with probability density function $f(y \mid \delta) = (1/\delta)\ e^{-y/\delta}$, $y, \delta > 0$.
- The conditional mean and variance of Y given δ are $E[Y \mid \delta] = \delta$ and $Var[Y \mid \delta] = \delta^2$.
- The mean claim amount, δ, follows an Inverse Gamma distribution with density function $p(\delta) = 4e^{-2/\delta} / \delta^4$, $\delta > 0$.

Determine the unconditional density of Y at y = 3.
A. Less than 0.01
B. At least 0.01, but less than 0.02
C. At least 0.02, but less than 0.04
D. At least 0.04, but less than 0.08
E. At least 0.08

24.29 (3, 5/00, Q.17) (2.5 points)
The future lifetimes of a certain population can be modeled as follows:
(i) Each individual's future lifetime is exponentially distributed with constant hazard rate θ.
(ii) Over the population, θ is uniformly distributed over (1, 11).
Calculate the probability of surviving to time 0.5, for an individual randomly selected at time 0.
(A) 0.05 (B) 0.06 (C) 0.09 (D) 0.11 (E) 0.12

24.30 (3, 5/01, Q.28) (2.5 points) For a population of individuals, you are given:
(i) Each individual has a constant force of mortality.
(ii) The forces of mortality are uniformly distributed over the interval (0, 2).
Calculate the probability that an individual drawn at random from this population dies within one year.
(A) 0.37 (B) 0.43 (C) 0.50 (D) 0.57 (E) 0.63

24.31 (SOA M, 5/05, Q.10 & 2009 Sample Q.163) The scores on the final exam in Ms. B's Latin class have a normal distribution with mean θ and standard deviation equal to 8.

θ is a random variable with a normal distribution with mean equal to 75 and standard deviation equal to 6.

Each year, Ms. B chooses a student at random and pays the student 1 times the student's score. However, if the student fails the exam (score $\le$ 65), then there is no payment.

Calculate the conditional probability that the payment is less than 90, given that there is a payment.

(A) 0.77 (B) 0.85 (C) 0.88 (D) 0.92 (E) 1.00

24.32 (SOA M, 11/05, Q.17 & 2009 Sample Q.204) (2.5 points)

The length of time, in years, that a person will remember an actuarial statistic is modeled by an exponential distribution with mean 1/Y.

In a certain population, Y has a gamma distribution with $\alpha = \theta = 2$.

Calculate the probability that a person drawn at random from this population will remember an actuarial statistic less than 1/2 year.

(A) 0.125 (B) 0.250 (C) 0.500 (D) 0.750 (E) 0.875

24.33 (SOA M, 11/05, Q.20) (2.5 points) For a group of lives age x, you are given:

(i) Each member of the group has a constant force of mortality that is drawn from the uniform distribution on [0.01, 0.02].

(ii) $\delta = 0.01$.

For a member selected at random from this group, calculate the actuarial present value of a continuous lifetime annuity of 1 per year.

(A) 40.0 (B) 40.5 (C) 41.1 (D) 41.7 (E) 42.3

Solutions to Problems:

24.1. E. The conditional probability of a loss of size 400 given δ is: $e^{-400/\delta} / \delta$.
The unconditional probability can be obtained by integrating the conditional probabilities versus the distribution of δ:

$$f(400) = \int_0^\infty f(400 \mid \delta)\, g(\delta)\, d\delta = \int_0^\infty \{e^{-400/\delta} / \delta\}\, \theta^\alpha\, \delta^{-(\alpha+1)}\, e^{-\theta/\delta} / \Gamma(\alpha)\, d\delta =$$

$$\{\theta^\alpha / \Gamma(\alpha)\} \int_0^\infty \delta^{-(\alpha+2)}\, e^{-(400+\theta)/\delta}\, d\delta = \{\{\theta^\alpha / \Gamma(\alpha)\}\, \Gamma(\alpha+1) / (400 + \theta)^{\alpha+1} = \boldsymbol{\alpha\theta^\alpha / (\theta+400)^{\alpha+1}}.$$

24.2. A. The conditional mean given δ is: δ. The unconditional mean can be obtained by integrating the conditional means versus the distribution of δ:

$$E[X] = \int_0^\infty E[X \mid \delta]\, g(\delta)\, d\delta = \int_0^\infty \delta\, \theta^\alpha\, \delta^{-(\alpha+1)}\, e^{-\theta/\delta} / \Gamma(\alpha)\, d\delta = \{\theta^\alpha / \Gamma(\alpha)\} \int_0^\infty \delta^{-\alpha}\, e^{\theta/\delta}\, d\delta$$

$$= \{\theta^\alpha / \Gamma(\alpha)\}\Gamma(\alpha-1) / \theta^{\alpha-1} = \boldsymbol{\theta / (\alpha-1)}.$$

Comment: The mean of a Pareto Distribution; the mixed distribution is a Pareto with scale parameter θ and shape parameter α.

24.3. D. The conditional mean given δ is δ. The conditional variance given δ is δ^2.
Thus the conditional second moment given δ is: $\delta^2 + \delta^2 = 2\delta^2$.
The unconditional second moment can be obtained by integrating the conditional second moments versus the distribution of δ:

$$E[X^2] = \int_0^\infty E[X^2 \mid \delta]\, g(\delta)\, d\delta = \int_0^\infty (2\delta^2)\, \theta^\alpha\, \delta^{-(\alpha+1)}\, e^{-\theta/\delta} / \Gamma(\alpha)\, d\delta =$$

$$\{2\theta^\alpha / \Gamma(\alpha)\} \int_0^\infty \delta^{(\alpha+1)}\, e^{\theta/\delta}\, d\delta = \{2\theta^\alpha / \Gamma(\alpha)\}\Gamma(\alpha-2) / \theta^{\alpha-2} = 2\theta^2 / \{(\alpha-1)(\alpha-2)\}.$$

Since the mean is $\theta/(\alpha-1)$, the variance is: $2\theta^2 / \{(\alpha-1)(\alpha-2)\} - \theta^2 / (\alpha-1)^2 =$
$(\theta^2 / \{(\alpha-1)^2(\alpha-2)\})\, \{2(\alpha-1) - (\alpha-2)\} = \boldsymbol{\theta^2\alpha / \{(\alpha-2)(\alpha-1)^2\}}$.

24.4. A. Weibull has density, $\tau(x/\theta)^\tau \exp(-(x/\theta)^\tau) / x = x^{-2/3}\theta^{-1/3} \exp(-x^{1/3}/\theta^{1/3})/3$.

The density of the mixed distribution is obtained by integrating the Weibull density times $g(\theta)$:

$$\int_0^\infty f(x)\, g(\theta)\, d\theta = \int_0^\infty \{x^{2/3}\,\theta^{-1/3} \exp[-x^{1/3}/\theta^{1/3}]/3\}\ 7^{2.5}(1/3) \exp[-7/\theta^{1/3}] / \{\theta^{11/6}\, \Gamma(2.5)\}\ d\theta =$$

$$\frac{7^{2.5}\, x^{-2/3}}{9\, \Gamma(2.5)} \int_0^\infty \theta^{-13/6} \exp[-(7+x^{1/3})/\theta^{1/3}]\ d\theta =$$

Make the change of variables, $y = (7+x^{1/3})/\theta^{1/3}$. $\theta = (7+x^{1/3})y^{-3}$. $d\theta = -3(7+x^{1/3})y^{-4}dy$.

$$\frac{7^{2.5}\, x^{-2/3}}{3\, \Gamma(2.5)} \int_0^\infty (7+x^{1/3})^{-13/2}\, y^{13/2} \exp(-y)\ (7+x^{1/3})y^{-4}\ dy =$$

$$\frac{7^{2.5}\, x^{-2/3}\, (7+x^{1/3})^{-7/2}}{3\, \Gamma(2.5)} \int_0^\infty y^{5/2} \exp(-y)\ dy = \frac{7^{2.5}\, x^{-2/3}\, (7+x^{1/3})^{-7/2}}{3\, \Gamma(2.5)}\ \Gamma(3.5)$$

$= (2.5)(1/3)(1/7)\, x^{-2/3} / \{1+(x/343)^{1/3}\}^{-7/2}$.

This is the density of a Burr Distribution, $\alpha\gamma(x/\theta)^\gamma(1+(x/\theta)^\gamma)^{-(\alpha+1)}/x$, with parameters:
$\alpha = 2.5$, $\theta = 343 = 7^3$, and $\gamma = 1/3$.

<u>Comment</u>: In general, if one mixes a Weibull with $\tau = t$ fixed, with its scale parameter varying via an Inverse Transformed Gamma Distribution, with parameters: α, θ, and $\tau = t$,
then the mixed distribution is a Burr with parameters: α, θ, and $\gamma = t$.
This Inverse Transformed Gamma Distribution has parameters: $\alpha = 2.5$, $\theta = 343 = 7^3$, and $\tau = 1/3$.

24.5. E. If the hazard rate of an Exponential Distribution, λ, is distributed via a Gamma(α, θ), then the mixed Pareto has parameters α and $1/\theta$.

$\pi(\lambda) = (4/3)\ \lambda^4\ e^{-2\lambda}$, $\lambda > 0$. Substituting x for λ, this is proportional to: $x^{5-1}\ e^{-x/(1/2)}$.

Thus the distribution of lambda is Gamma with $\alpha = 5$ and $\theta = 1/2$.

Thus the mixed distribution is Pareto with $\alpha = 5$ and $\theta = 2$.

Thus, $S(7) = 2^5 / (2+7)^5 = 32 / 9^5 =$ **0.00054**.

Alternately, this is the Exponential - Inverse Gamma, parameterized somewhat differently.

The mean of the Exponential is $\delta = 1/\lambda$, and δ follows an Inverse Gamma.

Since $\frac{d\lambda}{d\delta} = -1/\delta^2$, $g(\delta) = \pi(\lambda) \left|\frac{d\lambda}{d\delta}\right| = (4/3)\ \delta^{-4}\ e^{-2/\delta} / \delta^2 = (4/3)\ \delta^{-6}\ e^{-2/\delta}$.

Substituting x for δ, this is proportional to: $e^{-2/x} / x^{(5+1)}$.

Thus we have an Inverse Gamma has parameters $\alpha = 5$ and $\theta = 2$. Thus the mixed distribution is a Pareto, with $\alpha = 5$ and $\theta = 2$. $\Rightarrow S(x) = \{2/(2+x)\}^5$. $\Rightarrow S(7) = (2/9)^5 =$ **0.00054**.

Alternately, one can compute the unconditional survival function at x = 7 via integration:

$$S(7) = \int_0^\infty S(x \mid \lambda)\ \pi(\lambda)\ d\lambda = \int_0^\infty \exp(-7\lambda)\ (4/3)\ \lambda^4\ e^{-2\lambda}\ d\lambda = (4/3) \int_0^\infty \lambda^4\ e^{-9\lambda}\ d\lambda\ .$$

This is a "Gamma type" integral and thus: $S(7) = (4/3)\ \Gamma(5) / 9^5 = (4/3)\ (4!) / 9^5 =$ **0.00054**.

Alternately, one can compute the unconditional survival function at x via integration:

$$S(x) = \int_0^\infty S(7 \mid \lambda)\ \pi(\lambda)\ d\lambda = \int_0^\infty \exp(-x\lambda)\ (4/3)\ \lambda^4\ e^{-2\lambda}\ d\lambda = (4/3) \int_0^\infty \lambda^4\ e^{-(2+x)\lambda}\ d\lambda$$

$= (4/3)\ \Gamma(5) / (2+x)^5 = (4/3)(4!) / (2+x)^5 = 32 / (2+x)^5$. $\Rightarrow S(7) = 32 / 9^5 =$ **0.00054**.

Comment: If one recognizes the mixed distribution as a Pareto with scale parameter of 2 and shape parameter of 5, then one can determine the constant by looking in Appendix A of Loss Models without doing the Gamma type integral.

For α integer, $\Gamma[\alpha] = (\alpha-1)!$.

The Gamma density in the Appendix of Loss Models is: $\theta^{-\alpha}\ x^{\alpha-1}\ e^{-x/\theta} / \Gamma(\alpha)$, $x > 0$.

Since this probability density function must integrate to unity from zero to infinity:

$$\int_0^\infty t^{\alpha-1}\ e^{-t/\theta}\ dt = \Gamma(\alpha)\,\theta^\alpha, \text{ or for integer n: } \int_0^\infty t^n\ e^{-c\,t}\ dt = n! / c^{n+1}.$$

24.6. D. $A(x) = \int_0^x a(t)\,dt = x^4$.

$S(x) = M_\lambda[-A(x)]$.

The moment generating function of this Exponential Distribution is: $M(t) = \frac{1}{1 - 0.007t}$.

Thus $S(x) = \frac{1}{1 + 0.007x^4}$.

Thus, $S(6) = \frac{1}{1 + 0.007\,(6^4)}$ = **9.9%**.

Comment: See Exercise 5.13 in Loss Models.
The mixture is Loglogistic.
For the Weibull as per Loss Models, $h(x) = \tau x^{\tau-1}\theta^{-\tau}$.
Therefore, we can put the Weibull for fixed τ into the form of a frailty model;
$h(x) = \lambda\, a(x)$, by taking $a(x) = \tau x^{\tau-1}$ and $\lambda = \theta^{-\tau}$. $A(x) = x^\tau$.
Here $\tau = 4$, $a(x) = 4\,x^3$, and $\lambda = \theta^{-4}$.
Thus for a given value of theta or lambda, $S(x \mid \lambda) = \exp[-(x/\theta)^4] = \exp[-\lambda x^4]$.
Therefore, $S(6 \mid \lambda) = \exp[-\lambda 6^4] = \exp[-1296\lambda]$.

Thus, $S(6) = \int_0^\infty \exp[-1296\lambda]\,\exp[-\lambda/0.007]/0.007\,d\lambda = \int_0^\infty \exp[-1438.86\lambda]\,d\lambda\ /\ 0.007$

$= (1/1438.86)/0.007 = 9.9\%$.

24.7. D. The hazard rate for an Exponential is one over its mean. Therefore, the survival function is $S(t; \lambda) = e^{-\lambda t}$. Mixing over the different values of λ:

$$S(t) = \int_{0.01}^{0.11} S(t; \lambda)\, f(\lambda)\, d\lambda = \int_{0.01}^{0.11} e^{-t\lambda}\,(1/0.1)\, d\lambda = (-10/t)\, e^{-t\lambda} \Big]_{t=0.01}^{t=0.11} = (10/t)(e^{-0.01t} - e^{-0.11t}).$$

$S(3) = (10/3)(e^{-0.03} - e^{-0.33}) = 0.8384$. $S(5) = (10/5)(e^{-0.05} - e^{-0.55}) = 0.7486$.
The number of deaths expected between time 3 and time 5 is:
$(1000)\{S(3) - S(5)\} = (1000)(0.8384 - 0.7486) =$ **89.8**.
Comment: Similar to 3, 5/00, Q.17.

24.8. E. For this Inverse Gamma-Exponential, the mixed distribution is a Pareto with $\alpha = 3$ and $\theta = 25{,}000$. $S(20{,}000) = \{25{,}000/(25{,}000 + 20{,}000)\}^3 = (5/9)^3 =$ **17.1%**.

24.9. D. If the severity is Normal with fixed variance s^2, and the mixing distribution of their means is also Normal with mean μ and variance σ^2, then the mixed distribution is another Normal, with mean μ and variance: $s^2 + \sigma^2$.
In this case, the mixed distribution is Normal with mean 135 and variance: $15^2 + 10^2 = 325$.
Prob[145 ≤ score ≤ 155] = $\Phi[(155 - 135)/\sqrt{325}] - \Phi[(145 - 135)/\sqrt{325}] = \Phi[1.109] - \Phi[0.555]$ = 0.8663 - 0.7106 = **0.156**.

24.10. A. λ is the hazard rate of each Exponential, one over the mean. We are mixing λ via a Gamma. Therefore, the mixed distribution is Pareto with parameters $\alpha = 3$ and $\theta = 1/0.01 = 100$. (This is mathematically the same as the Inverse Gamma-Exponential described in this section or to the Gamma Exponential Conjugate Prior described in "Mahler's Guide to Conjugate Priors".)

Thus, $S(250) = \left(\frac{100}{100 + 250}\right)^3$ = **2.3%.**

Alternately, $h(x \mid \lambda) = \lambda\, a(x)$. For the Exponential, $h(x) = \lambda$.
Thus, this is a frailty model with $a(x) = 1$ and $A(x) = x$.
$S(x) = M_\lambda[-A(x)]$.

The moment generating function of this Gamma Distribution is: $M(t) = \left(\frac{1}{1 - 0.01t}\right)^3$.

Thus $S(x) = \left(\frac{1}{1 + 0.01x}\right)^3 = \left(\frac{100}{100 + x}\right)^3$.

Thus, $S(250) = \left(\frac{100}{100 + 250}\right)^3$ = **2.3%.**

Comment: Similar to ASTAM, 5/24, Q.6a.

24.11. A. Inverse Weibull has density: $4x^{-5}q^4\exp[-(q/x)^4]$.
The density of q is that of the Transformed Gamma Distribution:
$\tau(q/\theta)^{\tau\alpha}\exp(-(q/\theta)^\tau) / \{q\,\Gamma(\alpha)\} = 4\,q^{4.2}\,11^{-5.2}\exp[-(q/11)^4] / \Gamma(1.3)$.
The density of the mixed distribution is obtained by integrating the Inverse Weibull density times the density of q:

$$\int_0^\infty 4x^{-5}q^4\exp[-(q/x)^4]\,4\,q^{4.2}\,11^{-5.2}\exp[-(q/11)^4]\,dq\ /\ \Gamma(1.3) =$$

$$\frac{(16)\,11^{-5.2}\,x^{-5}}{\Gamma(1.3)}\int_0^\infty q^{8.2}\exp[-(q^4(11^{-4}+x^{-4})]\,dq =$$

Make the change of variables, $y = q^4(11^{-4} + x^{-4})$. $q = (11^{-4} + x^{-4})^{-1/4}y^{1/4}$.
$dq = (1/4)(11^{-4} + x^{-4})^{-1/4}y^{-3/4}dy$.

$$\frac{(16)\,11^{-5.2}\,x^{-5}}{\Gamma(1.3)}\int_0^\infty (11^{-4}+x^{-4})^{-2.05}\,y^{2.05}\exp(-y)\ (1/4)\,(11^{-4}+x^{-4})^{-1/4}\,y^{-3/4}\,dy =$$

$$\frac{(4)\,11^{-5.2}\,x^{-5}\,(11^{-4}+x^{-4})^{-2.3}}{\Gamma(1.3)}\int_0^\infty y^{1.3}\exp(-y)\,dy =$$

$$\frac{(4)\,11^{-5.2}\,x^{-5}\,(11^{-4}+x^{-4})^{-2.3}}{\Gamma(1.3)}\Gamma(2.3) = (4)(1.3)\,11^{-5.2}\,x^{-5}(11^{-4}+x^{-4})^{-2.3} =$$

$(4)(1.3)\,11^{-5.2}\,x^{-5}x^{9.2}(1 + (x/11)^4)^{-2.3} = (4)(1.3)\,(x/11)^{5.2\ 2}(1 + (x/11)^4)^{-2.3} / x$.
This is the density of an Inverse Burr Distribution, $\tau\gamma(x/\theta)^{\gamma\tau}(1+(x/\theta)^\gamma)^{-(\tau+1)}/x$,
with parameters $\tau = 1.3$, $\theta = 11$, and $\gamma = 4$. Therefore, the mixed distribution is:
$F(x) = \{(x/\theta)^\gamma / (1+(x/\theta)^\gamma)\}^\tau = \{1+(11/x)^4\}^{-1.3}$. $F(10) = \{1 + (11/10)^4\}^{-1.3} =$ **31.0%**.
<u>Comment</u>: In general, if one mixes an Inverse Weibull with $\tau = t$ fixed, with its scale parameter varying via a Transformed Gamma Distribution, with parameters α, θ, and $\tau = t$, then the mixed distribution is an Inverse Burr with parameters $\tau = \alpha$, θ, and $\gamma = t$.
For each Inverse Weibull, $S(10) = \exp[-(q / 10)^4)]$. One could instead average S(10) over the Inverse Weibulls, in order to get S(10) for the mixed distribution:

$$\int_0^\infty \exp[-(q/10)^4]\ 4\,q^{4.2}\,11^{-5.2}\exp[-(q/11)^4]\,dq\ /\ \Gamma(1.3) =$$

$$\frac{(4)\,(11^{-5.2})}{\Gamma(1.3)}\int_0^\infty \exp[-0.0001683q^4]\,q^{4.2}\,dq =$$

$$\frac{11^{-5.2}}{\Gamma(1.3)}\int_0^\infty \exp(-0.0001683y)\,y^{0.3}\,dy = \frac{11^{-5.2}}{\Gamma(1.3)}\Gamma(1.3)\,(0.0001683)^{-1.3} = 31.0\%.$$

24.12. B. Let the inflation factor be y. Then given y, in the year 2007 the losses have an Exponential Distribution with mean 3000y. Let z = 3000y. Then since y follows an Inverse Gamma with parameters $\alpha = 4$ and scale parameter $\theta = 3.5$, z follows an Inverse Gamma with parameters $\alpha = 4$ and $\theta = (3000)(3.5) = 10{,}500$.
Thus in the year 2007, we have a mixture of Exponentials each with mean z, with z following an Inverse Gamma. This is the (same mathematics as the) Inverse Gamma-Exponential.
For the Inverse Gamma-Exponential the mixed distribution is a Pareto, with
α = shape parameter of the Inverse Gamma and θ = scale parameter of the Inverse Gamma.
In this case the mixed distribution is a Pareto with $\alpha = 4$ and $\theta = 10{,}500$.
For this Pareto, $S(5500) = \{1 + (5500/10{,}500)\}^{-4}$ = **18.5**%.
Comment: This is an example of "parameter uncertainty." We assume that the loss distribution in year 2007 will also be an Exponential, we just are currently uncertain of its parameter.

24.13. B. $h(x \mid \lambda) = \lambda a(x)$. For an Exponential a(x) = 1, and $A(x) = \int_0^x a(t)\, dt = x$.

The moment generating function of this Inverse Gaussian Distribution is:
$M(t) = \exp[(\theta/\mu)(1 - \sqrt{1 - 2t\mu^2/\theta})] = \exp[(1/3)(1 - \sqrt{1 - 0.09t})]$.
$S(x) = M_\lambda[-A(x)] = \exp[(1/3)(1 - \sqrt{1 + 0.09x})]$.
Thus, $S(65) = \exp[(1/3)(1 - \sqrt{1 + (0.09)(65)})]$ = **58%.**

24.14. B. From the previous solution, $S(x) = \exp[(1/3)(1 - \sqrt{1 + 0.09x})]$.
$S(40) = \exp[(1/3)(1 - \sqrt{1 + (0.09)(40)})] = 0.683$.
Differentiating, $f(x) = \exp[(1/3)(1 - \sqrt{1 + 0.09x})]\ (1/3)(0.09)(1/2)/\sqrt{1 + 0.09x}$.
$f(40) = \exp[(1/3)(1 - \sqrt{1 + (0.09)(40)})]\ 0.015 / \sqrt{1 + (0.09)(40)} = 0.00478$.
h(40) = f(40)/S(40) = 0.00478/0.683 = **0.0070**.
Alternately, for a frailty model, $h(x) = a(x)\ \dfrac{d \ln M_\lambda[-A(x)]}{d\lambda}$.
$M(t) = \exp[(\theta/\mu)(1 - \sqrt{1 - 2t\mu^2/\theta})] = \exp[(1/3)(1 - \sqrt{1 - 0.09t})]$.
$\ln M(t) = (1/3)\ (1 - \sqrt{1 - 0.09t})$.
$\dfrac{d \ln M[t]}{dt} = (1/3)(0.09)(1/2) / \sqrt{1 - 0.09t}$.

$h(x \mid \lambda) = \lambda a(x)$. For an Exponential a(x) = 1, and $A(x) = \int_0^x a(t)\, dt = x$.

$h(x) = (1)\ (1/3)(0.09)(1/2) / \sqrt{1 + 0.09x}$.
$h(40) = 0.015 / \sqrt{1 + (0.09)(40)}$ = **0.0070**.

24.15. C. For the Inverse Gamma, $f(x \mid q) = q^\alpha e^{-q/x} / \{\Gamma[\alpha]\, x^{\alpha+1}\} = q^4 e^{-q/x} / \{6\, x^5\}$.
For the Exponential, $u(q) = e^{-q/100}/100$.

$$f(x) = \int f(x \mid q)\, u(q)\, dq = \int_0^\infty \frac{q^4 e^{-q/x}}{6x^5} \frac{e^{-q/100}}{100}\, dq = \int_0^\infty q^4 e^{-q(1/x + 1/100)}\, dq \,/\, (600x^5)$$

$= \{\Gamma[5] / (1/x + 1/100)^5\} / (600x^5) = \{(24)100^5/(x + 100)^5\} / 600 = (4)100^4/(x + 100)^5$.
This is the density of a Pareto Distribution with parameters $\alpha = 4$ and $\theta = 100$.
Therefore, $F(x) = 1 - \{\theta/(x+\theta)\}^\alpha = 1 - \{100/(x+ 100)\}^4$. $S(15) = (100/115)^4 =$ **57.2%**.
Comment: An example of an Exponential-Inverse Gamma.

24.16. For a Normal Distribution with mean zero and variance v:
$f(x \mid v) = \exp[-x^2/(2v)] / \sqrt{2\pi v}$.
An Inverse Gamma with $\alpha = 10$ and $\theta = 10$ has density:
$g(v) = 10^{10} e^{-10/v} v^{-11} / \Gamma(10)$, $v > 0$.
The mixed distribution has density:

$$\int_0^\infty \frac{\exp[-x^2/(2v)]}{\sqrt{2\pi v}} \frac{10^{10} e^{-10/v} v^{-11}}{\Gamma(10)}\, dv = \frac{10^{10}}{9!\,\sqrt{2\pi}} \int_0^\infty \exp[-(10+x^2/2)/v]\, v^{-11.5}\, dv$$

$$= \frac{10^{10}}{9!\,\sqrt{2\pi}} \frac{\Gamma(10.5)}{(10 + x^2/2)^{10.5}} = (1 + x^2/20)^{-10.5} \frac{\Gamma(10.5)}{\Gamma(10)\,\Gamma(1/2)\,\sqrt{2}}.$$

This is a **Student's t distribution with 20 degrees of freedom**.
Comment: Difficult! Since the Inverse Gamma Density integrates to one over its support,

$$\int_0^\infty \exp[-\theta/x]\, x^{-(\alpha+1)}\, dx = \Gamma(\alpha) / \theta^\alpha. \text{ Also, } \Gamma(1/2) = \sqrt{\pi}.$$

A Student's t distribution with ν degrees of freedom, has density:

$$f(t) = \frac{1}{\beta[\nu/2, 1/2]} \frac{1}{(t^2/\nu + 1)^{(\nu+1)/2}} \frac{1}{\sqrt{\nu}}, \text{ where } \beta[\nu/2, 1/2] = \Gamma[1/2]\Gamma[\nu/2] / \Gamma[(\nu+1)/2].$$

24.17. D. The mixed distribution is a Pareto with shape parameter = $\alpha = 3$ and
scale parameter = $\theta = 20$, with variance: $(2)(20^2)/\{(3-1)(3-2)\} - \{20 / (3-1)\}^2 = 400 - 100 =$ **300**.
Alternately, $\text{Var}[X \mid \delta]$ = Variance[Exponential Distribution with mean δ] = δ^2.
$f(\delta)$ is an Inverse Gamma Distribution, with $\theta = 20$ and $\alpha = 3$.
$E[\text{Var}[X \mid \delta]] = E[\delta^2]$ = 2nd moment of Inverse Gamma = $20^2/\{(3-1)(3-2)\} = 200$.
$\text{Var}[E[X \mid \delta]] = \text{Var}[\delta]$ = variance of Inverse Gamma =
2nd moment of Inverse Gamma - (mean Inverse Gamma)^2 = $200 - \{20 / (3-1)\}^2 = 100$.
$\text{Var}[X] = E[\text{Var}[X \mid \delta]] + \text{Var}[E[X \mid \delta]] = 200 + 100 =$ **300**.

24.18. E. $A(x) = \int_0^x a(t)\, dt = \sqrt{1 + 0.008x} - 1.$

The moment generating function of a Gamma Distribution with $\alpha = 6$ and $\theta = 1$ is:

$M(t) = \left(\frac{1}{1 - t}\right)^6.$

$S(x) = M_\lambda[-A(x)] = \left(\frac{1}{1 + A(x)}\right)^6 = \left(\frac{1}{\sqrt{1 + 0.008x}}\right)^6 = \left(\frac{1}{1 + 0.008x}\right)^3.$

Thus, $S(11) = \left(\frac{1}{1 + (0.008)(11)}\right)^3$ = **77.6%**.

Comment: See Exercise 5.14 in Loss Models.
The mixture is a Pareto Distribution with $\alpha = 3$, and $\theta = 1/0.008 = 125$.

24.19. D. Let the inflation factor be y. Then given y, in the next year the losses have an Exponential Distribution with mean 8000y. Let z = 8000y. Then since y follows an Inverse Gamma with parameters $\alpha = 2.5$ and scale parameter $\theta = 1.6$, z follows an Inverse Gamma with parameters $\alpha = 2.5$ and $\theta = (8000)(1.6) = 12{,}800$. Thus next year, we have a mixture of Exponentials each with mean z, with z following an Inverse Gamma. This is the (same mathematics as the) Inverse Gamma-Exponential. For the Inverse Gamma-Exponential the mixed distribution is a Pareto, with α = shape parameter of the Inverse Gamma, and θ = scale parameter of the Inverse Gamma.
In this case the mixed distribution is a Pareto with $\alpha = 2.5$ and $\theta = 12{,}800$.
For this Pareto Distribution, $S(10{,}000) = \{1 + (10{,}000/12{,}800)\}^{-2.5}$ = **23.6**%.

24.20. B. ln[x] follows a Normal with parameters μ and 0.3.
Therefore, we are mixing a Normal with fixed variance via another Normal.
Therefore, the mixture of ln[x] is Normal with parameters 5, and $\sqrt{0.3^2 + 0.4^2} = 0.5$.
Thus the mixture of x is LogNormal with parameters 5 and 0.5.
$S(200) = 1 - \Phi[\{\ln(200) - 5\}/0.5] = 1 - \Phi[0.60]$ = **27.43%**.

24.21. E. & 24.22. B.
For the Exponential, the loss elimination ratio is equal to the distribution function:
$LER(x) = 1 - e^{-x/\mu}$.
The mixed distribution of the size of loss is Pareto with the same parameters,
$\alpha = 3$ and $\theta = 1000$.

$E_\mu[LER(x)] = E_\mu[1 - e^{-x/\mu}] = E_\mu[F(x ; \mu)] = \int \pi[\mu]\ F(x ; \mu)\ d\mu$

$= \text{distribution function of the mixture} = \text{distribution function of the Pareto} = 1 - \left(\frac{1000}{x + 1000}\right)^3$.

$E_\mu[LER(500)] = 1 - (10/15)^3 = \mathbf{0.70370}$.

$E_\mu[LER(200)] = 1 - (10/12)^3 = 0.42130$.

$E_\mu[LER(x)\ LER(y)] = E_\mu[(1 - e^{-x/\mu})\ (1 - e^{-y/\mu})] = E_\mu[(1 - e^{-x/\mu} - e^{-y/\mu} + e^{-(x+y)/\mu}]$

$= 1 - E_\mu[S(x ; \mu)] - E_\mu[S(x ; \mu)] + E_\mu[S(x + y ; \mu)]$

$= 1 - \left(\frac{1000}{x + 1000}\right)^3 - \left(\frac{1000}{y + 1000}\right)^3 + \left(\frac{1000}{x + y + 1000}\right)^3$.

$E_\mu[LER(200)\ LER(500)] = 1 - (10/12)^3 - (10/15)^3 + (10/17)^3 = 0.32854$.

$Cov_\mu[LER(200)\ ,\ LER(500)] = E_\mu[LER(200)\ LER(500)] - E_\mu[LER(200)]\ E_\mu[LER(500)] =$
$0.32854 - (0.42130)(0.70370) = 0.03207$.

$E_\mu[LER(x)^2] = E_\mu[LER(x)\ LER(x)] = 1 - 2\left(\frac{1000}{x + 1000}\right)^3 + \left(\frac{1000}{2x + 1000}\right)^3$.

$E_\mu[LER(200)^2] = 1 - (2)(10/12)^3 + (10/14)^3 = 0.20702$.

$Var_\mu[LER(200)] = 0.20702 - 0.42130^2 = 0.02953$.

$E_\mu[LER(500)^2] = 1 - (2)(10/15)^3 + (10/20)^3 = 0.53241$.

$Var_\mu[LER(500)] = 0.53241 - 0.70370^2 = 0.03722$.

$Corr_\mu[LER(200)\ ,\ LER(500)] = \frac{0.03207}{\sqrt{(0.02953)\ (0.03722)}} = \mathbf{96.73\%}$.

<u>Comment</u>: The loss elimination ratios for deductibles of somewhat similar sizes are highly correlated across classes.
For a practical example for excess ratios, see Tables 3 and 4 in
"NCCI's 2007 Hazard Group Mapping," by John P. Robertson, Variance, Vol. 3, Issue 2, 2009, <u>not</u> on the syllabus of this exam.

24.23. D. Given β the mean severity is: $\sqrt{\beta}/2$.

Thus the mean of the mixed distribution is: $\int_{10}^{50} (\sqrt{\beta}/2) / 40 \ d\beta = \beta^{3/2}/120 \Big]_{\beta=10}^{\beta=50} = 2.683$.

Given β the second moment of severity is: $(\sqrt{\beta})^2/12 + (\sqrt{\beta}/2)^2 = \beta/3$.

Thus the 2nd moment of the mixed distribution is: $\int_{10}^{50} (\beta/3) / 40 \ d\beta = \beta^2/240 \Big]_{\beta=10}^{\beta=50} = 10$.

The variance of the mixed distribution is: $10 - 2.683^2 =$ **2.80**.

24.24. C. $A(x) = \int_0^x a(t) \ dt = 1.09^x / \ln(1.09) - 1 / \ln(1.09) = (1.09^x - 1) / \ln(1.09)$.

The moment generating function of a Gamma Distribution with α = 4 and θ = 1/50,000 is:

$M(t) = \left(\frac{1}{1 - t/50{,}000}\right)^4 = \left(\frac{50{,}000}{50{,}000 - t}\right)^4$.

$S(x) = M_\lambda[-A(x)] = \left(\frac{50{,}000}{50{,}000 + A(x)}\right)^4 = \left(\frac{50{,}000}{50{,}000 + (1.09^x - 1) / \ln(1.09)}\right)^4$.

Thus, $S(90) = \left(\frac{50{,}000}{50{,}000 + (1.09^{90} - 1) / \ln(1.09)}\right)^4 =$ **17.70%**.

Comment: The form of the hazard rate is that for Gompertz law; $h(x) = Bc^x$,
with B = λ and c = 1.09.
The B parameter varies across a group of individuals via a Gamma Distribution; some individuals in the group have higher hazard rates while others have lower hazard rates.
Although not mentioned in Loss Models, this "Gamma-Gompertz" frailty model is used in Survival Analysis.
In general, the mixed distribution has survival function: $S(x) = \{1 + \theta (c^x - 1)/ \ln[c]\}^{-\alpha}$.
The mean value of B is: 4/50,000 = 0.00008.
For Gompertz Law: $S(x) = \exp[(1 - c^x) B/ \ln(c)]$.
For B = 0.00008 and c = 1.09, $S(90) = \exp[(1 - 1.09^{90}) (0.00008) / \ln(1.09)] = 11.45\%$.
This is less than S(90) for the mixture; mixing has made the righthand tail heavier.

24.25. C. The mixed distribution is Normal with mean 6000 and variance: $\sigma^2 + \beta^2$.
Thus the 90th percentile of the lifetime of a randomly chosen transformer is:
$9200 = 6000 + 1.282\sqrt{\sigma^2 + \beta^2}$. ⇒ $\sqrt{\sigma^2 + \beta^2} = 2496$.

Thus the mixed distribution is Normal with mean 6000 and standard deviation 2496.
Thus Prob[lifetime ≤ 5000] = Φ[(5000 - 6000) / 2496] = Φ[-0.40] = **34.5%**.

24.26. The mean of each LogNormal Distribution is $\exp[m + v^2/2]$.

The density of the Normal Distribution of m is: $\exp[-\frac{(m-\mu)^2}{2\sigma^2}]\ \frac{1}{\sigma\sqrt{2\pi}}$.

Thus the overall mean is: $\int_{-\infty}^{\infty} \exp[m + v^2/2]\ \exp[-\frac{(m-\mu)^2}{2\sigma^2}] \frac{1}{\sigma\sqrt{2\pi}}\ dm$ =

$\frac{\exp[v^2/2]}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp[m]\ \exp[-\frac{m^2 - 2m\mu + \mu^2}{2\sigma^2}]\ dm$ =

$\frac{\exp[v^2/2]}{\sigma\sqrt{2\pi}}\ \exp[-\frac{\mu^2}{2\sigma^2}] \int_{-\infty}^{\infty} \exp[-\frac{m^2 - 2m(\mu + \sigma^2)}{2\sigma^2}]\ dm$.

However, using the hint: $\int_{-\infty}^{\infty} \exp[-\frac{m^2 - 2m(\mu + \sigma^2)}{2\sigma^2}]\ dm = \sqrt{2\pi}\ \sigma\ \exp[\frac{(\mu + \sigma^2)^2}{2\sigma^2}]$.

Thus, the overall mean is: $\frac{\exp[v^2/2]}{\sigma\sqrt{2\pi}}\ \exp[-\frac{\mu^2}{2\sigma^2}]\ \sqrt{2\pi}\ \sigma\ \exp[\frac{(\mu + \sigma^2)^2}{2\sigma^2}]$ =

$\exp[v^2/2]\ \exp[-\frac{\mu^2}{2\sigma^2}]\ \exp[\frac{\mu^2 + 2\mu\sigma^2 + \sigma^4}{2\sigma^2}]$ = $\mathbf{\exp[v^2/2 + \mu + \sigma^2/2]}$.

Comment: For example, if severity is LogNormal with parameters m and 0.3,
and m varies across the portfolio via a Normal Distribution with parameters 5 and 0.4,
then the overall mean is: $\exp[0.3^2/2 + 5 + 0.4^2/2] = 168.174$.

24.27. B. $f(\mu)$ is an Inverse Gamma Distribution, with $\theta = 20$ and $\alpha = 2$.
p(y) is an Exponential Distribution with $E[Y \mid \mu] = \mu$.
Therefore the mean severity = $E_\mu[\ E[Y \mid \mu]\ ] = E_\mu[\mu] = \int \mu\ f(\mu)\ d\mu$ =
mean of Inverse Gamma = $\theta / (\alpha-1) = 20 / (2-1)$ = **20**.
Alternately, the mixed distribution is a Pareto with shape parameter = α = 2,
and scale parameter = θ = 20. Therefore this Pareto has mean 20 / (2-1) = **20**.
Comment: One can do the relevant integral via the substitution $x = 1/\mu$, $dx = -d\mu/\mu^2$:

$$\int \mu\ f(\mu)\ d\mu = \int_0^{\infty} \mu\ (400 / \mu^3)\ e^{-20/\mu}\ d\mu = 400 \int_0^{\infty} \frac{e^{-20/\mu}}{\mu^2}\ d\mu = 400 \int_0^{\infty} e^{-20x}\ dx = 400/20 = 20.$$

24.28. C. This is an Exponential mixed via an Inverse Gamma. The Inverse Gamma has parameters $\alpha = 3$ and $\theta = 2$. Therefore the (prior) mixed distribution is a Pareto, with $\alpha = 3$ and $\theta = 2$. Thus $f(x) = (3)\ (2^3)\ (2+x)^{-4}$. $f(3) = (3)(8) / 5^4 =$ **0.0384**.
Alternately, one can compute the unconditional density at y = 3 via integration:

$$f(3) = \int_0^\infty f(3 \mid \delta)\ p(\delta)\ d\delta = \int_0^\infty (1/\delta) \exp(-3 / \delta)\ (4 / \delta^4) \exp(-2 / \delta)\ d\delta = \int_0^\infty 4\delta^{-5} \exp(-5 / \delta)\ d\delta.$$

Let $x = 5/\delta$ and $dx = (-5/\delta^2)d\delta$ in the integral:

$$f(3) = (4 / 5^4) \int_0^\infty x^3 \exp(-x)\ dx = (4/625)\ \Gamma(4) = (4/625)(3!) = (4/625)(6) = \mathbf{0.0384}.$$

Alternately, one can compute the unconditional density at y via integration:

$$f(y) = \int_0^\infty (1/\delta) \exp(-y / \delta)\ (4 / \delta^4) \exp(-2 / \delta)\ d\delta = \int_0^\infty 4\delta^{-5} \exp(-(2+y)/ \delta)\ d\delta.$$

Let $x = (2+y)/\delta$ and $dx = -((2+y)/\delta^2)d\delta$ in the integral:

$$f(y) = (4 / (2+y)^4) \int_0^\infty x^3 \exp(-x)\ dx = (4 / (2+y)^4)\ \Gamma(4) = (4/ (2+y)^4)(3!) = 24(2+y)^{-4}.\ \ f(3) = \mathbf{0.0384}.$$

Comment: If one recognizes this as a Pareto with $\theta = 2$ and $\alpha = 3$, then one can determine the constant by looking in Appendix A of Loss Models, rather than doing the Gamma integral.

24.29. E. The hazard rate for an Exponential is one over its mean. The mean is $1/\theta$ not θ.
The survival function is $S(t; \theta) = e^{-\theta t}$. $S(0.5; \theta) = e^{-0.5\theta}$. Mixing over the different values of θ:

$$S(0.5) = \int_1^{11} S(0.5; \theta)\ f(\theta)\ d\theta = \int_1^{11} e^{-0.5\theta}\ (1/10)\ d\theta = (-1/5)e^{-0.5\theta} \Big]_{\theta=1}^{\theta=11} =$$

$(e^{-0.5} - e^{-5.5})/5 = (0.607 - 0.004)/5 =$ **0.12**.
Comment: The mean future lifetime given θ is $1/\theta$. The overall mean future lifetime is:

$$\int_1^{11} (1/\theta)\ f(\theta)\ d\theta = \int_1^{11} (1/\theta)\ (1/10)\ d\theta = \ln(\theta) / 10 \Big]_{\theta=1}^{\theta=11} = 0.24.$$

24.30. D. For a constant force of mortality, λ, the distribution function is Exponential:
$F(t \mid \lambda) = 1 - e^{-\lambda t}$. $F(1 \mid \lambda) = 1 - e^{-\lambda}$.
The forces of mortality are uniformly distributed over the interval (0, 2). $\Rightarrow \pi(\lambda) = 1/2,\ 0 \le \lambda \le 2$.
Taking the average over the values of λ:

$$F(1) = \int_0^2 F(1 \mid \lambda)\ \pi(\lambda)\ d\lambda = \int_0^2 (1 - e^{-\lambda})/2\ d\lambda = 1 - (1 - e^{-2})(1/2) = \mathbf{0.568}.$$

Alternately, one can work with the means $\theta = 1/\lambda$, which is harder.
λ is uniform from 0 to 2. $\Rightarrow$ The distribution function of λ is: $F_\lambda(\lambda) = \lambda/2,\ 0 \le \lambda \le 2$.
$\Rightarrow$ The distribution function of θ is:
$F_\theta(\theta) = 1 - F_\lambda(\lambda) = 1 - F_\lambda(1/\theta) = 1 - 1/(2\theta),\ 1/2 \le \theta \le \infty$.
$\Rightarrow$ The density function of θ is: $1/(2\theta^2),\ 1/2 \le \theta \le \infty$.
Given θ, the probability of death by time 1 is: $1 - e^{-1/\theta}$.
Taking the average over the values of θ:

$$F(1) = \int_{1/2}^{\infty} (1 - e^{-1/\theta})/(2\theta^2)\ d\theta = 1 - \int_{1/2}^{\infty} e^{-1/\theta}/(2\theta^2)\ d\theta = 1 - (1/2)\ \left(e^{-1/\theta}\Big]_{\theta=1/2}^{\theta=\infty} \right)$$

$= 1 - (1 - e^{-2})/2 = \mathbf{0.568}$.
<u>Comment</u>: $F(1 \mid \lambda = 1) = 1 - e^{-1} = 0.632$. Thus choices A and B are unlikely to be correct.

24.31. D. If the severity is Normal with fixed variance s^2, and the mixing distribution of their means is also Normal with mean μ and variance σ^2, then the mixed distribution is another Normal, with mean μ and variance: $s^2 + \sigma^2$.
In this case, the mixed distribution is Normal with mean 75 and variance: $8^2 + 6^2 = 100$.
Prob[there is a payment] = Prob[Score > 65] = $1 - \Phi[(65 - 75)/10] = 1 - \Phi[-1] = 0.8413$.
Prob[90 > Score > 65] = $\Phi[(90 - 75)/10] - \Phi[(65 - 75)/10] = \Phi[1.5] - \Phi[-1]$
$= 0.9332 - 0.1587 = 0.7745$.
Prob[payment < 90 | payment > 0] = Prob[90 > Score > 65 | Score > 65] =
Prob[90 > Score > 65]/Prob[Score > 65] = 0.7745/0.8413 = **0.9206**.

24.32. D. Y is Gamma with $\alpha = \theta = 2$. Therefore, $F(y) = \Gamma[2; y/2]$.
Let the mean of each Exponential Distribution be $\delta = 1/y$. Then due to the change of variables, where large y corresponds to small δ: $F(\delta) = 1 - \Gamma[2; (1/2)/\delta]$.
Therefore, δ has an Inverse Gamma Distribution with $\alpha = 2$ and $\theta = 1/2$.
This is an Inverse Gamma - Exponential with mixed distribution a Pareto with $\alpha = 2$ and $\theta = 1/2$.
$F(x) = 1 - \{\theta/(x + \theta)\}^\alpha = 1 - \{0.5/(x + 0.5)\}^2$. $F(1/2) = 1 - (0.5/1)^2 =$ **0.75**.
Alternately, $\text{Prob}[T > t \mid y] = e^{-yt}$. $\text{Prob}[T > t] = \int e^{-yt} f(y)\, dy = M_Y[-t]$.

The moment generating function of a Gamma Distribution is: $1/(1 - \theta t)^\alpha$.
Therefore, the moment generating function of Y is: $1/(1 - 2t)^2$.
$\text{Prob}[T > 1/2] = M_Y[-1/2] = 1/2^2 = 1/4$. $\text{Prob}[T < 1/2] = 1 - 1/4 =$ **3/4**.
Alternately, $f(y) = y\, e^{-y/2} / (\Gamma(2)\, 2^2) = y\, e^{-y/2}/4$. Therefore, the mixed distribution is:

$$F(x) = \int_0^\infty (1 - e^{-xy})\, y\, e^{-y/2}/4 \; dy = 1 - (1/4) \int_0^\infty y\, e^{-y(x + 0.5)}\, dy = 1 - 0.25/(x + 0.5)^2.$$

$F(1/2) = 1 - 0.25 =$ **0.75**.
Alternately, the length of time until the forgetting is analogous to the time until the first claim. This time is Exponential with mean 1/Y and is mathematically the same as a Poisson Process with intensity Y. Since Y has a Gamma Distribution, this is mathematically the same as a Gamma-Poisson. Remembering less than 1/2 year, is analogous to at least one claim by time 1/2. Over 1/2 year, Y has a Gamma Distribution with $\alpha = 2$ and instead $\theta = 2/2 = 1$.
The mixed distribution is Negative Binomial, with $r = \alpha = 2$ and $\beta = \theta = 1$.
$1 - f(0) = 1 - 1/(1 + 1)^2 =$ **3/4**.

24.33. B. The present value of a continuous annuity of length t is: $(1 - e^{-\delta t})/\delta$.
Given constant force of mortality λ, the lifetimes are exponential with density $f(t) = \lambda e^{-\lambda t}$.

$$\text{For fixed } \lambda, \text{ APV} = \int_0^\infty \frac{1 - e^{-\delta t}}{\delta}\, \lambda e^{-\lambda t}\, dt = \{1 - \lambda/(\lambda + \delta)\}/\delta = 1/(\lambda + \delta) = 1/(\lambda + 0.01).$$

λ in turn is uniform from 0.01 to 0.02 with density 100.

$$\text{Mixing over } \lambda, \text{ Actuarial Present Value} = \int_{0.01}^{0.02} \frac{1}{\lambda + 0.01}\, 100\, d\lambda = 100\ln(3/2) = \mathbf{40.55}.$$

Section 25, Spliced Models[204]

A spliced model allows one to have different behaviors for different sizes of loss. For example as discussed below, one could splice together an Exponential Distribution for small losses and a Pareto Distribution for large losses. This would differ from a two-point mixture of an Exponential and Pareto, in which each distribution would contribute its density to all sizes of loss.

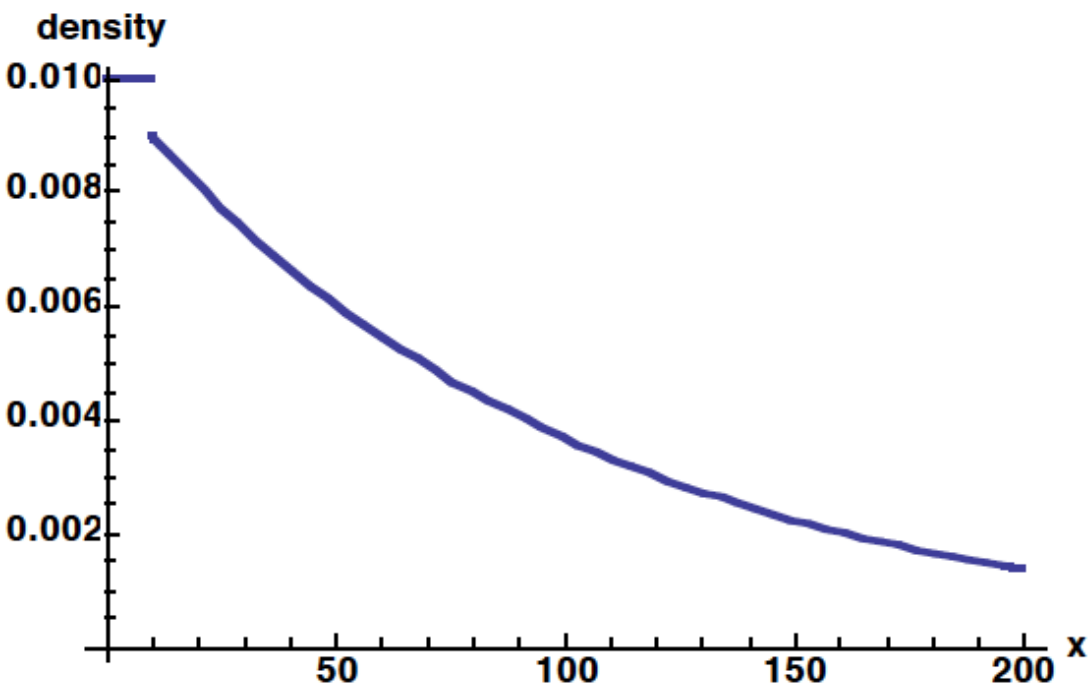
A Simple Example of a Splice:

Assume f(x) = 0.01 for 0 < x < 10, and f(x) = $0.009\ e^{0.1}\ e^{-x/100}$ for x > 10.

Exercise: Show that this f(x) is a density.
[Solution: f(x) ≥ 0.

$$\int_0^\infty f(x)\ dx = \int_0^{10} 0.01\ dx + \int_{10}^\infty 0.009\ e^{0.1}\ e^{-x/100}\ dx = 0.1 + 0.9\ e^{.1}\ e^{-10/100} = 1.]$$

Here is a graph of this density:



We note that this density is discontinuous at 10.

This is an example of a 2-component spliced model. From 0 to 10 it is proportional to a uniform density and above 10 it is proportional to an Exponential density.

[204] See Section 5.2.6 of Loss Models.

<u>Two-Component Splices</u>:

In general a **2-component spliced model** would have density:
$f(x) = w_1 f_1(x)$ on (a_1, b_1) and $f(x) = w_2 f_2(x)$ on (a_2, b_2), where $f_1(x)$ is a density with support (a_1, b_1), $f_2(x)$ is a density with support (a_2, b_2), and $w_1 + w_2 = 1$.

In the example, $f_1(x) = 1/10$ on (0, 10), $f_2(x) = e^{0.1}e^{-x/100}/100$ on (10, ∞), $w_1 = 0.1$, and $w_2 = 0.9$.
f_1 is the uniform distribution on (0, 10). f_2 is proportional to an Exponential with θ = 100.
In order to make f_2 a density on (10, ∞), we have divided by $S(10) = e^{-10/100} = e^{-0.1}$.[205]

<u>A Splice of an Exponential and a Pareto</u>:

Assume an Exponential with θ = 50.
On the interval (0, 100) this would have probability $F(100) = 1 - e^{-2} = 0.8647$.
In order to turn this into a density on (0, 100), we would divide this Exponential density by 0.8647: $(e^{-x/50}/50) / 0.8647 = 0.02313\ e^{-x/50}$.
This integrates to one from 0 to 100.

Assume a Pareto Distribution with α = 3 and θ = 200, with density
$(3)(200^3) / (200 + x)^4 = 0.015/(1 + x/200)^4$.
On the interval (100, ∞) this would have probability $S(100) = (\theta/(\theta + x))^\alpha = (200/300)^3 = 8/27$.
In order to turn this into a density on (100, ∞), we would multiply by 27/8:
$(27/8)\ \{0.015/(1 + x/200)^4\} = 0.050625 / (1 + x/200)^4$.
This integrates to one from 100 to ∞.

So we would have $f_1(x) = 0.02313e^{-x/50}$ on (0, 100), $f_2(x) = 0.050625/(1 + x/200)^4$ on (100, ∞).
We could use any weights w_1 and w_2 as long as they add to one, so that the spliced density will integrate one.

If we took for example, w_1 = 70% and w_2 = 30%, then the spliced density would be:
$(0.7)(0.02313\ e^{-x/50}) = 0.01619\ e^{-x/50}$ on (0, 100), and
$(0.3)\ \{0.050625/(1 + x/200)^4\} = 0.0151875 / (1 + x/200)^4$ on (100, ∞).

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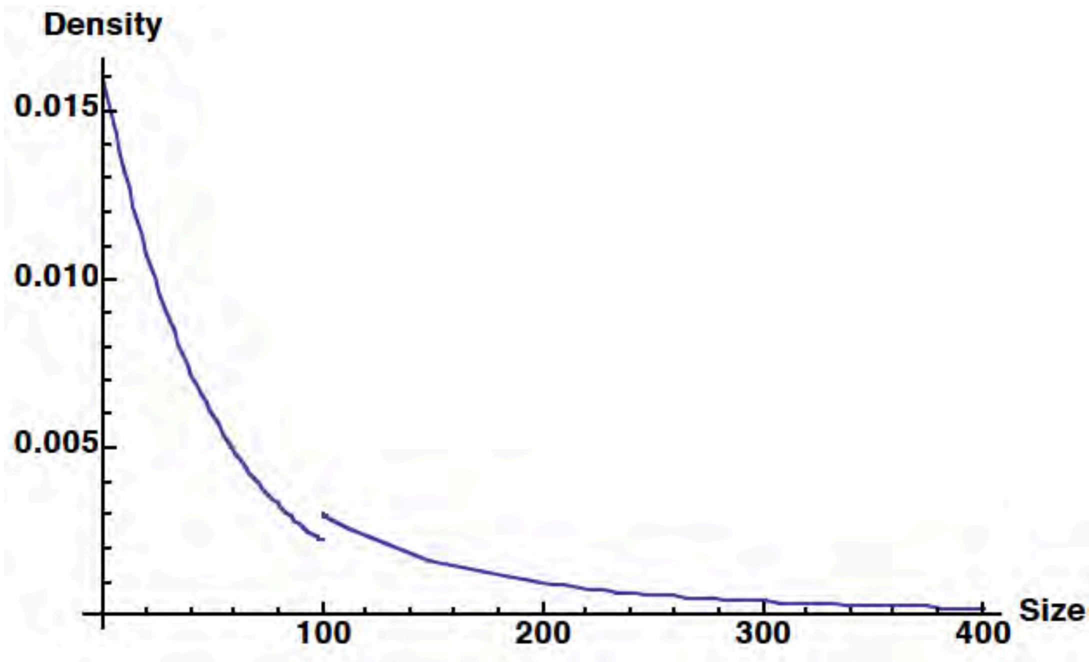
[205] This is how one alters the density in the case of truncation from below.

This 2-component spliced density looks as follows, with 70% of the probability below 100, and 30% of the probability above 100:



It is not continuous at 100.

This spliced density is: $0.01619e^{-x/50}$ on (0, 100), and $0.01519/(1 + x/200)^4$ on (100, ∞) ⇔
$(0.7)(0.02313e^{-x/50})$ on (0, 100), and $(0.3)(0.050625/(1 + x/200)^4)$ on (100, ∞) ⇔
$(0.7)(e^{-x/50}/50)/(1 - e^{-100/50})$ on (0, 100), $(0.3)\{(3)200^3/(200 + x)^4\}/\{(200/300)^3\}$ on (100, ∞) ⇔
(0.8096) (Exponential[50]) on (0, 100), and (1.0125) (Pareto[3, 200]) on (100, ∞).

Exercise: What is the distribution function of this splice at 80?
[Solution: (0.8096)(Exponential Distribution function at 80) = $(0.8096)(1 - e^{-80/50}) = 0.6461$.]

Exercise: What is the survival function of the splice at 300?
[Solution: (1.0125)(Pareto Survival function at 300) = $(1.0125)\{200/(200 + 300)\}^3 = 0.0648$.]

In general, it is easier to work with the distribution function of the first component of the splice below the breakpoint and the survival function of the second component above the breakpoint.
Note that at the breakpoint of 100, the distribution function of the splice is:
(0.8096)(Exponential Distribution function at 100) = $(0.8096)(1 - e^{-100/50}) = 0.700 =$
1 - (1.0125)(Pareto Survival function at 100) = $1 - (1.0125)\{200/(200 + 100)\}^3$.

Assume we had originally written the splice as:[206]
c_1 Exponential[50] on (0, 100), and c_2 Pareto[3, 200] on (100, ∞).

[206] As discussed, while this is mathematically equivalent, this is not the manner in which the splice would be written in Loss Models, which uses f_1, f_2, w_1, and w_2.

Since we chose weights of 70% & 30%, we want:

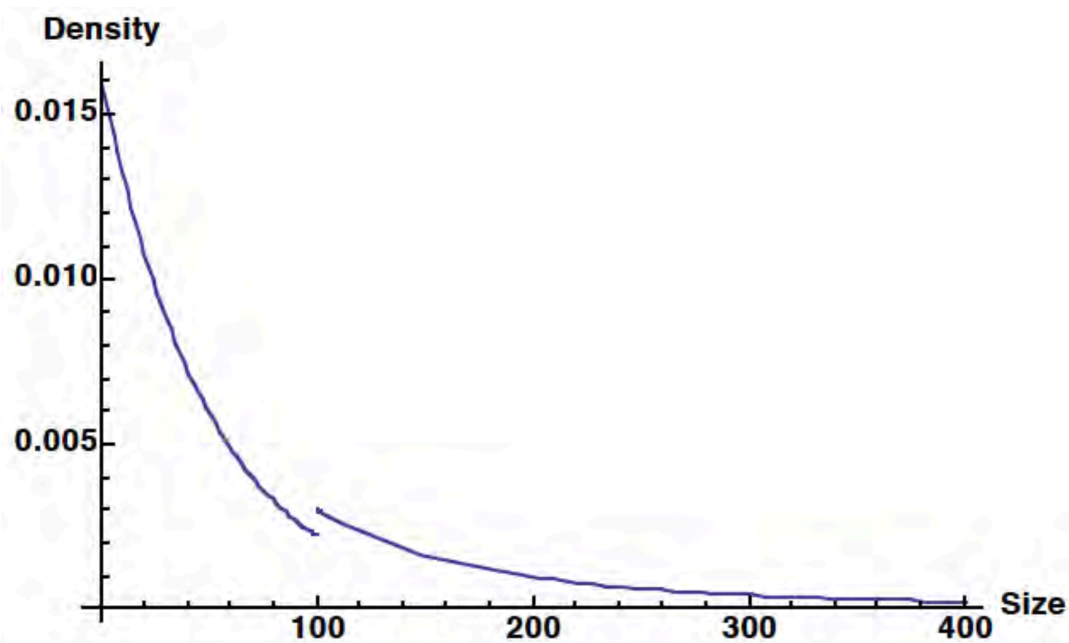
$$70\% = \int_0^{100} c_1 \text{ Exponential[50] } dx = c_1 (1 - e^{-100/50}).$$

$\Rightarrow c_1 = 70\% / (1 - e^{-100/50}) = 0.8096.$

$$30\% = \int_{100}^{\infty} c_2 \text{ Pareto[3, 200] } dx = c_2 \left(\frac{200}{200 + 100}\right)^3 = 0.29630\ c_2.$$

$\Rightarrow c_2 = 30\% / 0.29630 = 1.0125.$

Therefore, as shown previously, the spliced density can be written as:
(0.8096) (Exponential[50]) on (0, 100), and (1.0125) (Pareto[3, 200]) on (100, ∞).

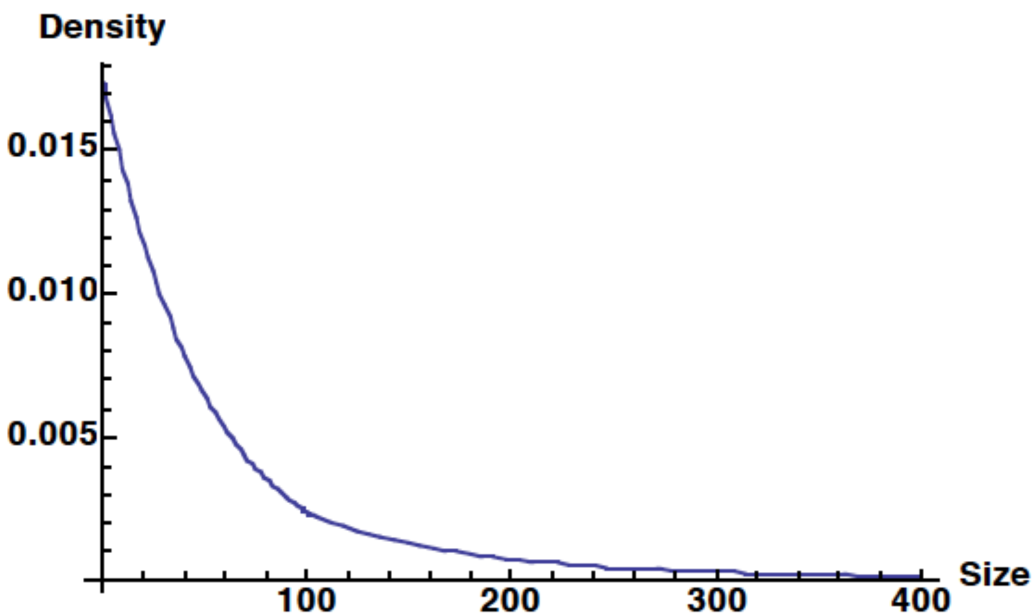


<u>Continuity</u>:

With appropriate values of the weights, a splice will be continuous at its breakpoint.

Exercise: Choose w_1 and w_2 so that the above spliced density would be continuous at 100.
[Solution: $f_1(100) = 0.02313e^{-100/50} = 0.00313$. $f_2(100) = 0.050625 / (1 + 100/200)^4 = 0.01$.
In order to be continuous at 100, we need $w_1 f_1(100) = w_2 f_2(100) = (1- w_1) f_2(100)$.
$w_1 = f_2(100) / \{f_1(100) + f_2(100)\} = 0.01 / (0.00313 + 0.01) = 0.762$. $w_2 = 1 - w_1 = 0.238$.]

If we take f(x): $(0.762)\ (0.02313e^{-x/50}) = 0.01763e^{-x/50}$ on (0, 100), and
$(0.238)\ \{0.050625 / (1 + x/200)^4\} = 0.01205 / (1 + x/200)^4$ on (100, ∞), then f(x) is continuous:



The density of an Exponential Distribution with mean 50 is: $e^{-x/50} / 50$.
The density of a Pareto Distribution with $\alpha = 3$ and $\theta = 200$ is: $\frac{(3)\ (200^3)}{(200 + x)^4} = \frac{3/200}{(1 + x/200)^4}$.

(0.01763)(50) = 0.8815. (0.01205)(200/3) = 0.8033.

Therefore, similar to the noncontinuous splice, we could rewrite this continuous splice as:
(0.8815) (Exponential[50]) on (0, 100), and (0.8033) (Pareto[3, 200]) on (100, ∞).

In general, a **2-component spliced density will be <u>continuous</u> at the breakpoint b**, provided the weights are inversely proportional to the component densities at the breakpoint:
$\mathbf{w_1 = f_2(b) / \{f_1(b) + f_2(b)\},\ \ w_2 = f_1(b) / \{f_1(b) + f_2(b)\}}$.[207]

[207] While this spliced density will be continuous at the breakpoint, it will not be differentiable at the breakpoint.

Moments:

One could compute the moments of a spliced density by integrating x^n f(x).

For example, the mean of this continuous 2-component spliced density is:

$$\int_0^{100} x\ 0.01763\ e^{-x/50}\ dx + \int_{100}^{\infty} \frac{x\ 0.01205}{(1 + x/200)^4}\ dx.$$

Exercise: Compute the integral $\int_0^{100} x\ 0.01763\ e^{-x/50}\ dx$.

[Solution: $\int_0^{100} x\ 0.01763\ e^{-x/50}\ dx = -50\ x\ 0.01763\ e^{-x/50} - 50^2\ 0.01763\ e^{-x/50} \Big]_{x=0}^{x=100} = 26.18.$

Alternately, as discussed previously, the first component of the continuous splice is:
(0.8815) Exponential[50], on (0, 100).

Now $\int_0^{100} x\ f_{exp}(x)\ dx = E[X \wedge 100] - 100\ S_{exp}(100) = 50(1 - e^{-100/50}) - 100\ e^{-100/50} = 29.700.$

Thus $\int_0^{100} x\ 0.01763\ e^{-x/50}\ dx = 0.8815 \int_0^{100} x\ f_{exp}(x)\ dx = (0.8815)(29.700) = 26.18.$

Comment: $\int x\ e^{-x/\theta}\ dx = -\theta\ x\ e^{-x/\theta} - \theta^2\ e^{-x/\theta}$.]

Exercise: Compute the integral $\int_{100}^{\infty} \frac{x\ 0.01205}{(1 + x/200)^4}\ dx$.

[Solution: One can use integration by parts.

$$\int_{100}^{\infty} \frac{x\ 0.01205}{(1 + x/200)^4}\ dx = 0.01205\ x \left.\frac{-200/3}{(1 + x/200)^3}\right]_{x=100}^{x=\infty} - 0.01205 \int_{100}^{\infty} \frac{-200/3}{(1 + x/200)^3}\ dx =$$

$$(0.01205)\ (20{,}000/3)\ \left(\frac{1}{1.5^3} + \frac{1}{1.5^2}\right) = 59.51.$$

Alternately, as discussed previously, the second component of the continuous splice is:
(0.8033) Pareto[3, 200], on (100, ∞).

Now $\int_{100}^{\infty} x\ f_{Pareto}(x)\ dx = \int_{100}^{\infty} (x - 100)\ f_{Pareto}(x)\ dx + 100 \int_{100}^{\infty} f_{Pareto}(x)\ dx =$

$$e_{Pareto}(100)\ S_{Pareto}(100) + 100\ S_{Pareto}(100) = \left(\frac{100 + 200}{3 - 1} + 100\right) \left(\frac{200}{200 + 100}\right)^3 = 74.074.$$

Thus $\int_{100}^{\infty} \frac{x\ 0.01205}{(1 + x/200)^4}\ dx = 0.8033 \int_{100}^{\infty} x\ f_{Pareto}(x)\ dx = (0.8033)(74.074) = 59.51.$

<u>Comment</u>: $e(x) = \int_x^{\infty} (t - x)\ f(t)\ dt\ / S(x)$. For a Pareto Distribution, $e(x) = \frac{x + \theta}{\alpha - 1}$.]

Thus the mean of this continuous splice is:

$$\int_0^{100} x\ 0.01763\ e^{-x/50}\ dx + \int_{100}^{\infty} \frac{x\ 0.01205}{(1 + x/200)^4}\ dx = 26.18 + 59.51 = 85.69.$$

More generally, assume we have a splice which is $w_1\ h_1(x)$ on (0, b) and $w_2\ h_2(x)$ on (b, ∞), where $h_1(x) = f_1(x) / F_1(b)$ and $h_2(x) = f_2(x) / S_2(b)$. Then the mean of this spliced density is:

$$\int_0^b x\ w_1\ h_1(x)\ dx + \int_b^\infty x\ w_2\ h_2(x)\ dx = \frac{w_1}{F_1(b)} \int_0^b x\ f_1(x)\ dx + \frac{w_2}{S_2(b)} \int_b^\infty x\ f_2(x)\ dx =$$

$$\frac{w_1}{F_1(b)} \{E[X_1 \wedge b] - bS_1(b)\} + \frac{w_2}{S_2(b)} \{E[X_2] - \int_0^b x\ f_2(x)\ dx\} =$$

$$\frac{w_1}{F_1(b)} \{E[X_1 \wedge b] + bF_1(b) - b\} + \frac{w_2}{S_2(b)} \{E[X_2] + bS_2(b) - E[X_2 \wedge b]\} =$$

$$\frac{w_1}{F_1(b)} \{E[X_1 \wedge b] - b\} + bw_1 + bw_2 + \frac{w_2}{S_2(b)} \{E[X_2] - E[X_2 \wedge b]\} =$$

$$b + \frac{w_1}{F_1(b)} \{E[X_1 \wedge b] - b\} + \frac{w_2}{S_2(b)} \{E[X_2] - E[X_2 \wedge b]\}.$$

For the example of the continuous splice, b = 100, $F_1(100) = 1 - e^{-100/50} = 0.8647$,
$E[X_1 \wedge b] = 50(1 - e^{-100/50}) = 43.235$,[208] $S_2(b) = (200/300)^3 = 8/27$, $E[X_2] = 200/(3-1) = 100$,
$E[X_2 \wedge b] = 100(1 - (2/3)^2) = 55.556$.[209] Therefore, for $w_1 = 0.762$ and $w_2 = 0.238$, the mean is:
100 + (0.762/0.8647)(43.235 - 100) + (0.238)(27/8)(100 - 55.556) = 85.68,
matching the previous result subject to rounding.

n-Component Splices:

In addition to 2-component splices, one can have 3-components, 4-components, etc.

In a three component splice there are three intervals and the spliced density is:
$w_1\ f_1(x)$ on (a_1, b_1), $w_2\ f_2(x)$ on (a_2, b_2), and $w_3\ f_3(x)$ on (a_3, b_3), where $f_1(x)$ is a density with support (a_1, b_1), $f_2(x)$ is a density with support (a_2, b_2), $f_3(x)$ is a density with support (a_3, b_3), and $w_1 + w_2 + w_3 = 1$.

Previously, when working with grouped data we had discussed assuming a uniform distribution on each interval. This is an example of an n-component splice, with n equal to the number of intervals for the grouped data, and with each component of the splice uniform.

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208 For the Exponential Distribution, $E[X \wedge d] = \theta(1 - e^{-d/\theta})$.

209 For the Pareto Distribution, $E[X \wedge d] = (\theta/(\alpha-1))(1 - (\theta/(d+\theta))^{\alpha-1})$.

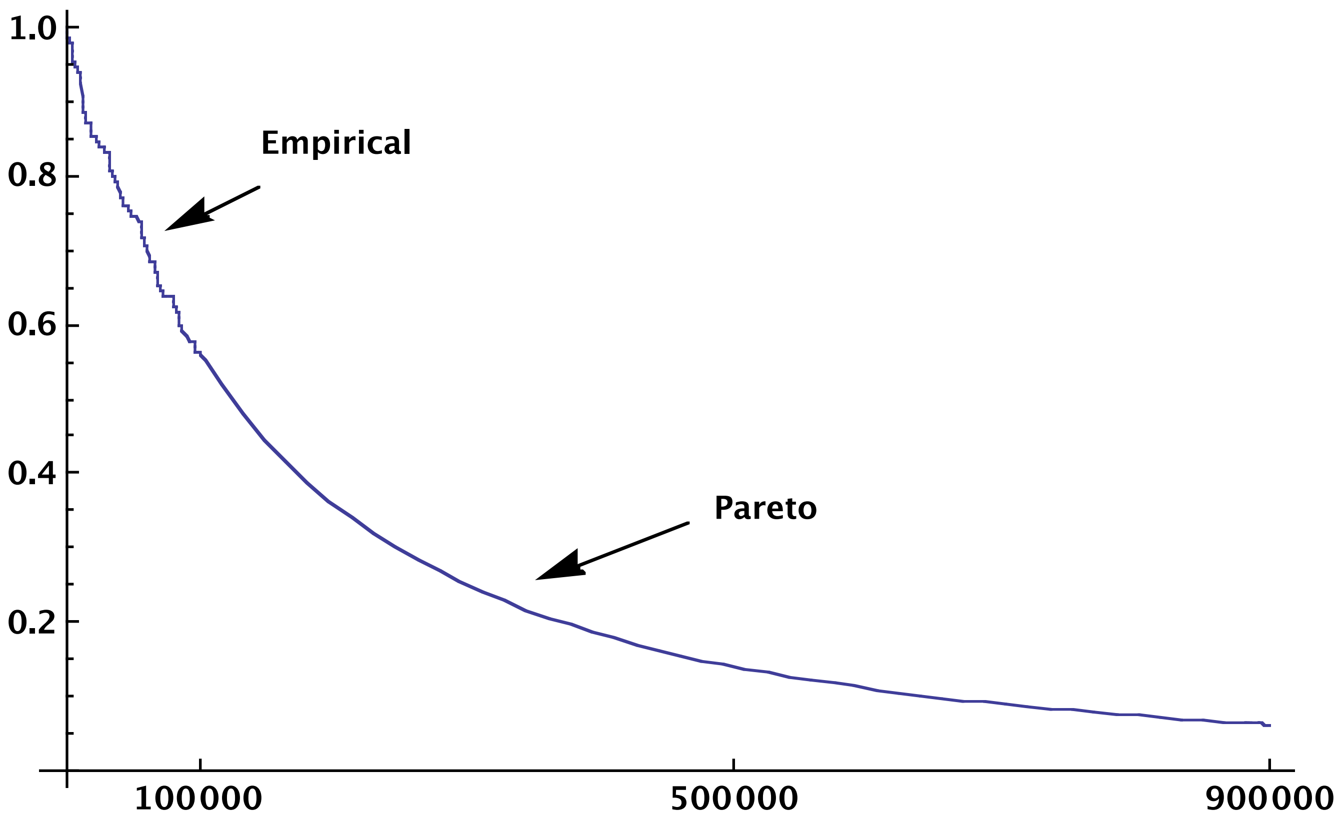
Using the Empirical Distribution:

One common use of splicing, is to use the Empirical Distribution function (or a smoothed version of it) for small losses, and some parametric distribution to model large losses.[210]

For example, take the ungrouped data in Section 1. We could model the losses of size less than 100,000 using the Empirical Distribution Function, and use a Pareto Distribution to model the losses of size greater than 100,000. There are 57 out of 130 losses of size less than 100,000.

Therefore, the Empirical Distribution Function at 100,000 is: 57/130 = 0.4385.
A Pareto Distribution with $\alpha = 2$ and $\theta = 298{,}977$, has $F(100{,}000) = 1 - (298{,}977/398{,}977)^2 = 0.4385$, matching the Empirical Distribution Function.

Thus one could splice together this Pareto Distribution from 100,000 to ∞, and the Empirical Distribution Function from 0 to 100,000. Here is what this spliced survival function looks like:

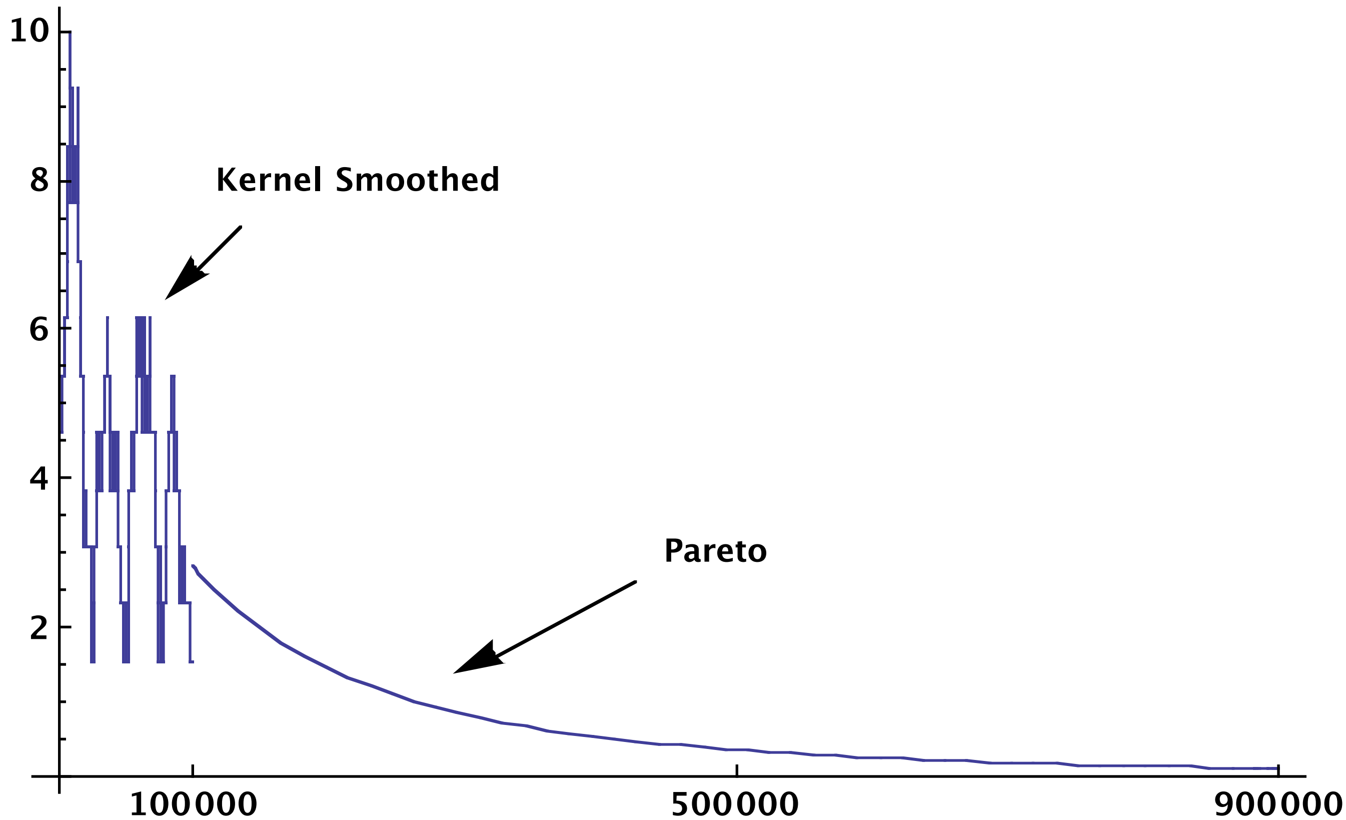


[210] *A variation of this technique is used in "Workers' Compensation Excess Ratios, an Alternative Method," by Howard C. Mahler, PCAS 1998.*

Using a Kernel Smoothed Density:[211]

Rather than use the Empirical Distribution, one could use a kernel smoothed version of the Empirical Distribution. For example, one could splice together the same Pareto Distribution above 100,000, and below 100,000 the kernel smoothed density for the ungrouped data in Section 1, using a uniform kernel with a bandwidth of 5000.

Here is one million times this spliced density:



[211] Kernel Smoothing is discussed in Section 14.6 of Loss Models, not on the syllabus.

Problems:

Use the following information for the next 11 questions:
f(x) = 0.12 for x ≤ 5, and f(x) = $0.06595e^{-x/10}$ for x > 5.

25.1 (1 point) What is the Distribution Function at 10?
A. less than 0.60
B. at least 0.60 but less than 0.65
C. at least 0.65 but less than 0.70
D. at least 0.70 but less than 0.75
E. at least 0.75

25.2 (3 points) What is the mean?
A. less than 5
B. at least 5 but less than 6
C. at least 6 but less than 7
D. at least 7 but less than 8
E. at least 8

25.3 (4 points) What is the variance?
A. less than 70
B. at least 70 but less than 75
C. at least 75 but less than 80
D. at least 80 but less than 85
E. at least 85

25.4 (5 points) What is the skewness?
A. less than 2.6
B. at least 2.6 but less than 2.7
C. at least 2.7 but less than 2.8
D. at least 2.8 but less than 2.9
E. at least 2.9

25.5 (3 points) What is $E[X \wedge 3]$?
A. less than 2.2
B. at least 2.2 but less than 2.3
C. at least 2.3 but less than 2.4
D. at least 2.4 but less than 2.5
E. at least 2.5

25.6 (3 points) What is the loss elimination ratio at 5?
A. less than 40%
B. at least 40% but less than 45%
C. at least 45% but less than 50%
D. at least 50% but less than 55%
E. at least 55%

25.7 (3 points) What is $E[(X-20)_+]$?
A. less than 0.75
B. at least 0.75 but less than 0.80
C. at least 0.80 but less than 0.85
D. at least 0.85 but less than 0.90
E. at least 0.90

25.8 (2 points) What is e(20)?
A.7 B. 8 C. 9 D. 10 E. 11

25.9 (1 point) What is the median?
A. 4.2 B. 4.4 C. 4.6 D. 4.8 E. 5.0

25.10 (2 points) What is the 90th percentile?
A. 18 B. 19 C. 20 D. 21 E. 22

25.11 (3 points) The size of loss for the Peregrin Insurance Company follows the given f(x).
The average annual frequency is 138.
Peregrin Insurance buys reinsurance from the Meriadoc Reinsurance Company for 5 excess of 15.
How much does Meriadoc expect to pay per year for losses from Peregrin Insurance?
A. 60 B. 70 C. 80 D. 90 E. 100

25.12 (3 points) For a two-component spliced model:
(i) Up to 500 it is proportional to a Weibull Distribution with θ = 600 and τ = 3.
(ii) Above 500 it is proportional to a Weibull Distribution with θ = 400 and τ = 2.
(iii) It is continuous.
Calculate the probability in the interval from 400 to 600.
A. 39% B. 41% C. 43% D. 45% E. 47%

Use the following information for the next 3 questions:
One has a two component splice, which is proportional to an Exponential Distribution with mean 3 for loss sizes less than 5, and is proportional to a Pareto Distribution with α = 4 and θ = 60 for loss sizes greater than 5. The splice is continuous at 5.

25.13 (2 points) What is the distribution function of the splice at 2?
A. less than 20%
B. at least 20% but less than 25%
C. at least 25% but less than 30%
D. at least 30% but less than 35%
E. at least 35%

25.14 (2 points) What is the survival function of the splice at 10?
A. less than 35%
B. at least 35% but less than 40%
C. at least 40% but less than 45%
D. at least 45% but less than 50%
E. at least 50%

25.15 (3 points) What is the mean of this splice?
A. less than 14.0
B. at least 14.0 but less than 14.5
C. at least 14.5 but less than 15.0
D. at least 15.0 but less than 15.5
E. at least 15.5

25.16 (4 points) In 2008, the size of monthly pension payments for a group of retired municipal employees follows a Single Parameter Pareto Distribution, with α = 2 and θ = \$1000.
The city announces that for 2009, there will be a 5% cost of living adjustment (COLA.)
However. the COLA will only apply to the first \$2000 in monthly payments.
What is the probability density function of the size of monthly pension payments in 2009?

25.17 (2 points) X follows a two-component splice with a constant density from zero to 200.
From 200 to infinity the splice is proportional to an Exponential Distribution with mean 500.
If the splice is continuous at 200, what is the probability that X is less than or equal to 300?
A. 40% B. 42% C. 44% D. 46% E. 48%

25.18 (3 points) You are given the following grouped data:

Range	# of claims	loss
0-1	6300	3000
1-2	2350	3500
2-3	850	2000
3-4	320	1000
4-5	110	500
over 5	70	500
	10,000	10,500

What is the mean of a 2-component splice between the empirical distribution below 4 and an Exponential with $\theta = 1.5$?

A. less than 1.045
B. at least 1.045 but less than 1.050
C. at least 1.050 but less than 1.055
D. at least 1.055 but less than 1.060
E. at least 1.060

Use the following information for the next 2 questions:

$$f(x) = \begin{cases} \dfrac{617{,}400}{218\,(10+x)^4}, & 0 < x \le 4 \\ \dfrac{3920}{25\,(10+x)^3}, & x > 4 \end{cases}.$$

25.19 (2 points) Determine the probability that X is greater than 2.

A. 54% B. 56% C. 58% D. 60% E. 62%

25.20 (4 points) Determine E[X].

A. 4 B. 5 C. 6 D. 7 E. 8

25.21 (SOA M, 11/05, Q.35 & 2009 Sample Q.211) (2.5 points)
An actuary for a medical device manufacturer initially models the failure time for a particular device with an exponential distribution with mean 4 years.
This distribution is replaced with a spliced model whose density function:
(i) is uniform over [0, 3]
(ii) is proportional to the initial modeled density function after 3 years
(iii) is continuous
Calculate the probability of failure in the first 3 years under the revised distribution.
(A) 0.43 (B) 0.45 (C) 0.47 (D) 0.49 (E) 0.51

25.22 (CAS3, 11/06, Q.18) (2.5 points) A loss distribution is a two-component spliced model using a Weibull distribution with θ_1 = 1,500 and τ = 1 for losses up to $4,000, and a Pareto distribution with θ_2 = 12,000 and α = 2 for losses $4,000 and greater.
The probability that losses are less than $4,000 is 0.60.
Calculate the probability that losses are less than $25,000.
A. Less than 0.900
B. At least 0.900, but less than 0.925
C. At least 0.925, but less than 0.950
D. At least 0.950, but less than 0.975
E. At least 0.975

Solutions to Problems:

25.1. E. $F(10) = \int_0^5 0.12\ dx + \int_5^{10} 0.06595\ e^{-x/10}\ dx = (5)(0.12) + 0.6595(e^{-5/10} - e^{-10/10})$

$= \mathbf{0.757}$.

Alternately, $S(10) = \int_{10}^{\infty} 0.06595\ e^{-x/10}\ dx = 0.6595e^{-10/10} = 0.243$.

$\Rightarrow F(10) = 1 - 0.243 = \mathbf{0.757}$.

25.2. D. mean $= \int_0^5 x\ 0.12\ dx + \int_5^{\infty} x\ 0.06595\ e^{-x/10}\ dx =$

$0.06\ x^2 \Big]_{x=0}^{x=5} - 0.06595\ \{10xe^{-x/10} + 100e^{-x/10}\} \Big]_{x=5}^{x=\infty} = 1.5 + 6.0 = \mathbf{7.5}$.

25.3. C. 2nd moment $= \int_0^5 x^2\ 0.12\ dx + \int_5^{\infty} x^2\ 0.06595\ e^{-x/10}\ dx =$

$0.04\ x^3 \Big]_{x=0}^{x=5} - 0.06595\{10x^2e^{-x/10} + 200xe^{-x/10} + 2000e^{-x/10}\} \Big]_{x=5}^{x=\infty} = 5 + 130 = 135$.

Variance $= 135 - 7.5^2 = \mathbf{78.75}$.

25.4. A. 3rd moment $= \int_0^5 x^3\ 0.12\ dx + \int_5^{\infty} x^3\ 0.06595\ e^{-x/10}\ dx =$

$0.03\ x^4 \Big]_{x=0}^{x=5} - 0.06595\ \{10x^3e^{-x/10} + 300x^2e^{-x/10} + 6000xe^{-x/10} + 60{,}000e^{-x/10}\} \Big]_{x=5}^{x=\infty}$

$= 18.75 + 3950 = 3968.75$.

Skewness $= \dfrac{3968.5 - (3)(7.5)(135) + (2)(7.53)}{78.75^{1.5}} = \mathbf{2.54}$.

25.5. D. $E[X \wedge 3] = \int_0^3 x\, 0.12\, dx + 3S(3) = 0.06\, x^2 \Big]_{x=0}^{x=3} - (3)\{1 - (0.12)(3)\} = 0.54 + 1.92 =$ **2.46**.

25.6. C. $E[X \wedge 5] = \int_0^5 x\, 0.12\, dx + 5S(5) = 0.06\, x^2 \Big]_{x=0}^{x=5} - (5)\{1 - (0.12)(5)\} = 1.5 + 2 = 3.5.$

$E[X \wedge 5] / E[X] = 3.5 / 7.5 =$ **46.7%**.

Alternately, since for x > 5, the density is proportional to an Exponential,
$f(x) = 0.06595e^{-x/10}$ for x > 5, $S(x) = 0.6595e^{-x/10}$ for x > 5.

The layer from 5 to infinity is: $\int_5^\infty 0.6595\, e^{-x/10}\, dx = 4.00.$

The loss elimination ratio at 5 is: 1 - 4/7.5 = **46.7%**.
Comment: $1 - (0.12)(5) = S(5) = 0.6595e^{-5/10} = 0.400.$

25.7. D. $E[X \wedge 20] = \int_0^5 x\, 0.12\, dx + \int_5^{20} x\, 0.06595\, e^{-x/10}\, dx + 20S(20) =$

$$0.06\, x^2 \Big]_{x=0}^{x=5} - 0.06595\, \{10xe^{-x/10} + 100e^{-x/10}\} \Big]_{x=5}^{x=20} + (20)(0.06595)(10e^{-2})$$

$= 1.5 + 3.323 + 1.785 = 6.608.$ $E[(X-20)_+] = E[X] - E[X \wedge 20] = 7.5 - 6.608 =$ **0.892**.

Alternately, $E[(X-20)_+] = \int_{20}^\infty 0.06595\, e^{-x/10}\, (x - 20)\, dx =$

$0.06595\, \{-10xe^{-x/10} + 100e^{-x/10}\} \Big]_{x=20}^{x=\infty} =$ **0.8925**.

Alternately, $E[(X-20)_+] = \int_{20}^\infty S(x)\, dx = \int_{20}^\infty 0.06595\, e^{-x/10}\, dx = -6.595e^{-x/10} \Big]_{x=20}^{x=\infty} =$ **0.8925**.

25.8. D. Beyond 5 the density is proportional to an Exponential density with mean 10, and therefore, beyond 5, the mean residual life is a constant **10**. Alternately,
$S(20) = (0.06595)(10e^{-2}) = 0.08925.$ $e(20) = E[(X-20)_+] / S(20) = 0.8925/0.08925 =$ **10.0**.

25.9. A. $f(x) = 0.12$ for $x \le 5$. ⇒ F(5) = 0.60. ⇒ median = 0.5 / 0.12 = **4.167**.

25.10. B. f(x) = 0.12 for x ≤ 5. ⇒ F(5) = 0.6. f(x) = $0.06595e^{-x/10}$ for x > 5.

⇒ $F(x) = 0.6 + \int_5^x 0.06595\, e^{-t/10}\, dt = 0.6 + 0.6595e^{-5/10} - 0.6595e^{-x/10}$, x > 5.

Require that: $0.9 = 0.6 + 0.6595e^{-5/10} - 0.6595e^{-x/10}$. ⇒ $e^{-x/10} = 0.15164$. ⇒ x = **18.86**.

25.11. C. $E[X \wedge 20] - E[X \wedge 15] = \int_{15}^{20} S(x)\, dx = \int_{15}^{20} 0.6595\, e^{-x/10}\, dx = -6.595e^{-x/10} \Big]_{x=15}^{x=20}$

= 0.579. Meriadoc reinsures the layer from 15 to 20, so it expects to pay: (138)(0.579) = **79.9**.

25.12. D. Let the splice be: α Weibull[600, 3] below 500, and β Weibull[400, 2] above 500.
We will need to solve for these proportionality constants.
For a Weibull Distribution with θ = 600 and τ = 3: $F(500) = 1 - \exp[-(500/600)^3] = 0.4394$.
For a Weibull Distribution with θ = 400 and τ = 2: $S(500) = \exp[-(500/400)^2] = 0.2096$.
Thus, for the total probability to be one: α 0.4394 + β 0.2096 = 1.
For a Weibull Distribution with θ = 600 and τ = 3:
$f(500) = (3)(500^2)\exp[-(500/600)^3] / 600^3 = 0.0019466$.
For a Weibull Distribution with θ = 400 and τ = 2:
$f(500) = (2)(500) \exp[-(500/400)^2] / 400^2 = 0.0013101$.
Thus, for the splice to be continuous at 500: α 0.0019466 = β 0.0013101. ⇒ β = 1.4858α.
⇒ α 0.4394 + (1.4858α) 0.2096 = 1. ⇒ α = 1.332. ⇒ β = 1.979.
For a Weibull Distribution with θ = 600 and τ = 3: $F(400) = 1 - \exp[-(400/600)^3] = 0.2564$.
For a Weibull Distribution with θ = 400 and τ = 2: $S(600) = \exp[-(600/400)^2] = 0.1054$.
Thus for the splice, the probability in the interval from 400 to 600 is:
(1.332)(0.4394 - 0.2564) + (1.979)(0.2096 - 0.1054) = **45.0%**.
Alternately, for the splice: F(400) = (1.332)(0.2564) = 0.3415.
S(600) = (1.979)(0.1054) = 0.2086. ⇒ F(600) = 0.7914
⇒ Probability in the interval from 400 to 600 is: 0.7914 - 0.3415 = **45.0%**.

25.13. C. Let the splice be: a(Exponential), x < 5, and b(Pareto), x > 5.
The splice must integrate to unity from 0 to ∞:
1 = a(Exponential Distribution at 5) + b(1 - Pareto Distribution at 5). ⇒
$1 = a(1 - e^{-5/3}) + b(60/65)^4$. ⇒ 1 = 0.8111a + 0.7260b.
The density of the Exponential is: $e^{-x/3}/3$. $f(5) = e^{-5/3}/3 = 0.06296$.
The density of the Pareto is: $(4)(60^4)/(x+60)^5$. $f(5) = (4)(60^4)/(65^5) = 0.04468$
Also in order for the splice to be continuous at 5:
a(Exponential density @ 5) = b(Pareto density @5) ⇒ a(0.06296) = b(0.04468).
⇒ b = 1.4091a. ⇒ 1 = 0.8111a + 0.7260(1.4091a). ⇒ a = 0.545.
⇒ the distribution function at 2 is: 0.545(Exponential Distribution at 2) = $0.545(1 - e^{-2/3})$ = **0.265**.

25.14. C. Continuing the previous solution, b = 1.4091a = 0.768.
⇒ the survival function at 10 is: 0.768(Pareto survival function at 10) = $0.768(60/70)^4$ = **0.415**.
Alternately, the distribution function at 10 is:

$$0.545 \int_0^5 \frac{e^{-x/3}}{3} \, dx + 0.768 \int_5^{10} \frac{(4)(60^4)}{(x+60)^5} \, dx =$$

0.545 (Exponential distribution function at 5) +
0.768 (Pareto distribution function at 10 - Pareto distribution function at 5) =
$0.545 (1 - e^{-5/3}) + 0.768\{(1 - (60/70)^4) - (1 - (60/65)^4)\}$
= (0.545)(0.811) + (0.768)(0.186) = 0.585.
Therefore, the survival function at 10 is: 1 - 0.585 = **0.415**.

25.15. E. mean = $0.545 \int_0^5 x \frac{e^{-x/3}}{3} \, dx + 0.768 \int_5^{\infty} x \frac{(4)(60^4)}{(x+60)^5} \, dx =$

The first integral is for an Exponential Distribution: $E[X \wedge 5] - 5S(5) = 3(1 - e^{-5/3}) - 5e^{-5/3} = 1.49$.

The second integral is for an Pareto Distribution: $E[X] - \int_0^5 x \, f_{Pareto}(x) \, dx =$

$E[X] - \{E[X \wedge 5] - 5S(5)\} = E[X] - E[X \wedge 5] + 5S(5) = (60/3)(60/65)^3 + (5)(60/65)^4 = 19.36$.
Thus the mean of the splice is: (0.545)(1.49) + (0.768)(19.36) = **15.68**.
Alternately, the second integral is for an Pareto:

$$\int_5^{\infty} x \, f_{Pareto}(x) \, dx = \int_5^{\infty} (x-5) \, f_{Pareto}(x) \, dx + 5 \int_5^{\infty} f_{Pareto}(x) \, dx = S_{Pareto}(5) \, e_{Pareto}(5) + 5 \, S_{Pareto}(5)$$

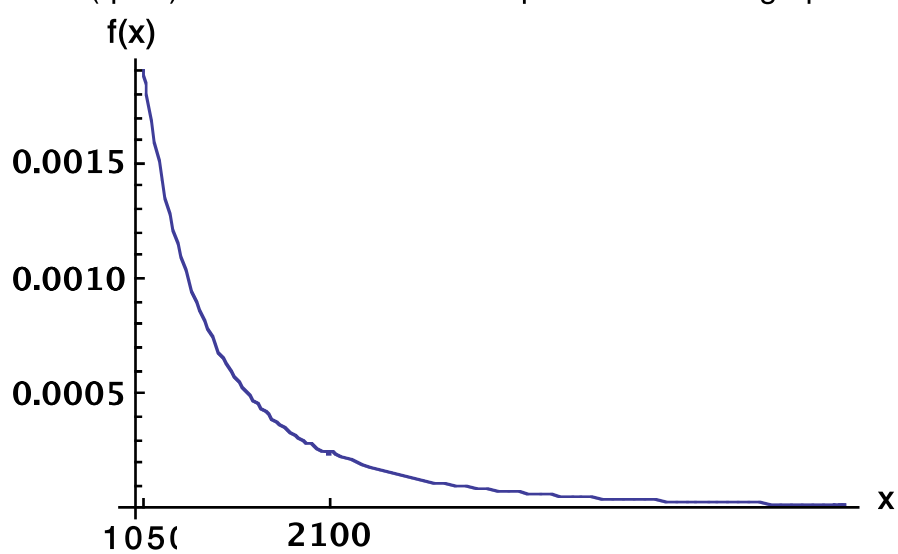
$= (60/65)^4 \frac{5 + 60}{4 - 1} + (5)(60/65)^4 = 19.36$. Proceed as before.

Comment: $e(x) = \int_x^{\infty} (t - x) f(t) \, dt \, / \, S(x)$. For a Pareto Distribution, $e(x) = \frac{x + \theta}{\alpha - 1}$.

25.16. In 2008, S(2000) = $(1000/2000)^2$ = 1/4.
For a Single Parameter Pareto Distribution, f(x) = $\alpha\theta^\alpha / x^{\alpha+1}$, x > θ.
For those whose payments are less than $2000 per month, the payment is multiplied by 1.05.
Thus in 2009 they follow a Single Parameter Pareto with α = 2 and θ = $1050.
The density is proportional to: f(x) = $2(1050^2) / x^3$, x > 1050.
For those whose payments are $2000 or more per month, their payment is increased by (2000)(5%) = 100. S(x) = $\{1000/(x-100)\}^2$, x > 2100.
The density is proportional to: f(x) = $2(1000^2) / (x - 100)^3$.
In 2009, the density is a splice, with 3/4 weight to the first component and 1/4 weight to the second component. Someone with $2000 in 2008, will get $2100 in 2009; $2100 is the breakpoint of the splice.
f(x) = $2(1050^2) / x^3$, x > 1050, would integrate from 1050 to 2100 to the distribution function of at 2100 of a Single Parameter Pareto with α = 2 and θ = $1050, 1 - $(1050/2100)^2$ = 3/4.
This is the desired weight for the first component, so this is OK.
f(x) = $2(1000^2) / (x - 100)^3$, would integrate from 2100 to ∞: $(1000^2) / (2100 - 100)^2$ = 1/4.
This is the desired weight for the second component, so this is OK.
The probability density function of the size of monthly pension payments in 2009 is a splice:

$$f(x) = \begin{cases} 2\,(1050^2) / x^3, & 2100 > x > 1050 \\ 2\,(1000^2) / (x - 100)^3, & x > 2100 \end{cases}.$$

<u>Comment</u>: Coming from the left, f(2100) = $2(1050^2) / 2100^3$ = 1/4200.
Coming from the right, f(2100) = $2(1000^2) / (2100 - 100)^3$ = 1/4000. Thus the density of this splice is not (quite) continuous at the breakpoint of 2100. A graph of this splice:



25.17. B. In order to integrate to one from 200 to infinity, the density of the exponential piece would have to be: (exp[-x/500] / 500) / exp[-200/500] = exp[-(x-200)/500] / 500.
Let w be the weight given to the uniform piece and 1- w be the weight given to the exponential piece. To be continuous at 200: w (1/200) = (1-w) exp[-(200-200)/500] / 500 = (1-w)/500. ⇒
w = 0.2857. ⇒ S(300) = (1 - 0.2857) exp[-(300-200)/500] = 0.5848. ⇒ F(300) = **0.4152**.

25.18. B. The empirical survival function at 4 is: (110 + 70)/10,000 = 0.018.
Above 4 the splice is proportional to an Exponential with survival function $e^{-x/1.5}$.
Let w be the weight applied to this Exponential. Matching S(4), set $0.018 = we^{-4/1.5}$. w = 0.259.
The contribution to the mean from the losses of size less than 4 is:
(3000 + 3500 + 2000 + 1000)/10,000 = 0.9500.
The contribution to the mean from the losses of size greater than 4 is:

$$0.259 \int_4^{\infty} x\, e^{-x/1.5}/1.5 \ dx = -0.259 \ (x\, e^{-x/1.5} + 1.5e^{-x/1.5}) \Big]_{x=4}^{x=\infty} = 0.0990.$$

mean = 0.9500 + 0.0990 = **1.0490**.

25.19. D. $F(2) = \int_0^2 \frac{617{,}400}{218\ (10 + x)^4} dx = \frac{617{,}400}{218} \left\{\frac{1}{(3)\ (10^3)} - \frac{1}{(3)\ (12^3)}\right\} = 0.3977.$

S(2) = 1 - 0.3977 = **0.6023**.
Comment: Via integration, one can determine that the first component of the splice has a total probability of 3/5, while the second component of the splice has a total probability of 2/5.

25.20. E. One can use integration by parts.

$$\int_0^4 \frac{x}{(10+x)^4}\,dx = \left.\frac{-x}{3\,(10+x)^3}\right]_{x=0}^{x=4} + \int_0^4 \frac{1}{3\,(10+x)^3}\,dx = -\frac{4}{(3)\,(14^3)} - \frac{1}{(2)\,(3)\,(14^2)} + \frac{1}{(2)\,(3)\,(10^2)} =$$

0.00033042.

Thus $\int_0^4 x\,\frac{617{,}400}{218\,(10+x)^4}\,dx = \frac{617{,}400}{218}\ 0.00033042 = 0.9358.$

$$\int_4^\infty \frac{x}{(10+x)^3}\,dx = \left.\frac{-x}{2\,(10+x)^2}\right]_{x=4}^{x=\infty} + \int_4^\infty \frac{1}{2\,(10+x)^2}\,dx = \frac{4}{(2)\,(14^2)} + \frac{1}{(2)\,(14)} = 9/196.$$

Therefore, $\int_4^\infty x\,\frac{3920}{25\,(10+x)^3}\,dx = \frac{3920}{25}\ (9/196) = 7.2000.$

Thus, $E[X] = \int_0^4 x\,\frac{617{,}400}{218\,(10+x)^4}\,dx + \int_4^\infty x\,\frac{3920}{25\,(10+x)^3}\,dx = 0.9358 + 7.2000 =$ **8.1358**.

Alternately, each component of the splice is proportional to the density of a Pareto Distribution.

The density of a Pareto with $\alpha = 3$ and $\theta = 10$ is: $\frac{(3)\,(10^3)}{(10+x)^4}$.

Thus the first component of the splice is: $\frac{617{,}400}{218}\ \frac{1}{3000}$ Pareto[3, 10] = $\frac{2058}{2180}$ Pareto[3, 10].

Now $\int_0^4 x\, f_{Pareto}(x)\,dx = E[X \wedge 4] - 4\,S(4) = (10/2)\,\{1 - (10/14)^2\} - (4)(10/14)^3 = 0.99125.$

Therefore, $\int_0^4 x\,\frac{617{,}400}{218\,(10+x)^4}\,dx = \frac{2058}{2180}\ 0.99125 = 0.9358.$

The density of a Pareto with $\alpha = 2$ and $\theta = 10$ is: $\frac{(2)\,(10^2)}{(10+x)^4}$.

Thus the second component of the splice is: $\frac{3920}{25}\ \frac{1}{200}$ Pareto[2, 10] = 0.784 Pareto[2, 10].

Now $\int_4^\infty x\, f_{Pareto}(x)\,dx = \int_4^\infty (x-4)\, f_{Pareto}(x)\,dx + \int_4^\infty 4\, f_{Pareto}(x)\,dx = e(4)\,S(4) + 4\,S(4)$

$= (14/1)\,(10/14)^2 + (4)\,(10/14)^2 = 9.1837.$

Therefore, $\int_4^{\infty} x \frac{3920}{25\,(10 + x)^3}\,dx = (0.784)\,(9.1837) = 7.2000.$

Thus, $E[X] = \int_0^4 x \frac{617{,}400}{218\,(10 + x)^4}\,dx + \int_4^{\infty} x \frac{3920}{25\,(10 + x)^3}\,dx = 0.9358 + 7.2000 = \mathbf{8.1358}.$

<u>Comment</u>: $e(x) = \int_x^{\infty} (t - x)\,f(t)\,dt \;/\; S(x)$. For a Pareto Distribution, $e(x) = \frac{x + \theta}{\alpha - 1}$.

25.21. A. A uniform on [0, 3] has density of 1/3.
On the interval 3 to ∞, we want something proportional to an Exponential with $\theta = 4$.
From 3 to ∞ this Exponential density would integrate to $S(3) = e^{-3/4}$.
Therefore, something proportional that would integrate to one is: $0.25e^{-x/4}/e^{-3/4} = 0.25e^{-(x-3)/4}$.
Thus the density of the splice is: $w(1/3)$ from 0 to 3, and $(1 - w)0.25e^{-(x-3)/4}$ from 3 to ∞.
In order to be continuous, the two densities must match at 3:
$w(1/3) = (1 - w)0.25e^{-(3-3)/4}$. ⇒ $4w = 3(1 - w)$. ⇒ $w = 3/7 = 0.429$.
Probability of failure in the first 3 years is the integral of the splice from 0 to 3: w = **0.429**.

25.22. C. For the Pareto, $S(4000) = \{12/(12 + 4)\}^2 = 9/16$.
The portion of the splice above $4000 totals 1 - 0.6 = 40% probability.
Therefore, the portion of the splice above 4000 is: (0.4) Pareto[2, 12,000] / (9/16).
For the Pareto, $S(25{,}000) = \{12/(12 + 25)\}^2 = 0.1052$.
Therefore, for the splice the probability that losses are greater than $25,000 is:
(.4)(0.1052)/(9/16) = 0.0748. 1 - 0.0748 = **0.9252**.
<u>Comment</u>: A Weibull with $\tau = 1$ is an Exponential.
It is easier to calculate S(25,000), and then F(25,000) = 1 - S(25,000).
When working above the breakpoint of $4000, work with the Pareto.
If working below the breakpoint of $4000, work with the Weibull.
It might have been better if the exam question had read instead:
"A loss distribution is a two-component spliced model using a density proportional to a Weibull distribution with $\theta_1 = 1{,}500$ and $\tau = 1$ for losses up to $4,000, and a density proportional to a Pareto distribution with $\theta_2 = 12{,}000$ and $\alpha = 2$ for losses $4,000 and greater."
The density above 4000 is proportional to a Pareto.
The original Pareto integrates to 9/16 from 4000 to infinity.
In order to get the density from 4000 to infinity to integrate to the desired 40%, we need to multiply the density of the original Pareto by: 40% / (9/16).

Section 26, Important Ideas & Formulas

Empirical Distribution Function (Section 4):

The Empirical Distribution Function at x: (# of losses ≤ x)/(# losses).
The Empirical Distribution Function has mean of F(x) and a variance of: F(x){1-F(x)}/N.

S(x) = 1 - F(x) = the Survival Function.

Limited Losses (Section 5):

$X \wedge L \equiv$ Minimum of x and L = Limited Loss Variable.

The **Limited Expected Value** at L = $E[X \wedge L]$ = E[Minimum[L, x]].

$$E[X \wedge L] = \int_0^L x\, f(x)\, dx + L S(L)$$

= contribution of small losses + contribution of large losses.

mean = $E[X \wedge \infty]$. $E[X \wedge x] \le x$. $E[X \wedge x] \le$ mean.

Losses Eliminated (Section 6):

N = the total number of accidents or loss events.

Losses Eliminated by a deductible of size d = $N \int_0^d x\, f(x)\, dx + N\, d\, S(d) = N\, E[X \wedge d]$.

Loss Elimination Ratio (LER) = $\dfrac{\textbf{Losses Eliminated by a deductible of size d}}{\textbf{Total Losses}}$.

LER(x) = $\dfrac{E[X \wedge x]}{E[X]}$.

Excess Losses (Section 7):

$(X - d)_+ \equiv 0$ when $X \le d$, $X - d$ when $X > d$ ⇔ left censored and shifted variable at d ⇔ amounts paid to insured with a deductible of d.

Excess Ratio = R(x) = (Losses Excess of x) / (total losses) = $E[(X - d)_+] / E[X]$.
R(x) = 1 - LER(x) = 1 - { $E[X \wedge x]$ / mean }.

Total Losses = Limited Losses + Excess Losses: $X = (X \wedge d) + (X - d)_+$.
$E[(X - d)_+] = E[X] - E[(X \wedge d)]$.

Excess Loss Variable (Section 8):

Excess Loss Variable for d ≡ X - d for X > d, undefined for X ≤ d ⇔
the nonzero payments excess of deductible d.

Mean Residual Life or Mean Excess Loss = e(x)
= the average dollars of loss above x on losses of size exceeding x.

$$\mathbf{e(x)} = \frac{E[X] - E[X \wedge x]}{S(x)}.$$

e(x) = (average size of those claims of size greater than x) - x.

Failure rate, force of mortality, or **hazard rate** = h(x) = f(x)/S(x) = $- \frac{d \ln(S(x))}{dx}$.

Layers of Loss (Section 9):

The percentage of losses in the layer from d to u =

$$\frac{\int_d^u (x - d) f(x)\, dx + S(u)(u - d)}{\int_0^\infty x\, f(x)\, dx} = \frac{E[X \wedge u] - E[X \wedge d]}{E[X]} = \text{LER}(u) - \text{LER}(d) = R(d) - R(u).$$

Layer Average Severity (LAS) for the layer from d to u =
The mean losses in the layer from d to u = E[X∧u] - E[X∧d] =
{LER(u) - LER(d)} E[X] = {R(d) - R(u)} E[X].

Average Size of Losses in an Interval (Section 10):

The average size of loss for those losses of size between a and b is:

$$\frac{\int_a^b x\, f(x)\, dx}{F(b) - F(a)} = \frac{\{E[X \wedge b] - b\, S(b)\} - \{E[X \wedge a] - a\, S(a)\}}{F(b) - F(a)}.$$

Proportional of Total Losses from Losses in the Interval [a, b] is:

$$\frac{\{E[X \wedge b] - b\, S(b)\} - \{E[X \wedge a] - a\, S(a)\}}{E[X]}.$$

<u>Policy Provisions (Section 11)</u>:

An <u>ordinary</u> deductible is a provision which states that when the loss is less than or equal to the deductible, there is no payment and when the loss exceeds the deductible, the amount paid is the loss less the deductible.

The Maximum Covered Loss is the size of loss above which no additional payments are made.

A **coinsurance factor** is the proportion of any loss that is paid by the insurer after any other modifications (such as deductibles or limits) have been applied.
A **coinsurance** is a provision which states that a coinsurance factor is to be applied.

The order to operations is:
1. Limit the size of loss to the maximum covered loss.
2. Subtract the deductible. If the result is negative, set the payment equal to zero.
3. Multiply by the coinsurance factor.

A policy limit is maximum possible payment on a single claim.
Policy Limit = c(u - d). Maximum Covered Loss = u = d + (Policy Limit)/c.
With no deductible and no coinsurance, the policy limit ⇔ the maximum covered loss.

Under a **franchise deductible** the insurer pays nothing if the loss is less than the deductible amount, but ignores the deductible if the loss is > the deductible amount.

<u>Name</u>	<u>Description</u>
ground-up loss	Losses prior to the impact of any deductible or maximum covered loss; the full economic value of the loss suffered by the insured regardless of how much the insurer is required to pay in light of any deductible, maximum covered loss, coinsurance, etc.

Truncated Data (Section 12):

Ground-up, unlimited losses have distribution function F(x).
G(x) is what one would see after the effects of either a deductible or maximum covered loss.

Left Truncated ⇔ Truncation from Below at d ⇔
deduct. d & record size of loss when size > d.

$G(x) = \frac{F(x) - F(d)}{S(d)}$, x > d   1 - G(x) = S(x) / S(d), x > d

g(x) = f(x) / S(d), x > d   x ⇔ the size of loss.

Truncation & Shifting from Below at d ⇔
deductible d & record non-zero payment ⇔ amount paid per (non-zero) payment.

$G(x) = \frac{F(x+d) - F(d)}{S(d)}$, x > 0.   g(x) = f(x+d) / S(d), x > 0

x ⇔ the size of (non-zero) payment.   x+d ⇔ the size of loss.

When data is truncated from above at the value L, claims of size greater than L are not in the reported data base. G(x) = F(x) / F(L), x ≤ L   g(x) = f(x) / F(L), x ≤ L.

Censored Data (Section 13):

Right Censored ⇔ Censored from Above at u ⇔
Maximum Covered Loss u & don't know exact size of loss, when ≥ u.

$$G(x) = \begin{cases} F(x) & x < u \\ 1 & x = u \end{cases}$$

$$g(x) = \begin{cases} f(x) & x < u \\ \text{point mass of probability } S(u) & x = u \end{cases}$$

The revised Distribution Function and density under censoring from above at u and truncation from below at d is:

$$G(x) = \begin{cases} \frac{F(x) - F(d)}{S(d)} & d < x < u \\ 1 & x = u \end{cases}$$

$$g(x) = \begin{cases} f(x)/S(d) & d < x < u \\ \text{point mass of probability } S(u)/S(d) & x = u \end{cases}$$

Left Censored and Shifted at d ⇔ $(X - d)_+$ ⇔ losses excess of d ⇔
0 when X ≤ d, X - d when X > d ⇔ amounts paid to insured with a deductible of d
⇔ payments per loss, including when the insured is paid nothing due to the deductible
⇔ amount paid per loss.
G(0) = F(d) ; G(x) = F(x+d), x > 0. g(0) point mass of F(d) ; g(x) = f(x+d), x > 0.

Average Sizes (Section 14):

Type of Data	Average Size
Ground-up, Total Limits	E[X]
Censored from Above at u	E[X ∧ u]
Truncated from Below at d	e(d) + d = {E[X] - E[X ∧ d]}/S(d) + d
Truncated and Shifted from Below at d	e(d) = {E[X] - E[X ∧ d]}/S(d)
Left Censored and Shifted	$E[(X - d)_+]$ = E[X] - E[X ∧ d]
Censored from Above at u and Truncated and Shifted from Below at d	{E[X ∧ u] - E[X ∧ d]} / S(d)

With Maximum Covered Loss of u and an (ordinary) deductible of d, the average amount paid by the insurer per loss is: E[X ∧ u] - E[X ∧ d].

With Maximum Covered Loss of u and an (ordinary) deductible of d, the average amount paid by the insurer per non-zero payment to the insured is:

$$\frac{E[X \wedge u] - E[X \wedge d]}{S(d)} = e(d).$$

A coinsurance factor of c, multiplies the average payment, either per loss or per non-zero payment by c.

Producing Additional Distributions (Section 15):

Introduce a scale parameter by "multiplying by a constant".
Let G(x) = 1 - F(1/x). One gets the Inverse Gamma from the Gamma.
Let G(x) = F(ln(x)). One gets the LogNormal from the Normal by “exponentiating.”
Add up independent identical copies. One gets the Gamma from the Exponential.
Let G(x) = $F(x^\tau)$. One gets a Weibull from the Exponential by "raising to a power."
One can get a new distribution as a continuous mixture of distributions.
The Pareto can be obtained as a mixture of Exponentials via an Inverse Gamma.
Another method of getting new distributions is via two-point or n-point mixtures.

Tails of Loss Distributions (Section 16):

If S(x) goes to zero slowly as x approaches ∞, this is a "heavy-tailed distribution."
The righthand tail is thick.
If S(x) goes to zero quickly as x approaches ∞, this is a "light-tailed distribution."
The righthand tail is thin.
The Pareto Distribution is heavy-tailed.
The Exponential distribution is light-tailed.
The Pareto Distribution is heavier-tailed than the LogNormal Distribution.

The Gamma, Pareto and LogNormal all have positive skewness.

Heavier Tailed	**Lighter Tailed**
f(x) goes to zero more slowly	**f(x) goes to zero more quickly**
Few Moments exist	**All (positive) moments exist**
Larger Coefficient of Variation	**Smaller Coefficient of Variation**
Higher Skewness	**Lower Skewness**
e(x) Increases to Infinity	**e(x) goes to a constant**
Decreasing Hazard Rate	**Increasing Hazard Rate**

Here is a list of some loss distributions, arranged in increasing heaviness of the tail:

Distribution	Mean Excess Loss	All Moments Exist
Weibull for $\tau > 1$	decreases to zero less quickly than 1/x	Yes
Gamma for $\alpha > 1$	decreases to a constant	Yes
Exponential	constant	Yes
Gamma for $\alpha < 1$	increases to a constant	Yes
Inverse Gaussian	increases to a constant	Yes
Weibull for $\tau < 1$	increases to infinity less than linearly	Yes
LogNormal	increases to infinity just less than linearly	Yes
Pareto	increases to infinity linearly	No

Let f(x) and g(x) be the two densities, then if:

$\lim_{x \to \infty} f(x) / g(x) = \infty$, f has a heavier tail than g

$\lim_{x \to \infty} f(x) / g(x) = 0$, f has a lighter tail than g

$\lim_{x \to \infty} f(x) / g(x) =$ positive constant, f has a similar tail to g.

Limited Expected Values (Section 17):

$$E[X \wedge x] = \int_0^x t\, f(t)\, dt + x\, S(x).$$

Rather than calculating this integral, make use of Appendix A of Loss Models, which has formulas for the limited expected value for each distribution.

mean = E[X∧ infinity].
e(x) = { mean - E[X∧x] } / S(x).

LER(x) = E[X∧x] / mean.
Layer Average Severity = E[X∧ top of Layer] - E[X∧ bottom of layer].

Expected Losses Excess of d: $E[(X - d)_+] = E[X] - E[X \wedge d]$.

Given **Deductible Amount d, Maximum Covered Loss u, and coinsurance factor c, then the average payment per non-zero payment by the insurer is:**

$$c\, \frac{E[X \wedge u] - E[X \wedge d]}{S(d)}.$$

Given **Deductible Amount d, Maximum Covered Loss u, and coinsurance factor c, then the insurer's average payment per loss to the insured is:**
c (E[X∧u] - E[X∧d]).

$$E[X \wedge x] = \int_0^x S(t)\, dt. \qquad E[X] = \int_0^\infty S(t)\, dt.$$

The Losses in a Layer can be written as an integral of the Survival Function from the bottom of the Layer to the top of the Layer:

$$E[X \wedge b] - E[X \wedge a] = \int_a^b S(t)\, dt.$$

The expected amount by which losses are less than b is: $\mathbf{E[(b - X)_+] = b - E[X \wedge b]}$.

E[Max[X, a]] = a + E[X] - E[X ∧ a]. E[Min[Max[X , a] , b]] = a + E[X ∧ b] - E[X ∧ a].

Limited Higher Moments (Section 18):

$$E[(X \wedge u)^2] = \int_0^u t^2 f(t)\, dt + S(u)\, u^2$$

The second moment of the average payment per loss under a Maximum Covered Loss u and a deductible of d = the second moment of the layer from d to u is:

$E[(X \wedge u)^2] - E[(X \wedge d)^2] - 2d\,\{E[X \wedge u] - E[X \wedge d]\}$.

Given a deductible of d and a Maximum Covered Loss u, the second moment of the non-zero payments is : (2nd moment of the payments per loss)/S(d).

If one has a coinsurance factor of c, then each payment is multiplied by c, therefore the second moment and the variance are each multiplied by c^2.

Mean Excess Loss (Mean Residual Life) (Section 19):

$$e(x) = E[X - x \mid X > x] = \frac{\int_x^\infty (t-x)\, f(t)\, dt}{S(x)} = \frac{\int_x^\infty t\, f(t)\, dt}{S(x)} - x = \frac{\int_x^\infty S(t)\, dt}{S(x)}.$$

e(x) = { mean - E[X∧x] } / S(x).

e(d) = average payment per payment with a deductible d.

It should be noted that for heavier-tailed distributions, just as with the mean, the Mean Excess Loss only exists for certain values of the parameters. Otherwise it is infinite.

Exponential Distribution: $e(x) = \theta$.

Pareto Distribution: $e(x) = \frac{\theta + x}{\alpha - 1}$, $\alpha > 1$.

Distribution	Behavior of e(x) as $x \to \infty$
Exponential	**constant**
Pareto	**increases linearly**
LogNormal	increases to infinity less than linearly
Gamma, $\alpha > 1$	decreases towards a horizontal asymptote
Gamma, $\alpha < 1$	increases towards a horizontal asymptote
Weibull, $\tau > 1$	decreases to zero
Weibull, $\tau < 1$	increases to infinity less than linearly

Hazard Rate (Section 20):

The **Hazard Rate**, force of mortality, or failure rate, is defined as:

$h(x) = f(x)/S(x)$, $x \geq 0$. $\quad h(x) = -\dfrac{d \ln(S(x))}{dx}$.

$S(x) = \exp[-\int_0^x h(t)\,dt]$.

For the Exponential, $h(x) = 1/\theta$ = constant.

$$\lim_{x \to \infty} e(x) = \lim_{x \to \infty} \frac{1}{h(x)}.$$

Loss Elimination Ratios and Excess Ratios (Section 21):

Loss Elimination Ratio = LER(x) = E[X∧x] / mean = 1 - R(x)
Excess Ratio = R(x) = (mean - E[X∧x]) / mean = 1 - { E[X∧x] / mean } = 1 - LER(x).

$$LER(x) = \frac{\int_0^x S(t)\,dt}{E[X]} = \frac{\int_0^x S(t)\,dt}{\int_0^\infty S(t)\,dt}.$$

The percent of losses in a layer can be computed either as the difference in Loss Elimination Ratios or the difference of Excess Ratios in the opposite order.

The Effects of Inflation (Section 22):

Uniform Inflation ⇔ Every size of loss increases by a factor of 1+r.

Under uniform inflation, for a fixed limit the excess ratio increases and for a fixed deductible amount the loss elimination ratio declines.
In order to keep up with inflation either the deductible or the limit must be increased at the rate of inflation, rather than being held constant.
Under uniform inflation the dollars limited by a fixed limit increase slower than the overall rate of inflation. Under uniform inflation the dollars excess of a fixed limit increase faster than the overall rate of inflation. Limited Losses plus Excess Losses = Total Losses.

Common ways to express the amount of inflation:
1. State the total amount of inflation from the earlier year to the later year.
2. Give a constant annual inflation rate.
3. Give the different amounts of inflation during each annual period between the earlier and later year.
4. Give the value of some consumer price index in the earlier and later year.
In all cases you want to determine the total inflation factor, (1+r), to get from the earlier year to the later year.

The Mean, Mode, Median, and the Standard Deviation are each multiplied by (1+r).
Any percentile of the distribution is multiplied by (1+r);
in fact this is the definition of inflation uniform by size of loss.
The Variance is multiplied by $(1+r)^2$. The nth moment is multiplied by $(1+r)^n$.
The Coefficient of Variation, the Skewness, and the Kurtosis are each unaffected by uniform inflation.

Provided the limit keeps up with inflation, the Limited Expected Value, in dollars, is multiplied by the inflation factor.
In the later year, the mean excess loss, in dollars, is multiplied by the inflation factor, provided the limit has been adjusted to keep up with inflation.
The Loss Elimination Ratio, dimensionless, is unaffected by uniform inflation, provided the deductible has been adjusted to keep up with inflation.
In the later year, the Excess Ratio, dimensionless, is unaffected by uniform inflation, provided the limit has been adjusted to keep up with inflation.

Most of the size of loss distributions are scale families; under uniform inflation one gets the same type of distribution. If there is a scale parameter, it is revised by the inflation factor.
For the Pareto, Single Parameter Pareto, Gamma, Weibull, and Exponential Distributions, θ becomes θ(1+r).
Under uniform inflation for the LogNormal, μ becomes μ + ln(1+r).

For distributions in general, one can determine the behavior under uniform inflation as follows.
One makes the change of variables Z = (1+r) X.
For the Distribution Function one just sets $F_Z(z) = F_X(x)$; one substitutes for x = z / (1+r).
Alternately, for the density function $f_Z(z) = f_X(x) / (1+r)$.

The domain [a, b] becomes under uniform inflation [(1+r)a, (1+r)b].
The uniform distribution on [a, b] becomes under uniform inflation the uniform distribution on [a(1+r), b(1+r)].

There are two alternative ways to solve many problems involving uniform inflation:
1. Adjust the size of loss distribution in the earlier year to the later year based on the amount of inflation. Then calculate the quantity of interest in the later year.
2. Calculate the quantity of interest in the earlier year at its deflated value, and then adjust it to the later year for the effects of inflation.

Given uniform inflation, with **inflation factor of 1+r, Deductible Amount d, Maximum Covered Loss u, and coinsurance factor c, then in terms of the values in the earlier year, the insurer's average payment per loss in the later year is:**

$$(1+r)\, c \,\{ E[X \wedge \frac{u}{1+r}] - E[X \wedge \frac{d}{1+r}] \}.$$

Given uniform inflation, with **inflation factor of 1+r, Deductible Amount d, Maximum Covered Loss u, and coinsurance factor c, then in terms of the values in the earlier year, the average payment per (non-zero) payment by the insurer in the later year is:**

$$(1+r)\, c \, \frac{E[X \wedge \frac{u}{1+r}] - E[X \wedge \frac{d}{1+r}]}{S\left(\frac{d}{1+r}\right)}.$$

Given uniform inflation, with inflation factor of 1+r, Deductible Amount d, Maximum Covered Loss u, and coinsurance factor c, then in terms of the values in the earlier year, the second moment of the insurer's payment per loss in the later year is:

$$(1+r)^2 c^2 \{E[(X \wedge \frac{u}{1+r})^2] - E[(X \wedge \frac{d}{1+r})^2] - 2\frac{d}{1+r}(E[X \wedge \frac{u}{1+r}] - E[X \wedge \frac{d}{1+r}])\}.$$

Given uniform inflation, with inflation factor of 1+r, Deductible Amount d, Maximum Covered Loss u, and coinsurance factor c, then in terms of the values in the earlier year, the second moment of the insurer's payment per (non-zero) payment by the insurer in the later year is:

$$(1+r)^2 c^2 \frac{E[(X \wedge \frac{u}{1+r})^2] - E[(X \wedge \frac{d}{1+r})^2] - 2\frac{d}{1+r}\{E[X \wedge \frac{u}{1+r}] - E[X \wedge \frac{d}{1+r}]\}}{S\left(\frac{d}{1+r}\right)}.$$

If one has a mixed distribution, then under uniform inflation each of the component distributions acts as it would under uniform inflation.

Lee Diagrams (Section 23):

Put the size of loss on the y-axis and probability on the x-axis.
The mean is the area under the curve.
Layers of loss correspond to horizontal strips.
Restricting attention to only certain sizes of loss corresponds to vertical strips.

Continuous Mixtures of Models (Section 24):

Mixture Distribution ⇔ Continuous Mixture of Models.
One takes a mixture of the density functions for specific values of the parameter ζ via some mixing distribution u: $g(x) = \int f(x; \zeta)\, u(\zeta)\, d\zeta$.

The nth moment of a mixed distribution is the mixture of the nth moments for specific values of the parameter ζ: $E[X^n] = E_\zeta[E[X^n \mid \zeta]]$.

If the severity is Exponential and the mixing distribution of their means is Inverse Gamma, then the mixed distribution is a Pareto, with
α = shape parameter of the Inverse Gamma and θ = scale parameter of the Inverse Gamma.

If the hazard rate of the Exponential, λ, is distributed via a Gamma(α, θ), then the mean $1/\lambda$ is distributed via an Inverse Gamma(α, $1/\theta$), and therefore the mixed distribution is Pareto.
If the Gamma has parameters α and θ, then the mixed Pareto has parameters α and $1/\theta$.

If the severity is Normal with fixed variance s^2, and the mixing distribution of their means is also Normal with mean μ and variance σ^2, then the mixed distribution is another Normal, with mean μ and variance: $s^2 + \sigma^2$.

In a Frailty Model, the hazard rate is of the form: $h(x \mid \lambda) = \lambda\, a(x)$, where λ is a parameter which varies across the portfolio, and a(x) is some function of x.

Let $A(x) = \int_0^x a(t)\, dt$.

Then $S(x) = M_\lambda[-A(x)]$.
For an Exponential Distribution: a(x) = 1, and A(x) = x.
For a Weibull Distribution: $\lambda = \theta^{-\tau}$, $a(x) = \tau x^{\tau-1}$, and $A(x) = x^\tau$.

Spliced Models (Section 25):

A **2-component spliced model** has: $f(x) = w_1\, f_1(x)$ on (a_1, b_1) and $f(x) = w_2\, f_2(x)$ on (a_2, b_2), where $f_1(x)$ is a density with support (a_1, b_1), $f_2(x)$ is a density with support (a_2, b_2), and $w_1 + w_2 = 1$.

A 2-component spliced density will be continuous at the breakpoint b, provided the weights are inversely proportional to the component densities at the breakpoint:

$$w_1 = \frac{f_2(b)}{f_1(b) + f_2(b)},\ w_2 = \frac{f_1(b)}{f_1(b) + f_2(b)}.$$

Relevant portions of the ASTAM Formula Sheet

ASTAM Formula Sheet

$P(z)$ denotes the probability generating function. $M(z)$ denotes the moment generating function.

Q_α denotes the α-quantile of a distribution, also known as the α-Value at Risk

$\mathrm{ES}_\alpha[X]$ denotes the α-Expected Shortfall of X. This is also known as the $\alpha-$TailVaR, or the α-CTE.

The distribution function of the standard normal distribution is denoted $\Phi(x)$. The probability density function of the standard normal distribution is denoted $\phi(x)$.

The q-quantile of the standard normal distribution is denoted z_q, that is $\Phi(z_q) = q$.

For counting distributions, p_k denotes the probability function.

Continuous distributions

Pareto(α, θ) Distribution

$$f(x) = \frac{\alpha\theta^\alpha}{(\theta + x)^{\alpha+1}}, \qquad F(x) = 1 - \left(\frac{\theta}{\theta + x}\right)^\alpha,$$

$$\mathrm{E}[X] = \frac{\theta}{\alpha - 1}, \quad \alpha > 1, \qquad \mathrm{Var}[X] = \left(\frac{\theta}{\alpha - 1}\right)^2 \frac{\alpha}{\alpha - 2}, \quad \alpha > 2,$$

$$\mathrm{E}[X \wedge x] = \frac{\theta}{\alpha - 1}\left(1 - \left(\frac{\theta}{x + \theta}\right)^{\alpha - 1}\right), \qquad \alpha > 1,$$

$$\mathrm{E}\left[X^k\right] = \frac{k!\,\theta^k}{(\alpha - 1)(\alpha - 2)\ldots(\alpha - k)}, \quad k = 1, 2, \cdots, \quad \alpha > k,$$

$$\mathrm{E}[X - Q | X > Q] = \frac{\theta + Q}{\alpha - 1}, \quad \alpha > 1.$$

Lognormal(μ, σ) Distribution

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{\log x - \mu}{\sigma}\right)^2\right\}, \qquad F(x) = \Phi\left(\frac{\log x - \mu}{\sigma}\right),$$

$$\mathrm{E}[X] = e^{\mu+\sigma^2/2}, \qquad \mathrm{Var}[X] = e^{2\mu+\sigma^2}\left(e^{\sigma^2} - 1\right),$$

$$\mathrm{E}\left[X^k\right] = e^{k\mu+k^2\sigma^2/2},$$

$$\mathrm{E}\left[(X \wedge x)^k\right] = \exp\left(k\mu + \frac{1}{2}k^2\sigma^2\right)\Phi\left(\frac{\log x - \mu - k\sigma^2}{\sigma}\right) + x^k\left(1 - \Phi\left(\frac{\log x - \mu}{\sigma}\right)\right),$$

$$\mathrm{ES}_\alpha[X] = \frac{e^{\mu+\sigma^2/2}}{1-\alpha}\Phi\big(z_{1-\alpha} + \sigma\big).$$

Exponential(θ) Distribution

$$f(x) = \frac{e^{-x/\theta}}{\theta}, \qquad F(x) = 1 - e^{-x/\theta},$$

$$\mathrm{E}[X] = \theta, \qquad \mathrm{Var}[X] = \theta^2,$$

$$\mathrm{E}\left[X^k\right] = k!\,\theta^k, \; k = 1, 2, \cdots,$$

$$\mathrm{E}\left[X \wedge x\right] = \theta\left(1 - e^{-x/\theta}\right),$$

$$\mathrm{E}[X - Q | X > Q] = \theta,$$

$$M_X(t) = (1 - t\theta)^{-1}, \quad t < \frac{1}{\theta}.$$

Gamma(α, θ) Distribution

$$f(x) = \frac{x^{\alpha-1}\,e^{-x/\theta}}{\theta^\alpha\,\Gamma(\alpha)},$$

$$\mathrm{E}[X] = \alpha\,\theta, \qquad \mathrm{Var}[X] = \alpha\,\theta^2,$$

$$\mathrm{E}\left[X^k\right] = \alpha(\alpha+1)...(\alpha+k-1)\theta^k, \qquad k = 1, 2, \cdots,$$

$$M_X(t) = (1 - t\theta)^{-\alpha}, \quad t < \frac{1}{\theta}.$$

Chi-squared(ν) Distribution

Gamma distribution with $\alpha = \nu/2$ and $\theta = 2$.
$\nu \in \mathbb{N}^+$ is the degrees of freedom parameter.

$$\mathrm{E}[X] = \nu, \qquad \mathrm{Var}[X] = 2\nu$$

$$M_X(t) = (1 - 2t)^{-\nu/2}, \quad t < \frac{1}{2}.$$

Beta(a, b) Distribution

$$f(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}, \qquad 0 < x < 1,$$

$$\mathrm{E}[X] = \frac{a}{a+b}, \qquad \mathrm{Var}[X] = \frac{a\,b}{(a+b)^2(a+b+1)}.$$

Normal(μ, σ^2) Distribution

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\}, \qquad F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right),$$

$$\mathrm{E}[X] = \mu, \qquad \mathrm{Var}[X] = \sigma^2,$$

$$\mathrm{ES}_\alpha[X] = \mu + \frac{\sigma}{1-\alpha}\phi\left(z_\alpha\right),$$

$$M_X(t) = e^{t\mu + t^2\sigma^2/2}.$$

Weibull(θ, τ) Distribution

$$f(x) = \frac{\tau x^{\tau-1}}{\theta^\tau} e^{-\left(\frac{x}{\theta}\right)^\tau}, \qquad F(x) = 1 - e^{-(x/\theta)^\tau},$$

$$\mathrm{E}[X] = \theta\,\Gamma\left(1+\frac{1}{\tau}\right), \qquad \mathrm{Var}[X] = \theta^2\left(\Gamma\left(1+\frac{2}{\tau}\right) - \left(\Gamma\left(1+\frac{1}{\tau}\right)\right)^2\right).$$

Counting Distributions

Poisson(λ) Distribution

$$p_k = \frac{\lambda^k e^{-\lambda}}{k!}, \qquad a = 0, \quad b = \lambda,$$

$$\mathrm{E}[N] = \lambda, \qquad \mathrm{Var}[N] = \lambda,$$

$$P_N(z) = \exp\{\lambda(z-1)\}, \qquad M_N(z) = \exp\{\lambda(e^z - 1)\}.$$

Binomial(m,q) Distribution

$$p_k = \binom{m}{k} q^k (1-q)^{m-k}, \qquad a = -\frac{q}{1-q}, \quad b = \frac{(m+1)q}{1-q},$$

$$\mathrm{E}[N] = mq, \qquad \mathrm{Var}[N] = mq(1-q)$$

$$P_N(z) = \left(1 + q(z-1)\right)^m, \qquad M_N(z) = \left(1 + q(e^z - 1)\right)^m.$$

Bernoulli(q) Distribution
Binomial Distribution with $m = 1$.

Negative Binomial(r,β) Distribution

$$p_0 = \left(\frac{1}{1+\beta}\right)^r, \qquad p_k = \frac{r(r+1)\cdots(r+k-1)}{k!}\left(\frac{\beta}{1+\beta}\right)^k\left(\frac{1}{1+\beta}\right)^r, \quad k = 1, 2, \dots ,$$

$$a = \frac{\beta}{1+\beta}, \quad b = \frac{(r-1)\beta}{1+\beta},$$

$$\mathrm{E}[N] = r\beta, \qquad \mathrm{Var}[N] = r\beta(1+\beta),$$

$$P_N(z) = \big(1 - \beta\,(z-1)\big)^{-r}, \qquad M_N(z) = \big(1 - \beta\,(e^z - 1)\big)^{-r}.$$

Geometric Distribution
Negative Binomial Distribution with $r = 1$;