Mahler's Guide to
Bernegger, Exposure Curves & the MBBEFD Distribution Class

CAS Exam 9

prepared by
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Mahler's Guide to Bernegger, Exposure Curves[1 2 3]

Information in bold or sections whose title is in bold are more important for passing the exam. Larger bold type indicates it is extremely important. Information presented in italics (including subsections whose titles are in italics) should rarely be needed to directly answer exam questions and should be skipped on first reading. It is provided to aid the reader's overall understanding of the subject, and to be useful in practical applications.

I have doubled underlined highly recommended questions to do on your first pass through the material, underlined recommended questions to do on your second pass, and starred additional questions to do on a third pass through the material.[4] No questions were labeled from the 2011 exam or later, in order to allow you to use them as practice exams.

Solutions to problems are at the end.[5]

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Please send me any suspected errors by Email.

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[1] "Swiss Re Exposure Curves and the MBBEFD Distribution Class," by Stefan Bernegger, ASTIN Bulletin, Vol. 27, No. 1, May 1997, pp. 99-111.
[2] Added to the syllabus for the 2011 exam.
[3] Prior to 2024, Bernegger's paper was on Exam 8.
The current material was part of my study guides for Exam 8.
[4] Obviously feel free to do whatever questions you want. This is just a guide for those who find it helpful.
[5] Note that problems include both some written by me and some from past exams. The latter are copyright by the Casualty Actuarial Society and are reproduced here solely to aid students in studying for exams. The solutions and comments are solely the responsibility of the author; the CAS bears no responsibility for their accuracy. While some of the comments may seem critical of certain questions, this is intended solely to aid you in studying and in no way is intended as a criticism of the many volunteers who work extremely long and hard to produce quality exams.

Bernegger discusses a class of distributions and how they can be used to help price reinsurance treaties. He is pricing excess of loss property reinsurance.[6]

In property reinsurance there is the possibility of a total loss or a Maximum Possible Loss (MPL). The insurer retains the loss in the layer from 0 to d, while the reinsurer pays the loss excess of d. Instead the reinsurer could just be reinsuring a layer of loss.

Bernegger is discussing exposure rating using an exposure curve.[7] Most of the paper is spent on a particular mathematical form of exposure curve and the corresponding distribution function.

Loss Elimination Ratios and Excess Ratios, Review:

The Loss Elimination Ratio (LER) is defined as the ratio of the losses eliminated by a deductible to the total losses prior to imposition of the deductible. The losses eliminated by a deductible d, are $E[X \wedge d]$, the Limited Expected Value at d.[8]

$$\mathbf{LER(x) = \frac{E[X \wedge x]}{E[X]}}.$$

The excess ratio R(x), is defined as the ratio of loss dollars excess of x divided by the total loss dollars. It is the complement of the Loss Elimination Ratio; they sum to unity.

$$R(x) = \frac{E[X] - E[X \wedge x]}{E[X]} = 1 - \frac{E[X \wedge x]}{E[X]} = 1 - LER(x).$$

The percentage of total losses in the layer from d to u is: LER(u) - LER(d) = R(d) - R(u).

For a distribution with support starting at zero, the Limited Expected Value can be written as an integral of the Survival Function from 0 to the limit:

$$E[X \wedge x] = \int_0^x S(t)\, dt.$$

[6] As discussed in “Basics of Reinsurance Pricing” by David R. Clark, “Property per-risk excess treaties provide a limit of coverage in excess of the ceding company’s retention. The layer applies on a “per risk” basis, which typically refers to a single property location. This is more narrow than a “per occurrence” property excess treaty which applies to multiple risks to provide catastrophe protection.”

[7] See pages 16 to 18 of “Basics of Reinsurance Pricing,” by David R. Clark.

[8] Bernegger uses the notation L(d) for the limited expected value at d. Other syllabus readings use E[X ; d].

$$\text{LER}(x) = \frac{E[X \wedge x]}{E[X]} = \frac{\int_0^x S(t)\,dt}{E[X]} = \frac{\int_0^x S(t)\,dt}{\int_0^\infty S(t)\,dt}.$$

Thus, **for a distribution with support starting at zero, the Loss Elimination Ratio is the integral from zero to the limit of S(x) divided by the mean**.

Since $R(x) = 1 - \text{LER}(x) = (E[X] - E[X \wedge x]) / E[X]$, the Excess Ratio can be written as:

$$R(x) = \frac{\int_x^\infty S(t)\,dt}{E[X]} = \frac{\int_x^\infty S(t)\,dt}{\int_0^\infty S(t)\,dt}.$$

So the excess ratio is the integral of the survival from the limit to infinity, divided by the mean.

For example, for the Shifted Pareto Distribution, $S(x) = \theta^a (\theta+x)^{-\alpha}$. So that:

$$R(x) = \frac{\theta^\alpha (\theta+x)^{1-\alpha} / (\alpha-1)}{\theta / (\alpha-1)} = \{\theta/(\theta+x)\}^{\alpha-1}.$$

$$\text{LER}(x) = \frac{\int_0^x S(t)\,dt}{E[X]}. \Rightarrow \frac{d\,\text{LER}(x)}{dx} = \frac{S(x)}{E[X]}.$$

Since $\frac{S(x)}{E[X]} \geq 0$, the loss elimination ratio is a increasing function of x.[9]

Also, if there is no point mass of probability for a loss of size zero, then S(0) = 1, and the derivative at zero of the LER is: LER'(0) = 1/E[X].

For a distribution with support starting at zero:

$$\frac{d\,\text{LER}(x)}{dx} = \frac{S(x)}{E[X]}. \Rightarrow \frac{d\,\text{LER}(0)}{dx} = \frac{1}{E[X]}. \Rightarrow S(x) = \frac{d\,\text{LER}(x)}{dx} \bigg/ \frac{d\,\text{LER}(0)}{dx}.$$

[9] If S(x) = 0, in other words there is no possibility of a loss of size greater than x, then the loss elimination is a constant 1, and therefore, more precisely the loss elimination is nondecreasing.

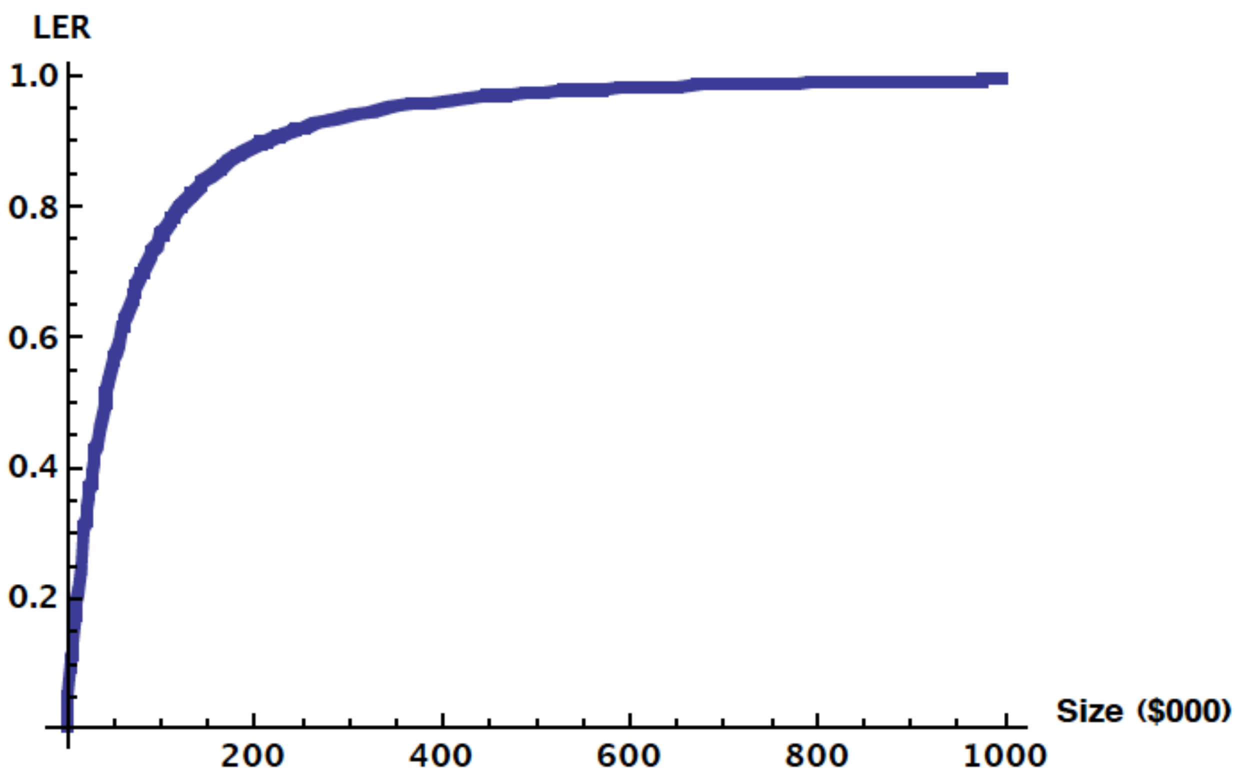
Therefore, **the loss elimination ratios (or the excess ratios) determine the distribution function**, as well as vice-versa.[10] [11]

$$\frac{d\ \mathrm{LER}(x)}{dx} = \frac{S(x)}{E[X]}. \Rightarrow \frac{d^2\ \mathrm{LER}(x)}{dx^2} = -\frac{f(x)}{E[X]}.$$

Since $\frac{f(x)}{E[X]} \geq 0$, $\frac{d^2\ \mathrm{LER}(x)}{dx^2} \leq 0$; the loss elimination ratio is a concave downwards function of x.

The loss elimination ratio as a function of x is increasing, concave downwards, and approaches one as x approaches infinity.

For example, here is a graph of the loss elimination ratio for a Shifted Pareto Distribution with parameters $\alpha = 3$ and $\theta = 100,000$:[12]



Since the loss elimination ratio is increasing and concave downwards, the excess ratio is decreasing and concave upwards (convex).

[10] "A distribution is characterized by its excess ratios and so there is no loss of information in working with excess ratios rather than with the size of loss density or distribution function."
Quoted from page 196 of "NCCI's 2007 Hazard Group Mapping" by John Robertson.
[11] Note that depending on the application, this could be either a distribution of size of loss or aggregate loss.
[12] As x approaches infinity, the loss elimination ratio approaches one. In this case it approaches the limit slowly.

Normalizing:

Bernegger normalizes everything with respect to the Maximum Possible Loss, M.[13]

If X is the loss in dollars, and M is the Maximum Possible Loss, then the normalized loss is: $x = X / M$. $0 \le x \le 1$.
If D is the retention in dollars, then the normalized retention is: $d = D / M$.

x has a distribution with support on 0 to 1, with a point mass of probability at 1, corresponding to a total loss or the Maximum Possible Loss (MPL).
Note that since x is restricted to the interval 0 to 1, its limited expected value at one equals the mean.

[13] Bernegger also deals with the mathematics of an unlimited distribution.
In that case, X can be normalized with respect to some reference loss X_0.

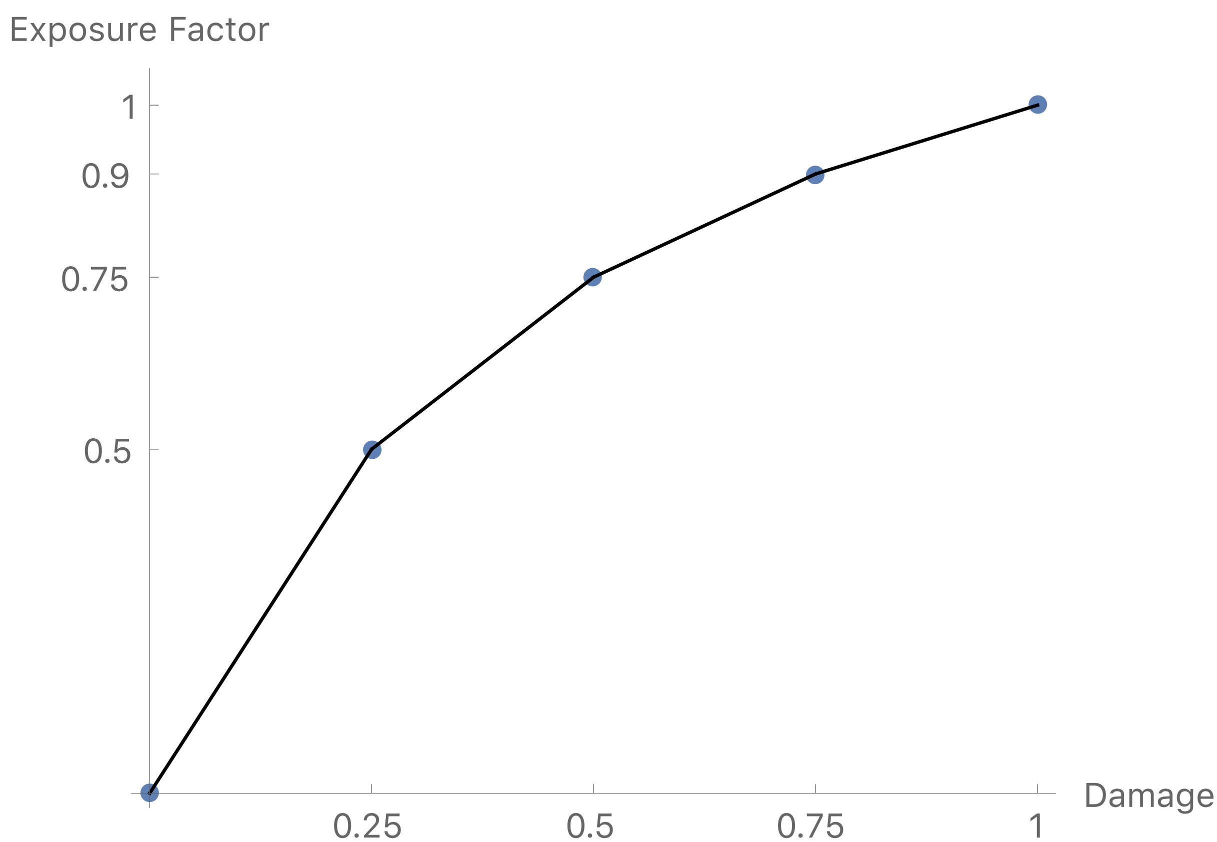
Exposure Curves:[14]

The exposure curve, G(x), is just the loss elimination ratio at x.

Exercise: Assume the following discrete severity distribution:

Percent of Maximum Possible Loss	Probability
25%	50%
50%	20%
75%	10%
100%	20%

Draw the corresponding exposure curve.
[Solution: The mean is: (50%)(25%) + (20%)(50%) + (10%)(75%) + (20%)(100%) = 50%.
At x = 25%, the limited expected value is: 25%.
The loss elimination ratio is: 25%/50% = 0.50.
At x = 50%, the limited expected value is: (50%)(25%) + (50%)(50%) = 37.5%.
The loss elimination ratio is: 37.5%/50% = 0.75.
At x = 75%, the limited expected value is: (50%)(25%) + (20%)(50%) + (30%)(75%) = 45%.
The loss elimination ratio is: 45%/50% = 0.90.
At x = 100%, the loss elimination ratio is 1.
The exposure curve consists of straight lines connecting these points:



[14] The use of exposure curves is discussed in "Basics of Reinsurance Pricing" by David R. Clark.
See also "An Exposure Rating Approach to Pricing Property Excess-of- Loss Reinsurance," by Stephen J. Ludwig, PCAS 1991, not on the syllabus.

$G(d) = E[X \wedge d] / E[X \wedge 1] = L(d) / L(1)$, $0 \le d \le 1$.[15]

$E[X \wedge 1] = E[X]$. $\Rightarrow G(d) = E[X \wedge d] / E[X] = \int_0^d S(t)\, dt \ / E[X]$.

$G'(d) = S(d) / E[X]$. $\qquad G'(d) \ge 0$.

$G'(0) = S(0) / E[X]$. $\Rightarrow$ $\mathbf{E[X] = 1/G'(0)}$.

$G'(x) / G'(0) = \{S(x) / E[X]\} / \{1 / E[X]\} = S(x)$. $\Rightarrow$ $\mathbf{S(x) = G'(x) / G'(0)}$.

$G'(1) = S(1) / E[X]$. $\Rightarrow S(1) = G'(1)/G'(0)$.

$\Rightarrow$ **The probability of having the Maximum Possible loss is: G'(1) / G'(0)**.

$G''(d) = -f(d) / E[X]$. $\qquad G''(d) \le 0$.

G(d) is an increasing and concave function on the interval [0, 1].
In other words, G'(d) ≥ 0 and G''(d) ≤ 0.
In addition, G(0) = 0 and G(1) = 1 by definition.[16]

Exercise: $G(x) = \dfrac{6x}{1 + 5x}$, $0 \le x \le 1$.

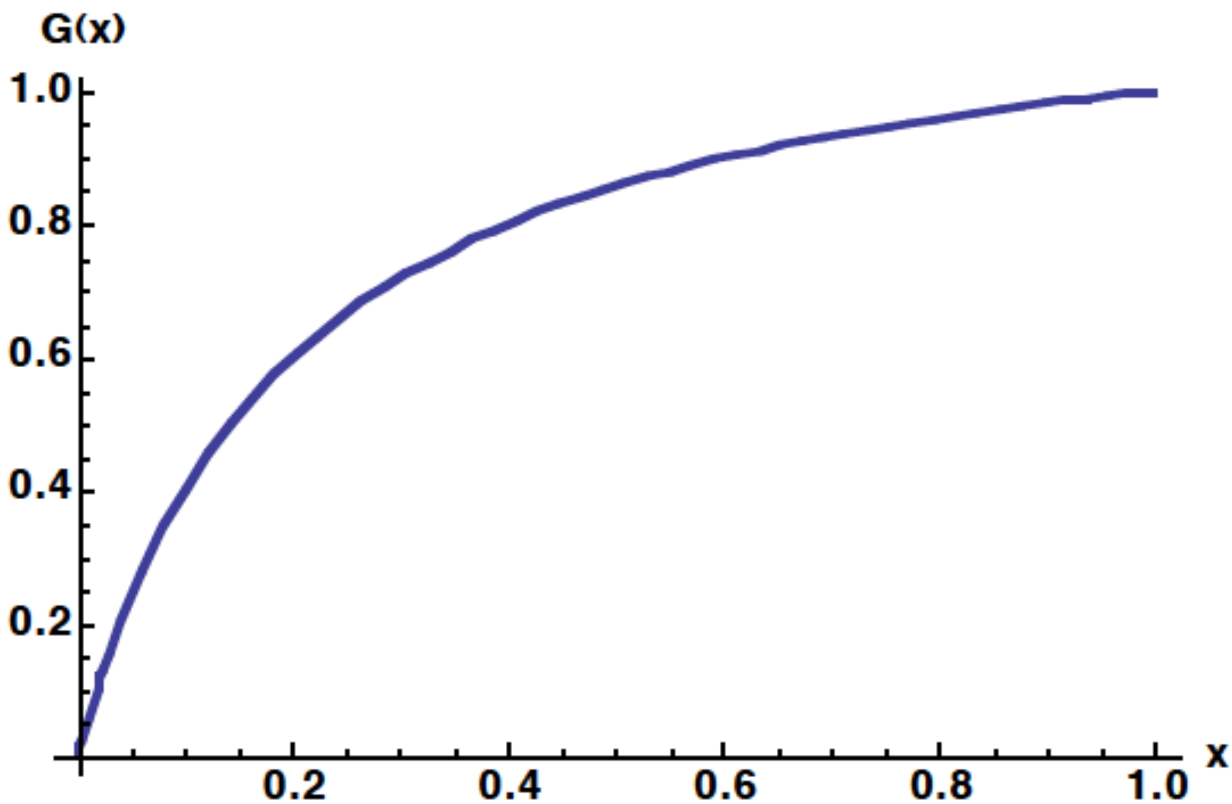
Demonstrate that this function is a valid exposure curve.
[Solution: G(0) = 0. G(1) = 1.

$G'(x) = \dfrac{(6)(1 + 5x) - (6x)(5)}{(1 + 5x)^2} = \dfrac{6}{(1 + 5x)^2} \ge 0$.

$G''(x) = \dfrac{-60}{(1 + 5x)^3} \le 0$.]

[15] Bernegger uses the notation L(d) for the limited expected value at d, $E[X \wedge d]$.
[16] See page 101 of Bernegger. Any severity distribution with mode of 0, censored at 1 will satisfy these conditions.

Here is a graph of $G(x) = \frac{6x}{1 + 5x}$, $0 \le x \le 1$, showing that it is increasing and concave downwards:



Exercise: The exposure curve is: $G(x) = \frac{6x}{1 + 5x}$, $0 \le x \le 1$. What is the mean size of loss? What is the probability of the Maximum Probable Loss?

[Solution: $G'(x) = \frac{6}{(1 + 5x)^2}$.

Mean = $1/G'(0) = 1/6$. The mean ground up loss is M/6.

$S(x) = G'(x) / G'(0) = \frac{1}{(1 + 5x)^2}$, $0 \le x \le 1$.

The probability of the Maximum Probable Loss = S(1) = 1/36.]

Exercise: The expected annual ground-up loss is $300,000. The MPL is $24 million.
The cedant's maximum retention under a reinsurance treaty is $8 million.

$G(x) = \frac{6x}{1 + 5x}$, $0 \le x \le 1$. Determine the expected annual losses paid by the reinsurer.

[Solution: x = 8/24 = 1/3. G(1/3) = 2 / (1 + 5/3) = 0.75.
The expected retained loss are: ($300,000) (0.75) = $225,000.
The expected losses paid by the reinsurer are: ($300,000) (0.25) = $75,000.
Comment: From the previous exercise, the mean size of loss is: M/6 = $4 million.
Thus the mean annual frequency must be: $300,000 / $4 million = 7.5%.]

The MBBEFD Distribution Class:[17]

The exposure curve that Bernegger discusses has the form:

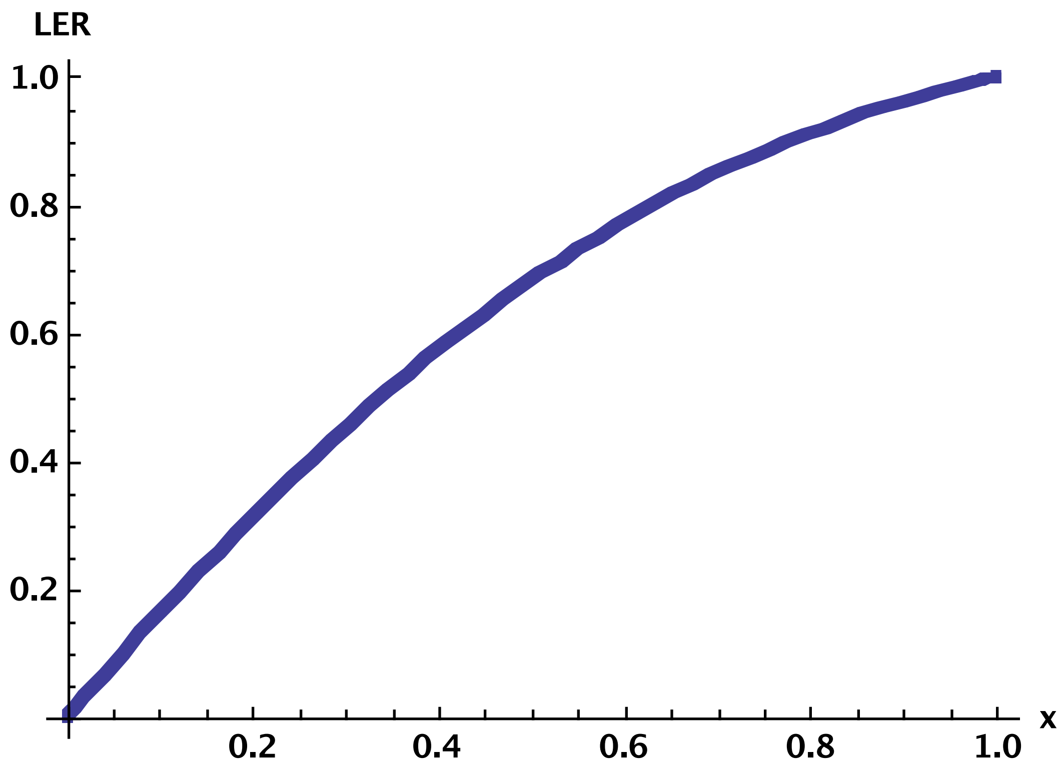
G(x) = $\frac{\ln(a + b^x) - \ln(a + 1)}{\ln(a + b) - \ln(a + 1)}$, $0 \le x \le 1$, where a and b are the two parameters.[18]

Note that $G(1) = \frac{\ln(a + b) - \ln(a+1)}{\ln(a+b) - \ln(a+1)} = 1$;

all of the loses would be eliminated by a retention equal to the Maximum Possible Loss.

$G(0) = \frac{\ln(a + 1) - \ln(a+1)}{\ln(a+b) - \ln(a+1)} = 0$; none of the loses would be eliminated by a retention of zero.

For example, here is a graph of G(x) for a = 0.2 and b = 0.04:



Note the general properties: G(0) = 0, G(1) = 1, G is increasing and concave downwards.[19]

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[17] Maxwell-Boltzmann, Bose-Einstein, and Fermi-Dirac distributions.
[18] See equation 3.1a in Bernegger.
[19] See the first paragraph of page 101 of Bernegger.

Exercise: For a = 0.2 and b = 0.04, determine G(0.5).

[Solution: $G(0.5) = \frac{\ln(0.2 + 0.04^{0.5}) - \ln(1.2)}{\ln(0.24) - \ln(1.2)} = 68.26\%$.]

Exercise: The expected annual ground-up loss is \$40,000. The MPL is \$2 million.
The cedant's maximum retention under the reinsurance treaty is \$1 million.
a = 0.2 and b = 0.04.
Determine the expected annual losses paid by the reinsurer.

[Solution: x = \$1 million/\$2 million = 0.5. $G(0.5) = \frac{\ln(0.2 + 0.04^{0.5}) - \ln(1.2)}{\ln(0.24) - \ln(1.2)} = 68.26\%$.

The expected retained loss are: (\$40,000) (68.26%) = \$27,304.
The expected losses paid by the reinsurer are: (\$40,000) (1 - 68.26%) = \$12,696.]

Let $y = b^x$. Then $G(y) = \frac{\ln(a + y) - \ln(a+1)}{\ln(a+b) - \ln(a+1)}$.

$$\frac{d\,G(y)}{dy} = \frac{1}{a + y} \frac{1}{\ln(a+b) - \ln(a+1)} = \frac{1}{a + b^x} \frac{1}{\ln(a+b) - \ln(a+1)}.$$

$$\frac{dy}{dx} = \ln(b)\, b^x.$$

$$\frac{d\,G(x)}{dx} = \frac{d\,G(y)}{dy} \frac{dy}{dx} = \frac{1}{a + b^x} \frac{1}{\ln(a+b) - \ln(a+1)} \ln(b)\, b^x.$$

Thus, this form of exposure curve has a derivative of:

$$G'(x) = \frac{\frac{\ln(b)\, b^x}{a + b^x}}{\ln(a+b) - \ln(a+1)}.$$

Exercise: For a = 0.2 and b = 0.04, determine the mean.

[Solution: $G'(0) = \frac{\frac{\ln(0.04)\, 0.04^0}{0.2 + 0.04^0}}{\ln(0.24) - \ln(1.2)} = 5/3$.

E[X] = 1/G'(0) = 3/5 = 0.6.
Comment: The mean loss is 60% of the Maximum Possible Loss, M.]

In general, $E[x] = 1/G'(0) = \frac{(a + 1)\{\ln(a+b) - \ln(a+1)\}}{\ln(b)}$.

Exercise: For a = 0.2 and b = 0.04, determine the survival function at 60% of the MPL.

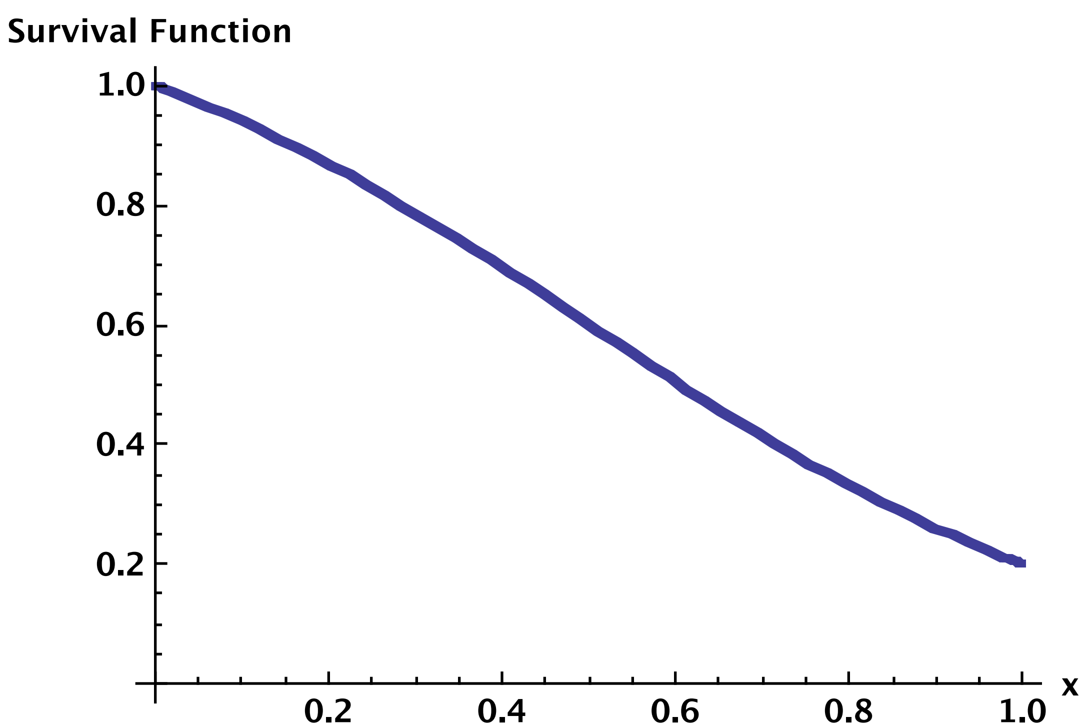
[Solution: $G'(0.6) = \dfrac{\dfrac{\ln(0.04)\ 0.04^{0.6}}{0.2 + 0.04^{0.6}}}{\ln(0.24) - \ln(1.2)} = 0.84043.$

$S(0.6) = G'(0.6)/G'(0) = (0.84043) / (5/3) = 50.43\%.$]

In general, $\mathbf{S(x) = G'(x) / G'(0)} = \dfrac{\dfrac{\ln(b)\ b^x}{a + b^x}}{\ln(a+b) - \ln(a+1)} \Big/ \left\{\dfrac{\dfrac{\ln(b)\ b^0}{a + b^0}}{\ln(a+b) - \ln(a+1)}\right\} = \dfrac{(a + 1)\ b^x}{a + b^x}$.[20]

For a = 0.2 and b = 0.04, $S(0.6) = \dfrac{(1.2)\ (0.04^{0.6})}{0.2 + 0.04^{0.6}}$ = 50.43%, matching the previous result.

Here is a graph of the survival function for a = 0.2 and b = 0.04:[21]



[20] See equation 3.1b in Bernegger.

[21] Note the point mass of probability of 20% at x = 1; S(1-ε) = 20%.

Exercise: For a = 0.2 and b = 0.04, determine the probability of having the MPL.

[Solution: $G'(1) = \frac{\frac{\ln(0.04)\ 0.04^1}{0.2 + 0.04^1}}{\ln(0.24) - \ln(1.2)} = 1/3$. S(1) = G'(1)/G'(0) = (1/3) / (5/3) = 1/5 = 20%.

Alternately, $S(1) = \frac{(a + 1)\ b}{a + b} = (1.2)(0.04) / 0.24 = 20\%$.]

In general, the probability of the maximum possible loss is: $S(1) = \frac{(a + 1)\ b}{a + b}$.

Exercise: For a = 0.2 and b = 0.04, find the 60th percentile of the distribution.

[Solution: $1 - 0.6 = 0.4 = S(x) = \frac{(1.2)\ (0.04^x)}{0.2 + 0.04^x}$. $\Rightarrow$ $0.08 + (0.4)(0.04^x) = (1.2)(0.04^x)$. $\Rightarrow$

$0.04^x = 0.08 / 0.80 = 0.1$. x = ln(0.1) / ln(0.04) = 0.7153.

Comment: In general, in order to determine the p^{th} percentile:

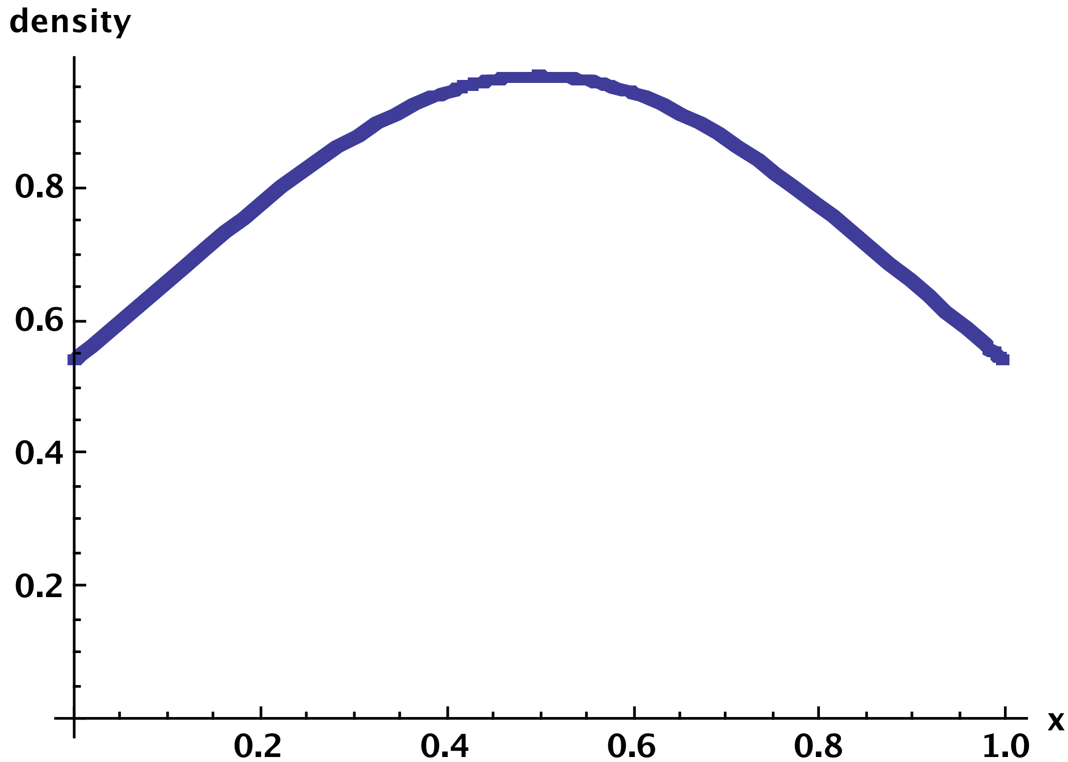
$1 - p = S(x) = \frac{(a + 1)\ b^x}{a + b^x}$. $\Rightarrow$ $\pi_p = x = \frac{\ln[\frac{a\ (1-p)}{a + p}]}{\ln(b)}$.]

$S(x) = \frac{(a + 1)\ b^x}{a + b^x}$. $\Rightarrow$ $f(x) = -S'(x) = -\frac{(a+1)\ a \ln[b]\ b^x}{(a + b^x)^2}$.

Exercise: For a = 0.2 and b = 0.04, determine the value of the density at 0.1.

[Solution: $f(0.1) = -\frac{(1.2)\ (0.2) \ln[0.04]\ 0.04^{0.1}}{(0.2 + 0.04^{0.1})^2} = 0.655$.]

Here is a graph of the density function for a = 0.2 and b = 0.04:[22]



Note that this density integrates from 0 to 1 to only 80%. As discussed previously, there is a point mass of probability of 20% at x = 1, corresponding to the Maximum Possible Loss.

Let p = Prob[x = 1]. Let μ = E[x].
μ = E[x] = (p)(1) + (1 - p) E[x | x < 1] $\geq$ p.
Since $0 \leq x \leq 1$, μ = E[x] $\leq$ 1.
Therefore, $0 \leq p \leq \mu \leq 1$.[23]

Also 1/G'(0) = E[x] $\leq$ 1. $\Rightarrow$ G'(0) $\geq$ 1.
p = G'(1)/G'(0) $\leq$ E[x] = 1/G'(0). $\Rightarrow$ G'(1) $\leq$ 1.
G'(x) = S(x) μ. $\Rightarrow$ G'(x) $\geq$ 0.
Thus G'(0) $\geq$ 1 $\geq$ G'(1) $\geq$ 0.[24]

This result also follows from the fact that g(x) is increasing and concave downwards, with G(0) = 0 and G(1) = 1.

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[22] In units where the MPL = 1.
[23] See equation 2.6 in Bernegger.
[24] See equation 2.6 in Bernegger.

Reparameterizing the Exposure Curve:

Let p = probability of the MPL = S(1) = $\frac{(a + 1)\,b}{a + b}$.

Then let **g = 1/p =** $\mathbf{\frac{a + b}{(a + 1)\,b}}$. [25]

Exercise: For a = 0.2 and b = 0.04, determine g.
[Solution: g = $\frac{0.2 + 0.04}{(1.2)(0.04)}$ = 5. p = 1/5 = 20%.]

Then, **a** = $\mathbf{\frac{(g - 1)\,b}{1 - gb}}$.

For b = 0.04 and g = 5, a = $\frac{(4)(0.04)}{1 - (5)(0.04)}$ = 0.20.

Exercise: Determine the form of the exposure curve G(x) in terms of b and g.
[Solution: a = $\frac{(g - 1)\,b}{1 - gb}$. ⇒ a + b = $\frac{g - gb}{1 - gb}$ b.

a + 1 = $\frac{1 - b}{1 - gb}$. ⇒ ln(a + b) - ln(a + 1) = ln[$\frac{g - gb}{1 - gb}$ b $\frac{1 - gb}{1 - b}$] = ln[gb].

a + b^x = $\frac{(g-1)b + (1 - gb)\,b^x}{1 - gb}$. ⇒ ln(a + b^x) - ln(a + 1) = ln[$\frac{a + b^x}{a + 1}$] = ln[$\frac{(g-1)b + (1 - gb)\,b^x}{1-b}$].

$$\Rightarrow G(x) = \frac{\ln(a + b^x) - \ln(a+1)}{\ln(a+b) - \ln(a+1)} = \frac{\ln\left[\frac{(g-1)b + (1 - gb)\,b^x}{1-b}\right]}{\ln[gb]}.]$$

In terms of b and g, the exposure curve is:[26]

$$\mathbf{G(x)} = \frac{\ln\left[\frac{(g - 1)b + (1 - gb)\,b^x}{1 - b}\right]}{\ln[gb]},\ 0 \le x \le 1,\ b \ge 0,\ g \ge 1.$$

For a = 0.2 and b = 0.04, and thus g = 5:

G(0.5) = $\frac{\ln[(5-1)(0.04) + (1 - 0.2)\,0.04^{0.5}] - \ln[1 - 0.04]}{\ln[0.2]}$ = 68.26%, matching a previous result.

25 See equation 3.2 in Bernegger.

26 See equation 3.3 in Bernegger. If bg = 1, then G(x) = (1 - b^x) / (1 - b).

Exercise: Determine G'(x) for the exposure curve parameterized in terms of b and g.

[Solution: Let $y = b^x$. Then $G(y) = \dfrac{\ln\left[\dfrac{(g-1)b + (1 - gb)\,y}{1-b}\right]}{\ln[gb]}$.

$$\frac{d\,G(y)}{dy} = \frac{1-gb}{(g-1)b+(1-gb)\,y}\ \frac{1}{\ln[gb]} = \frac{1-gb}{(g-1)b+(1-gb)\,b^x}\ \frac{1}{\ln[gb]}.$$

$\dfrac{dy}{dx} = \ln(b)\ b^x.$

$$\frac{d\,G(y)}{dy} = \frac{d\,G(y)}{dy}\ \frac{dy}{dx} = \frac{1-gb}{(g-1)b+(1-gb)\,b^x}\ \frac{1}{\ln[gb]}\ \ln(b)\ b^x = \frac{\ln(b)\ (1-gb)}{\ln(gb)\,\{(g-1)b^{1-x}+(1-gb)\}}.$$

<u>Comment</u>: One could instead substitute: $a = \dfrac{(g-1)\,b}{1-gb}$

into the previous form of $G'(x) = \dfrac{\dfrac{\ln(b)\ b^x}{a+b^x}}{\ln(a+b) - \ln(a+1)}$.]

Thus, this form of exposure curve has a derivative of:[27]

$$G'(x) = \frac{\ln(b)\ (1-gb)}{\ln(gb)\,\{(g-1)b^{1-x}+(1-gb)\}}.$$

p = probability of having the MPL = S(1) = G'(1)/G'(0)

$$= \frac{\ln(b)\ (1-gb)}{\ln(gb)\,\{(g-1)+(1-gb)\}}\ /\left\{\frac{\ln(b)\ (1-gb)}{\ln(gb)\,\{(g-1)b+(1-gb)\}}\right\} = \frac{(g-1)b+(1-gb)}{(g-1)+(1-gb)} = \frac{1-b}{g-gb} = 1/g,$$

as it has to be from the definition of g.

Exercise: For g = 5 and b = 0.04, determine the mean.

[Solution: $G'(0) = \dfrac{\ln(0.04)\ (1-0.2)}{\ln(0.2)\,\{(4)(0.04)+(1-0.2)\}} = 5/3$. E[X] = 1/G'(0) = 3/5 = 0.6.

<u>Comment</u>: Matches a previous result for a = 0.2 and b = 0.04.]

In general, $E[x] = 1/G'(0) = \dfrac{\ln(gb)\,\{(g-1)b+(1-gb)\}}{\ln(b)\ (1-gb)} = \dfrac{\ln(gb)\,(1-b)}{\ln(b)\ (1-gb)}$.

$E[x] = \dfrac{\ln(gb)\,(1-b)}{\ln(b)\ (1-gb)}$.[28]

[27] See equation 3.4 in Bernegger. For bg = 1, $G'(x) = \ln[b]\ b^x / (b-1)$.

[28] See equation 3.5 in Bernegger. For bg = 1, E[x] = (b-1) / ln[b] = (g-1) / {ln[g] g}.

Exercise: For g = 10 and b = 0.419, determine the mean.

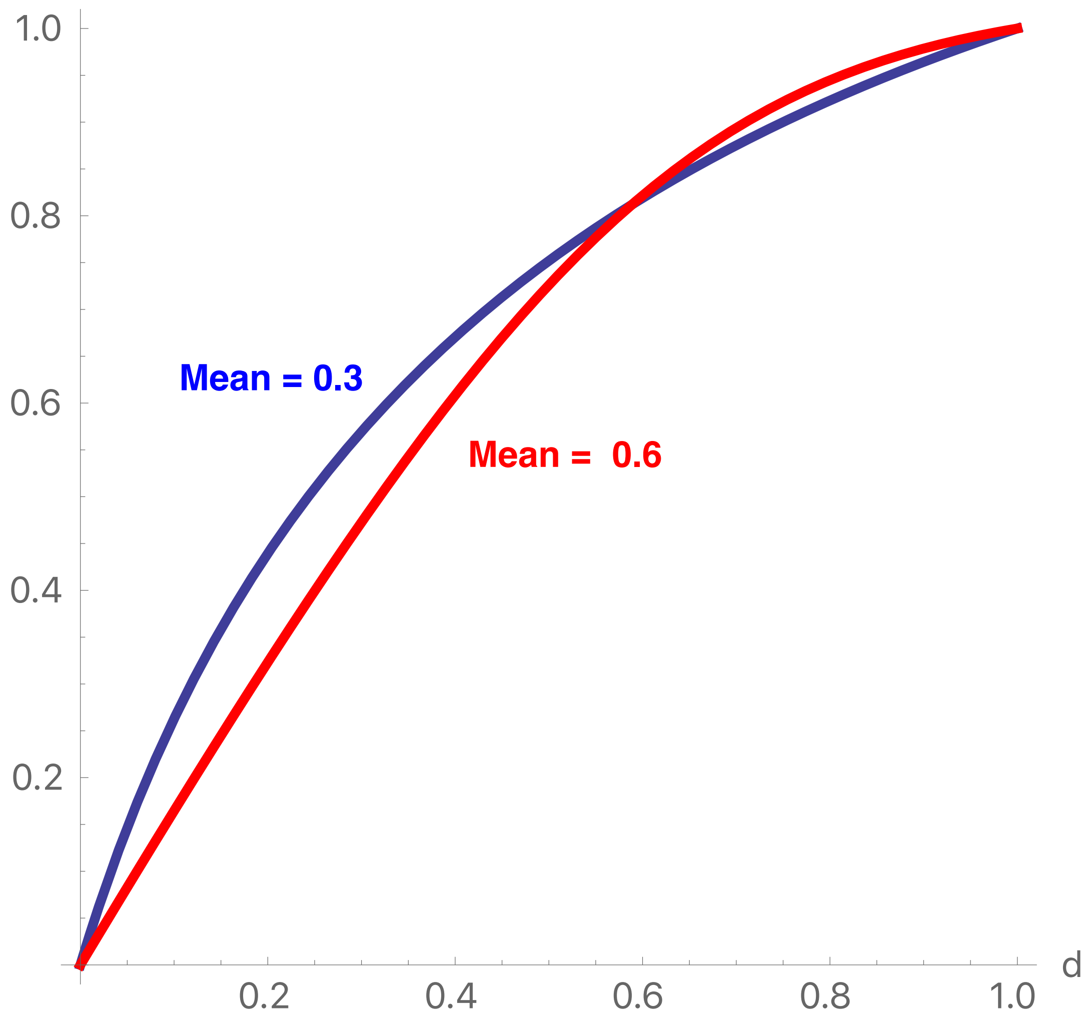
[Solution: $E[x] = \dfrac{\ln(gb)\,(1-b)}{\ln(b)\,(1-gb)} = \dfrac{\ln(4.19)\,(1-0.419)}{\ln(0.419)\,(1-4.19)} = 0.300.$

Comment: With the help of a computer, for g = 10, I found the value of b such that the mean is 0.3.]

Exercise: For g = 10 and b = 0.00436, determine the mean.

[Solution: $E[x] = \dfrac{\ln(gb)\,(1-b)}{\ln(b)\,(1-gb)} = \dfrac{\ln(0.0436)\,(1-0.00436)}{\ln(0.00436)\,(1-0.0436)} = 0.600.$]

Here is a comparison of the exposure curves for g = 10 and b = 0.419 or 0.00436:[29]



For the larger mean (b = 0.00436), there are more large losses, and for a given small deductible (retention) the loss elimination ratio is smaller than for the smaller mean (b = 0.419).

[29] Similar to Figure 3.1a in Bernegger.

Exercise: Determine G(0.2), for g = 10 and b = 0.419, and for g = 10 and b = 0.00436.

[Solution: $G(x) = \frac{\ln\left[\frac{(g-1)b + (1 - gb)\, b^x}{1-b}\right]}{\ln[gb]}$.

For g = 10 and b = 0.419, $G(0.2) = \frac{\ln\left[\frac{(9)(0.419) + (1 - 4.19)\, 0.419^{0.2}}{1-0.419}\right]}{\ln[4.19]} = 0.4394$.

For g = 10 and b = 0.00436, $G(0.2) = \frac{\ln\left[\frac{(9)(0.00436) + (1 - 0.0436)\, 0.00436^{0.2}}{1-0.00436}\right]}{\ln[0.0436]} = 0.3232$.

Comment: Matches what is shown in the previous graph.]

Exercise: For g = 25 and b = 0.0390, determine the mean.

[Solution: $E[x] = \frac{\ln(gb)\,(1-b)}{\ln(b)\,(1-gb)} = \frac{\ln(0.975)\,(1-0.039)}{\ln(0.039)\,(1-0.975)} = 0.300$.

Comment: With the help of a computer, for g = 25, I found the value of b such that the mean is 0.3.]

Exercise: For g = 10 and b = 0.419, determine G(0.4).

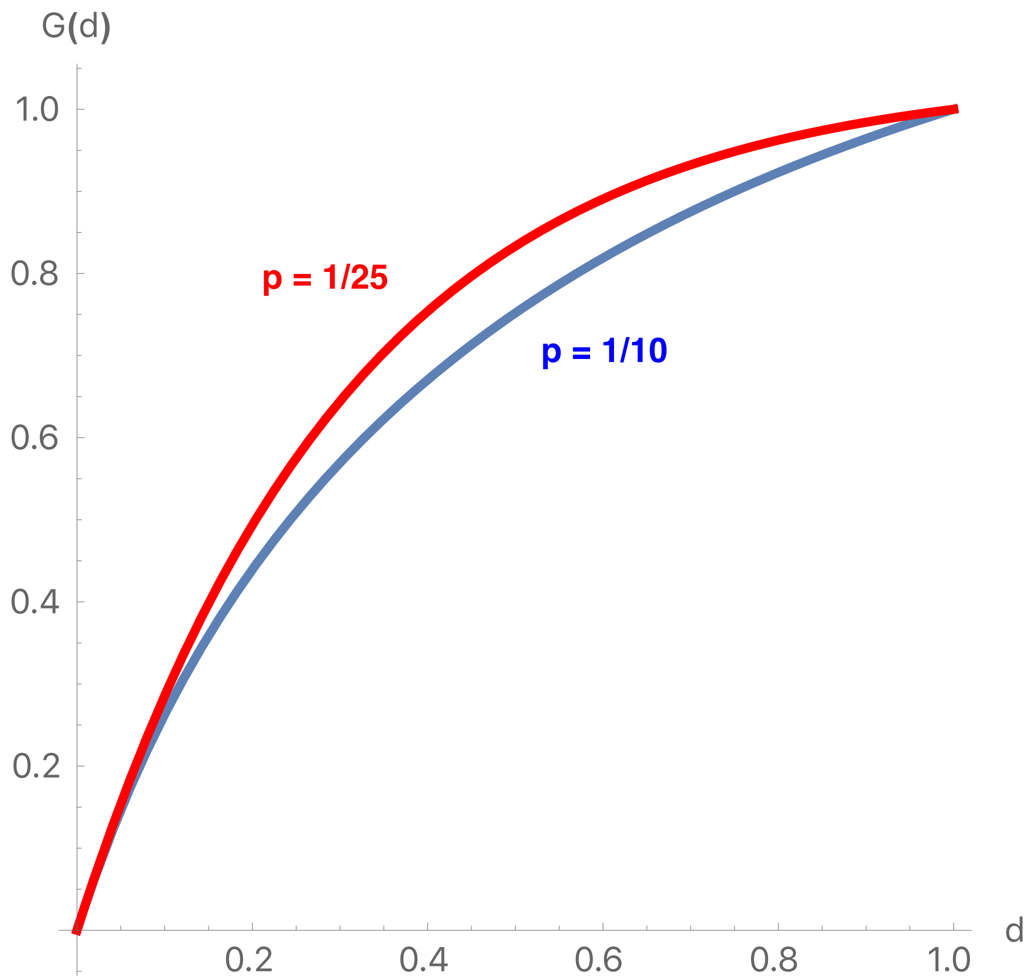
[Solution: $G(0.4) = \frac{\ln\left[\frac{(9)(0.419) + (1 - 4.19)\, 0.419^{0.4}}{1-0.419}\right]}{\ln[4.19]} = 0.6705$.]

Exercise: For g = 25 and b = 0.0390, determine G(0.4).

[Solution: $G(0.4) = \frac{\ln\left[\frac{(24)(0.039) + (1 - 0.975)\, 0.039^{0.4}}{1-0.039}\right]}{\ln[0.975]} = 0.7540$.

Comment: Two exposure curves each have a mean of 0.3, but they have different probabilities of the MPL. G(0.4) is smaller for the curve with bigger p (smaller g); for a fixed mean, the larger the chance of the MPL, the smaller the loss elimination ratio.]

Here is a comparison of the exposure curves for g = 10 and b = 0.419,
and g = 25 and b = 0.039:[30]



Both curves have the same mean of 0.3, but have different probabilities of the MPL, p.

The flatter the curve, the bigger the probability of the Maximum Possible Loss.
For p = 1, all losses would be total losses, and the exposure curve would be the diagonal line connecting (0, 0) and (1, 1).[31]

$$G(x) = \frac{\ln\left[\frac{(g-1)b + (1 - gb)\, b^x}{1-b}\right]}{\ln[gb]}. \Rightarrow G'(x) = \frac{\ln(b)\,(1-gb)}{\ln(gb)\,\{(g-1)b^{1-x} + (1-gb)\}}.$$

$$S(x) = G'(x) / G'(0) = \frac{(g-1)b + (1-gb)}{(g-1)b^{1-x} + (1-gb)} = \frac{1-b}{(g-1)b^{1-x} + (1-gb)}.$$ [32]

---

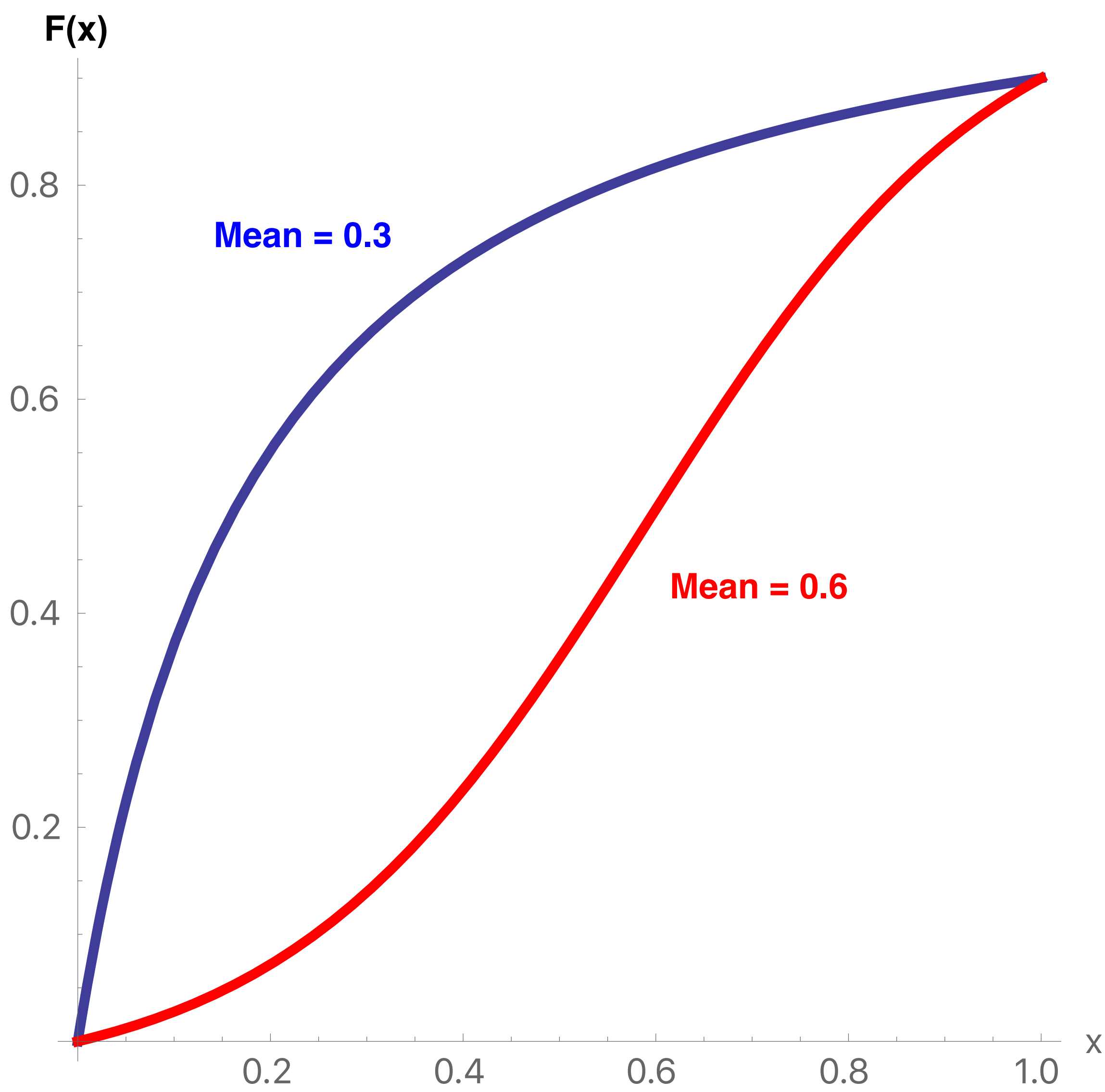
[30] Similar to Figure 3.1b in Bernegger.
[31] For p = 1, the mean loss would be 1.
[32] See equation 3.6 in Bernegger.

Exercise: Determine F(0.3) for g = 10 and b = 0.419, and for g = 10 and b = 0.00436.
[Solution:

For g = 10 and b = 0.419, $S(0.3) = \frac{1 - 0.419}{(9)0.419^{0.7} + 1 - 4.19} = 0.3407$. $\Rightarrow$ F(0.3) = 0.6593.

For g = 10 and b = 0.00436, $S(0.3) = \frac{1 - 0.00436}{(9)0.00436^{0.7} + 1 - 0.0436} = 0.8607$. F(0.3) = 0.1393.]

Here is a graph comparing the distribution functions for g = 10, with b = 0.419 and b = 0.00436:[33]

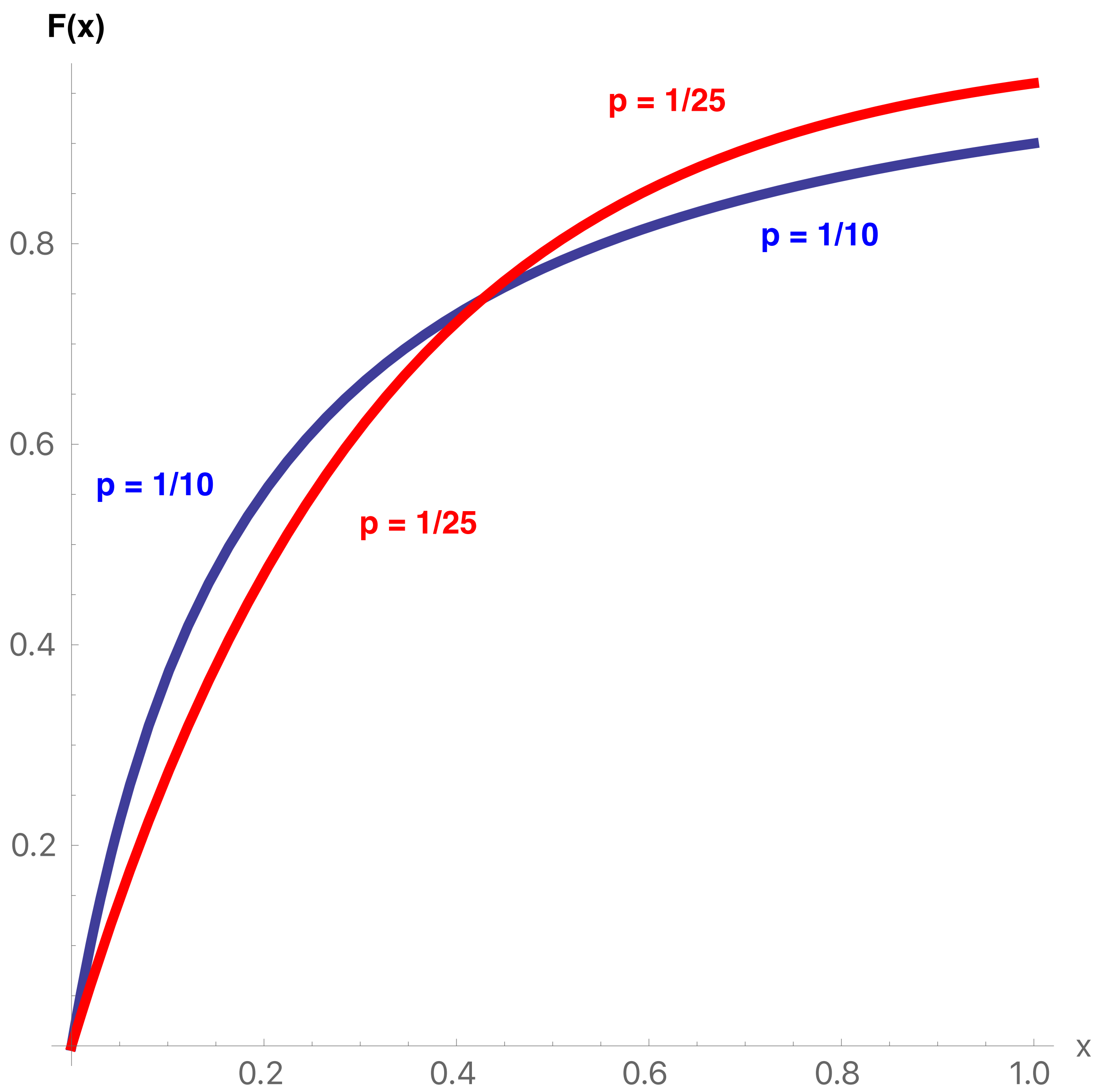


Exercise: For g = 25 and b = 0.0390, determine F(0.3).

[Solution: $S(0.3) = \frac{1 - 0.0390}{(24)0.0390^{0.7} + 1 - (25)(0.0390)} = 0.3841$. $\Rightarrow$ F(0.3) = 0.6159.]

[33] Similar to Figure 3.2a in Bernegger.

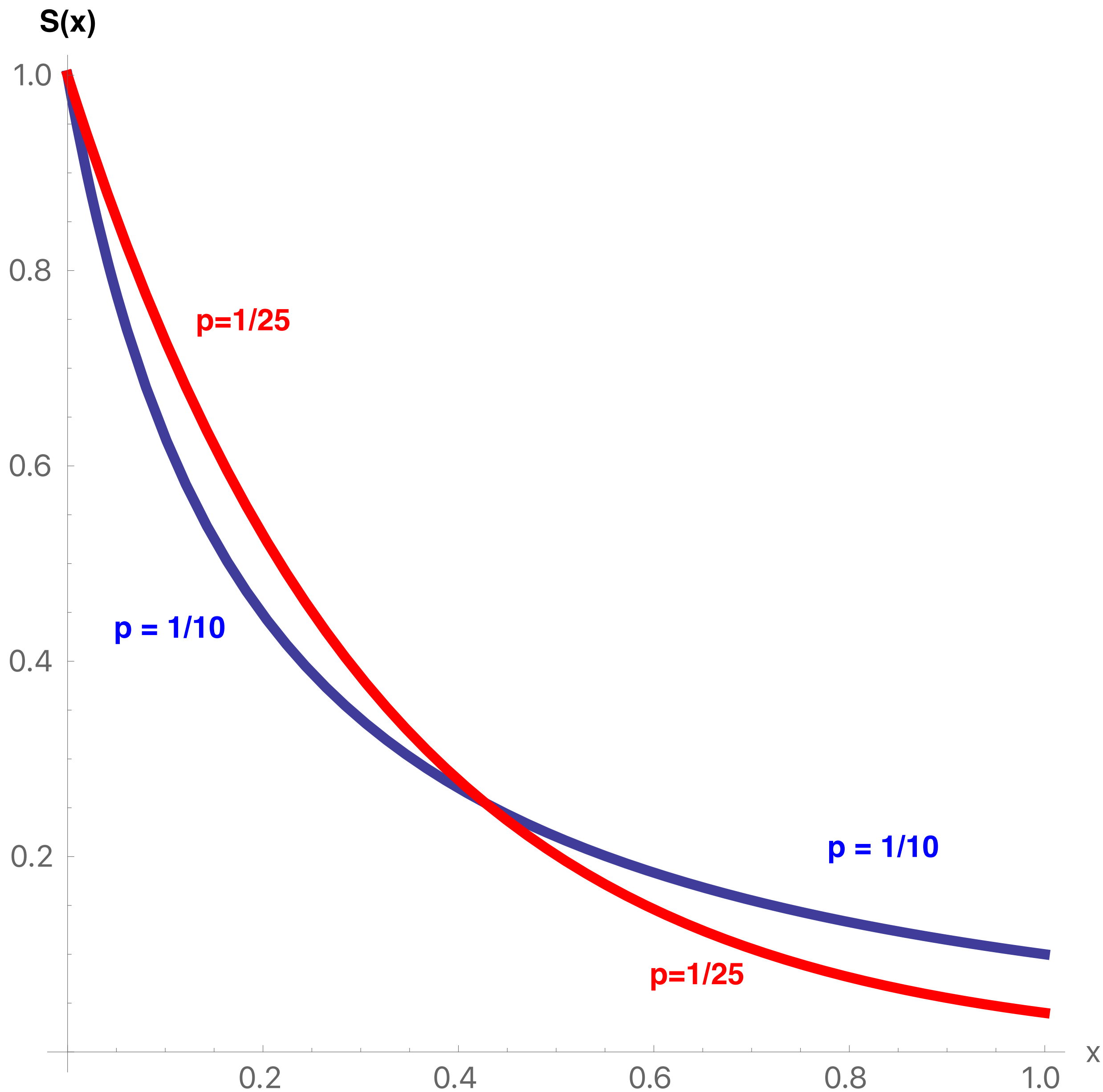
Here is a graph comparing the distribution functions for g = 10 and b = 0.419 versus g = 25 and b = 0.0390, both with a mean of 0.3:[34]



[34] Similar to Figure 3.2b in Bernegger.

$S(x) = \dfrac{1 - b}{(g - 1)b^{1-x} + (1 - gb)}$.

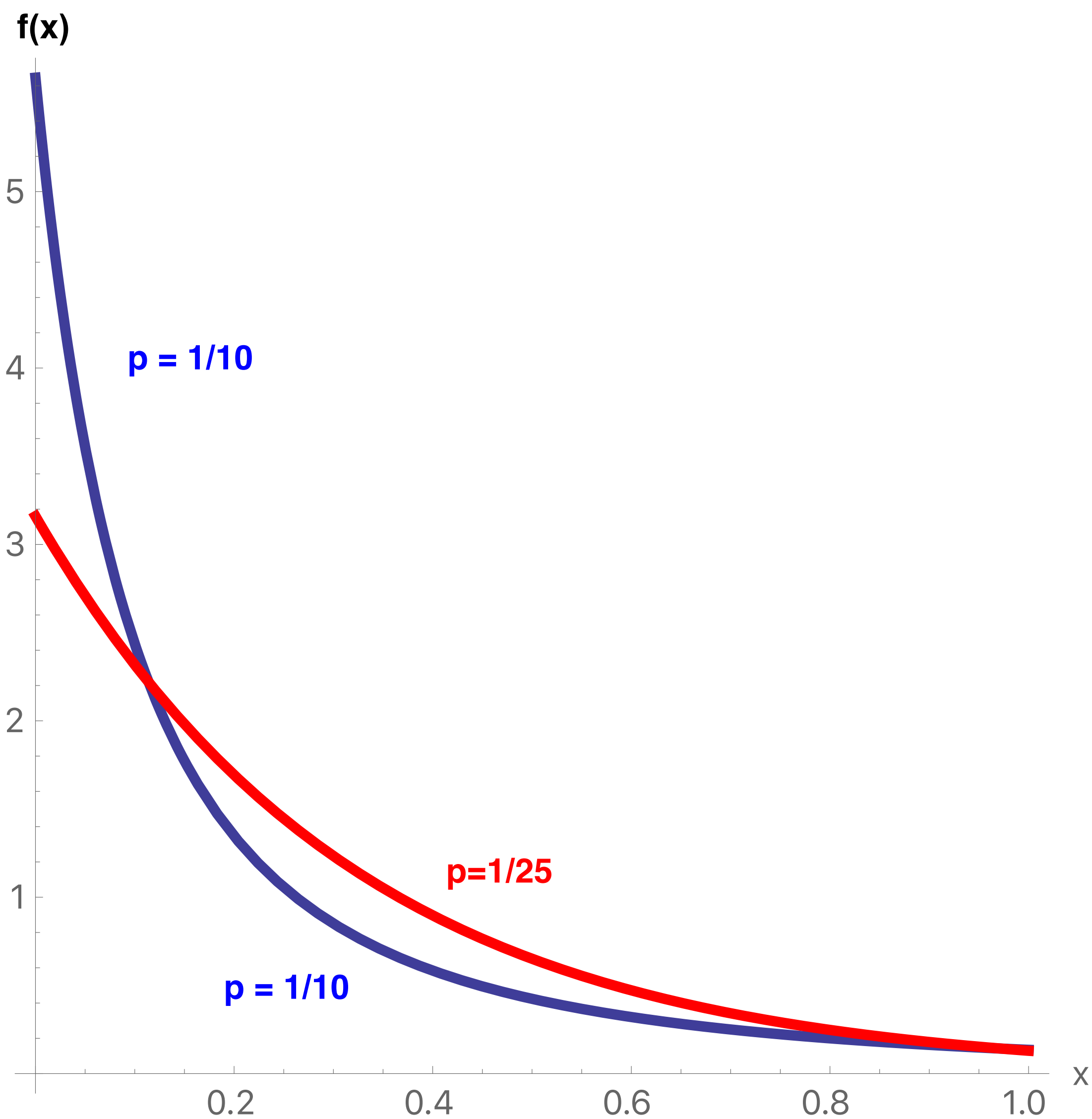
Here is a graph comparing the survival functions for g = 10 and b = 0.419 versus g = 25 and b = 0.0390, both with a mean of 0.3:



$$S(x) = \frac{1 - b}{(g-1)b^{1-x} + (1-gb)}.$$

$$\Rightarrow f(x) = -S'(x) = \frac{(b - 1)\ (g - 1) \ln[b]\ b^{1-x}}{\{(g-1)b^{1-x} + (1-gb)\}^2}.$$ [35]

Here is a graph comparing the probability density functions for g = 10 and b = 0.419 versus g = 25 and b = 0.0390, both with a mean of 0.3:

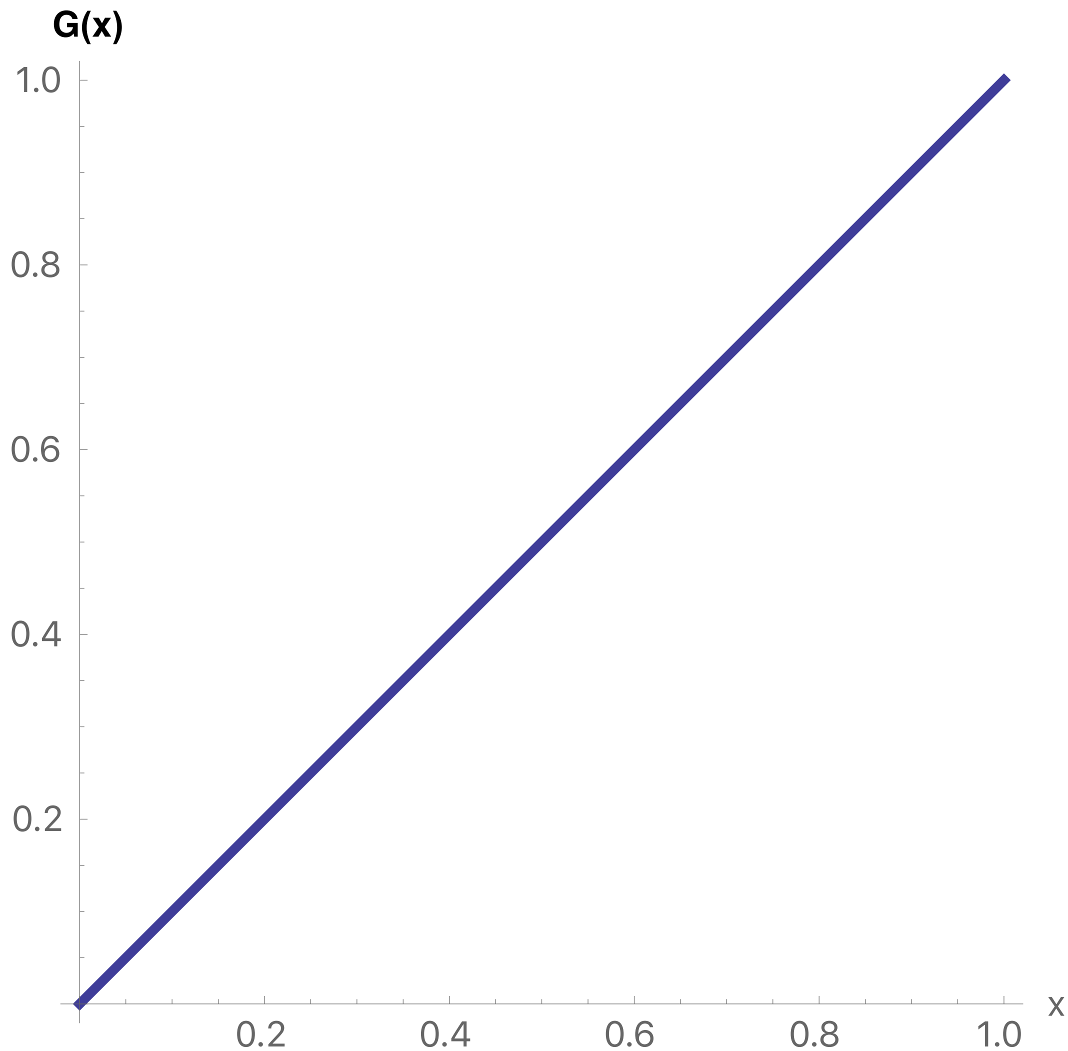


Recall that in each case, there is also a point mass of probability p at x = 1.

As x approaches one, f(x) approaches: $\frac{(b - 1)\ (g - 1) \ln[b]}{\{(g-1) + (1-gb)\}^2} = \frac{(g - 1) \ln[b]}{(b - 1)\ g^2}$.

[35] See equation 3.7 in Bernegger. If bg = 1, then f(x) = - ln(b) b^x.

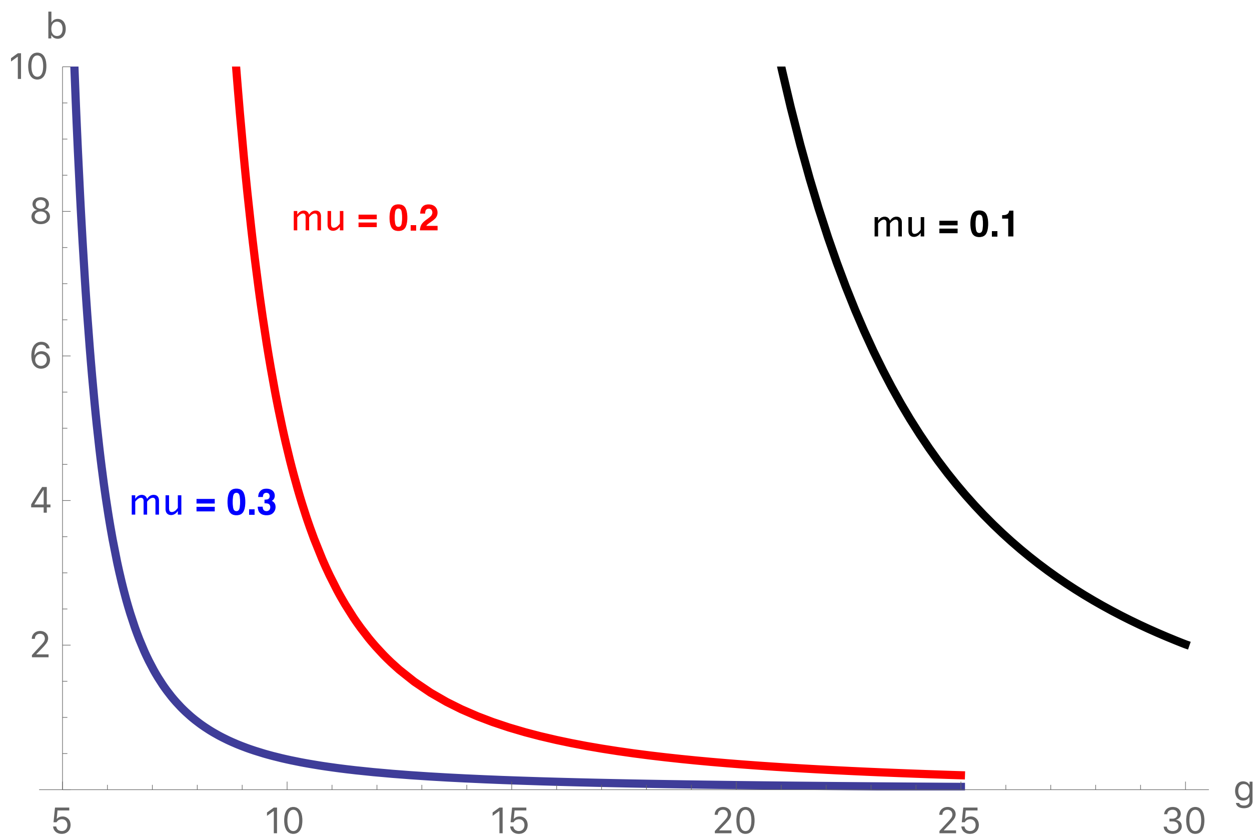
For $p = 1$, all losses would be total losses, and the exposure curve would be the diagonal line connecting (0, 0) and (1, 1):



For example, let M = 10 million. Then $x = 0.2$ corresponds to 2 million.
All losses are of size 10 million, so the loss elimination ratio at 2 million is: $2/10 = 0.2 = x$.

The Mean as a Function of the Parameters:

Here is a graph of the mean as a function of the parameters g and b:[36]



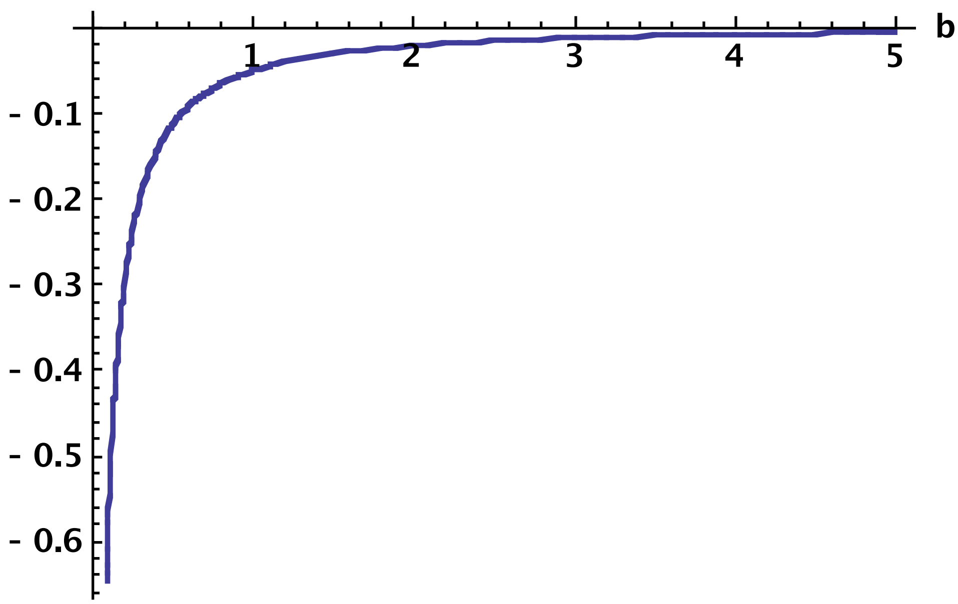
For fixed g > 1, as b increases the mean decreases:[37]
$\frac{\partial \mu}{\partial b} \le 0$.

For fixed b > 0, as g increases the mean decreases:[38]
$\frac{\partial \mu}{\partial g} \le 0$.

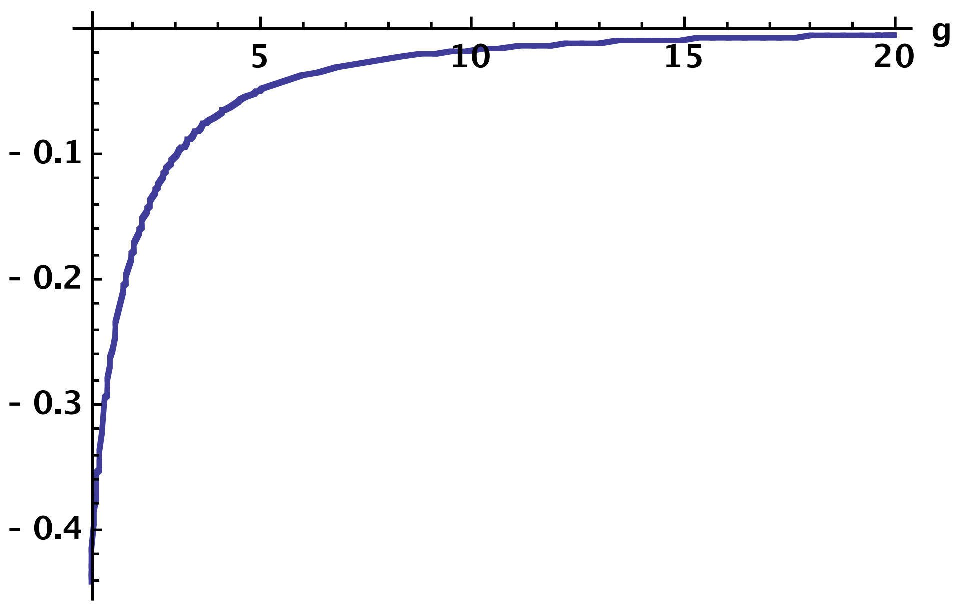
Since the mean is a decreasing function of g, the mean is an increasing function of p = 1/g.

[36] Similar to Figure 3.3 in Bernegger.
[37] See equation 3.8 in Bernegger.
[38] See equation 3.8 in Bernegger.

The fact that $\frac{\partial \mu}{\partial b} \le 0$ can be seen a little more clearly in the following graph of $\frac{\partial \mu}{\partial b}$ for g = 5:



The fact that $\frac{\partial \mu}{\partial g} \le 0$ can be seen a little more clearly in the following graph of $\frac{\partial \mu}{\partial g}$ for b = 0.5:



Limits:

$$\mu = E[x] = \frac{\ln(gb)\,(1 - b)}{\ln(b)\,(1 - gb)}.$$

$\mu = \dfrac{\ln(b) + \ln(g)}{\ln(b)} \dfrac{1 - b}{1 - gb}$. As $b \to 0$, $\mu \to \dfrac{\ln(b)}{\ln(b)} \dfrac{1}{1} = 1$.

For b = 0, there are only total losses. For b = 0, G(x) = x and E[x] = 1.[39]

As $b \to \infty$, $\mu \to \dfrac{\ln(b)}{\ln(b)} \dfrac{-b}{-gb} = 1/g = p$.

As $g \to 1$, $\mu \to \dfrac{\ln(b)}{\ln(b)} \dfrac{1 - b}{1 - b} = 1$.

For g = 1, p = 1, and there are only total losses. For g = 1, G(x) = x and E[x] = 1.[40]

$\mu = \dfrac{(1 - b)}{\ln(b)} \dfrac{\ln(gb)}{1 - gb}$. As $g \to \infty$, gb increases faster than ln[gb], and $\mu \to 0$.

In summary:[41]

As $b \to 0$, $\mu \to 1$.

As $b \to \infty$, $\mu \to p = 1/g$.

As $g \to 1$, $\mu \to 1$.

As $g \to \infty$, $\mu \to 0$.

Also, as will be discussed subsequently:
For bg = 1 and g > 1: E[x] = (b - 1) / ln(b).
For g > 1 and b = 1, E[x] = ln[g] / (g - 1).

[39] b = 0 corresponds to a = 0.
[40] g = 1 corresponds to a = 0.
[41] See equation 3.9 in Bernegger.

Special Cases:

For bg = 1 and g > 1:[42]

For $x \le 1$, $S(x) = \frac{1-b}{(g-1)b^{1-x} + (1-gb)} = \frac{1-b}{(1/b-1)b^{1-x}} = b^x = \exp[-x/\theta]$, with $\theta = -1/\ln[b]$.[43]

Thus the distribution is an Exponential with mean -1/ln[b], censored from above at 1.

For the Exponential with mean θ, $E[X \wedge x] = \theta\,(1 - e^{-x/\theta})$.

The mean of the distribution censored from above at 1 is: $E[X \wedge 1] = \theta\,(1 - e^{-\theta}) = (b - 1) / \ln(b)$.

For bg = 1 and g > 1: $E[x] = (b - 1) / \ln(b)$.[44]

The loss elimination ratio for the Exponential distribution censored from above at 1 is:

$E[X \wedge x] / E[X \wedge 1] = (1 - e^{-x/\theta}) / (1 - e^{-\theta}) = (1 - b^x) / (1 - b)$.

For bg = 1 and g > 1: $G(x) = \frac{1 - b^x}{1 - b}$.[45]

$S(x) = \frac{1-b}{(g-1)b^{1-x} + (1-gb)}$. For g > 1, as $b \to 1$, $S(x) \to \frac{0}{0}$.

For g > 1, using L'Hospital's Rule, as $b \to 1$, $S(x) \to \dfrac{\frac{\partial(1-b)}{\partial b}}{\frac{\partial\{(g-1)b^{1-x} + (1-gb)\}}{\partial b}} =$

$$\frac{-1}{(g - 1)\,(1 - x)\,b^{-x} - g} \to \frac{-1}{(g - 1)\,(1 - x) - g} = \frac{1}{1 + (g - 1)\,x}.$$

Thus for g > 1 and b = 1, $S(x) = \frac{1}{1 + (g - 1)\,x}$, $x \le 1$.[46]

This is a Shifted Pareto Distribution, censored from above at one, with $\alpha = 1$ and $\theta = 1/(g-1)$.

---

[42] bg = 1 corresponds to a = ∞. bg = 1 corresponds to the Maxwell-Boltzmann Distribution.
bg > 1 (a < 0) corresponds to the Bose-Einstein Distribution.
bg < 1 (a > 0) corresponds to the Fermi-Dirac Distribution.

[43] See equation 3.6 in Bernegger. Since g = 1/p > 1, b < 1, and thus $\theta = -1/\ln[b] > 0$.

[44] See equation 3.5 in Bernegger.

[45] See equation 3.3 in Bernegger.
For the uncensored Exponential Distribution, LER(x) = 1 - exp[-x/q].

[46] See equation 3.6 in Bernegger.

Exercise: For g > 1 and b = 1, compute the mean.

[Solution: $E[x] = \int_0^1 S(t)\ dt = \left.\frac{\ln[1 + (g-1)(x)]}{g-1}\right]_{x=0}^{x=1} = \ln[g] / (g - 1)$.

Comment: For a Shifted Pareto Distribution with $\alpha = 1$, $E[X \wedge x] = -\theta \ln[\frac{\theta}{x + \theta}]$.

The mean of the distribution censored from above at 1 is:

$E[X \wedge 1] = -\theta \ln[\frac{\theta}{1 + \theta}] = \frac{-1}{g - 1} \ln[\frac{1/(g-1)}{1 + 1/(g-1)}] = -\ln[1/g] / (g - 1) = \ln[g] / (g - 1)$.]

For g > 1 and b = 1, E[x] = ln[g] / (g - 1).[47]

Exercise: For g > 1 and b = 1, determine the form of the exposure curve, G(x).

[Solution: $G(x) = \frac{\int_0^x S(t)\ dt}{\int_0^1 S(t)\ dt} = \frac{\ln[1 + (g-1)(x)] / (g - 1)}{\ln[g] / (g - 1)} = \frac{\ln[1 + (g-1)(x)]}{\ln[g]}$.]

For g > 1 and b = 1, $G(x) = \frac{\ln[1 + (g-1)(x)]}{\ln[g]}$.[48]

[47] See equation 3.5 in Bernegger. b = 1 corresponds to a = -1.
[48] See equation 3.3 in Bernegger.

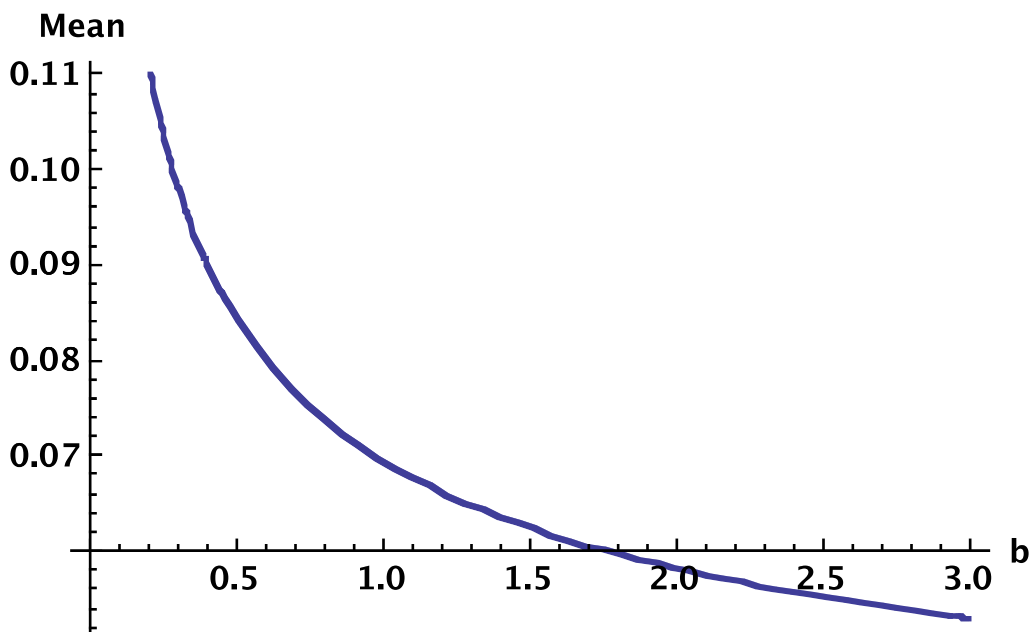
Method of Moments:[49]

Given a value of g, or p = 1/g, and the mean, one can solve for the remaining parameter b using:

$$E[x] = \frac{\ln(gb)\,(1-b)}{\ln(b)\,(1-gb)}.$$

However, you can not solve this equation in closed form.

For example, let us assume that g = 60 and the mean is 0.06.
Here is a graph of the mean as a function as b:



We can see that the mean is 0.06 for b equal to about 1.7 or 1.8.

We also see that the mean is a decreasing function of b.[50]
Thus we can try a value of b, and iterate.[51]

In this example, for b = 2 the mean is 0.0580.[52]
Try a smaller b. For b = 1.5 the mean is 0.0623.
For b = 1.75 the mean is 0.0600.
Thus b = 1.75.[53]

---

[49] See Section 4 of Bernegger. See 8, 11/17, Q.18b.
[50] As discussed previously, this is general result. See equation 3.8 in Bernegger.
[51] This is probably too long to be asked on your exam.
[52] For b = 1, the mean is ln[g] / (g - 1) = ln(60) / 59 = 0.694.
[53] To more decimal places, b = 1.74691.

$$\mu = E[x] = \int_0^{1-} x\ f(x)\ dx + p.$$

$$E[x^2] = \int_0^{1-} x^2\ f(x)\ dx + p. \Rightarrow E[x^2] \geq p.$$

$0 \leq x \leq 1. \Rightarrow x^2 \leq x. \Rightarrow E[x^2] \leq E[x].$
Also, $0 \leq \sigma^2 = E[x^2] - E[x]^2. \Rightarrow E[x^2] \geq E[x]^2$.

Therefore:[54]
$\mu^2 \leq E[x^2] \leq \mu$, and $p \leq E[x^2]$.

If instead of being given g, one were fitting two parameters by matching the first two moments, or equivalently the mean and the variance, this would be much more difficult.[55]

First, there is no convenient closed form formula for the second moment of MBBEFD class.[56] Bernegger suggests one should calculate the second moment by approximating via a discrete distribution with the same losses in a set of narrow layers as the continuous distribution.[57]
For particular values of the parameters, Mathematica had no trouble doing the integral needed to determine the second moment; I assume there are other software packages that would also do this integral.

Let the given first two empirical moments be μ and μ_2.
Here is Bernegger's iterative scheme to fit via method of moments (2 parameters, via computer):
1. Try $p = \mu_2$.[58]
2. $g = 1/p$. As described above, find the b such that the first moments match.
3. Determine the second moment for these values of b and g.
4. Compare the results of step 3 to μ_2. If they match, you are done.
5. Use the fact that the second moment is an increasing function of p to choose
   a new value of p.
   Return to step 2.

---

[54] See equation 4.4 in Bernegger.
[55] This is way too long for you to be asked to do a calculation on your exam.
If one were using this for practical applications on a regular basis, it would only need to be programmed once.
[56] The second moment involves the polylogarithm function $Li_2[x] = x + x^2/2^2 + x^3/3^2 + x^4/4^2 + \ldots$.
[57] One needs to remember to include the contribution to the second moment from the point mass of probability p at 1.
[58] $p \leq E[x^2]$, so this is an upper bound on the value of p.

For example, let the empirical first moment be 0.14 and the empirical second moment be 0.09.

Take p = 0.09. (g = 1/0.09).
Find the b such that the mean is 0.14. b = 64.54.
Compute the second moment for g = 1/0.09 and b = 64.54: 0.10156.
The second moment is too big, so reduce p.

Try p = 0.07.
Find the b such that the mean is 0.14. b = 8.35.
Compute the second moment for g = 1/0.07 and b = 8.35: 0.09070.
The second moment is still a little too big, so reduce p.

Try p = 0.07 - (0.09070 - 0.09) (0.09 - 0.07) / (0.10156 - 0.09070) = 0.0687.
Find the b such that the mean is 0.14. b = 7.483.
Compute the second moment for g = 1/0.0687 and b = 7.483: 0.09002.

Close enough. The fitted parameters are: p = 0.0687 and b = 7.483.

Simulation:[59]

As discussed previously, the p^{th} percentile can be obtained by setting S(x) = 1 - p:

$$\pi_p = \frac{\ln\left[\frac{a\,(1-p)}{a+p}\right]}{\ln(b)} = \frac{\ln\left[\frac{(g-1)\,b\,(1-p)}{(1-gb)\,p + (g-1)b}\right]}{\ln(b)}, \text{ where } \pi_p \le 1.$$

Similarly, we can simulate a random draw given a random number u from [0, 1], setting S(x) = 1 - u:

$$x = \text{Min}\left[1, \frac{\ln\left[\frac{a\,(1-u)}{a+u}\right]}{\ln(b)}\right] = \text{Min}\left[1, \frac{\ln\left[\frac{(g-1)\,b\,(1-u)}{(1-gb)\,u + (g-1)b}\right]}{\ln(b)}\right].$$

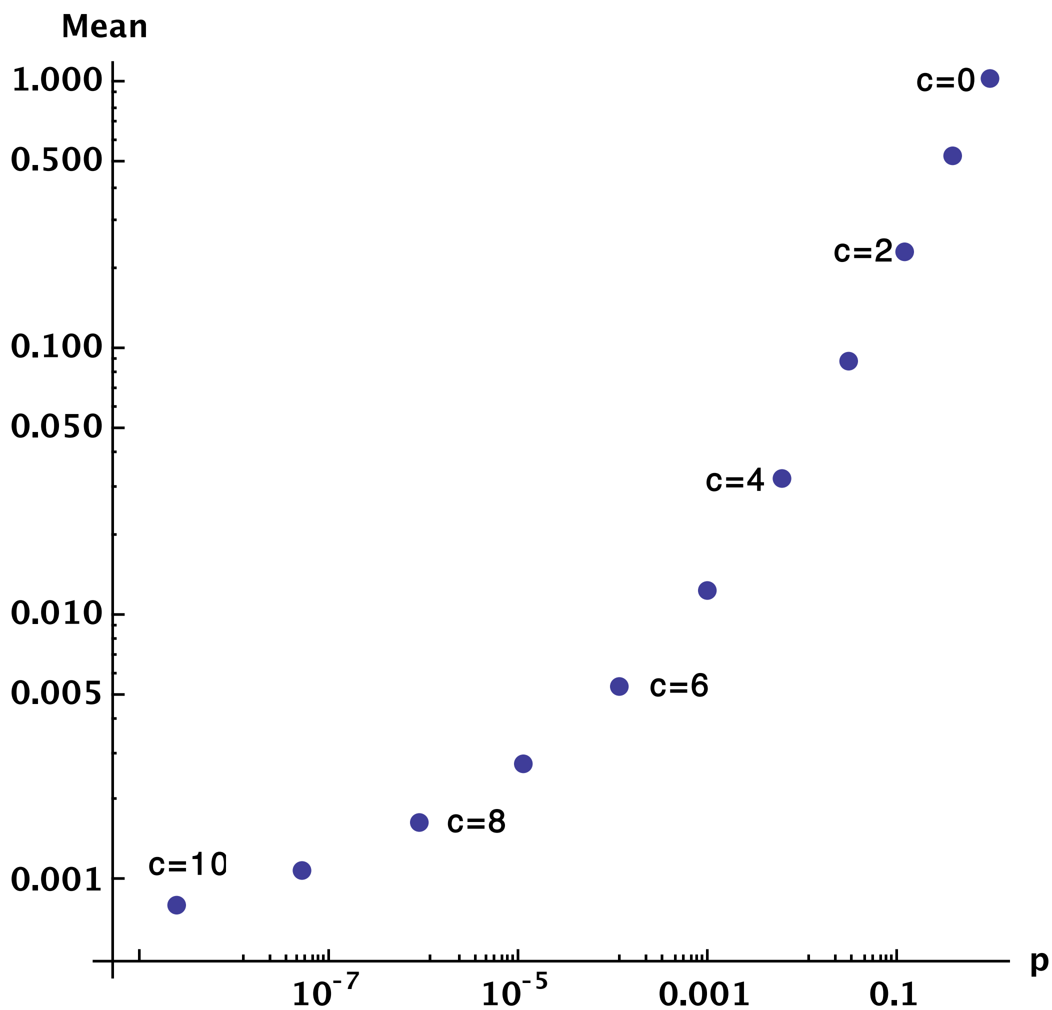
59 Bernegger does not discuss simulation.

SwissRe Curves:

Bernegger discusses a subset of exposure curves based on the MBBEFD Distribution Class. This subset is based on one parameter c, and closely matches exposure curves used by and named after SwissRe and Lloyds of London.

b = exp[3.1 - 0.15 c (1+c)].
g = exp[c (0.78 + 0.12c)].

Here is a graph on a log-log scale of the probability of the MPL, and the mean, as a function of c:[60]



Note that for c = 0, g = exp[0] = 1, and thus p = 1/1 = 1; this corresponds to only total losses. As c increases, the probability of a total loss, p, decreases.

[60] Similar to Figure 4.1 in Bernegger. c does not have to be an integer.

Exercise: For c = 4, determine b and g.
[Solution: b = exp[3.1 - 0.15 c (1+c)] = exp[3.1 - (0.15)(4)(5)] = 1.105.
g = exp[c(0.78 + 0.12c)] = exp[(4) {0.78 + (0.12)(4)}] = 154.47.
Comment: p = 1/g = 0.65%. c = 4 corresponds to the fourth SwissRe exposure curve Y_4.]

Exercise: For c = 4, determine the mean.
[Solution: $E[x] = \frac{\ln(gb)\,(1-b)}{\ln(b)\,(1-gb)} = \frac{\ln(170.7)\,(1-1.105)}{\ln(1.105)\,(1-170.7)} = 0.0319.$]

Bernegger started with the four SwissRe exposure curves, Y_1, Y_2, Y_3, and Y_4, which were based on empirical data rather than a distribution class.[61] He determined the values of b and g that best corresponded to each exposure curve. He plotted the values of b and g and was able to come up with the above parameterization in terms of c, which closely matches the popular exposure curves used to price excess of loss property reinsurance.

c = 1.5 corresponds to Y_1, c = 2 corresponds to Y_2, c = 3 corresponds to Y_3,
c = 4 corresponds to Y_4, and c = 5 corresponds to a Lloyd's curve used to rate industrial risks.[62]

Bernegger has managed to synthesize what were different empirical exposure curves into one theoretical framework. The actuary by varying the value of c continuously can hopefully produce an exposure curve that is appropriate for a given type of insured property.[63]

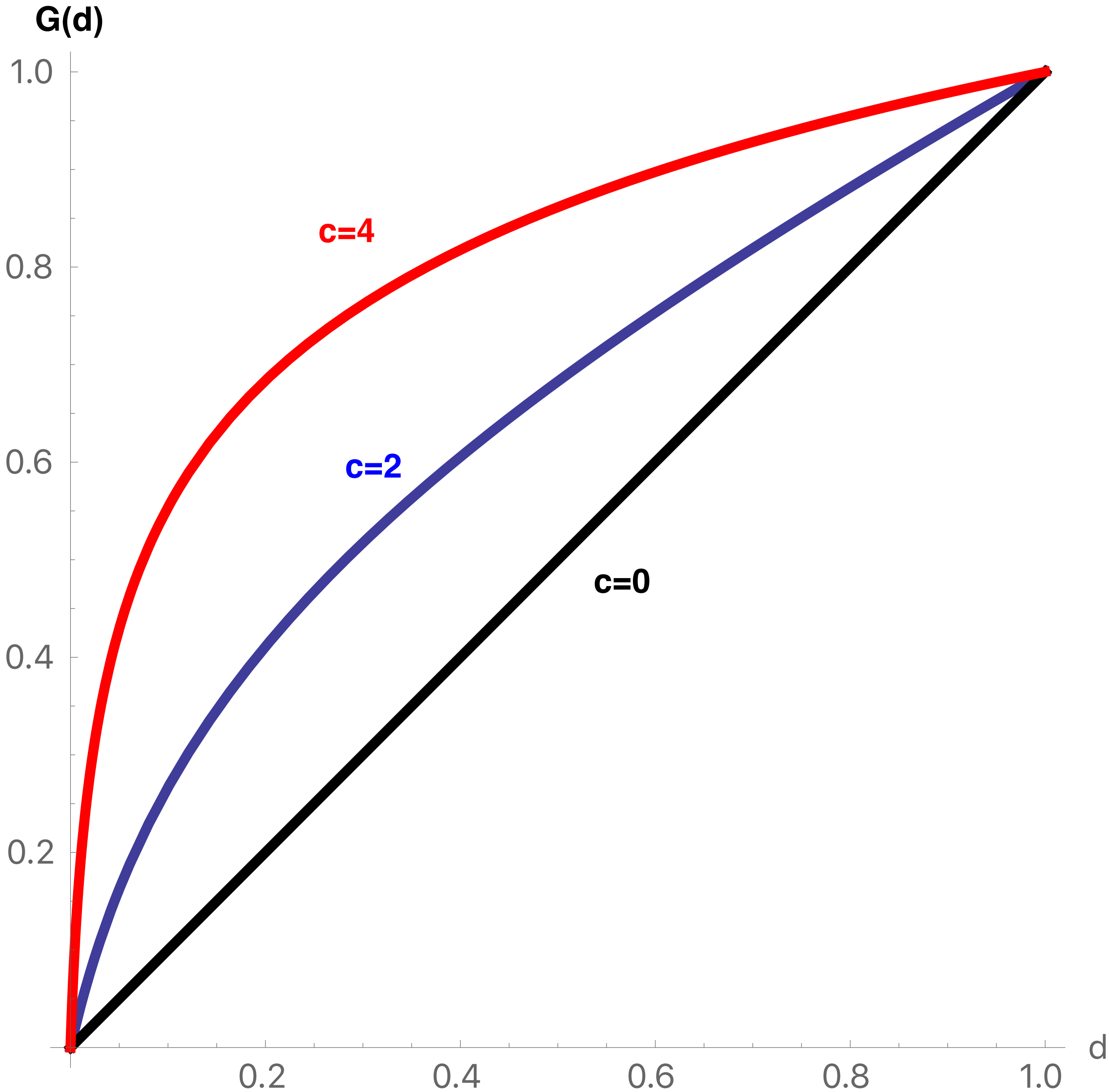
Exercise: For c = 4, determine G(0.3).
[Solution: $G(x) = \frac{\ln\left[\frac{(g-1)b + (1 - gb)\, b^x}{1-b}\right]}{\ln[gb]}$.

$$G(0.3) = \frac{\ln\left[\frac{(153.47)(1.105) + (1 - 170.7)\, 1.105^{0.3}}{1 - 1.105}\right]}{\ln[170.7]} = 76.2\%.]$$

---

[61] He does not show the SwissRe exposure curves.
[62] Sometimes referred to as Gasser curves. Y_1 is used for Personal Lines,
Y_2 is used Commercial Lines (small scale), Y_3 is used for Commercial Lines (medium scale),
and Y_4 is used for Industrial (not large scale) and Commercial Lines (large scale).
The Lloyd's curve, sometimes called Y_5 is used for large scale Industrial risks.
[63] While the exposure curve is normalized in terms of the MPL, which for small and medium scales is usually the insured value, different exposure curves may also be appropriate for different size properties.
While these curves are used for excess of loss property reinsurance, different exposure curves would apply to reinsurance of homeowners. For homeowners, one could develop different exposures curves by major peril for different construction types of home. For wind, one would have different exposure curves for hurricanes and non-hurricane losses.

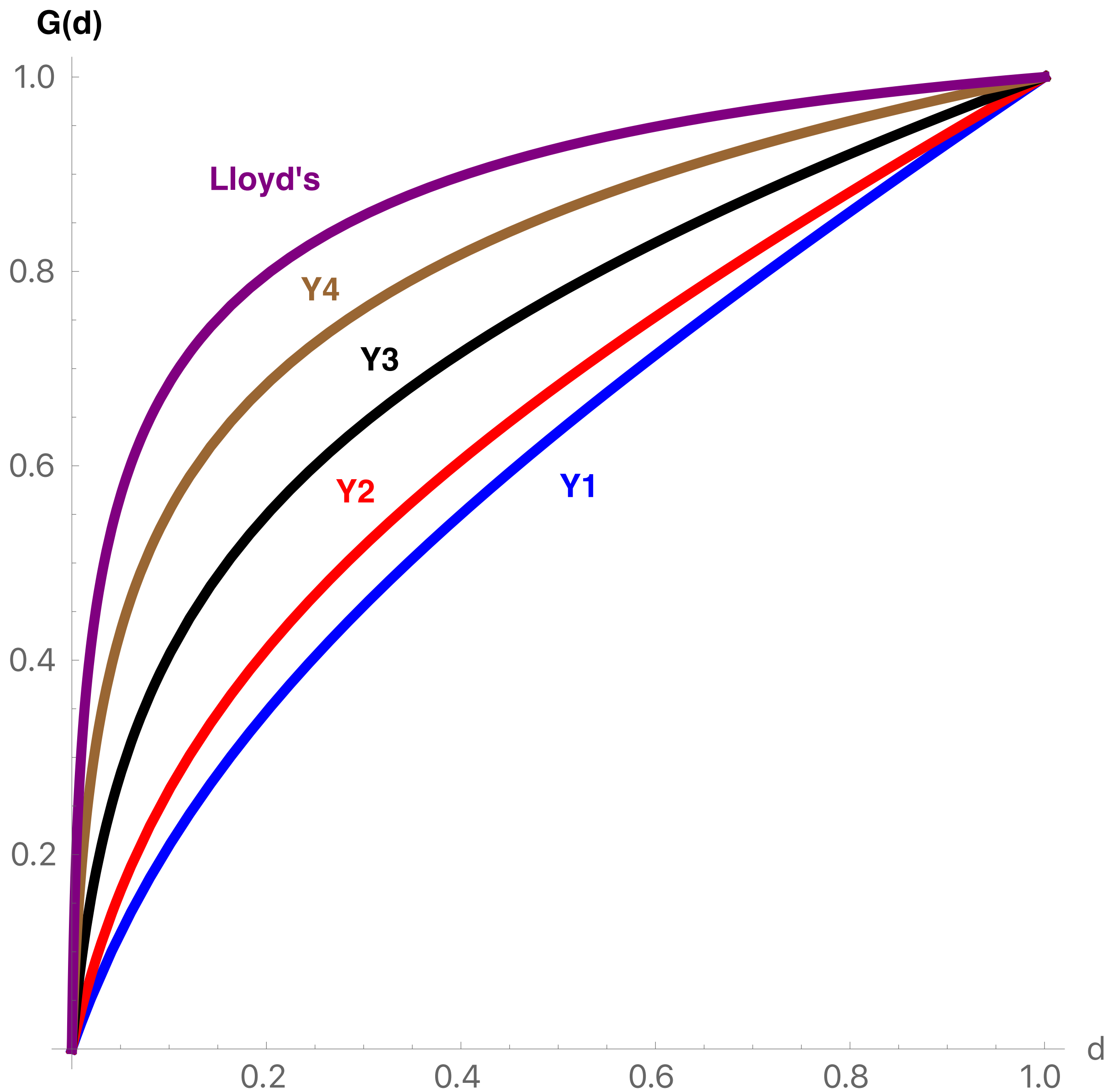
Here is a graph of the exposure curves for c = 0, 2, and 4:[64]



Recall that there is a point mass of probability of size p = 1/g at a total loss, in other words at 1.
For very small c, the probability of a total loss is very large.
For c = 0, there are only total losses and the exposure curve is a straight line along the diagonal.
For c = 0.5, p = 65.7%. For c = 3, p = 3.3%. For c = 5, p = 0.1%.

[64] Similar to figure 4.2 in Bernegger. For c = 0, p = 1 and there are only total losses.

Here are the different exposure curve used by SwissRe referred to by Bernegger:[65]



Y_1 is used for Personal Lines, Y_2 is used for Commercial Lines (small scale),
Y_3 is used for Commercial Lines (medium scale),
and Y_4 is used for Industrial Risks (not large scale) and Commercial Lines (large scale).
The Lloyd's curve, sometimes called Y_5, is used for large scale Industrial Risks.

Y_1 being closest to the diagonal has the largest probability of a total loss, while the Lloyd's curve has the smallest probability of a maximum probable loss.

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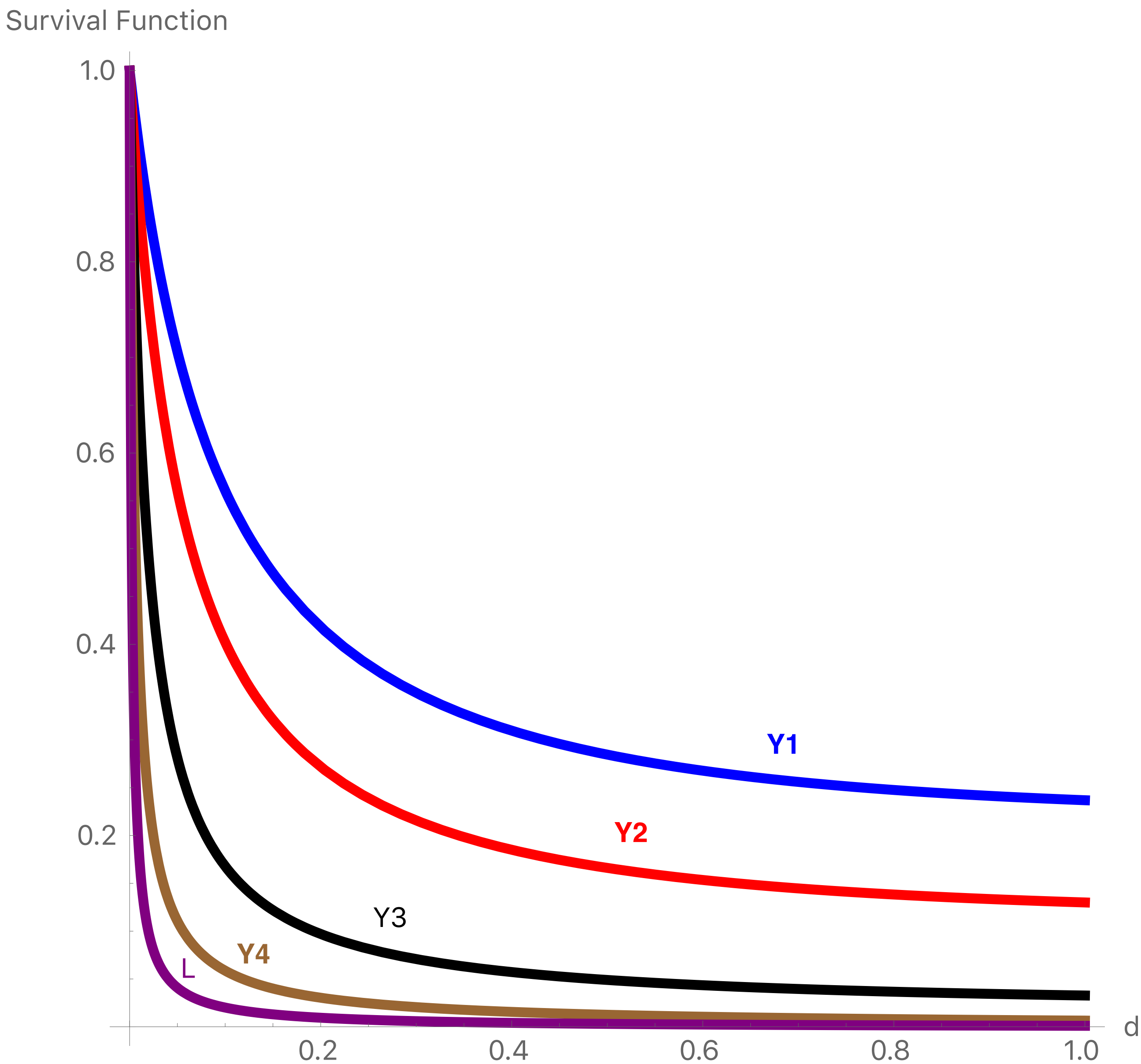
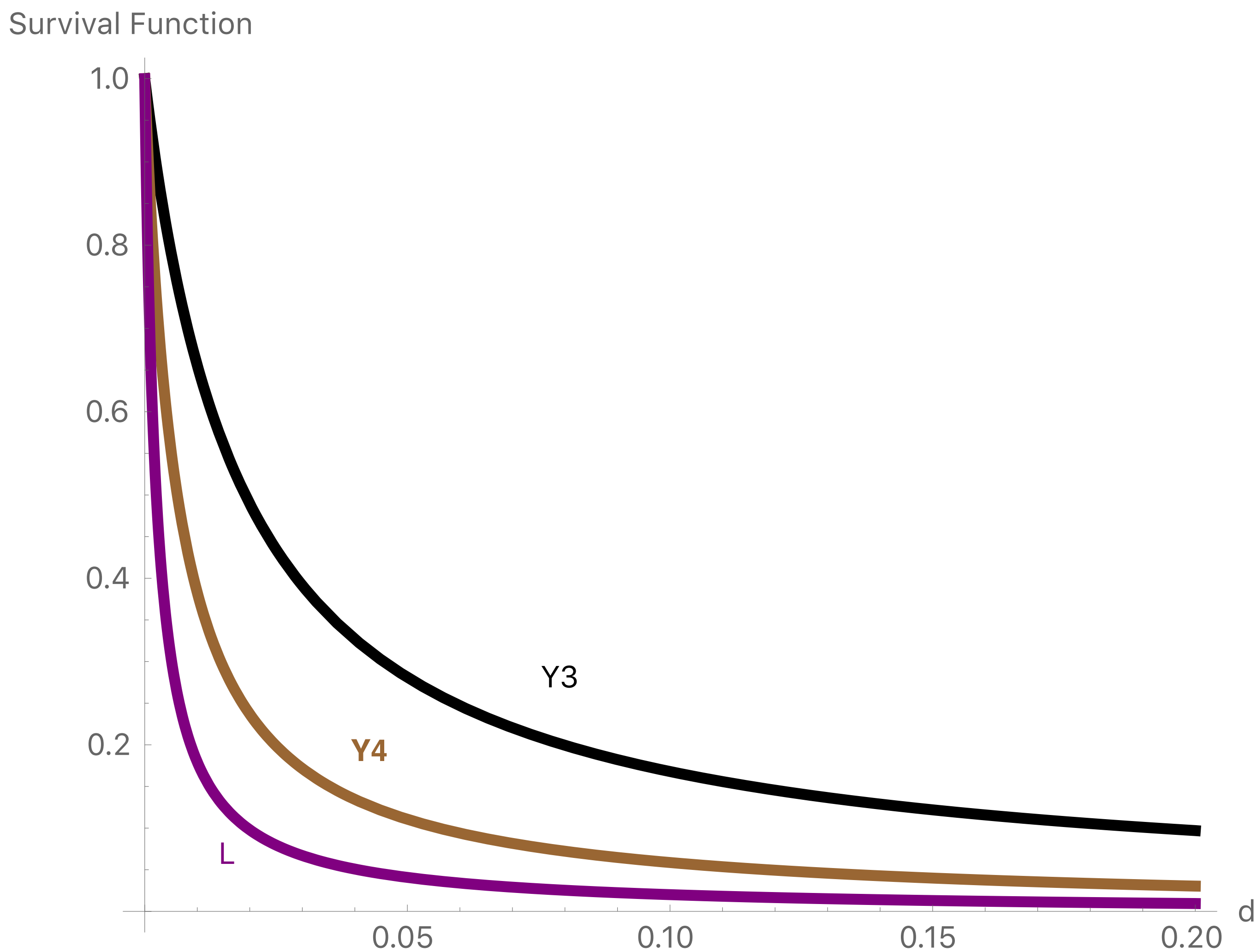
[65] Sometimes referred to as Gasser curves, after Peter Gasser.
c = 1.5 corresponds to Y_1, c = 2 corresponds to Y_2,
c = 3 corresponds to Y3, c = 4 corresponds to Y4, and c = 5 corresponds to a Lloyd's curve.

The survival functions for the SwissRe curves Y_1, Y_2, Y_3, and Y_4, and the Lloyd's curve:[66]

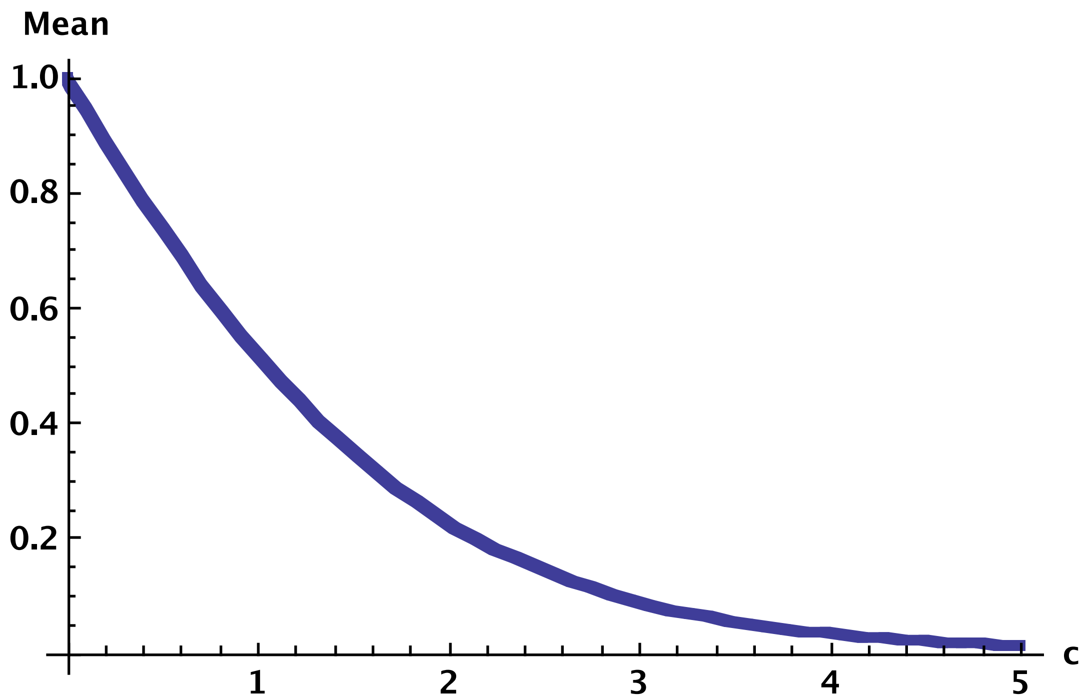


[66] Recall that there is point mass of probability at the Maximum Possible Loss, corresponding to d = 1.

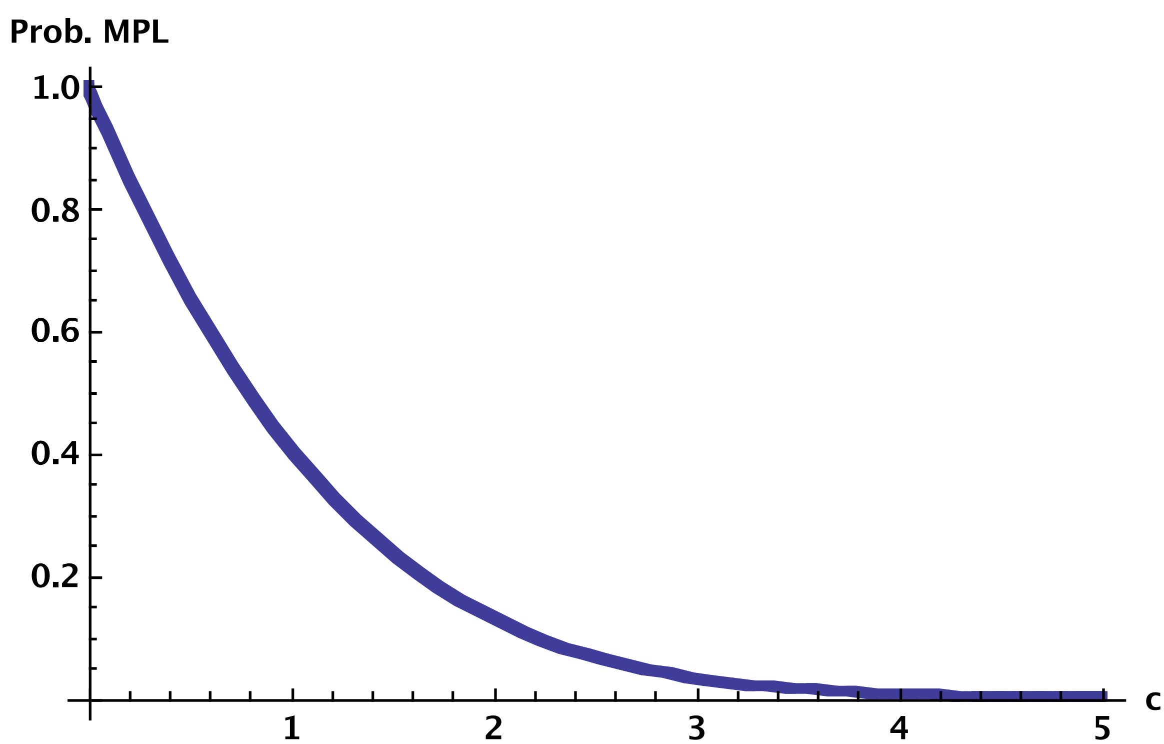
Only graphed from 0 to 0.2, the survival functions for the SwissRe curves Y_3, and Y_4, and the Lloyd's curve:



As c increases, the mean decreases:[67]

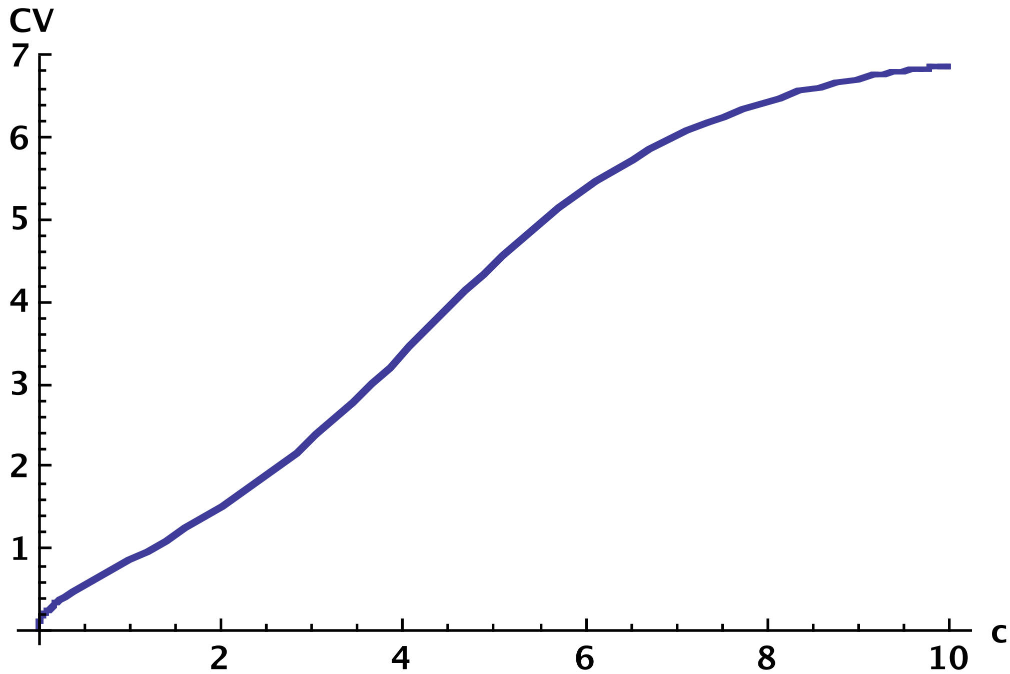


As c increases, the probability of having the maximum possible loss (MPL) decreases:

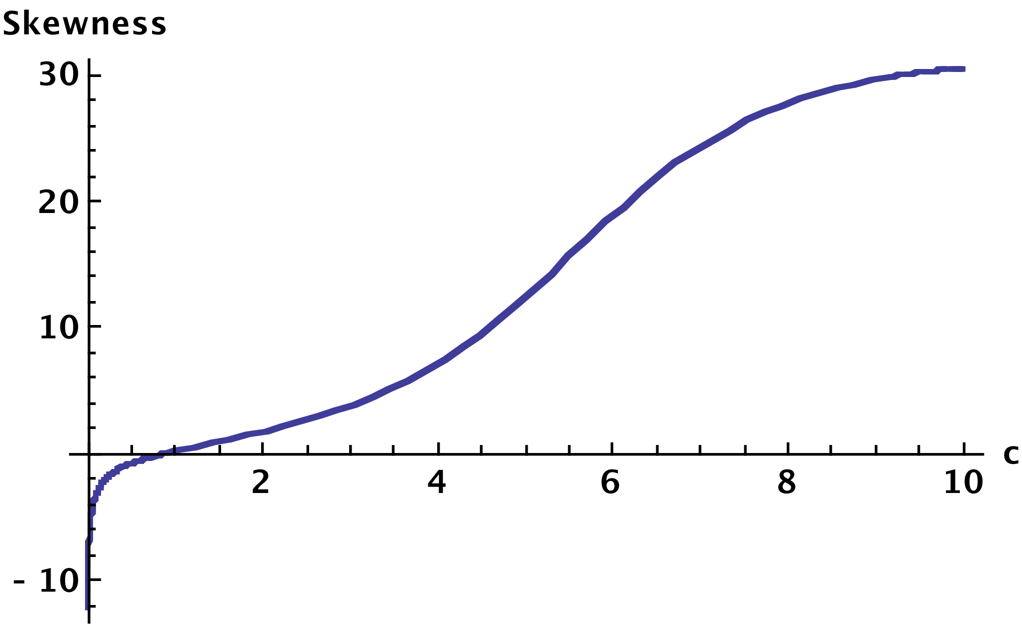


[67] The mean is stated as a fraction of the MPL.

Using a computer, one can calculate the moments of the MBBEFD class of distributions as parameterized by Bernegger.[68] From the moments, one can compute the coefficient of variation and skewness. I graphed the results for the SwissRe curves as a function of c.
As c increases, the coefficient of variation (standard deviation / mean) increases:



For c small, the probability of a total loss is big, and the skewness is negative. For larger c the skewness is positive. As c increases, the skewness increases:



68 One has to be careful to include the point mass of probability at 1. I did all calculations in Mathematica.

Here is a table summarizing some information on the SwissRe curves and the Lloyd's curve:[69] [70] [71] [72] [73] [74] [75]

Curve	c	b	g	p	Mean	CV	Skewness
Y_1	1.5	12.648	4.22	23.7%	0.349	1.14	0.86
Y_2	2	9.025	7.69	13.0%	0.226	1.48	1.63
Y_3	3	3.669	30.57	3.3%	0.087	2.30	3.64
Y_4	4	1.105	154.47	0.6%	0.032	3.34	6.98
Lloyd's	5	0.247	992.27	0.1%	0.012	4.43	12.23

[69] $b = \exp[3.1 - 0.15\ c\ (1+c)]$.
[70] $g = \exp[c\ (0.78 + 0.12c)]$.
[71] $p = 1/g$ = probability of the Maximum Possible Loss.
[72] $p = 1/g$ = probability of the Maximum Possible Loss.
[73] The mean is stated as a fraction of the MPL.
[74] Coefficient of Variation = Standard Deviation / Mean.
[75] Lloyd's curve used to rate industrial risks.

Instead a Distribution from Zero to Infinity:[76]

The MBBEFD class can also be used to model the distribution of losses on 0 to infinity.[77]

The survival function has the same form as was used on the interval [0, 1]:[78]

$$S(x) = \frac{1-b}{(g-1)b^{1-x} + (1-gb)}, \; x > 0, \; 1 > b > 0, \; g > 1.$$

Exercise: For b = 0.2 and g = 10, determine S(4).

[Solution: $S(4) = \frac{1-0.2}{(9)(1/0.2^3) + (1-2)} = 0.071\%$.]

As x approaches infinity, since $b < 1$, $b^{1-x} = b/b^x$, approaches infinity.
Thus $S(\infty) = 0$. There is no point mass of probability as there was previously.

$$S(1) = \frac{1-b}{(g-1) + (1-gb)} = 1/g.$$

So it is still the case that $g = 1/S(1)$.[79]

$$\int S(x)\, dx = (1-b)\,\frac{\ln[(g-1)b + (1-gb)b^x]}{\ln(b)\,(1-gb)}.$$ [80]

$$E[x] = \int_0^\infty S(x)\, dx = (1-b)\,\frac{\ln[(g-1)b]}{\ln(b)\,(1-gb)} - (1-b)\,\frac{\ln[(g-1)b + (1-gb)]}{\ln(b)\,(1-gb)}$$

$$= \frac{(1-b)}{\ln(b)\,(1-gb)}\,\ln\left[\frac{(g-1)b}{(g-1)b + (1-gb)}\right] = \frac{\ln\left[\frac{(g-1)b}{1-b}\right](1-b)}{\ln(b)\,(1-gb)}.$$

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[76] Section 3.8 of Bernegger.

[77] One can normalize everything with respect to a reference loss X_0, but this is not necessary.
For example, one could put everything in units of $1000.

[78] If gb = 1, then $S(x) = b^x$, an Exponential Distribution with mean: 1/ln[1/b].
As b approaches 1, S(x) approaches 1/{1 + (g-1)x}, a Shifted Pareto Distribution with $\alpha = 1$ and $\theta = 1/(g-1)$.

[79] If the losses have been normalized, then x = 1 corresponds to a loss of size X_0.

[80] One can check this via differentiation.

Exercise: For b = 0.2 and g = 10, determine the mean.

[Solution: $E[x] = \dfrac{\ln[\frac{(g-1)b}{1-b}]\,(1-b)}{\ln(b)\,(1-gb)} = \dfrac{\ln[\frac{1.8}{0.8}]\,(0.8)}{\ln(0.2)\,(1-2)} = 0.403.$]

$$\int_0^x S(t)\,dt = (1-b)\,\frac{\ln[(g-1)b + (1-gb)\,b^x]}{\ln(b)\,(1-gb)} - (1-b)\,\frac{\ln[(g-1)b + (1-gb)]}{\ln(b)\,(1-gb)}$$

$$= \frac{(1-b)}{\ln(b)\,(1-gb)}\,\ln[\frac{(g-1)b + (1-gb)b^x}{(g-1)b + (1-gb)}] = \frac{\ln[\frac{(g-1)b + (1-gb)\,b^x}{1-b}]\,(1-b)}{\ln(b)\,(1-gb)}.$$

$$G(x) = \frac{\int_0^x S(t)\,dt}{E[x]} = \frac{\ln[\frac{(g-1)b + (1-gb)\,b^x}{1-b}]}{\ln[\frac{(g-1)b}{1-b}]}.$$[81]

Exercise: For b = 0.3 and g = 5, determine G(4).

[Solution: $G(4) = \dfrac{\ln[\frac{1.2 + (1-1.5)\,0.3^4}{0.7}]}{\ln[\frac{1.2}{0.7}]} = 99.37\%.$]

One can derive the results for the previous case where x is on the interval [0, 1] from those here. The previous case is mathematically the same as the current distribution of losses with each loss censored from above at 1; in other words any loss greater than one is limited to 1.
The mean of the losses censored from above at one is the limited expected value at 1: $E[X \wedge 1]$.

Let $\tilde{G}(x)$ be the exposure curve for the previous case, losses censored from above at 1.

Then for d < 1, $\tilde{G}(d) = \dfrac{\text{losses eliminated by a deductible of size d}}{\text{mean of losses censored from above at 1}} = \dfrac{E[X \wedge d]}{E[X \wedge 1]} = \dfrac{E[X \wedge d]\,/\,E[X]}{E[X \wedge 1]\,/\,E[X]} =$

$$\frac{G(d)}{G(1)} = \frac{\ln[\frac{(g-1)b + (1-gb)b^d}{1-b}]}{\ln[\frac{(g-1)b}{1-b}]} \Big/ \frac{\ln[\frac{(g-1)b + (1-gb)b}{1-b}]}{\ln[\frac{(g-1)b}{1-b}]} = \frac{\ln[\frac{(g-1)b + (1-gb)\,b^d}{1-b}]}{\ln[gb]}.$$[82]

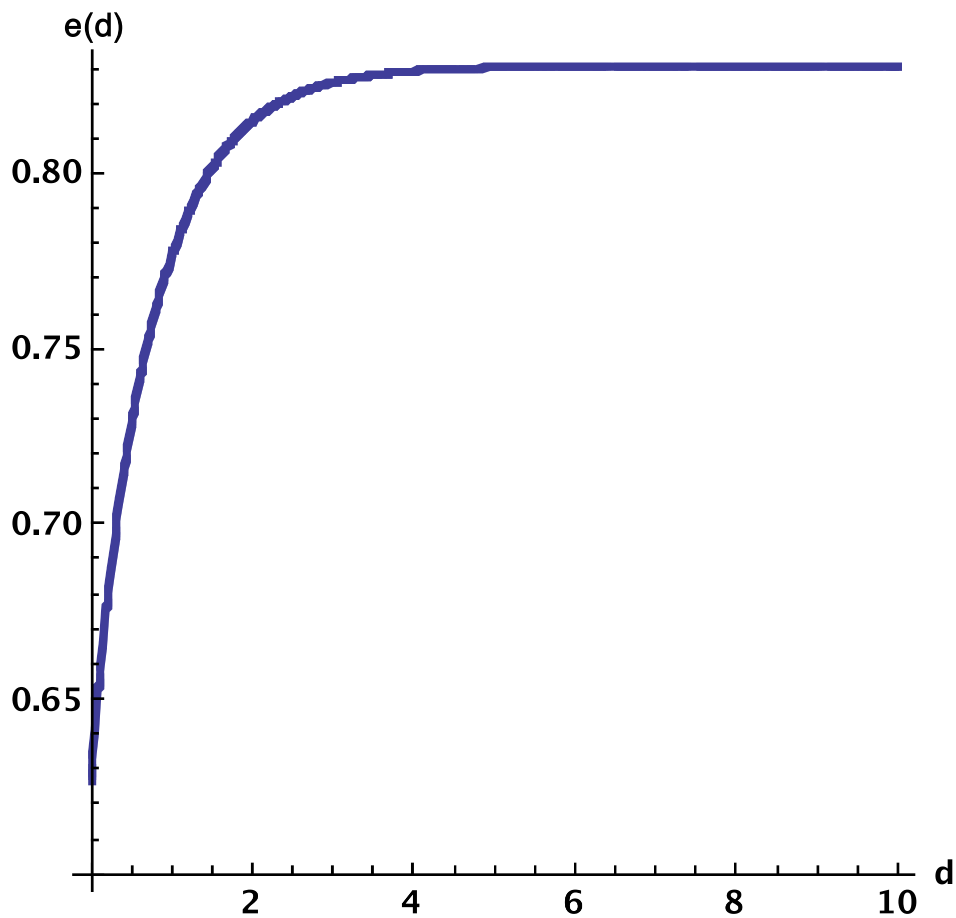
[81] See equation 3.10 in Bernegger. If gb = 1, then $G(x) = 1 - b^x$.

[82] Matching the previous result, equation 3.3 in Bernegger.

Define the mean residual life (mean excess loss) as:[83] [84]

$$e(d) = \frac{\text{losses excess of d}}{S(d)} = \frac{\{1 - G(d)\}\ E[x]}{S(d)}.$$

Here is a graph of the mean residual life as a function of d, for g = 5 and b = 0.3:



As d approaches infinity, e(d) approaches a constant, 1 / ln[1/b] = 0.8306.
In general, the righthand tail is similar to that of an Exponential Distribution with mean: 1 / ln[1/b].[85]

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83 Bernegger does not discuss the mean residual life.
See Loss Models, where it is called the mean excess loss.

84 The mean residual life is a useful way to examine the behavior in the righthand tail.
See "Workers Compensation Excess Ratios: An Alternate Method of Estimation," by Howard C. Mahler.
e(d) would not be useful to analyze the righthand tail behavior of a distribution with support from 0 to 1.

85 If gb = 1, then $S(x) = b^x$, an Exponential Distribution with mean: 1/ln[1/b].

Important Ideas and Formulas:

Bernegger normalizes everything with respect to the Maximum Possible Loss, M.
If X is the loss in dollars, and M is the Maximum Possible Loss, then the normalized loss is: $x = X / M$. $0 \le x \le 1$.
If D is the retention in dollars, then the normalized retention is: $d = D / M$.

The exposure curve, G(x), is just the loss elimination ratio at x.
$G(d) = E[X \wedge d] / E[X \wedge 1] = L(d) / L(1)$, $0 \le d \le 1$.

$E[X] = 1/G'(0)$. $S(x) = G'(x) / G'(0)$.
The probability of having the Maximum Possible loss is: $G'(1) / G'(0)$.

G(d) is an increasing and concave function on the interval [0, 1].
$G'(d) \ge 0$ and $G''(d) \le 0$. In addition, $G(0) = 0$ and $G(1) = 1$ by definition.

The MBBEFD Distribution Class has of the form:

$$\mathbf{G(x)} = \frac{\ln(a + b^x) - \ln(a + 1)}{\ln(a + b) - \ln(a + 1)},\ 0 \le x \le 1, \text{ where a and b are the two parameters.}$$

p = probability of the MPL = $S(1) = G'(1) / G'(0) = \frac{(a + 1)\, b}{a + b}$.

The flatter the curve, the bigger the probability of the Maximum Possible Loss. For p = 1, all losses are total losses, and the exposure curve is the diagonal line connecting (0, 0) and (1, 1).

$$\mathbf{g = 1/p} = \frac{a + b}{(a + 1)\, b}. \qquad \mathbf{a} = \frac{(g - 1)\, b}{1 - gb}.$$

$$\mathbf{G(x)} = \frac{\ln\left[\frac{(g - 1)b + (1 - gb)\, b^x}{1 - b}\right]}{\ln[gb]},\ 0 \le x \le 1,\ b \ge 0,\ g \ge 1. \quad G'(x) = \frac{\ln(b)\ (1 - gb)}{\ln(gb)\, \{(g - 1)b^{1-x} + (1 - gb)\}}.$$

$$E[x] = 1/G'(0) = \frac{\ln(gb)\, (1 - b)}{\ln(b)\ (1 - gb)}.$$

$$S(x) = G'(x) / G'(0) = \frac{(g - 1)b + (1 - gb)}{(g - 1)b^{1-x} + (1 - gb)} = \frac{1 - b}{(g - 1)b^{1-x} + (1 - gb)}.$$

A subset of exposure curves based on one parameter c, and closely matches exposure curves used by and named after SwissRe and Lloyds of London:
$b = \exp[3.1 - 0.15\, c\, (1+c)]$. $g = \exp[c\, (0.78 + 0.12c)]$.

Items That Help Determine the Appropriate Exposure Curve:[86]

Whether a lot of total losses occur, or partial and small losses are the rule, depends on various factors, some of which are in turn dependent on each other. The decisive factors are: the peril covered, the class of risk, the size of risk, and the fire protection measures.

Perils covered: A fire normally causes more damage to an individual building than a windstorm; typical windstorm losses in Europe amount to only a few per thousand of the sum insured. While a gas explosion can completely destroy a house, lightning strikes generally cause only partial damage. Earthquakes, on the other hand, cause minor to devastating damage to buildings, depending on their strength.

Class of risks: The class of risk has a decisive influence. Gunpowder factories are obviously more likely to suffer total losses than food processing plants. Depending on the class of risk, the average degree of loss varies considerably, as the following statistical details illustrate:

Class of risk	Average degree of loss
Residential building	1.9%
Administration building	0.5%
Farm building	4.9%
Industrial building	4.4%

Size of risk: A fire often causes only partial damage to a large building, whereas small buildings are more likely to suffer total destruction in the event of a fire. The obvious measure for the size of a risk is the sum insured. However, this is only a good indicator for the risk as long as the insured property can be destroyed by a single fire event. This is the case with detached houses, for example.

Large industrial plants, in contrast, often consist of several groups of buildings which are clearly separated from each other and therefore cannot be affected simultaneously by a fire. The sum insured may give an indication of the size of the entire industrial plant but does not allow any conclusion to be drawn on the effective size of the risk. The MPL (Maximum Possible Loss) is therefore often used as a measure of the size of the risk. The larger a risk, the smaller the MPL usually is as a percentage of the sum insured.

Fire protection measures: Fire protection has a considerable influence on the course of the loss distribution function. Fire protection measures make it possible to stop hostile fires at an earlier stage, thereby reducing the total overall claims burden which is to be expected over a certain period and increasing the share of the overall claims burden accounted for by minor losses.

[86] From "Exposure Rating," by SwissRe.

Problems:

1. (5 points) A reinsurer uses the following exposure curve under a non-proportional treaty:

$$G(x) = \frac{\ln(12^x - 0.7) - \ln(0.3)}{\ln(11.3) - \ln(0.3)}.$$

a. (1 point) Calculate the probability of a total loss.
b. (1 point) Calculate the mean loss.
c. (1 point) Calculate the 40^{th} percentile of the size of loss distribution.
d. (1 point) Calculate the 70^{th} percentile of the size of loss distribution.
e. (1 point) Determine the retention such that the cedant retains 75% of expected losses.

2. (5 points) The Maximum Possible Loss for property damage to a skyscraper, the John Adams Tower, is $900 million.
There is a 4% mean annual frequency.
There is a 0.2% chance of a total loss each year.
If there is a partial loss, its average size is $135 million.
Assume the MBBEFD Distribution Class for the severity distribution.
The owner of the building buys property insurance with a $50 million deductible.
The insurer buys excess of loss reinsurance with a $200 million retention.
(The reinsurance retention applies per location per occurrence.)
Determine the expected annual losses retained by the building owner, paid by the insurer, and paid by the reinsurer. Use a computer to aid you.

3. (2.5 points) The loss elimination ratio at $1 \geq x \geq 0$ is: $\dfrac{\ln[a + b^x] - \ln[a+1]}{\ln[a + b] - \ln[a+1]}$, $1 > b > 0$, $a > 0$.

Determine the form of the distribution function.

4. (2 points) Discuss why Bernegger proposes using exposure curves based on analytical distributions and why he uses the MBBEFD class of distributions.

5. (1.5 points) You are given the following exposure curve: $G(x) = 1 - (1-x)^4$, $0 \leq x \leq 1$.
Determine the probability density function.

***6*.** (3 points) A reinsurer is writing a property non-proportional treaty.
A reinsurer uses an exposure curve based on the MBBEFD Distribution Class as per Bernegger, with b = 9 and p = 5%.
The cedant's maximum retention under the treaty is $200,000 and the treaty limit is $800,000.
In the reinsurance treaty, the attachment point and limit apply per occurrence per location.
The insurer's expected loss ratio is 68%.
For the portfolio of risks to be reinsured, it is assumed that all locations within the range are exactly equal to the midpoint of the range.

Range of Insured Values ($000s)	Subject Premium ($000s)
25 to 100	400
100 to 200	200
200 to 500	300
500 to 1000	200
1000 to 2000	100
2000 to 5000	200

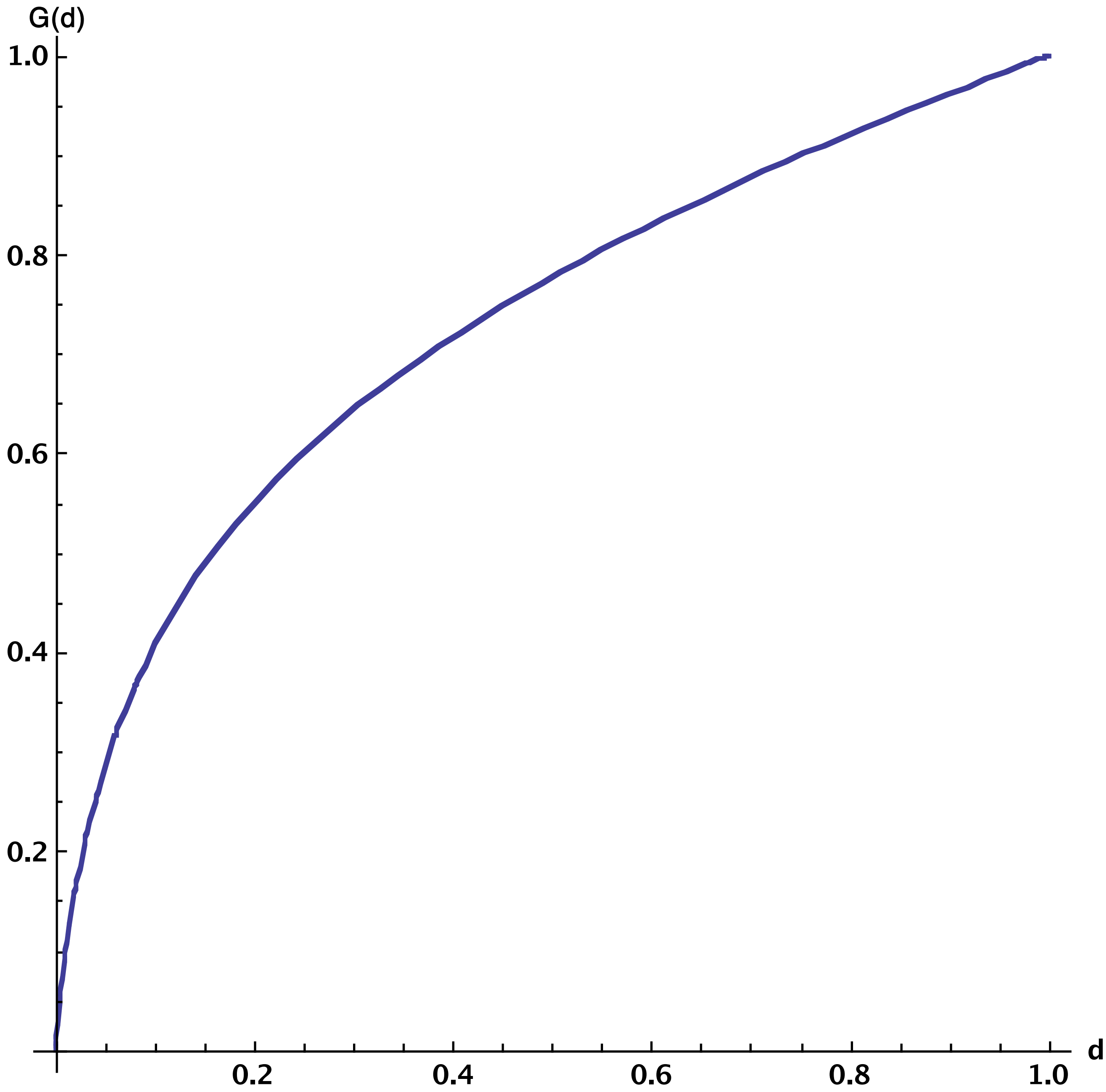
Calculate the ratio of the reinsurer's loss cost to the subject premium.

7. (2.5 points) For the following discrete distributions, graph the exposure curve.
(a) (0.5 point) All losses are total, in other words equal to the maximum possible loss (MPL).
(b) (2 points) 60% of losses are 10% of the MPL, 30% of losses are 40% of the MPL, and the remaining 10% of losses are equal to the MPL.

8. (4 points) For an insured property, the probability of the Maximum Possible Loss is 2.5%.
The median loss is 1.6% of the maximum possible loss.
An exposure curve is based on the MBBEFD Distribution Class as per Bernegger.
Determine the value of the exposure curve at 30% of the MPL.

9. (1.5 points) $G(x) = x^\alpha$, $0 \le x \le 1$.
For what values of α is this function is a valid exposure curve? Show all work.

10. (2 points) A property has a Maximum Possible Loss of $250 million.
Reinsurer A covers the layer $40 million excess of $10 million.
Reinsurer B covers the layer $50 million excess of $50 million.
Reinsurer C covers the layer $150 million excess of $100 million.
Using the following exposure curve, determine the percentage of total losses expected to be paid by each of the reinsurers.



11. (1 point) Urban the Underwriter is working on an excess of loss property reinsurance treaty, to cover a very large office building.
Urban and you agree on the Maximum Possible Loss.
Name two items you can ask Urban to select, so that you can help him choose an exposure curve to use that is based on the MBBEFD class.

***12*.** (2.5 points) The loss elimination ratio at x ≥ 0 is: $\frac{\ln[1 + x/10]}{\ln[2 + x/10]}$.
Determine the value of the distribution function at 8.

13. (3 points) An actuary decides to use the following exposure curve to price a risk.

$$G(x) = \frac{16}{7}\left\{1 - \left(\frac{3}{3+x}\right)^2\right\},\ 0 \le x \le 1.$$

The maximum possible loss is \$5 million.
a. (1.5 points) Demonstrate that this function is a valid exposure curve.
b. (0.5 point) Determine the mean size of loss.
c. (0.5 point) Determine the probability of the maximum possible loss.
d. (0.5 point) Determine the ratio of pure risk premium in the layer \$1 million excess of \$1 million.

<u>14</u>. (3 points) A reinsurer uses an exposure curve under a non-proportional treaty, with as per Bernegger c = 4.5.
Hint: b = exp[3.1 - 0.15 c (1+c)], and g = exp[c(0.78 + 0.12c)].
a. (1 point) Calculate the probability of a total loss.
b. (1 point) Calculate the mean loss.
c. (1 point) The cedant's maximum retention under the treaty is \$100 million and the maximum possible first-dollar loss is \$400 million.
Calculate the percentage of pure risk premium ceded to the reinsurer by the cedant.

***15*.** (3 points) An actuary decides to use the following exposure curve to price a risk.

$$G(x) = \frac{4\sqrt{x}}{3 + \sqrt{x}},\ 0 \le x \le 1.$$

The maximum possible loss is \$10 million.
a. (2.5 points) Demonstrate that this function is a valid exposure curve.
b. (0.5 point) Determine the mean as a percentage of the maximum possible loss.

16. (4 points) In order to price a non-proportional treaty,
a reinsurer uses an exposure curve with the following form:

$$G(x) = \frac{\ln\left[\frac{(g-1)b + (1 - gb)\, b^x}{1-b}\right]}{\ln[gb]}$$

has a derivative of:

$$G'(x) = \frac{\ln(b)\ (1 - gb)}{\ln(gb)\ \{(g-1)b^{1-x} + (1 - gb)\}}\ ..$$

b = exp[3.1 - 0.15 c (1+c)]. g = exp[c (0.78 + 0.12c)].
(a) (3 points) Find the mean of the distribution for c = 1 and c = 2.
(b) (0.5 point) Find the probability of a total loss for c = 1 and c = 2.
(c) (0.5 point) Briefly discuss what happens in general to the exposure curve as c increases.

17. (2 points) According to Bernegger, “Often, underwriters have only a finite number of discrete exposure curves at their disposal. These curves are available in graphical or tabulated form, and are also implemented in computerized underwriting tools. One of the curves must be selected for each risk band, but it is not always clear which curve should be used. In such cases, the underwriter might also want to use a virtual curve lying between two of the discrete curves available to him.
This can be achieved by replacing the discrete curves with analytical exposure curves. Each set of parameters then defines another curve. If a continuous set of parameters is available, the exposure curves can be varied smoothly within the whole range of available curves.”
Discuss some potential problems with this approach and how Bernegger proposes to overcome them.

18. (3 points) Assume the following discrete severity distribution:

Percent of Maximum Possible Loss	Probability
25%	30%
50%	40%
75%	20%
100%	10%

Draw the corresponding exposure curve.

19. (5 points) Bernegger discusses a subset of exposure curves based on the MBBEFD Distribution Class. This subset is based on one parameter c, and closely matches exposure curves used by and named after SwissRe and Lloyds of London.
b = exp[3.1 - 0.15 c (1+c)]. g = exp[c (0.78 + 0.12c)].
For c = 4.5, with the aid of a computer, graph the percent of losses expected to be paid by a treaty that covers the layer from y to 2y, where y is as a percent of the MPL.

20. (1.5 points) Bernegger describes two ways to fit to an MBBEFD class distribution.
Briefly describe each of these two methods.

21. (1 point) For the peril of theft, draw approximate exposure curves for:
- household contents for Homeowners
- an individual valuable art object

22. (1 point) According to Bernegger, "The Swiss Re Y_i exposure curves (i = 1 ... 4) are very well known and widely used by non proportional property underwriters. As will be shown in this section, all these curves can be approximated very well with the help of a subclass of the MBBEFD exposure curves."
Briefly discuss the steps Bernegger used to accomplish this.

23. (1.5 points) You are given the following exposure curve: $G(x) = 1.5x - 0.5x^3$, $0 \le x \le 1$.
Determine the probability density function.

24. (4 points) An actuary is using the following exposure curve to rate a non-proportional reinsurance treaty: $G(x) = (1 - 0.06^x) / 0.94$.
The actuary is also given the following information:

Maximum Possible Loss	\$10,000,000
Insured Value	\$10,000,000
Gross Premium	\$160,000
Expected Loss Ratio	65%
Retention of non-proportional reinsurance treaty	\$500,000
Limit of non-proportional reinsurance treaty	\$3,500,000

(a) (1.5 points) Determine the expected ceded risk premium.
(b) (0.5 point) Determine the probability of the maximum possible loss.
(c) (1 point) Determine the expected annual claim frequency for the insurer.
(d) (1 point) Determine the expected annual claim frequency for the reinsurer.

25. (2 points) The aggregate loss experience of an insurer's book of business is described by the following distribution function:

$F(x) = x^{0.5}$ where $0 \le x \le 1$

a. (1 point) Derive an exposure curve from the above cumulative distribution function.
b. (1 point) Given that the maximum possible loss is \$10,000,000, use the derived exposure curve in part a. above to determine the ratio of pure risk premium in the layer \$4,000,000 excess of \$1,000,000.

26. (2 points) An actuary for a reinsurer uses the following exposure curve to price a non-proportional treaty with the assumption:

$$G(x) = \frac{1 - b^x}{1 - b}.$$

The probability of a total loss is 20%.
The maximum possible loss for the reinsurer is \$100 million and the ratio of pure risk premium retained by the cedant is 75%.
Calculate the cedant's maximum retention under the treaty.

27. (3 points) A portfolio of industrial risks of similar size each have insured value of \$100 million.
A reinsurance actuary is pricing an excess of loss property treaty.
The actuary has trended and developed historical losses:

Loss Size	Number of Claims	Ground-up Loss
At most \$10 million	100	\$300 million
More than \$10 million and at most \$50 million	40	\$650 million
Greater than \$50 million	12	\$1050 million
Total	152	\$2000 million

The actuary is unsure if a Swiss Re Y_3 or Y_4 exposure curve is a better fit.
The following MBBEFD exposure curve formulas are available:

$$G(x) = \frac{\ln\left[\frac{(g-1)b + (1 - gb)\, b^x}{1-b}\right]}{\ln[gb]}$$

$b(c) = \exp[3.1 - 0.15c(1+c)]$
$g(c) = \exp[c\,(0.78 + 0.12c)]$
$c = 3$ for Y_3 and $c = 4$ for Y_4.

Which of the two curves do you recommend be used and why?

28. (4 points) A homogenous group of property risks exhibit the following risk profile:

Exposure Distribution	Losses as a percent of the Maximum Possible Loss (MPL)
90%	0%
3%	25%
2%	50%
1%	75%
4%	100%

a. (2 points) Plot the exposure curve, G(x). Label the axes and the points on the curve.
b. (0.5 point) A reinsurer uses this exposure curve to price a property excess of loss treaty where the underlying limit is \$200 million.
Calculate the proportion of total losses in the layer \$100 million excess of \$50 million.
c. (0.5 point) Determine the parameters b and g for the 2 parameter MBBEFD distribution that fits this exposure curve using the following information:

Calculated value of parameter b

Parameter g	μ				
	55%	60%	65%	70%	75%
4.0	0.2156	0.0950	0.0405	0.0154	0.0050
3.5	0.4058	0.1714	0.0719	0.0280	0.0093
3.0	0.9860	0.3780	0.1514	0.0585	0.0199
2.5	4.4342	1.2709	0.4411	0.1600	0.0544
2.0	985.31	19.909	3.2182	0.8426	0.2500

d. (1 point) Using the fitted 2 parameter MBBEFD distribution from part (c), calculate the proportion of total losses in the layer \$100 million excess of \$50 million.

$$G(x) = \frac{\ln\left[\frac{(g-1)b + (1 - gb)\, b^x}{1-b}\right]}{\ln[gb]}.$$

29. (8, 11/11, Q.9) (2 points)
A reinsurer uses the following exposure curve under a non-proportional treaty:

$$G(x) = \frac{\ln(0.1 + 0.01^x) - \ln(1.1)}{\ln(0.11) - \ln(1.1)}.$$

The cedant's maximum retention under the treaty is \$50 million and the maximum possible first-dollar loss is \$100 million.

A function with the form of $G(x) = \frac{\ln(a + b^x) - \ln(1+a)}{\ln(a+b) - \ln(a+1)}$

has a derivative of $G'(x) = \frac{\frac{\ln(b)\, b^x}{a + b^x}}{\ln(a+b) - \ln(a+1)}$.

a. (0.5 point) Calculate the ratio of pure risk premium retained by the cedant.
b. (1.5 points) Calculate the probability of a total loss.

30. (8, 11/12, Q.8) (3 points) An actuary decides to use the following exposure curve to price a risk and has determined that the appropriate b parameter is 0.15.

$$G(x) = \frac{1 - b^x}{1 - b}, \ 0 \le x \le 1.$$

a. (1.5 points) Demonstrate that this function is a valid exposure curve.
b. (1 point) Given that the maximum possible loss is \$2,000,000, use the selected exposure curve above to determine the ratio of pure risk premium in the layer \$1,000,000 excess of \$500,000.
c. (0.5 point) Discuss the appropriateness of the ratio of pure risk premium calculated above if the b parameter that the actuary selected was too high. State whether the actuary has underestimated or overestimated the probability of a total loss.

31. (1.5 points) Using the information in the previous question, 8, 11/12, Q.8, answer the following questions:
a. (1 point) Given that the maximum possible loss is \$2,000,000, use the selected exposure curve above to determine the ratio of pure risk premium in the layer \$100,000 excess of \$100,000.
b. (0.5 point) Discuss the appropriateness of the ratio of pure risk premium calculated above if the b parameter that the actuary selected was too high.

32. (8, 11/13, Q.20) (2 points) The aggregate loss experience of an insurer's book of business is described by the following distribution function:

$F(x) = x^{0.25}$ where $0 \le x \le 1$

a. (1 point) Derive an exposure curve from the above cumulative distribution function.
b. (1 point) Given that the maximum possible loss is \$2,000,000, use the derived exposure curve in part a. above to determine the ratio of pure risk premium in the layer \$1,000,000 excess of \$500,000.

33. (8, 11/13, Q.22) (1.5 points) An actuary for a reinsurer uses the following exposure curve to price a non-proportional treaty with the assumption that b = 0.1:

$$G(x) = \frac{1 - b^x}{1 - b}.$$

The maximum possible loss for the reinsurer is \$50 million and the ratio of pure risk premium retained by the cedant is 65%.
Calculate the cedant’s maximum retention under the treaty.

34. (8, 11/15, Q.20) (2.5 points)
An actuary is using the following exposure curve to rate a non-proportional reinsurance treaty:

$$G(x) = \frac{1 - b^x}{1 - b} .$$

The actuary is also given the following information:

Maximum Possible Loss	\$5,000,000
Insured Value	\$5,000,000
Gross Premium	\$6,000
Expected Loss Ratio	60%
Retention of non-proportional reinsurance treaty	\$150,000
Expected Ceded Risk Premium	\$2,705

a. (0.5 point) Briefly describe a method to allocate gross premium for the non-proportional reinsurance treaty between the ceding company and the reinsurer.
b. (0.5 point) Given the probability of a total loss is 0.03, calculate the parameter b in the formula above.
c. (1.5 points)
Given the answer in part b. above, calculate the limit of the non-proportional reinsurance treaty.

35. (8, 11/16, Q.21) (4 points) An insurance company insures high value homes and plans to increase the maximum property value they will insure next year. The company is considering purchasing a new \$4,000,000 excess of \$4,000,000 reinsurance treaty.
The reinsurer is given the following limit profile:

Insured Value Range	Experience Period On-Level Premium	Treaty Subject Premium
\$1,000,000 to \$4,000,000	\$100,000,000	\$25,000,000
\$4,000,001 to \$8,000,000	\$0	\$5,000,000

A reinsurance actuary has trended and developed the portfolio's historical losses for the Experience Period:

Loss Size	Number of Claims	Experience Period Ground-up Loss
Less than \$1,000,000	200	\$22,000,000
Greater than \$1,000,000	10	\$18,000,000
Total	210	\$40,000,000

The actuary is unsure if a Swiss Re Y_3 or Y_4 exposure curve is a better fit.
The following MBBEFD exposure curve formulas are available along with the following information:

$$G(x) = \frac{\ln\left[\frac{(g-1)b + (1-gb)\,b^x}{1-b}\right]}{\ln[gb]}$$

$$b(c) = \exp[3.1 - 0.15c(1+c)]$$
$$g(c) = \exp[c\,(0.78 + 0.12c)]$$

% of Insured	% of Cumulative Loss	
Value	Y_3	Y_4
25%	60%	73%
40%	72%	82%
100%	100%	100%

Given a proposed treaty rate of 1% of total subject premium, calculate the expected ceded loss ratio for the new treaty.

36. (8, 11/17, Q.18) (3.25 points)
A homogenous group of property risks exhibit the following risk profile:

Exposure Distribution	Losses as a percent of the Maximum Possible Loss (MPL)
80%	0%
6%	25%
8%	50%
4%	75%
2%	100%

a. (2 points) Plot the exposure curve, G(x). Label the axes and the points on the curve.
b. (0.75 point) Determine the parameters b and g for the 2 parameter MBBEFD distribution that fits this exposure curve using the following information:

Calculated value of parameter b

Parameter g	μ								
	40.0%	42.5%	45.0%	47.5%	50.0%	52.5%	55.0%	57.5%	60.0%
50.0	0.0023	0.0017	0.0013	0.0009	0.0005	0.0003	0.0002	0.0001	0.0001
25.0	0.0096	0.0073	0.0055	0.0041	0.0022	0.0015	0.0010	0.0006	0.0004
16.7	0.0239	0.0181	0.0137	0.0103	0.0057	0.0039	0.0026	0.0017	0.0010
12.5	0.0483	0.0365	0.0276	0.0209	0.0118	0.0081	0.0055	0.0036	0.0023
10.0	0.0877	0.0659	0.0497	0.0375	0.0212	0.0147	0.0100	0.0067	0.0044
8.3	0.1498	0.1116	0.0836	0.0628	0.0354	0.0245	0.0168	0.0113	0.0075
7.1	0.2470	0.1817	0.1347	0.1004	0.0561	0.0389	0.0267	0.0181	0.0121
6.3	0.4000	0.2892	0.2116	0.1561	0.0861	0.0594	0.0408	0.0277	0.0185
5.6	0.6446	0.4557	0.3277	0.2385	0.1291	0.0885	0.0605	0.0411	0.0276
5.0	1.0469	0.7190	0.5055	0.3616	0.1910	0.1297	0.0882	0.0597	0.0400

c. (0.5 point) A reinsurer uses this exposure curve to price a property excess of loss treaty where the underlying limit is \$100 million. Calculate the proportion of total losses in the layer \$25 million excess of \$25 million.

Solutions:

1. a. This is a member of the MBBEFD Distribution Class with a = -0.7 and b = 12.

$$G(x) = \frac{\ln(a + b^x) - \ln(a+1)}{\ln(a+b) - \ln(a+1)}.$$

$$g = \frac{a + b}{(a + 1)\, b} = \frac{12 - 0.7}{(1 - 0.7)\,(12)} = 3.139.$$ p = 1/g = **31.9%**.

Alternately, $$G'(x) = \frac{\ln(12)\, 12^x}{\{\ln(11.3) - \ln(0.3)\}\,(12^x - 0.7)}.$$

$$G'(0) = \frac{\ln(12)\, 1}{\{\ln(11.3) - \ln(0.3)\}\,(1 - 0.7)} = 2.283.$$

$$G'(1) = \frac{\ln(12)\, 12}{\{\ln(11.3) - \ln(0.3)\}\,(12 - 0.7)} = 0.7272.$$

p = S(1) = G'(1) / G'(0) = 0.7272 / 2.283 = **31.9%**.

b. $$E[x] = \frac{\ln(gb)\,(1 - b)}{\ln(b)\,(1 - gb)} = \frac{\ln[(3.139)(12)]\,(1 - 12)}{\ln(12)\,\{1 - (3.139)(12)\}} = \mathbf{0.438}.$$ (43.8% of the MPL.)

Alternately, $$E[x] = \frac{(a + 1)\,\{\ln(a+b) - \ln(a+1)\}}{\ln(b)} = \frac{(0.3)\,\{\ln(11.3) - \ln(0.3)\}}{\ln(12)} = \mathbf{0.438}.$$

Alternately, E[x] = 1/G'(0) = 1/2.283 = **0.438**.

c. $$S(x) = G'(x) / G'(0) = \frac{(0.3)\,(12^x)}{(12^x - 0.7)}.$$

$$F(x) = 0.4. \Rightarrow S(x) = 0.6. \Rightarrow 0.6 = \frac{(0.3)\,(12^x)}{(12^x - 0.7)}. \Rightarrow 0.42 = (0.3)\,(12^x).$$

$\Rightarrow \ln(0.42/0.30) = x \ln(12). \Rightarrow x =$ **0.135**. (13.5% of the MPL.).

d. Since there is a pointmass of probability of 31.9% at x = 1, the 70th percentile of the distribution function is **1**, in other words the MPL.

e. $$0.75 = G(x) = \frac{\ln(12^x - 0.7) - \ln(0.3)}{\ln(11.3) - \ln(0.3)}. \Rightarrow \ln(12^x - 0.7) = 1.5176. \Rightarrow 12^x = 5.2613.$$

$\Rightarrow x =$ **0.668**. (66.8% of the MPL.)

2. p = probability of a total loss assuming there is a loss = 0.2/4 = 5%. g = 1/p = 20.
The average size of loss is: (19/20)(135) + (1/20)(900) = \$173.25 million.
The average size of loss in units of the MPL is: 173.25 / 900 = 0.1925.

Thus, $0.1925 = E[x] = \dfrac{\ln[gb]\,(1\text{-}b)}{\ln(b)\,(1\,\text{-}gb)} = \dfrac{\ln[20b]\,(1\text{-}b)}{\ln(b)\,(1\,\text{-}20b)}$.

Solving with the aid of a computer, b = 0.418.

$$G(x) = \frac{\ln[(g\text{-}1)b + (1 - gb)\,b^x] - \ln[1\text{-}b]}{\ln[gb]} = \frac{\ln[7.942 - (7.36)(0.418^x)] - \ln[0.582]}{\ln[8.36]}.$$

$$G(50/900) = G(1/18) = \frac{\ln[7.942 - (7.36)(0.418^{1/18})] - \ln[0.582]}{\ln[8.36]} = 0.2208.$$

The reinsurer pays excess of a ground up loss of 250 million.

$$G(250/900) = G(5/18) = \frac{\ln[7.942 - (7.36)(0.418^{5/18})] - \ln[0.582]}{\ln[8.36]} = 0.6188.$$

The expected ground up annual loss is: (4%)(\$173.25 million) = \$6.930 million.
The owner of the building is expected to retain: (0.2208)(\$6.930 million) = **\$1.530 million**.
The insurer is expected to pay: (0.6198 - 0.2208)(\$6.930 million) = **\$2.765 million**.
The reinsurer is expected to pay: (1 - 0.6198)(\$6.930 million) = **\$2.635 million**.
<u>Comment</u>: One can <u>not</u> fit via Method of Moments in closed form.

3. $LER(x) = \dfrac{\int_0^x S(t)\,dt}{E[X]}$. $\Rightarrow \dfrac{d\,LER(x)}{dx} = \dfrac{S(x)}{E[X]}$.

$\Rightarrow \dfrac{d\,LER(0)}{dx} = \dfrac{1}{E[X]}$. $\Rightarrow S(x) = \dfrac{d\,LER(x)}{dx} \Big/ \dfrac{d\,LER(0)}{dx}$. $\Rightarrow F(x) = 1 - \dfrac{d\,LER(x)}{dx} \Big/ \dfrac{d\,LER(0)}{dx}$.

$\dfrac{d\,LER(x)}{dx} = \dfrac{1}{\ln[a + b] - \ln[a+1]}\,\dfrac{\ln[b]\,b^x}{a + b^x}$. $\dfrac{d\,LER(0)}{dx} = \dfrac{1}{\ln[a + b] - \ln[a+1]}\,\dfrac{\ln[b]}{a + 1}$.

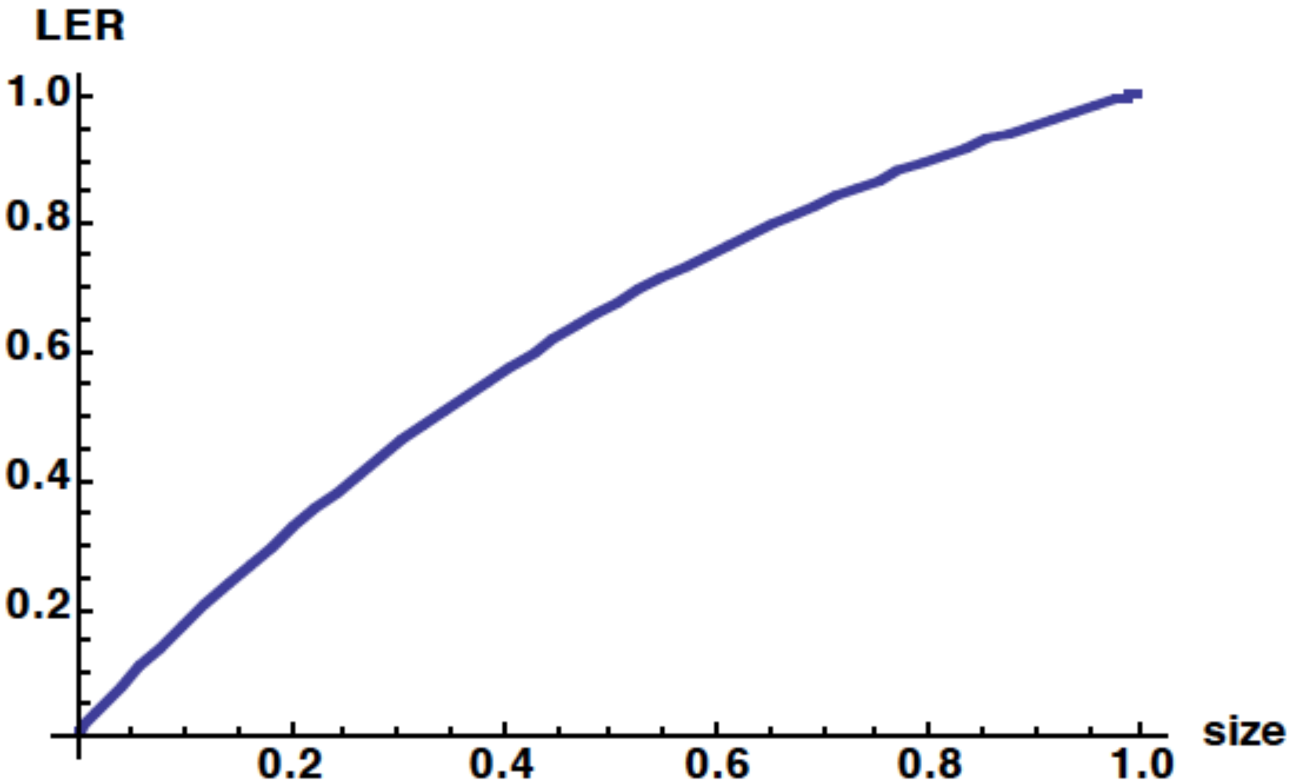
$\Rightarrow F(x) = 1 - (a+1)\dfrac{b^x}{a + b^x}$, $0 \le x < 1$.

$S(1^-) = (a+1)b / (a + b) > 0$.
Thus there is a point mass of probability at 1 of size: $(a+1)b / (a + b)$.

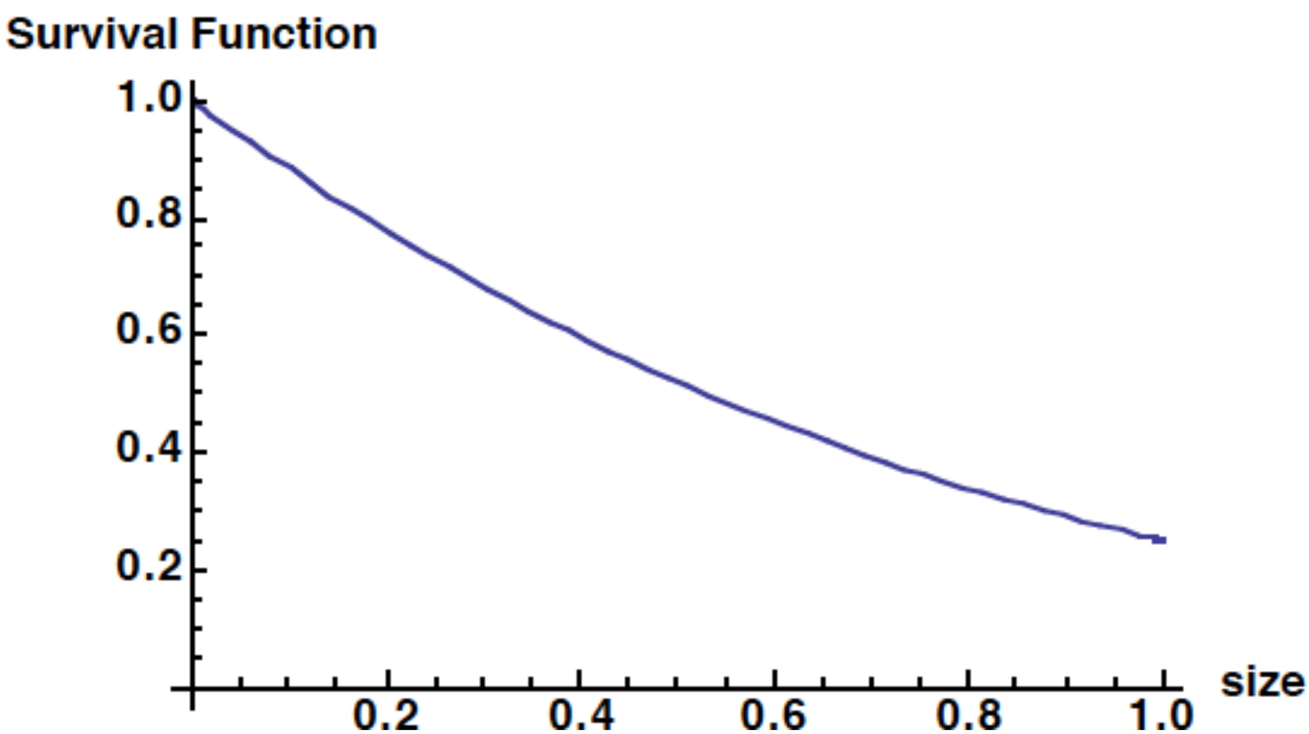
<u>Comment</u>: $E[X] = 1/LER'(0) = (a+1)\,\dfrac{\ln[a + b] - \ln[a+1]}{\ln[b]}$.

Note that $F(1^-) = a\,(1 - b) / (a + b) < 1$.
This is a member of the MBBEFD Distribution Class.

Here is a graph of the loss elimination ratio for b = 0.2 and a = 3:



As it should, the LER is increasing, concave downwards, and approaches 1 as x approaches 1.
Here is a graph of the Survival Function for b = 0.2 and a = 3:



There is a point mass of probability at x = 1 of size: (a+1)b / (a + b) = (4)(0.2)/3.2 = 25%.

4. Exposure curves are used in exposure rating excess of loss property reinsurance. Often, underwriters have only a finite number of discrete exposure curves at their disposal. These curves are available in graphical or tabulated form, and are also implemented in computerized underwriting tools. One of the curves must be selected for each risk band, but it is not always clear which curve should be used. In such cases, the underwriter might also want to use a virtual curve lying between two of the discrete curves available to him.
This can be achieved by replacing the discrete curves with analytical exposure curves. Each set of parameters then defines another curve. If a continuous set of parameters is available, the exposure curves can be varied smoothly within the whole range of available curves.
Practical problems can arise if a curve family with more than two parameters is used. It might then become very difficult to find a set of parameters which can be associated with the information available for a class of risks. This problem can be overcome if a curve family is restricted to a one or two parameter subclass and if new parameters are introduced which can easily be interpreted by the underwriters.
The MBBEFD class of distributions has two parameters. One can parameterize it in terms of the probability of the Maximum Single Loss and the mean loss, which can be easily be interpreted by the underwriters. A subset of the resulting exposure curves closely matches exposure curves, the SwissRe curves, that are currently used for exposure rating of excess of loss property reinsurance.
One can then vary the parameters continuously to get curves similar to the SwissRe curves.
Comment: My answer is longer than should be needed for full credit.

5. $G'(x) = 4\ (1-x)^3$. $E[X] = 1/G'(0) = 1/4$.
$S(x) = E[X]\ G'(x) = (1-x)^3$.
$f(x) = -S'(x) = \mathbf{3\ (1-x)^2}$, $0 \le x \le 1$.
Comment: A Beta Distribution as per Loss Models with $a = 1$, $b = 3$, and $\theta = 1$.

6. The reinsurer is covering the layer 800 Xs 200, or from $200,000 to $1 million.

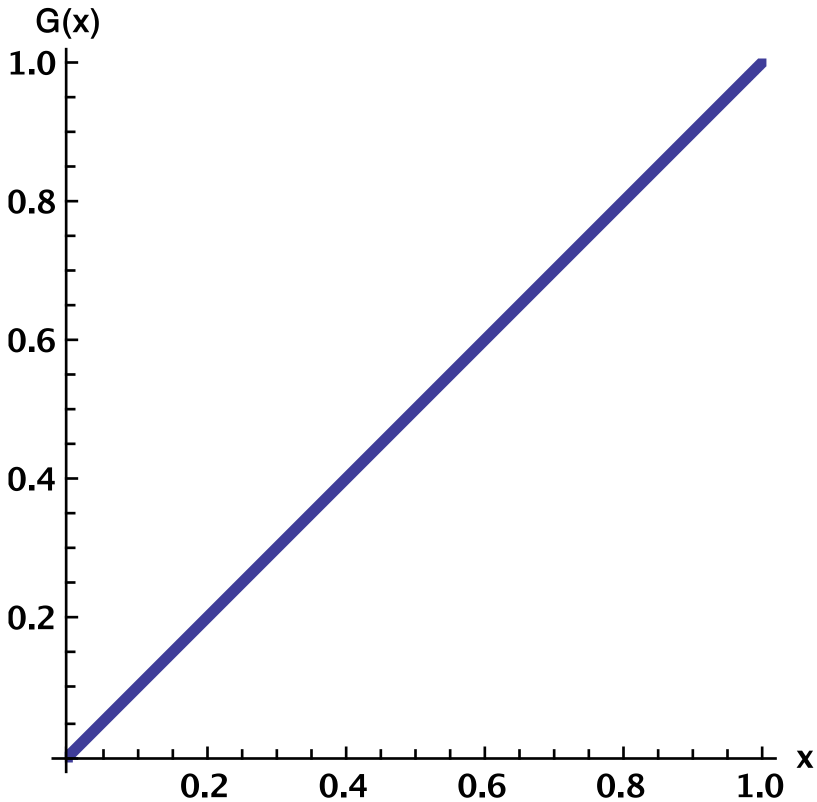
$$g = 1/p = 20. \quad G(x) = \frac{\ln\left[\frac{(g-1)b + (1 - gb)\, b^x}{1-b}\right]}{\ln[gb]} = \frac{\ln[(22.375)(9^x) - 21.375]}{\ln[180]}.$$

The properties in the first two intervals contribute nothing to the reinsurer's loss cost.

$$G(200/350) = \frac{\ln[(22.375)(9^{200/350}) - 21.375]}{\ln[180]} = 77.91\%.$$

The third interval contributes: 100% - 77.91% = 22.09%.
G(200/750) = 56.52%.
The fourth interval contributes: 100% - 56.52% = 43.48%.
G(200/1500) = 41.47%. G(1000/1500) = 83.25%.
The fifth interval contributes: 83.25% - 41.47% = 41.78%.
G(200/3500) = 26.66%. G(1000/3500) = 58.20%.
The sixth interval contributes: 58.20% - 26.66% = 31.54%.
The reinsurer's expected loss cost is:
(68%)($100,000) {(22.09%)(3) + (43.48%)(2) + (41.78%)(1) + (31.54%)(2)} = $175,501.
The total subject premium is: ($100,000) (4 + 2 + 3 + 2 + 1 + 2) = $1,400,000.
The ratio of the reinsurer's loss cost to the subject premium is:
$175,501 / $1,400,000 = **12.54%**.
Comment: Similar to the pricing example at page 19 of "Basics of Reinsurance Pricing" by David R. Clark.
For each size of property we use the exposure curve to estimate what percent of total losses is in the reinsured layer. The percent of losses in a layer depends on the severity distribution which is the basis of the exposure curve, which represents the loss elimination ratios.
Percent of total losses in a layer is: LER(top) - LER(bottom).
Since the reinsurance limit and attachment point are on a per location per occurrence basis, the percent of losses in a layer depends on the severity distribution not the frequency.
For example, for a $3.5 million property, the expected percent of total losses in the layer from $200,000 to $1 million is LER(1m) - LER(200K) = 58.20% - 26.66% = 31.54%.
This includes the (unlikely) possibility of more than one loss in a year at this given location, since if this occurs the same layer is reinsured for each loss at this property.

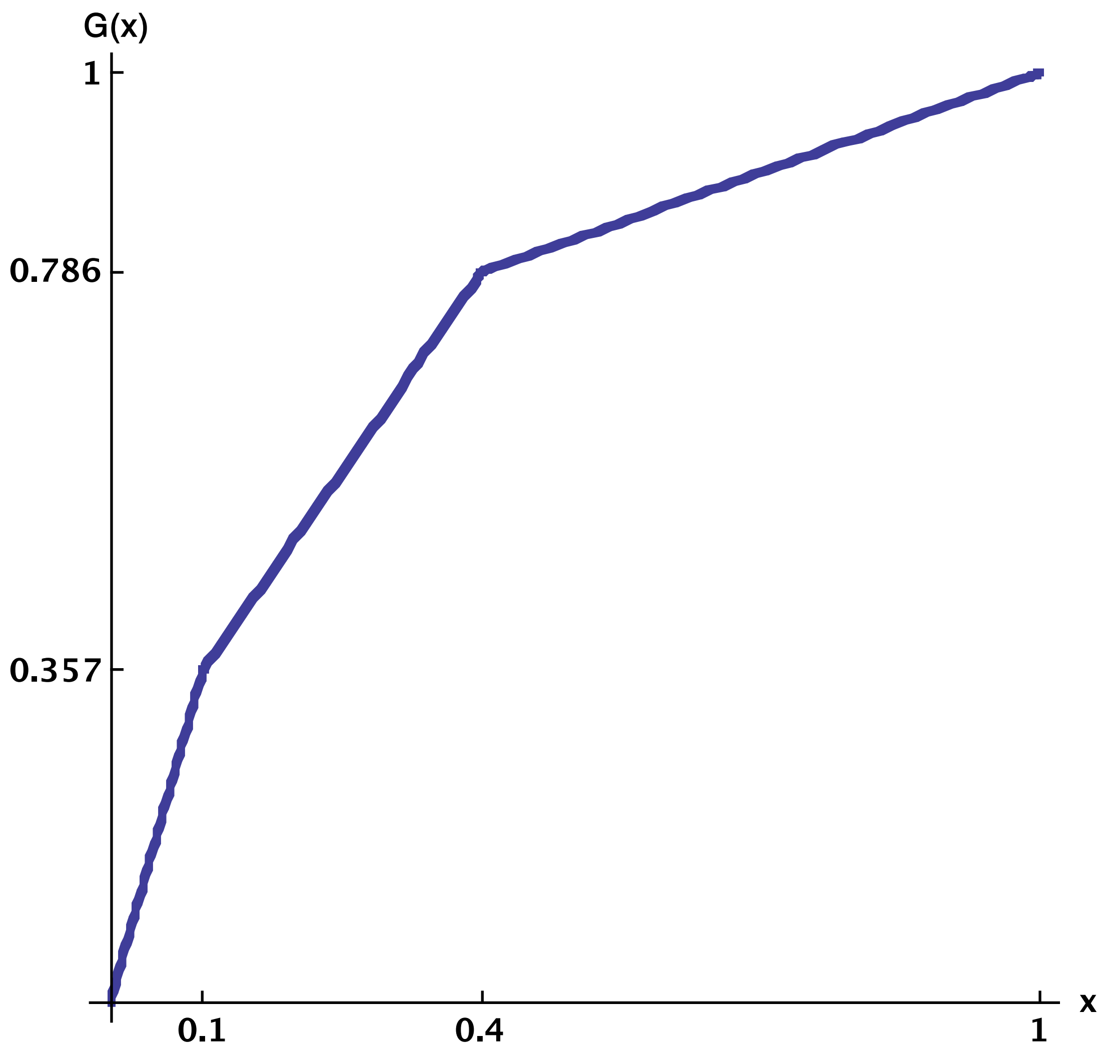
7. (a) S(x) = 1 for x < 1. ⇒ Mean = 1. ⇒ $G(x) = \int_0^x S(t)\, dt$ / mean = x.



b. Mean = (60%)(10%) + (30%)(40%) + (10%)(1) = 0.28.
For d ≤ 10%, the losses eliminated are d.
For 10% < d ≤ 40%, d is eliminated from a loss of size greater than 10% of the MPL;
the probability of such a large loss is: 30% + 10% = 40%.
Thus, the losses eliminated are: (60%)(10%) + (40%) d = 0.06 + 0.4d.
For 40% < d, the losses eliminated are: (60%)(10%) + (30%)(40%) + (10%) d = 0.18 + 0.1d.

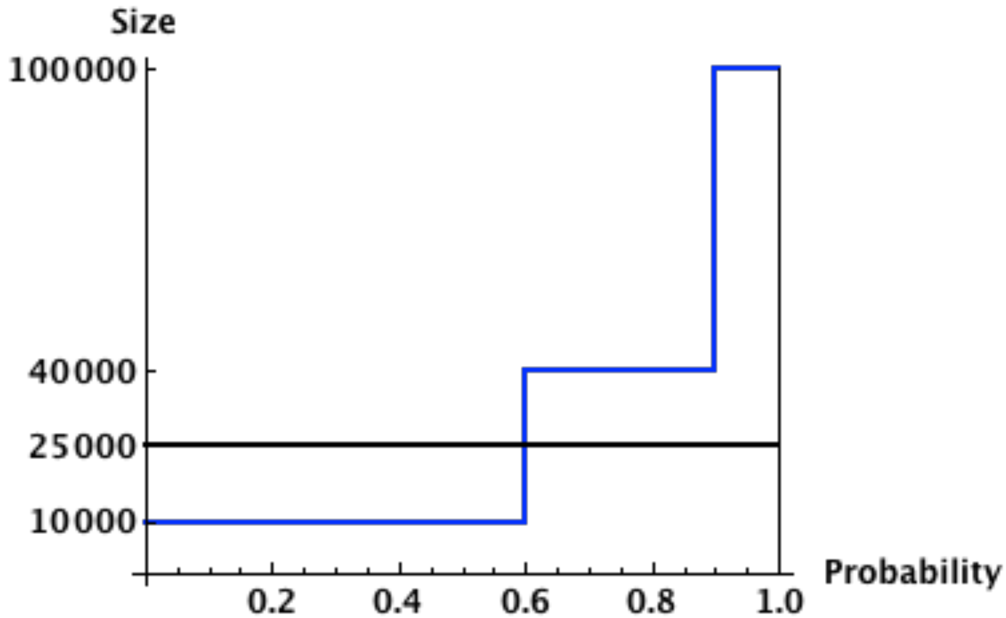
$$\text{Thus } G(x) = \begin{cases} x/0.28, \ x \le 0.1 \\ (0.06 + 0.4x) / 0.28, \text{ for } 0.1 < x \le 0.4 \\ (0.18 + 0.1x) / 0.28, \text{ for } 0.4 < x \le 1 \end{cases}.$$

G(0.1) = 0/1.0.28 = 0.357. G(0.4) = 0.22/0.28 = 0.786.



<u>Comment</u>: Let us assume for example an MPL of $100,000.
Then losses are $10,000 with probability 60%, $40,000 with probability 30%, and $100,000 with probability 10%. The mean loss is $28,000.
The expected losses eliminated for d = $70,000, corresponding to x = 0.7 is:
(60%)(10,000) + (30%)(40,000) + (10%)(70,000) = 25,000.
Thus the loss elimination ratio is: 25,000 / 28,000 = 89.3%.
In my formula for G(x), G(0.7) = {0.18 + (0.1)(0.7)} / 0.28 = 0.25/0.28 = 89.3%, matching.

For example, assume instead that d = $25,000.
Then in the following Lee Diagram, the losses eliminated are represented by the area below both the horizontal line at 25,000 and the cumulative distribution function.



This area consists of two rectangles: one of height 10,000 and width 0.6,
and another of height 25,000 and width 0.4.
Thus the losses eliminated are: (10,000)(0.6) + (25,000)(0.4) = 16,000.
The loss elimination ratio is: 16,000/28,000 = 57.1%.
In my formula for G(x), G(0.25) = {0.06 + (0.4)(0.25)} / 0.28 = 0.16/0.28 = 57.1%, matching.

8. g = 1/p = 1/2.5% = 40.

$$G(x) = \frac{\ln\left[\frac{(g-1)b + (1 - gb)\, b^x}{1-b}\right]}{\ln[gb]}.$$

$$G'(x) = \frac{\ln(b)\,(1-gb)}{\ln(gb)\,\{(g-1)b^{1-x} + (1-gb)\}}.$$

$$S(x) = G'(x) / G'(0) = \frac{(g-1)b + (1-gb)}{(g-1)b^{1-x} + (1-gb)} = \frac{1-b}{(g-1)b^{1-x} + (1-gb)} = \frac{1-b}{39\, b^{1-x} + 1 - 40b}.$$

Set $0.5 = S(0.016) = \dfrac{1-b}{39\, b^{0.984} + 1 - 40b}$.

$\Rightarrow 1 - b = 19.5\, b^{0.984} + 0.5 - 20b. \Rightarrow 39\, b^{0.984} - 38\, b = 1.$

Try some values of b, for example for b = 3: $39\, b^{0.984} - 38\, b = 0.961$.

For b = 2: $39\, b^{0.984} - 38\, b = 1.140$.

For b = 2.86: $39\, b^{0.984} - 38\, b = 1.000$.

For g = 40 and b = 2.86: $G(0.3) = \dfrac{\ln\left[\frac{(39)(2.86) + (1 - 114.4)\, 2.86^{0.3}}{1-2.86}\right]}{\ln[114.4]}$ = **66.69%**.

9. If $\alpha > 0$, G(0) = 0 and G(1) = 1.

$G'(x) = \alpha\, x^{\alpha-1} \geq 0$.

$G''(x) = \alpha\,(\alpha-1)\, x^{\alpha-1}$. For this second derivative to be nonpositive, we require that $\alpha \leq 1$.

Thus for this function to be a valid exposure curve, **$0 < \alpha \leq 1$**.

<u>Comment</u>: If G(x) = x, then S(1) = G'(1)/G'(0) = 1/1 = 1, so that every loss is the maximum possible loss.

For $\alpha < 1$, $G'(0) = \infty$, so that the mean is $1/\infty = 0$. So this is <u>not</u> a useful exposure curve for $\alpha < 1$, even though it satisfies Bernegger's conditions at page 101.

10. Reinsurer A covers the layer from \$10 million to \$50 million,
corresponding to from d = 10/250 = 0.04 to d = 50/250 = 0.20.
Using the exposure curve graph (as best as I can) G(0.04) = 25% and G(0.2) = 55%.
Reinsurer A is responsible for: 55% - 25% = 30% of total expected losses.
Reinsurer B covers the layer from \$50 million to \$100 million,
corresponding to from d = 50/250 = 0.20 to d = 100/250 = 0.4.
Using the exposure curve graph (as best as I can) G(0.2) = 55% and G(0.4) = 70%.
Reinsurer B is responsible for: 70% - 55% = 15% of total expected losses.
Reinsurer C covers the layer from \$100 million to \$250 million,
corresponding to from d = 100/250 = 0.4 to d = 250/250 = 1.
Reinsurer C is responsible for: 100% - 70% = 30% of total expected losses.
Comment: Bernegger's exposure curve with c = 3, corresponding to the third SwissRe curve.
G(10/250) = G(0.04) = 24.8%. G(50/250) = G(0.2) = 54.9%. G(100/250) = G(0.4) = 71.6%.
The ceding insurer is responsible for 24.8% of total expected losses.
Reinsurer A is responsible for: 54.9% - 24.8% = 30.1% of total expected losses.
Reinsurer B is responsible for: 71.6% - 54.9% = 16.7% of total expected losses.
Reinsurer C is responsible for: 100% - 71.6% = 28.4% of total expected losses.

11. Given the probability of the Maximum Possible Loss and the mean loss as a percent of the MPL, you can fit a curve via method of moments.
Alternately, given the first and second moments of the size of loss distribution (in units of the Maximum Possible Loss), or equivalently the mean and variance, you can fit a curve via method of moments.
Comment: Given p and the mean size of a partial loss, one could then calculate
μ = (1-p)(mean size of a partial loss) + p (1).

12. $\text{LER}(x) = \dfrac{\int_0^x S(t)\,dt}{E[X]}$. $\Rightarrow \dfrac{d\,\text{LER}(x)}{dx} = \dfrac{S(x)}{E[X]}$.

$\Rightarrow \dfrac{d\,\text{LER}(0)}{dx} = \dfrac{1}{E[X]}$. $\Rightarrow S(x) = \dfrac{1}{E[X]} \Big/ \dfrac{d\,\text{LER}(0)}{dx}$. $\Rightarrow F(x) = 1 - \dfrac{d\,\text{LER}(x)}{dx} \Big/ \dfrac{d\,\text{LER}(0)}{dx}$.

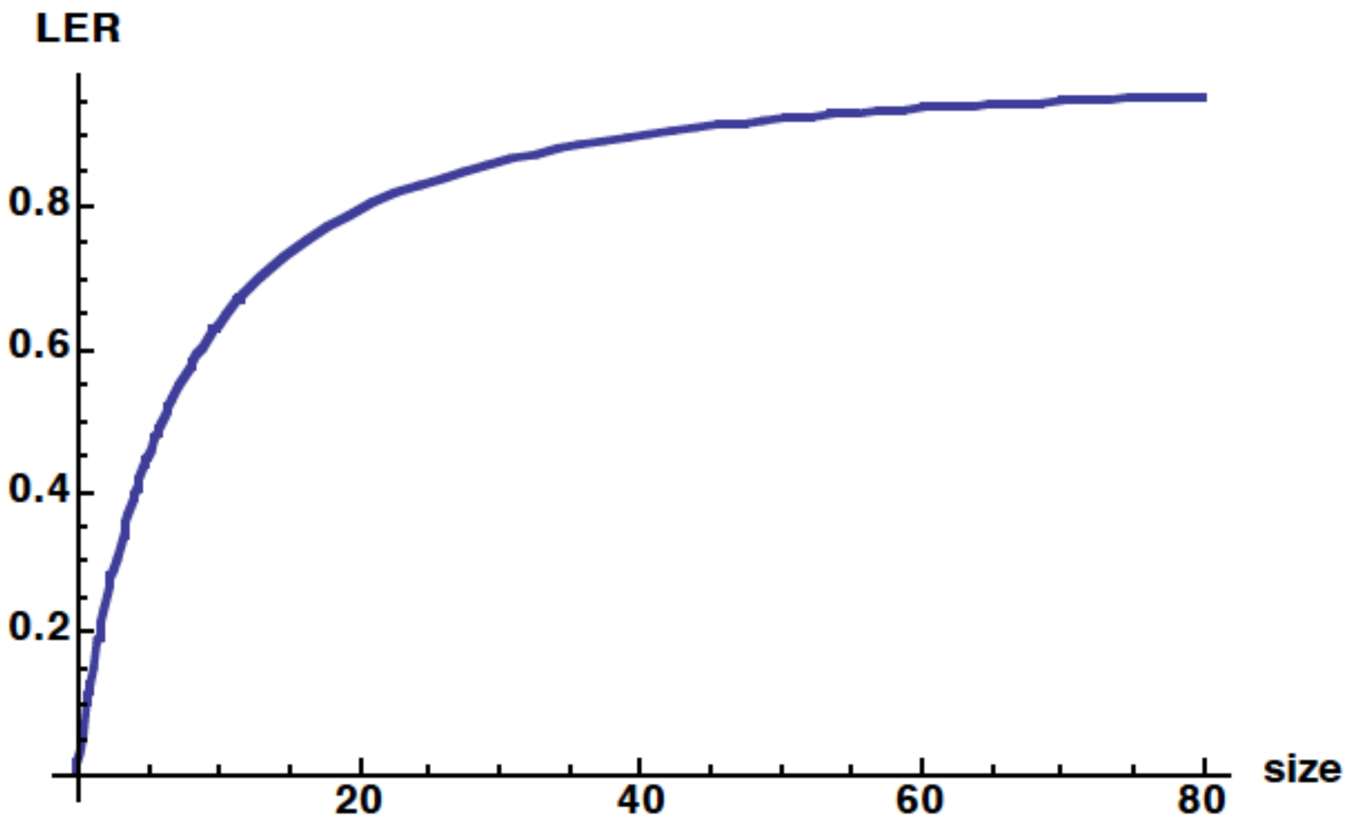
$\dfrac{d\,\text{LER}(x)}{dx} = \dfrac{\dfrac{\ln[2 + x/10]}{10 + x} - \dfrac{\ln[1 + x/10]}{20 + x}}{\ln[2 + x/10]^2}$. $\dfrac{d\,\text{LER}(0)}{dx} = \dfrac{1}{10\ln[2]}$.

$\Rightarrow F(x) = 1 - 10\ln[2]\,\dfrac{\dfrac{\ln[2 + x/10]}{10 + x} - \dfrac{\ln[1 + x/10]}{20 + x}}{\ln[2 + x/10]^2}$. $F(8) =$ **76.3%**.

<u>Comment</u>: If the loss elimination ratio at $x \ge 0$ is: $\dfrac{\ln[1 + x/\theta]}{\ln[\beta + x/\theta]}$, $\theta > 0$, $\beta > 1$, then

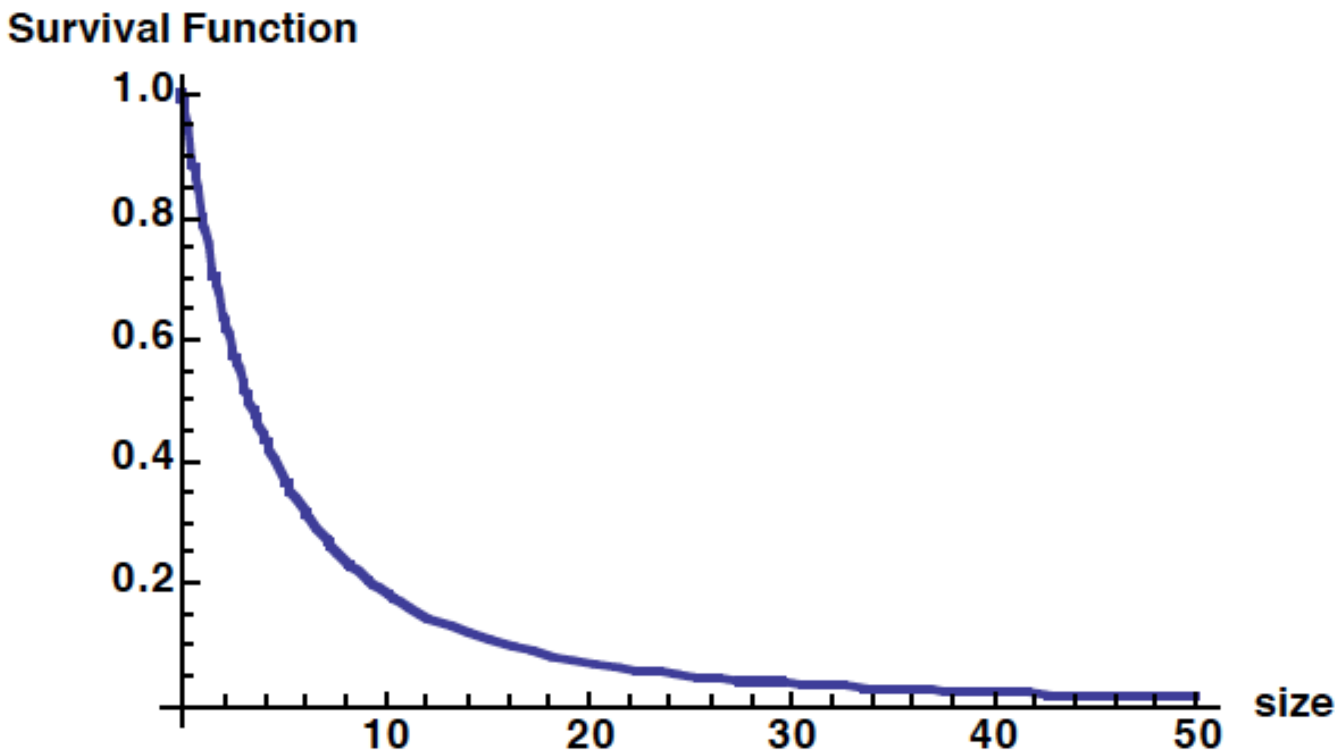
$F(x) = 1 - \theta\ln[\beta]\,\dfrac{\dfrac{\ln[\beta + x/\theta]}{\theta + x} - \dfrac{\ln[1 + x/\theta]}{\beta\theta + x}}{\ln[\beta + x/\theta]^2}$.

As far as I know, not a distribution that is used in actuarial work.
Here is a graph of the loss elimination ratio for $\theta = 10$ and $\beta = 2$:

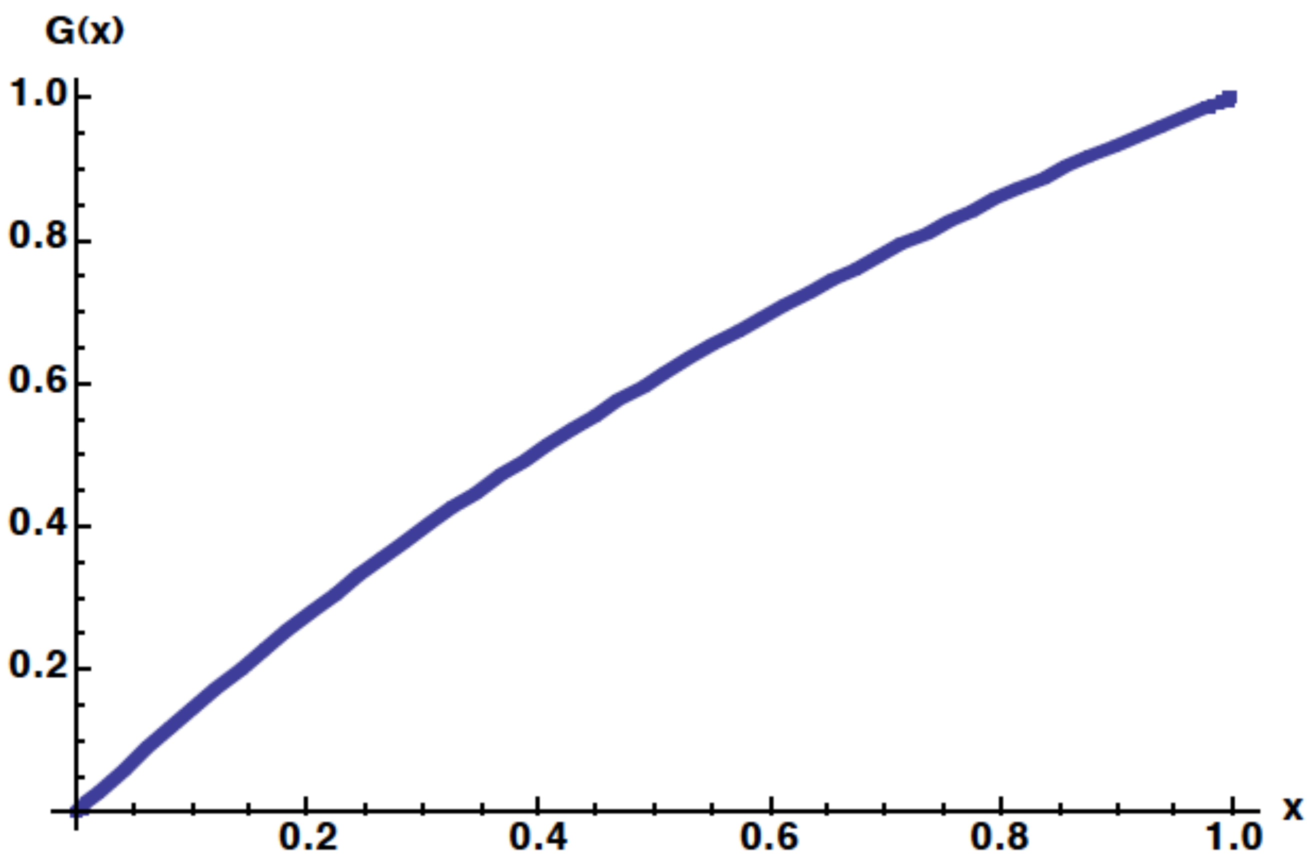


As it should, the LER is increasing, concave downwards, and approaches 1 as x approaches infinity.

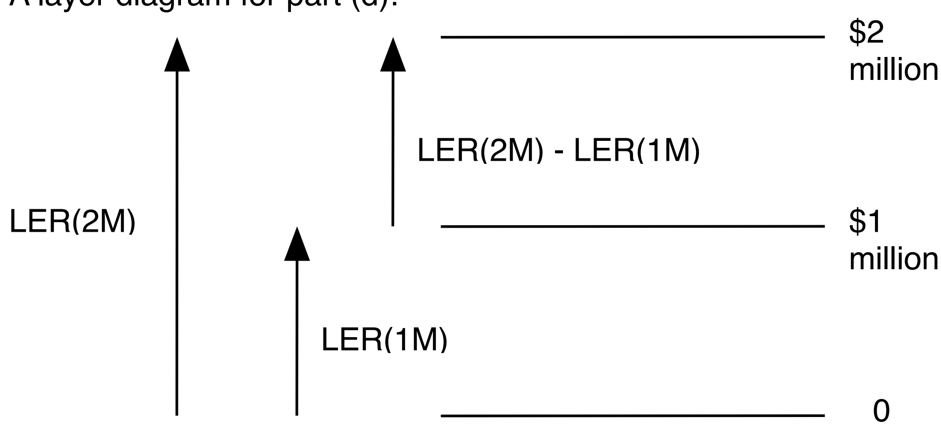
Here is a graph of the Survival Function for $\theta = 10$ and $\beta = 2$:



13. a) "G(d) is an increasing and concave function on the interval [0, 1].
In addition, G(0) = 0 and G(1) = 1 by definition."
$G(x) \geq 0$. $G(0) = (16/7)(0) = 0$. $G(1) = (16/7)(1 - 9/16) = 1$.
$G'(x) = (288/7)(3+x)^{-3} \geq 0$. $G''(x) = -(864/7)(3+x)^{-4} \leq 0$.
b) Mean = $1/G'(0) = (7/288)\ 3^3 = 21/32 = 0.656$.
Mean loss is: (0.656)($5 million) = **$3.28 million**.
c) $S(1) = G'(1)/G'(0) = 3^3 / 4^3 = 27/64 =$ **42.2%**.
d) We want the layer from $1 million to $2 million.
1 million ⇔ 1 million / 5 million = 1/5. 2 million ⇔ 2 million / 5 million = 2/5.
G(2/5) - G(1/5) = (16/7)(1 - 0.7785) - (16/7)(1 - 0.8789) = **22.9%**.
Comment: Here is a graph of G, showing that it is an increasing and concave function:



A layer diagram for part (d):



14. a. g = exp[(4.5){0.78 + (0.12)(4.5)}] = 380. p = 1/g = **0.263%**.
b. b = exp[3.1 - (0.15)(4.5)(5.5)] = 0.542.

$$E[x] = \frac{\ln[gb]\ (1-b)}{\ln(b)\ (1-gb)} = \frac{\ln[(380)(0.542)]\ (1 - 0.542)}{\ln(0.542)\ \{1 - (380)(0.542)\}} = \mathbf{1.94\%}.$$

c. $$G(x) = \frac{\ln\left[\dfrac{(g-1)b + (1 - gb)\ b^x}{1-b}\right]}{\ln[gb]}.$$

$$G(100/400) = G(0.25) = \frac{\ln\left[\dfrac{(379)(0.542) + \{1 - (380)(0.542)\}\ 0.542^{0.25}}{1 - 0.542}\right]}{\ln[380)(0.542)]} = 78.2\%.$$

The percentage of pure risk premium ceded to the reinsurer is: 1 - 78.2% = **21.8%**.

Alternately, $a = \dfrac{(g - 1)\ b}{1 - gb} = \dfrac{(379)\ (0.542)}{1 - (380)(0.542)} = -1.0022.$

$$G(x) = \frac{\ln(a + b^x) - \ln(a+1)}{\ln(a+b) - \ln(a+1)} = \frac{\ln\left[\dfrac{a + b^x}{a+1}\right]}{\ln\left[\dfrac{a + b}{a+1}\right]}.$$

$$G(0.25) = \frac{\ln\left[\dfrac{-1.0022 + 0.542^{0.25}}{-1.0022+1}\right]}{\ln\left[\dfrac{-1.0022 + 0.542}{-1.0022+1}\right]} = 78.3\%.$$ 1 - 78.3% = **21.7%**.

15. a) "G(d) is an increasing and concave function on the interval [0, 1].
In addition, G(0) = 0 and G(1) = 1 by definition."
$G(x) \geq 0$. $G(0) = 0$. $G(1) = 4/4 = 1$.

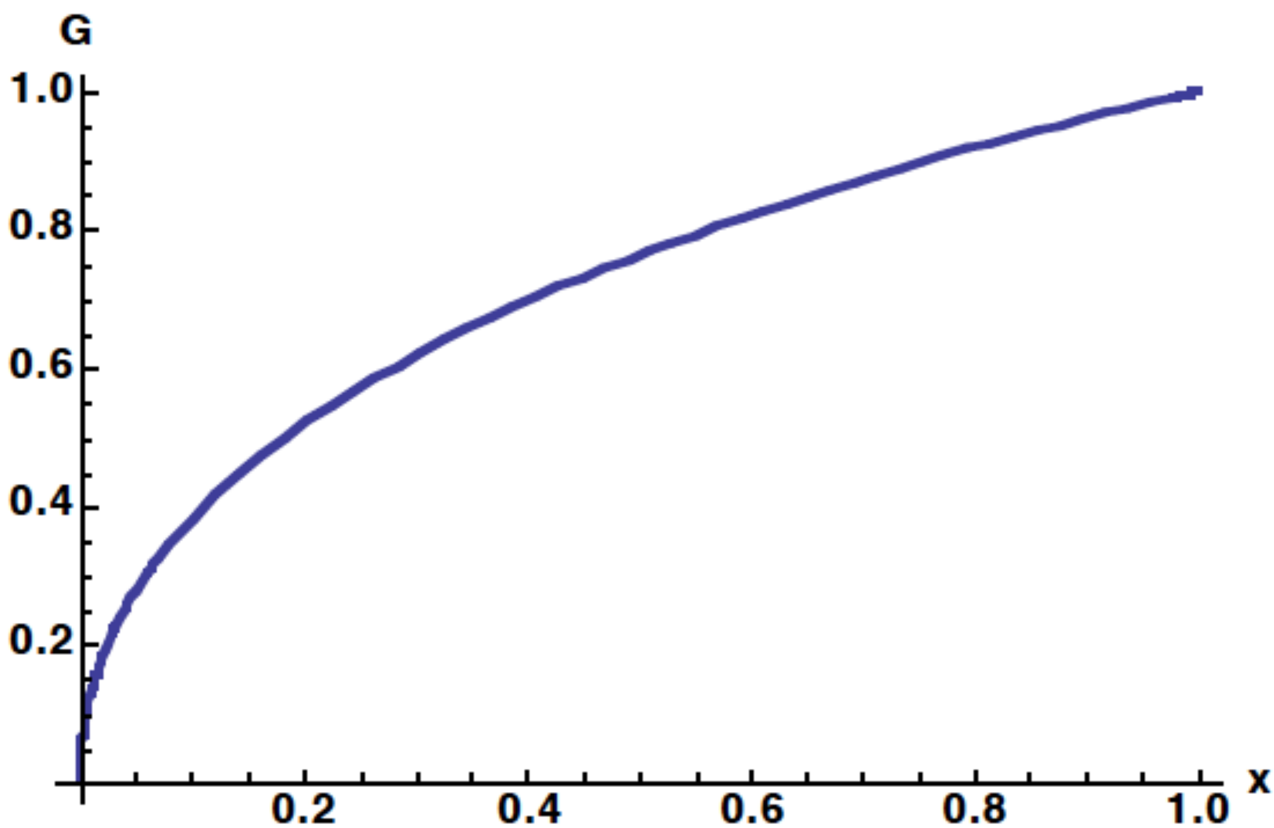
$$G'(x) = \frac{(3+\sqrt{x})\,(2/\sqrt{x}) - (4\,\sqrt{x})(0.5/\sqrt{x})}{(3+\sqrt{x})^2} = \frac{6/\sqrt{x}}{(3+\sqrt{x})^2} \geq 0.$$

$$G''(x) = \frac{(3+\sqrt{x})^2(-3/x^{3/2}) - (6/\sqrt{x})(0.5/\sqrt{x})2(3+\sqrt{x})}{(3+\sqrt{x})^4} =$$

$$-\frac{(3+\sqrt{x})^2\,(3/x^{3/2}) + (6/x)\,(3+\sqrt{x})}{(3+\sqrt{x})^4} \leq 0.$$

b) $1/G'(x) = \frac{(3+\sqrt{x})^2\,\sqrt{x}}{6}$. As x approaches zero, 1/G'(x) approaches zero.

Mean = 1/G'(0) = 0.
Thus in spite of satisfying Bernegger's conditions for a valid exposure curve, the given function would not be of any practical use.
Comment: Here is a graph of G(x), showing that it is an increasing and concave function:



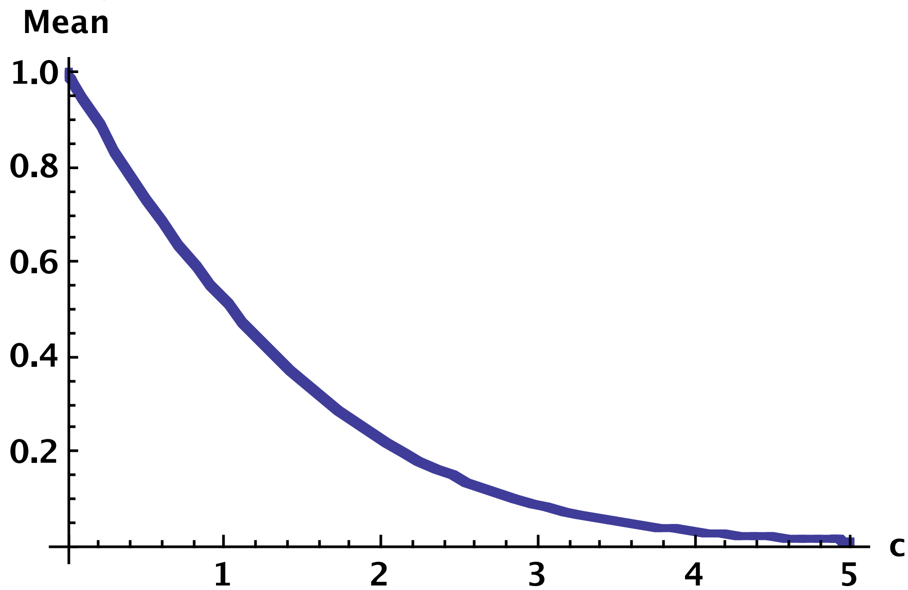
16. (a) For c = 1, b = 16.445 and g = 2.4596.

mean = 1/G'(0) = $\frac{\ln[(2.4596)(16.445)]\ \{(2.4596-1)(16.445) + 1 - (2.4596)(16.445)\}}{\ln(16.445)\ \{1 - (2.4596)(16.445)\}}$ = **0.517**.

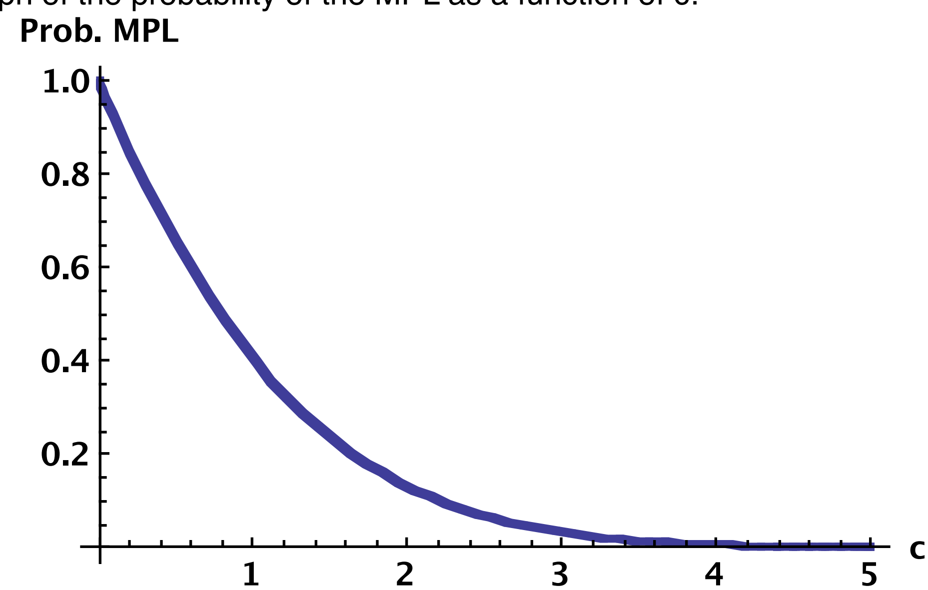
For c = 2, b = 9.0250 and g = 7.6906.

mean = 1/G'(0) = $\frac{\ln[(7.6906)(9.0250)]\ \{(7.6906-1)(9.0250) + 1 - (7.6906)(9.0250)\}}{\ln(9.0250)\ \{1 - (7.6906)(9.0250)\}}$ = **0.226**.

(b) p = G'(1)/G'(0) = 1/g . For c = 1, p = 1/2.4596 = **40.7%**.
For c = 2, p = 1/7.6906 = **13.0%**.
(c) As c increases, the expected loss as a percentage of the maximum loss decreases, as does the probability of a total loss. Large values of c may be appropriate for industrial risks.
Comment: A graph of the mean (as a percent of MPL) as a function of c:

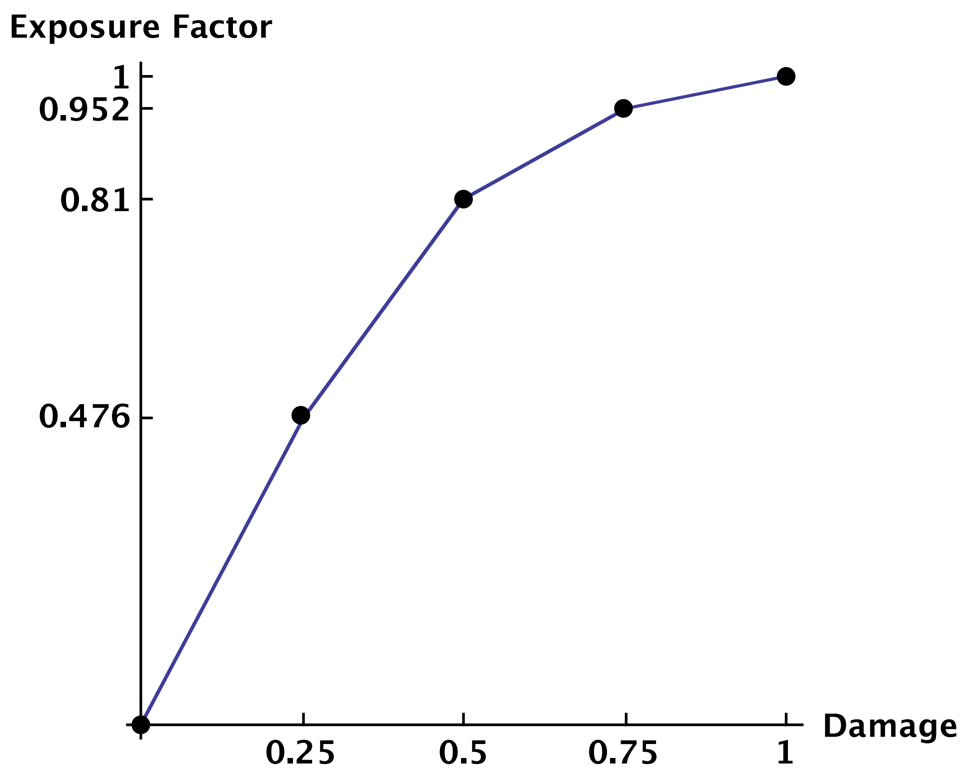


A graph of the probability of the MPL as a function of c:



17. "However, the curves must fulfill certain conditions which restrict the range of the parameters. In addition, practical problems can arise if a curve family with many (more than two) parameters is used. It might then become very difficult to find a set of parameters which can be associated with the information available for a class of risks. This problem can be overcome if a curve family is restricted to a one- or two-parameter subclass and if new parameters are introduced which can easily be interpreted by the underwriters."
Comment: See page 100 of Bernegger.
The MBBEFD class of analytical exposure curves will be introduced, and Bernegger will eventually reparameterize them in terms b and g, and then in terms of c.

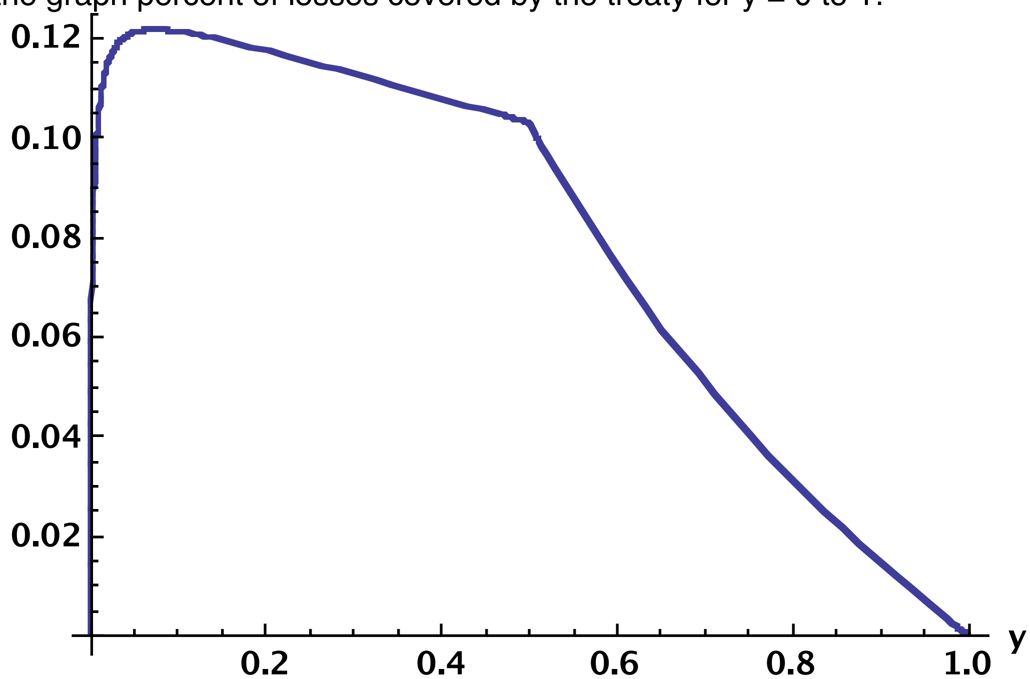
18. The mean is: (30%)(25%) + (40%)(50%) + (20%)(75%) + (10%)(100%) = 52.5%.
At x = 25%, the limited expected value is: 25%.
The loss elimination ratio is: 25%/52.5% = 0.476.
At x = 50%, the limited expected value is: (30%)(25%) + (70%)(50%) = 42.5%.
The loss elimination ratio is: 42.5%/52.5% = 0.810.
At x = 75%, the limited expected value is: (30%)(25%) + (40%)(50%) + (30%)(75%) = 50%.
The loss elimination ratio is: 50%/52.5% = 0.952.
At x = 100%, the loss elimination ratio is 1.
The exposure curve consists of straight lines connecting these points:



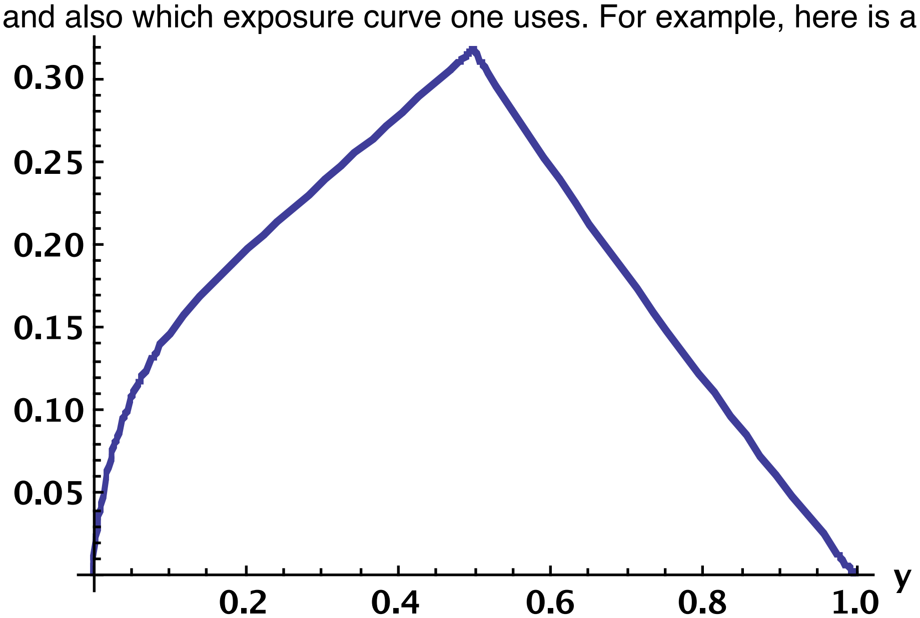
19. b = exp[3.1 - 0.15 c (1+c)] = exp[3.1 - (0.15)(4.5)(5.5)] = 0.5420.
g = exp[c (0.78 + 0.12c)] = exp[(4.5){0.78 + (0.12)(4.5)}] = 379.9.

$$G(x) = \frac{\ln\left[\frac{(g-1)b + (1 - gb)\, b^x}{1-b}\right]}{\ln[gb]} = \ln[448.4 - (447.4)\,(0.5420^x)] \,/\, \ln[205.9].$$

For example, G(0.4) = 0.861 and G(0.8) = 0.969. Thus for y = 0.4, the percent of losses expected to be paid by a treaty that covers the layer from y to 2y is: 0.969 - 0.861 = 0.108.
Here the graph percent of losses covered by the treaty for y = 0 to 1:

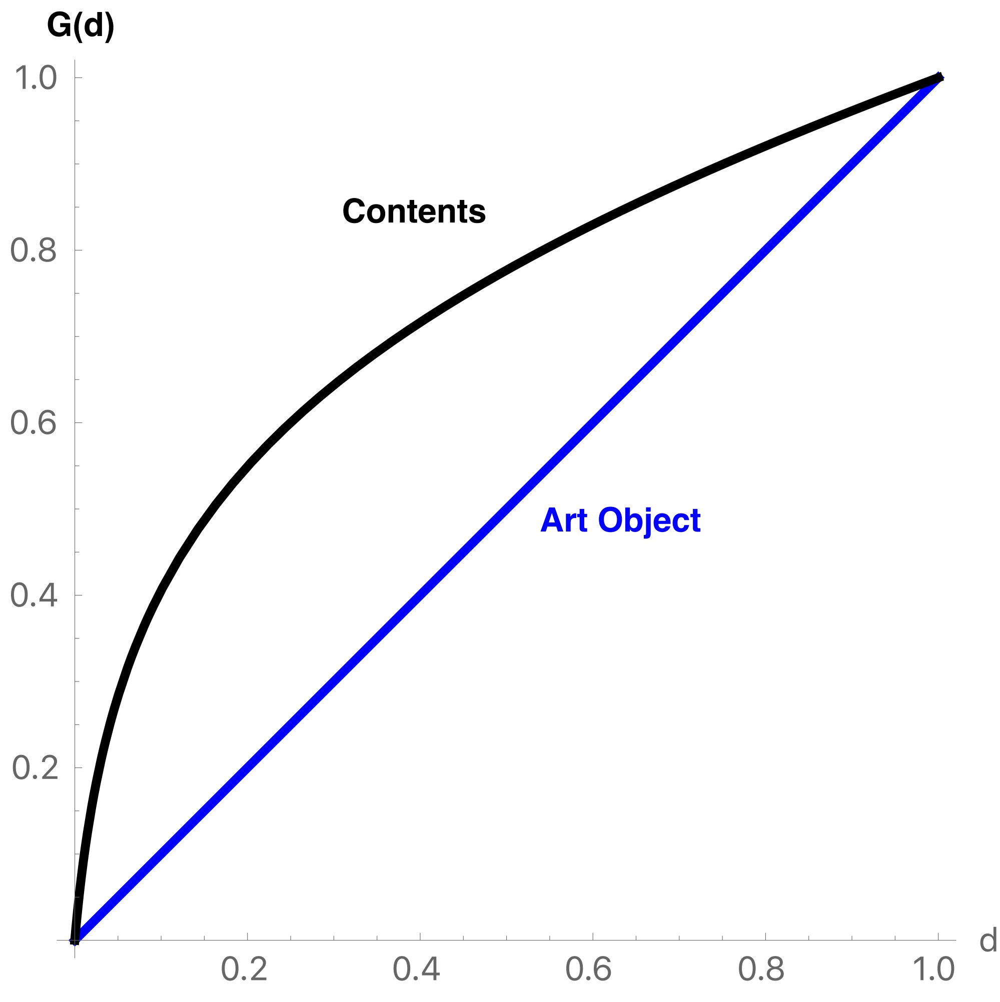


Comment: The graph depends on the ratio of the upper end of the layer to the bottom end of the layer, and also which exposure curve one uses. For example, here is a similar graph for c = 2:



20. 1. Match the mean and the probability of a total loss.
There exists exactly one distribution function belonging to the MBBEFD class for each given pair of p and μ, provided that $0 \le p \le \mu \le 1$.
($g = 1/p$ and then one can derive b from $\mu = 1/G'(0)$.)
2. Match the expected value and standard deviation, in other words the first two moments. While one can not solve for the method of moments in closed form, one can use an iterative scheme.
Comment: See Sections 4.1 and 4.2 of Bernegger.

21. The individual art object will either not be stolen or will be a total loss.
Thus its exposure curve is the diagonal line connecting (0, 0) and (1, 1).
In contrast, household contents are unlikely to be stolen in their entirety. Thus we would expect lots of partial losses.



22. "In a first step, the parameters b_i and g_i have been evaluated for each curve i. By plotting the points belonging to these pairs of parameters in the (b, g) plane, we found that the points were lying on a smooth curve in the plane. In a next step, this curve was modeled as a function of a single curve parameter c. Finally, the parameters c, representing the curves Y_i were evaluated."
Comment: See Section 4.3 of Bernegger.

23. $G'(x) = 1.5 - 1.5x^2$. $E[X] = 1/G'(0) = 1/1.5 = 2/3$.
$S(x) = E[X]\, G'(x) = 1 - x^2$.
$f(x) = -S'(x) = \mathbf{2x}$, $0 \le x \le 1$.
Comment: A Beta Distribution as per Loss Models with a = 2, b = 1, and $\theta = 1$.

24. (a) Expected Losses = (65%)(160,000) = \$104,000.
$G(500K / 10M) = G(0.05) = (1 - 0.06^{0.05}) / 0.94 = 0.1396$.
$G(\{500K + 3500K\} / 10M) = G(0.4) = (1 - 0.06^{0.4}) / 0.94 = 0.7186$.
Expected portion of total pure premium ceded is: 0.7186 - 0.1396 = 0.5790.
Expected ceded risk premium is: (0.5790) (104,000) = **\$60,216**.
(b) $G'(x) = -\ln 0.06\ 0.06^x / 0.94$.
$G'(0) = -\ln 0.06 / 0.94$. $G'(1) = -(\ln 0.06)(0.06)/ 0.94$.
Probability of a total loss = $G'(1)/G'(0)$ = **0.06**.
Alternately, as shown in equation 3.3 in Bernegger, this is a special case of MBBEFD with bg = 1, and b = 0.06. The probability of a total loss is: 1/g = **0.06**.
(c) $G'(0) = -\ln 0.06 / 0.94 = 2.9930$.
The mean size of loss is: $M / G'(0)$ = 10,000,000 / 2.9930 = \$3.341 million.
Thus, the expected annual claim frequency for the insurer is:
\$104,000 / \$3.341 million = **3.11%**.
(d) $S(x) = G'(x)/G'(0) = 0.06^x$. $S(500K / 10M) = S(0.05) = 0.06^{0.05} = 0.8688$.
Thus 86.88% of the losses are big enough to pierce the reinsured layer.
The expected annual claim frequency for the reinsurer is: (0.8688)(3.11%) = **2.70%**.

25. (a) The exposure curve is the loss elimination ratio.

$S(x) = 1 - x^{0.5}$. $E[X \wedge x] = \int_0^x S(t)\, dt = x - 2x^{1.5}/3$. $E[X \wedge 1] = 1/3$.

Thus $G(x) = E[X \wedge x] / E[X \wedge 1] = \mathbf{3x - 2x^{1.5}}$, $0 \le x \le 1$.
(b) We want the layer from \$1,000,000 to \$5,000,000.
$G(1{,}000{,}000 / 10{,}000{,}000) = G(0.1) = (3)(0.1) - (2)(0.1^{1.5}) = 0.2368$.
$G(5{,}000{,}000 / 10{,}000{,}000) = G(0.5) = (3)(0.5) - (2)(0.5^{1.5}) = 0.7929$.
Thus the percent of expected losses in the layer is: 0.7929 - 0.2368 = **0.5561**.
Comment: Similar to 8, 11/13, Q.20.

26. $G(x) = \frac{1 - b^x}{1 - b}$. $G'(x) = -\ln b\, b^x / (1-b)$.

$G'(0) = -\ln b / (1-b)$. $G'(1) = -\ln b\, b / (1-b)$.

0.2 = Probability of a total loss = $G'(1)/G'(0) = b$. ⇒ b = 0.2.

$0.75 = G(x) = \frac{1 - b^x}{1 - b} = (1 - 0.2^x) / 0.8$. ⇒ x = ln(0.4) / ln(0.2) = 0.5693.

The reinsurer's maximum loss is: 100 million = M (1-x) = 0.4307 M. ⇒ M = 232.2 million.

The cedant's maximum loss is: x M = 0.5693 M = (0.5693)(232.2) = **$132.2 million**.

Alternately, as shown in equation 3.3 in Bernegger, this is a special case of MBBEFD with bg = 1.

The probability of a total loss is 0.2. ⇒ g = 1/0.2. ⇒ b = 0.2. Proceed as before.

Comment: Similar to 8, 11/13, Q.22 and 8, 11/15, Q. 20b.

G(x) is the loss elimination ratio at x, where is x is as a percent of the maximum possible loss M.

G(x) is the percent of pure risk premium retained by the cedant if the reinsurer covers the layer from x M to the maximum possible loss M.

Note that: maximum amount ceded + maximum amount retained = 100 million + 132.2 million = 232.2 million = maximum possible loss.

27. The empirical exposure factor for 10% (10M) is: {300 + (52)(10)} / 2000 = 41%.

The empirical exposure factor for 50% (50M) is: {300 + 650 + (12)(50)} / 2000 = 77.5%.

For Y_3, c = 3. b = exp[3.1 - (0.15)(3)(1+3)] = 3.669. g = exp[(3){0.78 + (0.12)(3)}] = 30.57.

$$G(0.1) = \frac{\ln\left[\frac{(30.57-1)3.669 + (1 - (30.57)(3.669))\, 3.669^{0.1}}{1-3.669}\right]}{\ln[(30.57)(3.669)]} = 40.56\%.$$

$$G(0.5) = \frac{\ln\left[\frac{(30.57-1)3.669 + (1 - (30.57)(3.669))\, 3.669^{0.5}}{1-3.669}\right]}{\ln[(30.57)(3.669)]} = 77.69\%.$$

For Y_4, c = 4. b = exp[3.1 - (0.15)(4)(1+4)] = 1.1052. g = exp[(4){0.78 + (0.12)(4)}] = 154.47.

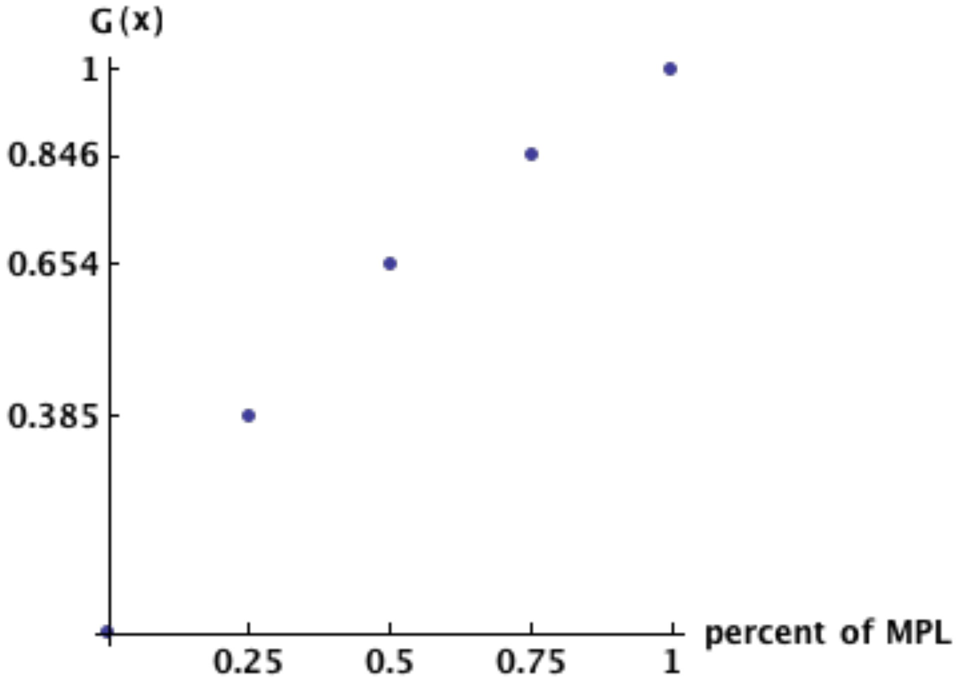
$$G(0.1) = \frac{\ln\left[\frac{(154.47-1)1.1052 + (1 - (154.47)(1.1052))\, 1.1052^{0.1}}{1-1.1052}\right]}{\ln[(154.47)(1.1052)]} = 55.37\%.$$

$$G(0.5) = \frac{\ln\left[\frac{(154.47-1)1.1052 + (1 - (154.47)(1.1052))\, 1.1052^{0.5}}{1-1.1052}\right]}{\ln[(154.47)(1.1052)]} = 86.14\%.$$

Since Y_3 much more closely matches the empirical exposure factors, I will use Y_3 rather than Y_4.

Comment: Similar to 8, 11/16, Q. 21.

28. (a)
We want the limited expected values; we only look at the 10% of the time there is a loss.
E[X; 25%] = (10%)(25%) / 10% = 25%.
E[X; 50%] = {(3%)(25%) + (7%)(50%)} / 10% = 42.5%.
E[X; 75%] = {(3%)(25%) + (2%)(50%) + (5%)(75%)} / 10% = 55%.
E[X] = {(3%)(25%) + (2%)(50%) + (1%)(75%) + (4%)(100%) } / 10% = 65%.
Exposure curve is a graph of loss elimination ratios.
G(0) = 0. G(25%) = E[X ; 25%] / E[X] = 25/65 = 0.385. G(50%) = 42.5/65 = 0.654.
G(75%) = 55/65 = 0.846. G(100%) = 1.
The Exposure Curve:



(b) We want the layer from 25% to 75% of the MPL.

$$\frac{E[X; 75\%] - E[X; 25\%]}{E[X]} = \frac{55\% - 25\%}{65\%} = \mathbf{0.462}.$$

Alternately, G(75%) - G(25%) = 0.846 - 0.385 = **0.461**.

(c) g = 1 / Prob[MPL | there is a loss] = 1 / (4%/10%) = **2.5**.
μ = mean = 65%.
Reading off the table, b = **0.4411**.

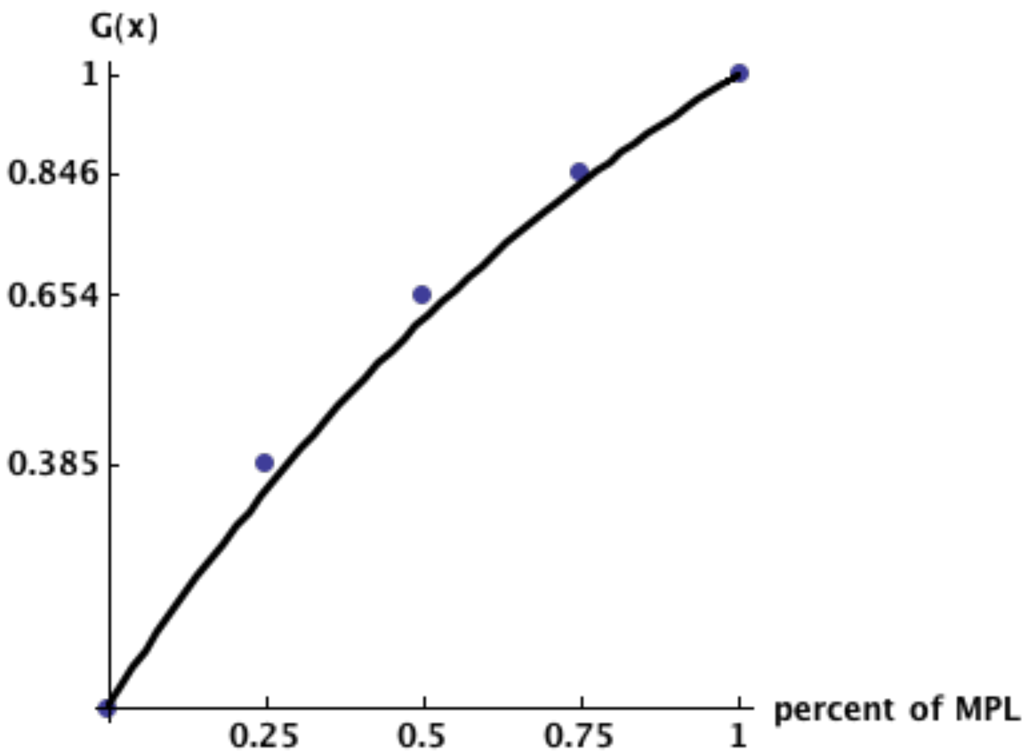
(d) $G(0.25) = \dfrac{\ln\left[\dfrac{(2.5-1)(0.4411) + \{1 - (2.5)(0.4411)\}\ 0.4411^{0.25}}{1-0.4411}\right]}{\ln[(2.5)(0.4411)]}$

= ln(1.03402) / ln(1.10275) = 0.342.

$G(0.75) = \dfrac{\ln\left[\dfrac{(2.5-1)(0.4411) + \{1 - (2.5)(0.4411)\}\ 0.4411^{0.75}}{1-0.4411}\right]}{\ln[(2.5)(0.4411)]}$

= ln(1.08434) / ln(1.10275) = 0.828.
G(0.75) - G(0.25) = 0.828 - 0.342 = **0.486**.

Comment: Similar to 8, 11/17, Q. 18.
Using the fitted curve results in a somewhat different answer in part (d) than part (b).
A plot of the fitted exposure curve and the exposure curve from part (a):



29. a. Dividing by the \$100 million maximum possible loss, \$50 million corresponds to: x = 1/2.

$$G(1/2) = \frac{\ln(0.1 + 0.01^{1/2}) - \ln(1.1)}{\ln(0.11) - \ln(1.1)} = \mathbf{74.0\%}.$$

b. $G'(1) = \dfrac{\dfrac{\ln(0.01)\ 0.01^1}{0.1 + 0.01^1}}{\ln(0.11) - \ln(1.1)} = 0.1818.$ $G'(0) = \dfrac{\dfrac{\ln(0.01)\ 0.01^0}{0.1 + 0.01^0}}{\ln(0.11) - \ln(1.1)} = 1.1818.$

The probability of a total loss = S(1) = G'(1)/G'(0) = 0.1818/1.1818 = **10.0%**.

Alternately, $g = \dfrac{a + b}{(a+1)\ b} = \dfrac{0.1 + 0.01}{(1.1)\ (0.01)} = 10.$

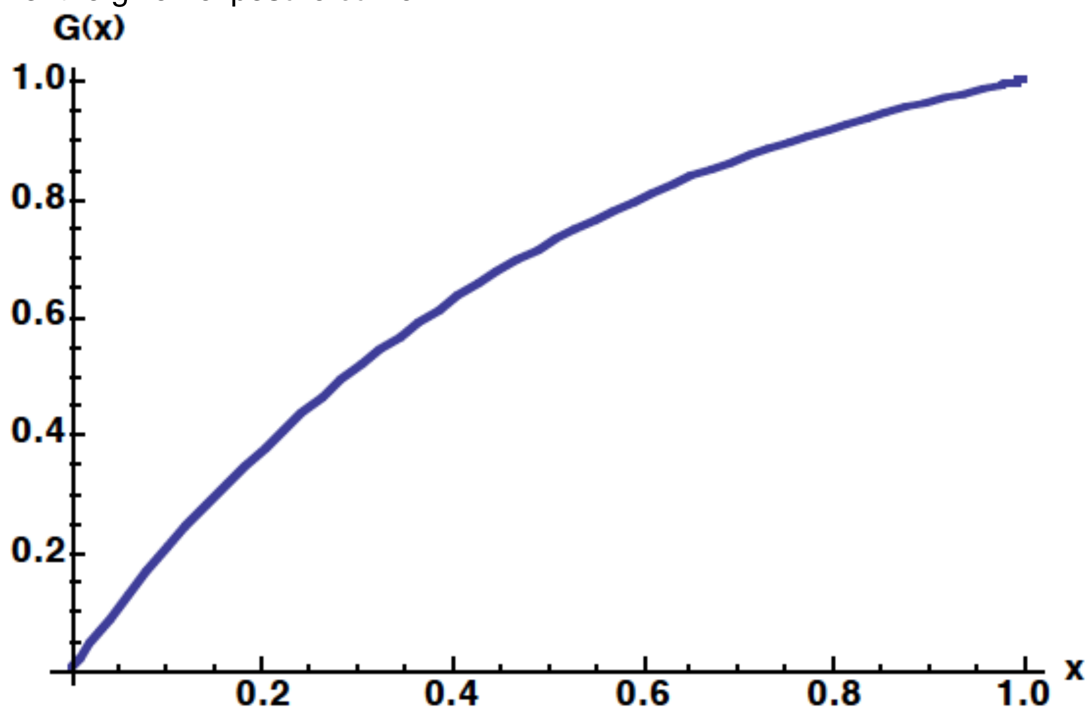
The probability of a total loss = p = 1/g = **10.0%**.

<u>Comment</u>: G(x) is the Loss Elimination Ratio, with x normalized; in this case x = 1 at the maximum possible first dollar loss of \$100 million.

$$G(x) = \int_x^1 S(t)\ dt \ / \int_0^1 S(t)\ dt. \Rightarrow G'(x) = S(x) \ / \int_0^1 S(t)\ dt. \Rightarrow G'(0) = 1 \ / \int_0^1 S(t)\ dt.$$

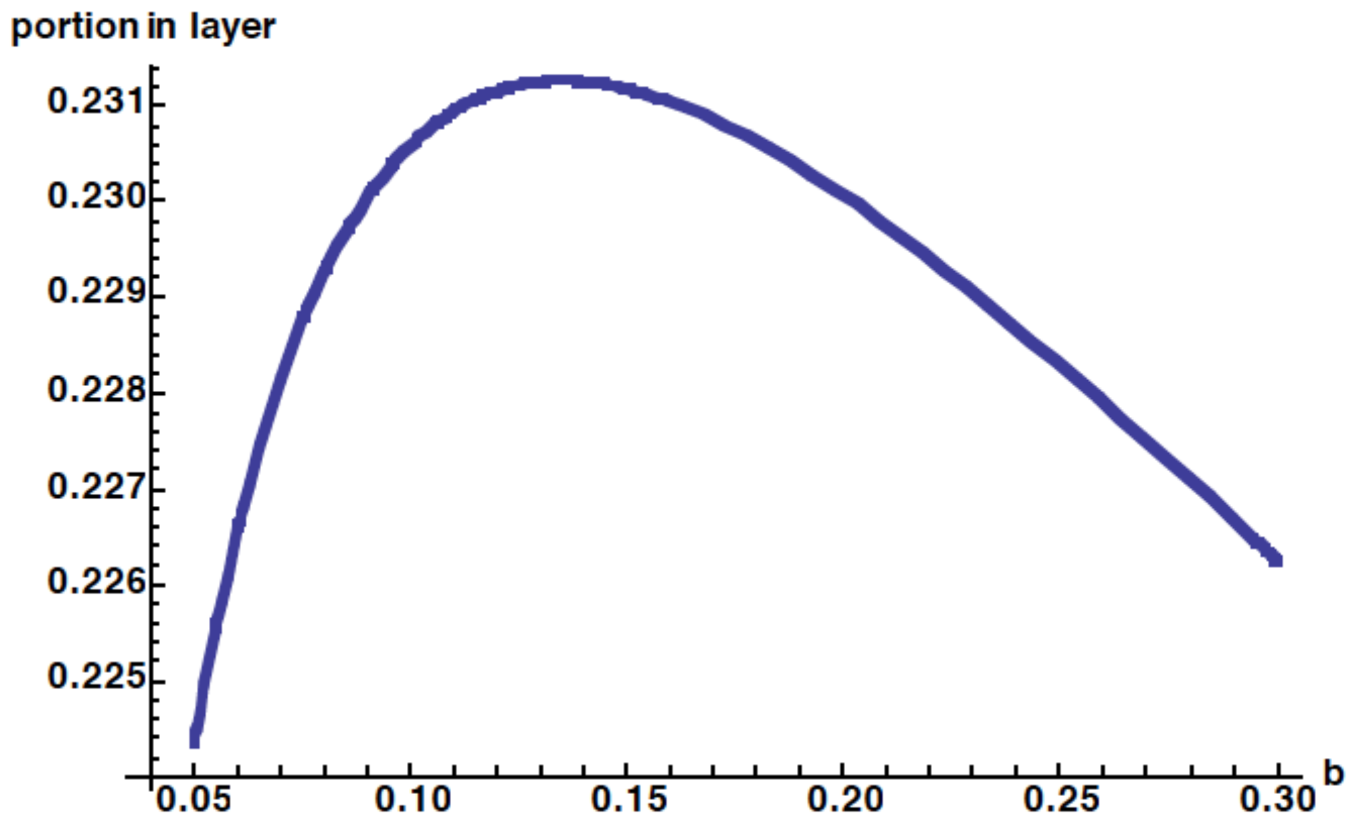
$\Rightarrow$ G'(x) = S(x) G'(0). $\Rightarrow$ G'(x)/G'(0) = S(x).

30. a) "G(d) is an increasing and concave function on the interval [0, 1].
In addition, G(0) = 0 and G(1) = 1 by definition."
$G(x) \geq 0$. $G(0) = 0$.
$G'(x) = -b^x \ln b / (1-b) \geq 0$. Note that since b = 0.15 < 1, ln(b) < 0.
$G''(x) = -b^x (\ln b)^2 / (1-b) \leq 0$.
Note that G(1) = 1, so that there is a maximum possible loss, and
x = (size of loss) / (maximum possible loss).
Therefore this function is a valid exposure curve.
b) We are looking at the layer from $500,000 to $1,500,000.
x = 500 /2000 = 0.25, and x = 1500/2000 = 0.75.
$G(0.25) = (1 - 0.15^{0.25}) / (1 - 0.15) = 0.4443$.
$G(0.75) = (1 - 0.15^{0.75}) / (1 - 0.15) = 0.8929$.
The portion of total losses in this layer is: G(0.72) - G(0.25) = 0.8929 - 0.4443 = **0.4486**.
c) Assume for example, that instead the correct b is 0.10.
Then $G(0.75) - G(0.25) = (1 - 0.1^{0.75}) / (1 - 0.1) - (1 - 0.1^{0.25}) / (1 - 0.1) = 0.4272 < 0.4486$.
Thus the actuary would have <u>overestimated</u> the portion of total losses in the layer.
$G'(0) = -\ln(b) / (1 - b)$. $G'(1) = -b \ln b / (1-b)$.
The probability of a total loss is: G'(1) / G'(0) = b.
Thus the actuary would have <u>overestimated</u> the probability of a total loss.
<u>Comment</u>: See page 101 of Bernegger for part (a).
The given function is a member of the MBBEFD family of curves with gb =1 and g > 1;
see equation 3.3 in Bernegger. Thus this must be a valid exposure curve.
A graph of the given exposure curve:



G(d) is an increasing and concave function on the interval [0, 1].
In addition, G(0) = 0 and G(1) = 1. Thus this is a valid exposure curve.
$G(x) = (1 - 0.15^x) / 0.85 = \{1 - \exp[-x/\theta]\} / 0.85$, with $\theta = -1/\ln(0.15) = 0.527$.
Thus, G is proportional to an Exponential Distribution, and thus is an increasing and concave function. In fact, G(x) is an Exponential Distribution, censored from above at 1.

31. a) We are looking at the layer from \$100,000 to \$200,000.
x = 100 /2000 = 0.05, and x = 200/2000 = 0.1.
$G(0.1) - G(0.05) = (1 - 0.15^{0.1}) / (1 - 0.15) - (1 - 0.15^{0.05}) / (1 - 0.15) = 0.09683$.
b) Assume that the correct value of b is for example 0.1.
$G(0.1) - G(0.05) = (1 - 0.1^{0.1}) / (1 - 0.1) - (1 - 0.1^{0.05}) / (1 - 0.1) = 0.10769 > 0.09683$.
Thus the actuary would have underestimated the portion of total losses in the layer.
Comment: If the b parameter that the actuary selected was too high, whether the layer was overestimated or underestimated depends on what layer we look at. The exam question looked at a high layer, while here we looked at a low layer.
For a medium layer, the behavior with b can be complicated. For example, here is a graph of the portion of losses in the layer from \$500,000 to \$900,000 as a function of b:



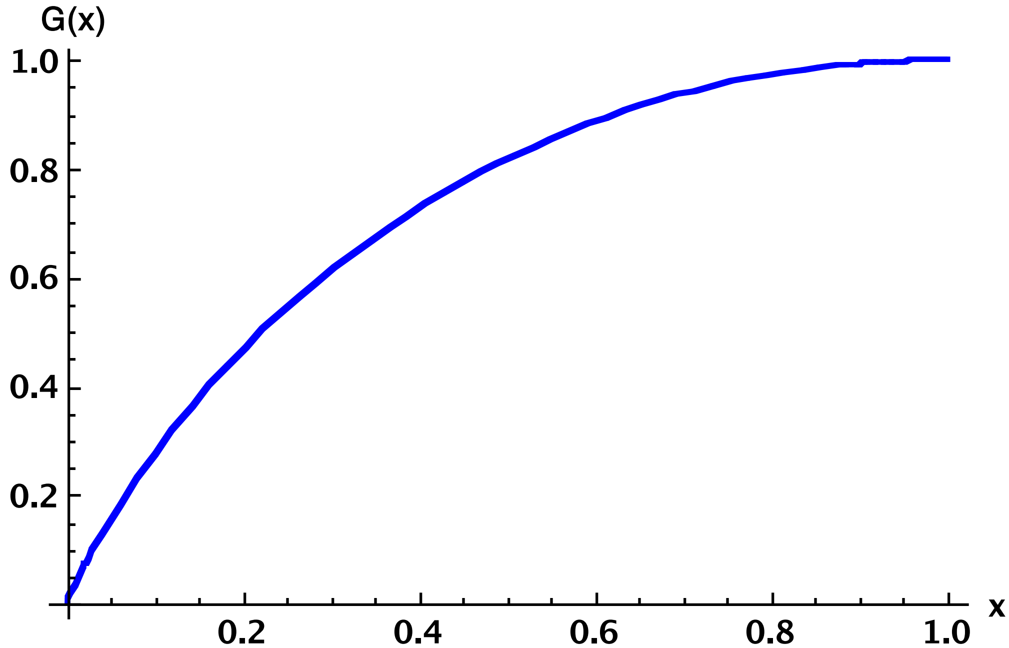
32. (a) The exposure curve is the loss elimination ratio.

$S(x) = 1 - x^{0.25}$. $E[X \wedge x] = \int_0^x S(t)\,dt = x - 0.8\,x^{1.25}$. $E[X \wedge 1] = 0.2$.

Thus $G(x) = E[X \wedge x] / E[X \wedge 1] = \mathbf{5x - 4x^{1.25}}$.

(b) We want the layer from \$500,000 to \$1,500,000.
$G(500,000 / 2,000,000) = G(0.25) = (5)(0.25) - (4)\,(0.25^{1.25}) = 0.5429$.
$G(1,500,000 / 2,000,000) = G(0.75) = (5)(0.75) - (4)\,(0.75^{1.25}) = 0.9582$.
Thus the percent of expected losses in the layer is: 0.9582 - 0.5429 = **0.4153**.

Comment: Note that for the given distribution function, $F(0) = 0$ and $F(1) = 1$.
A graph of the exposure curve, $G(x) = 5x - 4x^{1.25}$, $0 \le x \le 1$:



33. $0.65 = G(x) = \frac{1 - b^x}{1 - b} = (1 - 0.1^x) / 0.9. \Rightarrow x = \ln(0.415) / \ln(0.1) = 0.382$.

The reinsurer's maximum loss is: 50 million = M (1-x) = 0.618 M. ⇒ M = 80.9 million.
The cedant's maximum loss is: x M = 0.382 M = (0.382)(80.9) = **\$30.9 million**.

Comment: G(x) is the loss elimination ratio at x, where is x is as a percent of the maximum possible loss M. G(x) is the percent of pure risk premium retained by the cedant if the reinsurer covers the layer from x M to the maximum possible loss M.

34. (a) First estimate the total expected pure premium under the underlying business.
(In this case that is: (60%)(6000) = 3600.)
Then apportion the pure premium between the reinsurer and ceding company by using exposure curves.
The expected percent ceded will be: G(A + L) - G(L), where A is the retention and L is the limit.

(b) $G(x) = \dfrac{1 - b^x}{1 - b}$. $\qquad G'(x) = -\ln b\ b^x / (1-b)$.

$G'(0) = -\ln b / (1-b)$. $\qquad G'(1) = -\ln b\ b / (1-b)$.

0.03 = Probability of a total loss = G'(1)/G'(0) = b. ⇒ b = **0.03**.
Alternately, as shown in equation 3.3 in Bernegger, this is a special case of MBBEFD with bg = 1.
The probability of a total loss is 0.03. ⇒ g = 1/0.03. ⇒ b = **0.03**.
(c) The ceded premium divided by the total expected losses is: 2705 / 3600 = 75.14%.
Let the limit be L, then the ceded layer is from 150K to L + 150K.
Thus we have: 75.14% = G({L + 150K}/5000K) - G(150K/5000K)

$$= \frac{1 - 0.03^{(L + 150K)/5000K}}{1 - 0.03} - \frac{1 - 0.03^{(150K/5000K)}}{1 - 0.03} = \frac{1 - 0.03^{(L + 150K)/5000K}}{0.97} - 10.29\%.$$

$$\Rightarrow 85.43\% = \frac{1 - 0.03^{(L + 150K)/5000K}}{0.97}. \Rightarrow 0.03^{(L+150K)/5000K} = 0.1713.$$

⇒ (L + 150K)/5000K = ln(0.1713) / ln(0.03) = 0.5032. ⇒ L = **$2.366 million**.

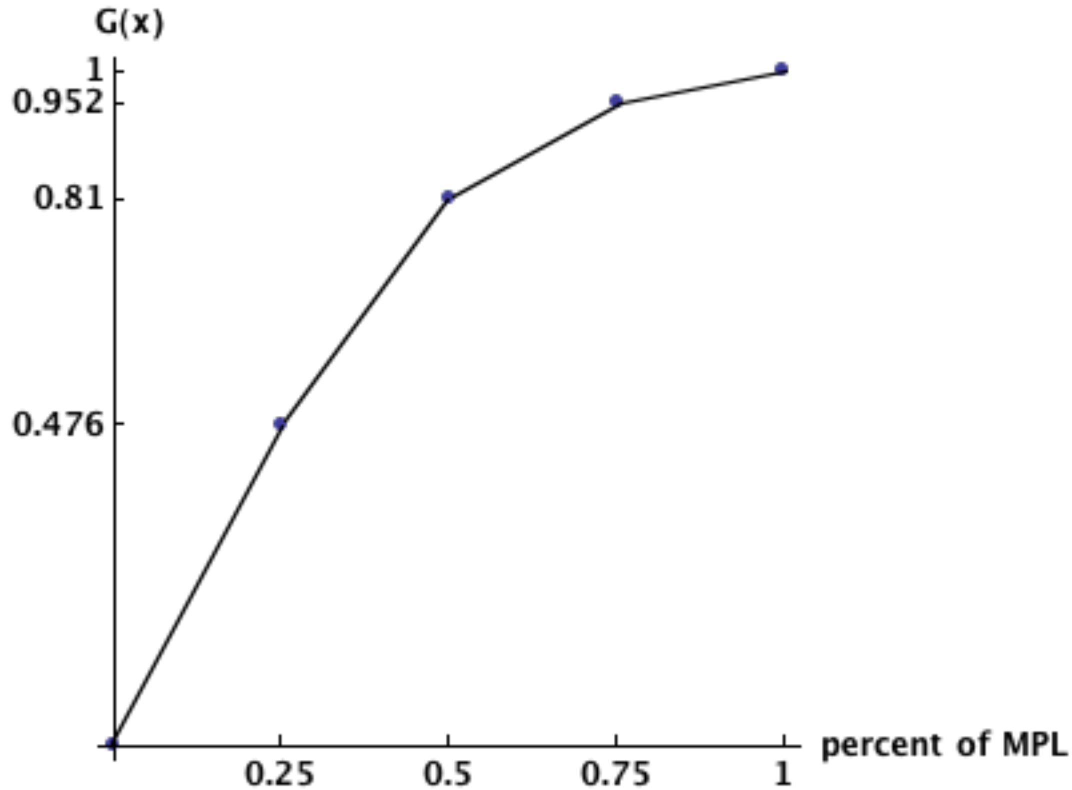
Comment: In part (a), "allocate gross premium for the non-proportional reinsurance treaty between the ceding company and the reinsurer" is language that is not used in the syllabus readings. Personally, it took me a while to figure out what the questioner was getting at.

35. The trended loss ratio during the experience period is: 40/100 = 40%.
I will assume this is the expected loss ratio in the future.
For the first interval of insured values, the middle is 2.5M.
Thus during the experience period, 1M is 1/2.5 = 40% of insurance to value.
During the experience period, the percent of cumulative losses in the layer from 0 to 1M is:
{22M + (10)(1M)} / 40M = 80%.
This more closely matches Swiss Re Y_4 exposure curve, G(40%) = 82%, so I will use Y_4.
For Y_4, c = 4. b = exp[3.1 - (0.15)(4)(1+4)] = 1.105. g = exp[(4){0.78 + (0.12)(4)}] = 154.47.
During the future treaty period, the second interval of insured value has a midpoint of 6M.
Thus the 4M xs 4M treaty covers from 4/6 = 2/3 to 1.

$$G(2/3) = \frac{\ln\left[\frac{(154.47-1)1.105 + (1 - (154.47)(1.105))\ 1.105^{2/3}}{1-1.105}\right]}{\ln[(154.47)(1.105)]}$$

= ln[112.2] / ln[170.7] = 0.918.
G(1) - G(2/3) = 1 - 91.8% = 8.2%.
The expected ceded losses are: (8.2%)(40%)(5M) = $164,000.
The ceded premium is: (1%)($30M) = $300,000.
Thus the expected ceded loss ratio is: 164/300 = **54.7%**.
Comment: Bernegger divides by the Maximum Possible Loss (MPL) in order to normalize losses; I have assumed for each property that the MPL is its insured value.
Using the midpoint of each interval of insured value is an approximation.
For example, for a property worth $1M, $1M is 100% of insurance to value; however, for a property worth $4M, $1M is only 25% of insurance to value. Thus $1M ranges from 25% to 100% of the insurance to value for these properties. Using for the whole set of such properties 1/2.5 = 40% of insurance to value is some sort of approximate average.
The same situation applies to the second interval of insured values.
You need to remember that c = 3 for Y_3 and c = 4 for Y_4.
For Y_3, G(2/3) = 86.2%.
One could interpolate between Y_3 and Y_4, giving more weight to Y_4.
I believe which curve is selected would often be based on the type of risks being covered and underwriting judgement, rather than one empirical exposure factor based on limited data, particularly when one is specifically going to make major changes to the book of business.
Y_3 is usually used for Commercial Lines (medium scale), and Y_4 is usually used for Industrial (not large scale) and Commercial Lines(large scale). Thus I am not sure why we are looking at these curves for use with homes, although these are high value homes.
Y_1 is usually used for Personal Lines. Y_2 is usually used for Commercial Lines (small scale).

36. (a) We want the limited expected values;
we only look at the 20% of the time there is a loss.
E[X; 25%] = (20%)(25%) / 20% = 25%.
E[X; 50%] = {(6%)(25%) + (14%)(50%)} / 20% = 42.5%.
E[X; 75%] = {(6%)(25%) + (8%)(50%) + (6%)(75%)} / 20% = 50%.
E[X] = {(6%)(25%) + (8%)(50%) + (4%)(75%) + (2%)(100%) } / 20% = 52.5%.
Exposure curve is a graph of loss elimination ratios.
G(0) = 0. G(25%) = E[X ; 25%] / E[X] = 25/52.5 = 0.476. G(50%) = 42.5/52.5 = 0.810.
G(75%) = 50/52.5 = 0.952. G(100%) = 1.
The Exposure Curve:



(b) g = 1 / Prob[MPL | there is a loss] = 1 / (2%/20%) = **10**.
μ = mean = 0.525.
Reading off the table, b = **0.0147**.
(c) We want the layer from 25% to 50% of the MPL.

$$\frac{E[X; 50\%] - E[X; 25\%]}{E[X]} = \frac{42.5\% - 25\%}{52.5\%} = \mathbf{1/3}.$$

Alternately, G(50%) - G(25%) = 0.810 - 0.476 = **0.334**.

<u>Comment</u>: $G'(0) = \frac{0.476 - 0}{0.25 - 0} = 1.904. \Rightarrow \mu = 1/G'(0) = 1/1.904 = 0.525.$

$G'(1) = \frac{1 - 0.952}{1 - 0.75} = 0.192. \Rightarrow p = G'(1) / G'(0) = 0.192 / 1.904 = 0.10. \Rightarrow g = 1/p = 10.$