To Buyers of Mahler's Guide to Advanced Ratemaking

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This study guide is split into two volumes: Sections 1 to 10, and Sections 11 and following.

In the electronic version use the bookmarks / table of contents in the Navigation Panel in order to help you find what you want.
You may find it helpful to print out selected portions, such as the Table of Contents.

Information in bold is more important to pass your exam. Information presented in italics (and subsections whose titles are in italics) should not be needed to directly answer exam questions. It is provided to aid the reader's overall understanding of the subject.

I have doubled underlined highly recommended questions to do on your first pass through the material, underlined recommended questions to do on your second pass, and starred additional questions to do on a third pass through the material.[1] No questions were labeled from the 2011 exam or later, in order to allow you to use them as practice exams.

Changes to the Exam 8 syllabus for Fall 2022:
Update the National Council on Compensation Insurance, Experience Rating Plan Manual.
At the time of writing, I do not have this updated version of the plan.
I assume it will have updated tables, but be otherwise similar.
Please check my webpage for further information.

Exam 8 will not contain multiple-choice questions.[2] All questions will require an essay response.

The CAS is not releasing exams starting with Fall 2020.

The CAS used computer based testing for the Fall 2020 Exam,
and expects to do so going forward. Be sure to check the CAS webpage for information.

My solutions to questions are intended to be model solutions.[3] Often they are more detailed and contain more explanation than would be needed in order to get full credit. This was done in order to give you a clearer and better understanding of the subject material.[4]

After you have done one of the more recent released exams, be sure to look closely at the CAS Examiner's Report. **See the sample solutions in the Examiner's Reports, and read the comments of the examiners.**

On numerical questions clearly label your final answer and show enough intermediate steps so someone can follow what you did. If there are a series of parallel calculations you can show just one of them. Of course, if writing down more helps you to do the problem, do so.

[1] Obviously feel free to do whatever questions you want. This is just a guide for those who find it helpful.

[2] I have removed the multiple choices from most past exam questions that had them.

[3] In some cases, I even quote a reading word for word. This does not mean you need to be able to do so!

[4] Sometimes much of what I say is directed at those students who did not answer a question correctly and need to learn more. In any case, you want to know as much as possible to help answer the question that will be on your exam, as opposed to whatever they happened to have asked on some past exam.

In the case of verbal questions, do not concentrate on grammar or complete sentences. Feel free to list or outline your ideas. The selected use of abbreviations can save some time.

In any case, remember that spending considerable additional time to increase 80% credit to 100% credit on a single question is usually not a good use of your limited exam time. You can come back later to a question, if you have the time.
As stated in the CAS Syllabus: "The model response to the typical essay question is brief, less than one-half of a written page. Be concise — candidates do not need to answer in complete sentences when a well-composed outline format is more appropriate. Candidates should not waste time on obscure details. They should show that they have learned the relevant material and that they understand it. They should state the obvious, if it is part of the answer."

Also read "The Importance of Adverbs on Exams," in which the Exam Committee notes the difference between: briefly discuss, discuss, and fully discuss.[5]
"Brief descriptions, discussions, etc., are worth 1/4 point.
(Unmodified) discussions or descriptions are worth 1/2 point.
Full descriptions or discussions are worth at least 1 point.
Please look carefully for these word choices and point values on all CAS upper-level exams. Most importantly, answer the question in accordance with the amount of information being asked."

The CAS is gradually moving towards an integrative testing framework.
Integrative Questions (IQs) will require candidates to understand multiple facets of the syllabus material and concepts in addressing complex business problems in a single exam question. IQs will differ from a typical exam question in three significant ways.

1. An IQ will be worth more points. One IQ could be worth 10-15% of the total exam.
2. Each IQ will require candidates to draw from multiple syllabus learning objectives in order to answer the question.
3. IQs will test at a higher average Bloom's Taxonomy level than a standard exam question.

To assist candidates with preparing to answer an IQ, t**he CAS released sample IQs and responses**.[6] It should be noted that while the samples were constructed in parallel with the IQ that will appear on the exam, they may not be structured in the same manner nor cover the same learning objectives as the exam question. It is advised that candidates use the samples to validate preparation and identify potential areas for improvement after completing the majority of their study, rather than using them during their initial study as one might use text book exercises.

Exam 8 featured one IQ on the Fall 2017 exam,
and two each on the Fall 2018 and Fall 2019 exams.

It is expected that Exams 7, 8, and 9 will continue to include IQs in future sittings, and the number of IQs that will appear on the exams will gradually increase over time. At the same time, there will be fewer exam questions overall to account for the presence of IQs in order to avoid any increase in the time length of the exam. There will be no change to the normal grading process, as described in the Syllabus, for IQs.

[5] http://www.casact.org/admissions/index.cfm?fa=adverbs
[6] See the CAS webpage for updated information.

In March 2022 the CAS added New Questions to the CBT Sample Exams 5-9.[7]
Sample questions 12 and 13 are from the unreleased Fall 2021 Exam 8.[8]
I have included them in my Sections 1 and 22. .

My study guide includes question written by me, and some by Sholom Feldblum.[9] In addition, the former exam questions are arranged in chronological order. The more recent exam questions are on average more similar to what you will be asked on your exam, than are less recent questions.

Note that In some cases, numerical values shown in one of my spreadsheets are unrounded, while the corresponding value in my text may be rounded.

It is important that you **do problems when learning a subject and then some more problems a few weeks later.**
As you get closer to the exam, the portion of time spent doing problems should increase.

There are two manners in which you should be doing problems. First you can do problems in order to learn the material. Take as long on each problem as you need to fully understand the concepts and the solution. Reread the relevant syllabus material. Carefully go over the solution to see if you really know what to do. Think about what would happen if one or more aspects of the question were revised. This manner of doing problems should be gradually replaced by the following manner as you get closer to the exam.

The second manner is to do a series of problems under exam conditions, with the items you will have when you take the exam. Take in advance a number of points to try based on the time available. For example, if you have an uninterrupted hour, then one might try 60 / 4 = 15 points of problems. Do problems as you would on an exam in any order, skipping some and coming back to some, until you run out of time. Leave time to double check your work.

It is important that you develop the skill of quickly and clearly writing down what you know. Many of you will benefit by giving some of your solutions to questions to someone else to “grade”.[10] They should give you feedback on whether they were able to follow what you did.[11] They should point out where you wrote more than was necessary or not enough.

Read the “Hints on Study and Exam Techniques” in the CAS Syllabus.

The CAS has posted a pdf on Bloom’s Taxonomy of question writing.
You might want to look at it.
http://www.casact.org/admissions/syllabus/Blooms-Taxonomy.pdf

[7] https://abe-prd-1.pvue2.com/st2/driver/startDelivery?sessionUUID=972271b1-7c06-4e8f-8a4b-499d4e047cd0
[8] Also in the Sample Questions are Fall 2019 questions 17 and 19, which are in my study guide.
[9] I thank Sholom Feldblum for the kind permission to use his material. Any mistakes are my responsibility.
[10] Someone else taking this exam or who has just passed this exam would be a good choice.
[11] On average you get less credit on essay questions when graded by someone else then when you self-grade.

I thank Sholom Feldblum for the kind permission to use his material.
I thank Roger Wilk for kindly providing me with his notes on Clark, and Grossi & Kunreuther.
Finally, thanks to the many past students who have helped me to improve these study guides.

Sold separately are my seminar style slides. They are electronic.

Feel free to send me any questions or suggestions: hmahler@mac.com
Please send me any suspected errors by Email.
(Please specify as carefully as possible the page and Exam number.)

I will post a list of errata on my webpage: www.howardmahler.com/Teaching

Preparing for a CAS Exam--what to do with hard material
by Dr. J. Eric Brosius, FCAS

The syllabus for a typical CAS exam includes both easy and hard material. Many students learn the easy material well, but adopt less-than-optimal strategies for learning the hard material. Some spend a lot of time trying to understand syllabus readings that are nearly incomprehensible. Others ignore the more difficult readings altogether. Neither approach is a good idea, not if you hope to pass! I will suggest a better way to approach these readings. Your goal in studying is not to understand the material in general but to be able to answer the questions. Do not study the syllabus readings in a vacuum; consider also what types of questions are likely to be asked. Each exam contains both easy problems and hard problems.

We can divide the problems into four categories based on the difficulty of the material and the difficulty of the problem, as follows:

	Easy Problems	Hard Problems
Hard Material	4	3
Easy Material	1	2

Box 1 contains easy problems on easy material. These are easy to answer; unfortunately, there are not enough of them!

Box 2 contains hard problems on easy material. You can prepare for these by practicing problems from old tests and other sources of sample problems.

Box 3 contains hard problems on hard material. Few students can afford to spend the time required to answer all of these. Fortunately, the Examination Committee does not ask many of these question: even if they understand the reading well enough to do so, there isn't much point in a question that no one can answer. Be prepared to skip Box 3 problems if necessary.

Box 4 contains easy problems on hard material. These problems can supply the extra points you need to change a "5" into a “6". They appear often, because the Examination Committee tends to ask easy questions about hard readings. When a reading is technically difficult, and especially if it was recently added to the syllabus, even the simplest question poses a challenge. Study these readings with an eye to answering the obvious questions. It is a shame not to get points for a question that could have been answered if only you had read the first paragraph of the reading.

Plan for your exam in such a way that you **focus on Box 2 and Box 4**. Prepare for Box 2 questions by studying the easy material in detail, and by doing many sample problems. Prepare for Box 4 questions by outlining the high points of the material, and by trying to guess, alone or with other students, what questions on this material might appear on the exam.

Use whatever order to go through the material that works best for you.
Here is a schedule that may work for some people.
Modify it to meet your own needs.
In any case, leave plenty of time to go back and review material.

A 14 week Study Schedule for Exam 8:

Week	Sections of Study Guide
1	1-2
2-3	3
4	4-6
5-6	7
7	8-9
8	10-12
9-10	13-14
11	15-16
12	17-18
13	19-20
14	21-24

Since 2011, the points on exam questions are similar to the present. Going back a few more years further in time, a 5 point exam question might only be worth 3 points today.[12]

Exam 8	Points	Number of Questions	Integrated Questions	Average % of Exam per Integrated Question
2011	59	25		
2012	54.75	23		
2013	57.5	25		
2014	60.25	25		
2015	59.5	23		
2016	53.25	21		
2017	53.75	20	1	15.8%
2018	52	17	2	17.8%
2019	52.5	19	2	10.5%

The CAS has stopped releasing pass marks:

Exam 8	Pass Mark	Percent of Available Points	95th Percentile	75th Percentile
2011	43.75	74.15%	47.38	43.00
2012	37.75	68.95%	44.25	39.75
2013	40.75	70.87%	47.50	43.63
2014	37.50	62.24%	44.50	40.63
2015	40.75	68.49%	48.50	43.13
2016	37.25	69.95%	42.88	38.88
2017	37.5	69.77%	43.00	39.50
2018	33.75	64.91%	47.38	43.00
2019	37	70.48%	42.88	38.50

[12] For my problems, it depends on when I wrote them.
My older ones are probably more like the older exam questions as far as points go.
I am sorry that my study guides are not more consistent with respect to “points”.
The CAS stopped releasing exams with the Fall 2020 exam.

Exam 8	Exams Taken	Passed	Raw Pass Ratio	Effective Pass Ratio
2011	418	93	22.2%	23.9%
2012	519	218	42.0%	43.7%
2013	592	283	47.8%	49.3%
2014	729	350	48.0%	50.2%
2015	771	313	40.60%	42.18%
2016	791	301	38.05%	40.13%
2017	945	376	39.8%	41.7%
2018	953	314	32.9%	35.1%
2019	1080	376	34.8%	37.0%
2020	228	86	37.7%	41.1%
S2021	174	66	37.9%	42.0%
F2021	900	316	35.1%	38.5%
2022	870	329	37.8%	39.8%

One measure of the difficulty of an exam is the ratio of the 75th percentile to the available points:

Exam 8	Points	75th Percentile	Ratio
2011	59	43.00	72.9%
2012	54.75	39.75	72.6%
2013	57.5	43.63	75.9%
2014	60.25	40.63	67.4%
2015	59.5	43.13	72.5%
2016	53.25	38.88	73.0%
2017	53.75	39.50	73.5%
2018	52	35.00	67.3%
2019	52.5	38.50	73.3%

The lower the ratio of the 75th percentile to the available points, the harder the exam.

Mahler's Guide to

Advanced Ratemaking

CAS Exam 8

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Study Aid 2023-**8**

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Mahler's Guide to Advanced Ratemaking

Information in bold or sections whose title is in bold are more important for passing the exam. Larger bold type indicates it is extremely important. Information presented in italics (including subsections whose titles are in italics) should rarely be needed to directly answer exam questions and should be skipped on first reading. It is provided to aid the reader's overall understanding of the subject, and to be useful in practical applications.

I have doubled underlined highly recommended questions to do on your first pass through the material, underlined recommended questions to do on your second pass, and starred additional questions to do on a third pass through the material.[1] No questions were labeled from the 2011 exam or later, in order to allow you to use them as practice exams.

Solutions to problems are at the end of each section.[2]

[1] Obviously feel free to do whatever questions you want. This is just a guide for those who find it helpful.

[2] Note that problems include both some written by me and some from past exams. The latter are copyright by the Casualty Actuarial Society and are reproduced here solely to aid students in studying for exams. The solutions and comments are solely the responsibility of the author; the CAS bears no responsibility for their accuracy. While some of the comments may seem critical of certain questions, this is intended solely to aid you in studying and in no way is intended as a criticism of the many volunteers who work extremely long and hard to produce quality exams. There are also some past exam questions copyright by the Society of Actuaries.

Volume	Section #	Pages	Section Name
one	1	9-110	Mahler, An Example of Credibility and Shifting Risk Parameters
one	2	111-213	Bailey & Simon, Credibility of a Single Car
one	3	214-589	Goldburd, Khare and Tevet, Generalized Linear Models
one	4	590-621	ASOP 12: Risk Classification
one	5	622-713	Robertson, NCCI's 2007 Hazard Group Mapping
one	6	714-797	Couret & Venter, Class Frequency Vectors
one	7	798-1140	Clark, Reinsurance Pricing
one	8	1141-1226	Bernegger, Exposure Curves
one	9	1227-1407	Grossi & Kunreuther, Catastrophes
one	10	1408-1530	Experience Rating
two	11	1531-1613	NCCI Experience Rating Plan
two	12	1614-1717	ISO Experience Rating Plan
two	13	1718-1818	Frequency and Loss Distributions
two	14	1819-2144	Bahnemann, Distributions for Actuaries
two	15	2145-2269	Lee Diagrams, Loss Distributions
two	16	2270-2409	Retrospective Rating
two	17	2410-2507	Table M Construction
two	18	2508-2599	NCCI Retrospective Rating
two	19	2600-2691	Table L
two	20	2692-2790	Lee Diagrams, Retrospective Rating
two	21	2791-2820	Limited Table M
two	22	2821-2852	Other Loss Sensitive Plans
two	23	2853-2970	Pricing Large Dollar Deductible Policies
two	24	2971-2988	Concluding Remarks, Individual Risk Rating

For Fall 2020, the CAS went back to computer based testing.
The CAS stopped releasing exams, starting with the 2020 Exam.

CAS Sample Q.11 (from the Fall 2021 Exam 8) is in my Section 1.
CAS Sample Q.6 (from the Fall 2021 Exam 8) is in my Section 22.

Past Exam Questions by Section

Sec.		1995 Exam 9	1996 Exam 9	1997 Exam 9	1998 Exam 9
1	Mahler, Shifting Risk Parameters	10, 31	20	44, 45, 46	13, 14, 25
2	Bailey & Simon, Cred. Single Car	6, 30, 32	50	19	26
3	Goldburd, Khare and Tevet, GLMs				
4	ASOP 12: Risk Classification			18	15, 22
5	Robertson, Hazard Group Mapping				
6	Couret & Venter, Class Freq.				
7	Clark, Reinsurance Pricing				
8	Bernegger, Exposure Curves				
9	Grossi & Kunreuther, Catastrophes				
10	Experience Rating	20, 40, 42	4, 27, 28c&d	31a, 32	18, 37b, 38, 39
11	NCCI Experience Rating Plan	16, 41	24, 25	10, 34	17, 20, 36
12	ISO Experience Rating Plan	17	1, 21, 22, 23	9, 33	41
13	Frequency and Loss Distributions				
14	Bahnemann, Distrib. for Actuaries	11, 33, 35	36, 38, 41, 42	13, 36a, 40a	30a, 31, 33, 34
15	Lee Diagrams, Loss Distributions		39	37	29
16	Retrospective Rating	21, 22, 24, 44, 46, 47	29, 31, 32, 34	1, 27	4, 44c, 42, 47
17	Table M Construction	45	10	22, 23	
18	NCCI Retro. Rating				46
19	Table L	25	30, 35		43
20	Lee Diagrams, Retro. Rating	50		4, 26	
21	Limited Table M				
22	Other Loss Sensitive Plans				
23	Pricing LDD Policies				
24	Conclud. Remarks, Indiv. Risk Rat.				

Some questions are based on more than one syllabus reading, particularly on recent exams.[3]
In any case, sometimes it is unclear what is the best section in which to put a question.
In those cases, I have made one of the possible reasonable choices of where to put a question.

[3] Integrated questions involve several different syllabus readings.

Sec.		1999 Exam 9	2000 Exam 9	2001 Exam 9	2002 Exam 9
1	Mahler, Shifting Risk Parameters	48	34	1	
2	Bailey & Simon, Cred. Single Car	1	32	2, 22	47
3	Goldburd, Khare and Tevet, GLMs				
4	ASOP 12: Risk Classification	2, 43b			48
5	Robertson, Hazard Group Mapping				
6	Couret & Venter, Class Freq.				
7	Clark, Reinsurance Pricing				
8	Bernegger, Exposure Curves				
9	Grossi & Kunreuther, Catastrophes				
10	Experience Rating	12, 13, 31	1, 4, 40		
11	NCCI Experience Rating Plan	28	17, 42	25	33
12	ISO Experience Rating Plan	30	2	27	11, 12, 34
13	Frequency and Loss Distributions				
14	Bahnemann, Distrib. for Actuaries	35, 38, 40, 41	39	11, 35, 37c	41, 42
15	Lee Diagrams, Loss Distributions	34, 39	37		43
16	Retrospective Rating	5, 6, 9, 21, 22, 23, 25	5, 6, 44	8, 9, 10, 31, 32, 34	14, 15, 16, 35, 40
17	Table M Construction		19, 48	30	36
18	NCCI Retro. Rating				
19	Table L	26	45		38, 39
20	Lee Diagrams, Retro. Rating				17
21	Limited Table M				
22	Other Loss Sensitive Plans				
23	Pricing LDD Policies	42	38		1
24	Conclud. Remarks, Indiv. Risk Rat.				

Sec.		2003 Exam 9	2004 Exam 9	2005 Exam 9	2006 Exam 9
1	Mahler, Shifting Risk Parameters	21	3	2	
2	Bailey & Simon, Cred. Single Car	22	2	3	2
3	Goldburd, Khare and Tevet, GLMs	25			5
4	ASOP 12: Risk Classification		23		
5	Robertson, Hazard Group Mapping			9	
6	Couret & Venter, Class Freq.				
7	Clark, Reinsurance Pricing				
8	Bernegger, Exposure Curves				
9	Grossi & Kunreuther, Catastrophes				
10	Experience Rating	2, 6, 26, 28	15, 16, 39	26	23, 27
11	NCCI Experience Rating Plan	27		24, 27	24
12	ISO Experience Rating Plan	3, 4, 5	14, 41	28	28
13	Frequency and Loss Distributions				
14	Bahnemann, Distrib. for Actuaries	13, 37, 38, 43	5, 6, 19 25, 26	6, 7, 10 23a, 35	6, 8
15	Lee Diagrams, Loss Distributions				
16	Retrospective Rating	7, 10, 31, 32, 33	18, 20, 45, 47	31, 32	30, 32, 35
17	Table M Construction		43	8	9
18	NCCI Retro. Rating				
19	Table L	30	44		7
20	Lee Diagrams, Retro. Rating	8, 9, 29	4, 17	33	29, 34
21	Limited Table M				
22	Other Loss Sensitive Plans				
23	Pricing LDD Policies	35	46, 48	34, 36	31, 33, 36
24	Conclud. Remarks, Indiv. Risk Rat.				

Sec.		2007 Exam 9	2008 Exam 9	2009 Exam 9	2010 Exam 9
1	Mahler, Shifting Risk Parameters	6			
2	Bailey & Simon, Cred. Single Car	2	5	4	5
3	Goldburd, Khare and Tevet, GLMs	4a	3	3	3
4	ASOP 12: Risk Classification				
5	Robertson, Hazard Group Mapping				
6	Couret & Venter, Class Freq.				
7	Clark, Reinsurance Pricing				
8	Bernegger, Exposure Curves				
9	Grossi & Kunreuther, Catastrophes				
10	Experience Rating	26	23	20	23
11	NCCI Experience Rating Plan	25, 28	25	21	20
12	ISO Experience Rating Plan	27	24	22	21
13	Frequency and Loss Distributions				
14	Bahnemann, Distrib. for Actuaries	7, 8, 10	26, 27	17, 18, 26	17, 26
15	Lee Diagrams, Loss Distributions			24	
16	Retrospective Rating	32, 35	36	28, 30, 31	27, 29
17	Table M Construction	30, 34	28		
18	NCCI Retro. Rating				
19	Table L		32, 33	32	
20	Lee Diagrams, Retro. Rating	31	29		25, 31
21	Limited Table M				
22	Other Loss Sensitive Plans				
23	Pricing LDD Policies	33, 36	30, 31	29a	28
24	Conclud. Remarks, Indiv. Risk Rat.			27	24

Sec.		2011 Exam 8	2012 Exam 8	2013 Exam 8	2014 Exam 8
1	Mahler, Shifting Risk Parameters		3		
2	Bailey & Simon, Cred. Single Car	1	6		5
3	Goldburd, Khare and Tevet, GLMs	*3*	2, 4	2	3
4	ASOP 12: Risk Classification				
5	Robertson, Hazard Group Mapping	4	1	4	2
6	Couret & Venter, Class Freq.	2	5	3	1, 4
7	Clark, Reinsurance Pricing	7, 8	7, 10	21, 23, 25	20, 21, 22 23, 25
8	Bernegger, Exposure Curves	9	8	20, 22	
9	Grossi & Kunreuther, Catastrophes	5, 6	9	24	24
10	Experience Rating	15, 16b&c	11, 16a&c	9, 10b	9, 11
11	NCCI Experience Rating Plan	12	13		10
12	ISO Experience Rating Plan	14	14	8	8
13	Frequency and Loss Distributions				
14	Bahnemann, Distrib. for Actuaries	10, 17	15	6	7
15	Lee Diagrams, Loss Distributions	11	22		6
16	Retrospective Rating	20, 21, 25	19, 23	14	17
17	Table M Construction			12	13
18	NCCI Retro. Rating				
19	Table L		18	13	
20	Lee Diagrams, Retro. Rating	22	21	15	12, 18
21	Limited Table M				
22	Other Loss Sensitive Plans				
23	Pricing LDD Policies	18, 19	20	16, 19	16, 19
24	Conclud. Remarks, Indiv. Risk Rat.	23			

Added for the 2011 Exam: Bernegger, Robertson, Couret & Venter, Grossi & Kunreuther.
Clark Reinsurance Pricing was on Exam 6 prior to 2011.

For the 2016 exam, Goldburd, M.; Khare, A.; and Tevet, D., “Generalized Linear Models for Insurance Rating,” replaced Anderson, D.; Feldblum, S; Modlin, C; Schirmacher, D.; Schirmacher, E.; and Thandi, N., “A Practitioner’s Guide to Generalized Linear Models”

Sec.		2015	2016	2017	2018	2019
		Exam 8	Exam 8	Exam 8	Exam 8	Exam 8
1	Mahler, Shifting Risk Parameters	4				
2	Bailey & Simon, Cred. Single Car	1	1	3	3	3
3	Goldburd, Khare and Tevet, GLMs	3	4, 5, 6, 7	4, 5, 6	5, 6, 7	2, 5, 6
4	ASOP 12: Risk Classification		3		4	
5	Robertson, Hazard Group Mapping	6	2	2		4
6	Couret & Venter, Class Freq.	5				1
7	Clark, Reinsurance Pricing	21, 23	20	19	15	17, 18
8	Bernegger, Exposure Curves	20	21	18		
9	Grossi & Kunreuther, Catastrophes	22	18, 19	20	16, 17	19
10	Experience Rating	10, 11, 12	11	11	9, 10	9, 10, 11
11	NCCI Experience Rating Plan		9, 10			
12	ISO Experience Rating Plan	9		9, 10	11	
13	Frequency and Loss Distributions				1*	
14	Bahnemann, Distrib. for Actuaries	8a		7, 8, 14	8, 13	13
15	Lee Diagrams, Loss Distributions	7		12		
16	Retrospective Rating	15, 16, 17		13, 15, 17	14	14, 15*
17	Table M Construction		12	16		7*, 12
18	NCCI Retro. Rating					
19	Table L		14			
20	Lee Diagrams, Retro. Rating		13			
21	Limited Table M					16
22	Other Loss Sensitive Plans			1*		
23	Pricing LDD Policies	13, 14, 18, 19	15, 16		2*, 12	8
24	Conclud. Remarks, Indiv. Risk Rat.					

ASOP No. 12 Risk Classification was added to the syllabus for 2017.
It replaced American Academy of Actuaries "Risk Classification Statement of Principles."
For the 2017 exam, many previous readings were replaced by:
a CAS Study Note "Individual Risk Rating," by Fisher, McTaggart, Petker, and Pettingell,
and a CAS Monograph "Loss Distributions for Actuaries," by Bahnemann.
For the 2020 Exam, the NCCI Retro Manual was replaced by the NCCI Circular CIF-2018-28.

Integrated questions (which cover material in more than one section) are marked with a star.[4]

The CAS stopped releasing exams, starting with the 2020 Exam.

[4] The 2017 Exam 8 Sample Integrative Question is in Section 23.

Section 3, Generalized Linear Models, Goldburd, Khare, and Tevet[1]

Generalized Linear Models are widely used by actuaries in ratemaking, loss reserving, etc.

GLMs can be thought of as a generalization of multiple linear regressions.
However, **the distribution of random errors need not be Normal**.
Common distributions for the errors are:
Normal, Poisson, Gamma, Binomial, Negative Binomial, Inverse Gaussian, and Tweedie.

Also **there is a link function that connects the linear combination of variables and the thing to be modeled.**
Common link functions are: identity, inverse, logarithmic, logit, and inverse square.
In a linear model, the link function is equal to the identity function.
In a multiplicative model, the link function is logarithmic; this is analogous to an Exponential regression.

Generalized Linear Models are fit via maximum likelihood.

Our goal in modeling is to find the right balance where we pick up as much of the systematic effects (called the signal) as possible and as little of the randomness in the data (called the noise).

Based on the syllabus reading, I do not expect you to be asked to fit a model. Rather you should concentrate on how to set up a GLM, choose between different models, and how to interpret computer output.

Therefore, do not get bogged down in the mathematical details of some of the examples I give, which are provided for those who find that concrete examples help them to learn the material.

This CAS Study Note also discusses some things that apply to most modeling and actuarial work, rather than just to GLMs.

[1] Generalized Linear Models for Insurance Rating, by Mark Goldburd, Anand Khare, and Dan Tevet,
CAS monograph.
Only Chapter 1 to 9 are on the syllabus.
A previous edition was added to the syllabus for 2016; this updated edition was added for 2020.
The updated edition has added: Section 5.4.5 Natural Cubic Splines,
Section 6.3.2 Working Residuals (and associated Appendix),
Chapter 8 Model Documentation. (The old Chapter 8 is the new Chapter 9.)
Other small additions were made here and there in the new edition.
You might also find it useful to glance at “Predictive Models, A Practical Guide for Practitioners and Regulators”,
by Don Closter and Caryn Carmean, a short 2019 CAS White Paper, not on the syllabus.

Types of Variables:

Variables can be continuous: size of loss, height, weight, Body Mass Index (BMI), etc.

Variables can be discrete: number of children, number of claims in the last three years, etc.

Variables can be categorical; there are a discrete number of categories.
The different possible values that a categorical variable can take on are called its levels.

In the case of nominal variables, the categories do not have a natural order.
For example, type of vehicle: sedan, SUV, truck, van.

Sometimes however, the categories have a natural order; such variables are called ordinal.
For example injuries may be categorized as: minor, serious, catastrophic, and fatal.
This also occurs when a continuous variable is grouped into categories.

Additive and Multiplicative Models:

When one uses the identity function, the model is additive:
$\mu = \beta_0 + \beta_1 x_1 + ... + \beta_p x_p$.
This is analogous to a linear regression.

For example, $\mu = 100 + 5x_1 - 3x_2$.
Each increase of 1 in x_1 results in an increase of 5 in m.
Each increase of 1 in x_2 results in an decrease of 3 in m.

When one uses the log link function, the model is multiplicative:
$\ln[\mu] = \beta_0 + \beta_1 x_1 + ... + \beta_p x_p. \Leftrightarrow \mu = \exp[\beta_0 + \beta_1 x_1 + ... + \beta_p x_p]$.
This is analogous to an exponential regression.

For example, $\mu = \exp[5 + 0.2x_1 - 0.1x_2]$.
Each increase of 1 in x_1 results in μ being multiplied by $e^{0.2} = 1.221$.
Each increase of 1 in x_2 results in μ being multiplied by $e^{-0.1} = 0.905$.

Advantages of Multiplicative Rating Structures:[2]

1. **A multiplicative plan guarantees positive premium**.
Having additive terms in a model can result in negative premiums, which doesn't make sense; you may have to implement clunky patches like minimum premium rules.

2. **A multiplicative model has more intuitive appeal**.

"It doesn't make much sense to say that having a violation should increase your auto premium by $500, regardless of whether your base premium is $1,000 or $10,000. Rather it makes more sense to say that the surcharge for having a violation is 10%."[3] [4]

"For these and other reasons, **log link models, which produce multiplicative structures, are usually the most natural model for insurance risk**."

Nevertheless, sometimes a multiplicative model (model using a log link function) does not do a good job of modeling the data, while a different link function does a better job. This is an empirical issue. Most factors in insurance rating algorithms are multiplicative, however it is not uncommon to also have additive elements as well.[5]

Even if one uses a log link function, when interaction terms are included in a model, the structure of the model will no longer have all of the nice features of a multiplicative model.
For example: $\mu = \exp[5 + 0.2x_1 - 0.1x_2 + 0.03x_1x_2]$.
Now the effect of a change in x_1 depends on the value of x_2, while the effect of a change in x_2 depends on the value of x_1. For example, in private passenger auto insurance, the effect on expected pure premiums of gender varies by age.

Also keep in mind that for a binary or binomial target variable, for example whether or not a policy is renewed, a logit link function is commonly used as will be discussed.

[2] See page 5 of Generalized Linear Models for Insurance Rating.
[3] This is an empirical question. For example, a more complicated surcharge such as $100 plus 10% of base premium might be a better prediction of the extra future expected costs.
[4] It would be extremely unusual to pay $10,000 or more as a base premium for private passenger automobile. Perhaps they are referring to commercial automobile. In any case, the $10,000 is just for illustrative purposes.
[5] Chapter 2 of "Basic Ratemaking" by Werner and Modlin has some examples.

Other Uses of GLMs:[6]

While GLMs are commonly used for classification ratemaking, the benefits of GLMs are not restricted to the application of pricing.
The following are a few of the other applications for which insurance companies are using GLMs:

- Practitioners are using GLMs to reduce a variety of risk variables into one score. This has obvious application in regards to creating underwriting tiers, credit scores, fire protection scores, vehicle symbols, etc.

- Many companies have begun to perform elasticity modeling. By building elasticity models for new and renewal business, companies can predict the impact of various actions on market share. A few companies are already linking the profitability and elasticity models to find the optimal pricing decision.

- Claims handlers are starting to see the advantages of GLMs and are using them to help set more accurate reserves and to provide early identification of claims that may be fraudulent or are most likely to end up in a lawsuit.

- Competitive analysis units are using GLMs to reverse-engineer competitors' rates given a large sample of rating quotes.

[6] Quoted from “GLM Basic Modeling: Avoiding Common Pitfalls,” by Geoff Werner and Serhat Guven, CAS Forum Winter 2007, not on the syllabus.

Common Link Functions:

$g(\mu) = \Sigma\beta_i x_i$. $\Leftrightarrow \mu = g^{-1}(\Sigma\beta_i x_i)$.

The x_i are the predictor or explanatory variables.

The β_i are the coefficients, which are to be fit.

$\beta x = \Sigma\beta_i x_i$, is the linear predictor.

g is the link function, whose form needs to be specified.

The link function must satisfy the condition that it be differentiable and monotonic (either strictly increasing or strictly decreasing). Common link functions to use include:

Identity	$g(\mu) = \mu$	$g^{-1}(y) = y$	$\mu = \beta x$
Log	$g(\mu) = \ln(\mu)$	$g^{-1}(y) = e^y$	$\mu = e^{\beta x}$
Logit	$g(\mu) = \ln[\mu/(1 - \mu)]$	$g^{-1}(y) = \dfrac{e^y}{e^y + 1}$	$\mu = \dfrac{e^{\beta x}}{e^{\beta x} + 1}$
Reciprocal	$g(\mu) = 1/\mu$	$g^{-1}(y) = 1/y$	$\mu = 1 / (\beta x)$

With more than one variable, the use of the log link function results in a familiar multiplicative model for classification relativities.

One can also use other powers as a link function, such as $g(\mu) = 1/\mu^2$ or $g(\mu) = \sqrt{\mu}$.

Let p be the probability of policy renewal. Then $0 < p < 1$.
Thus, $0 < p / (1 - p) < \infty$.
Applying the logit link function, $-\infty < \ln[p / (1 - p)] < \infty$.
So we have converted the domain from 0 to 1 to a range of minus infinity to infinity.

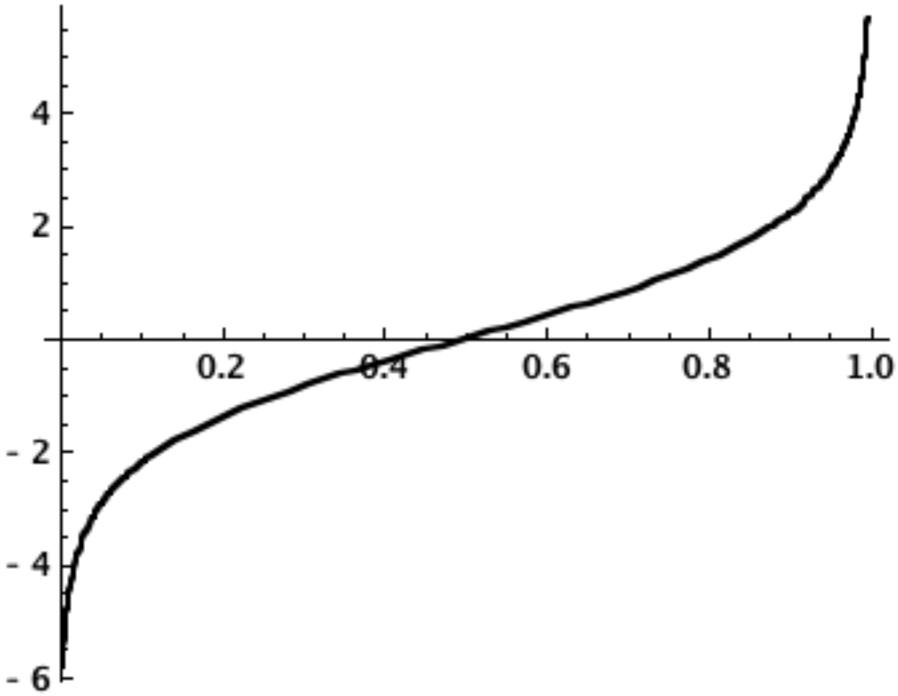
The inverse of the logit link function, $\dfrac{e^y}{e^y + 1}$, converts the interval from minus infinity to infinity to the interval from zero to one, which would be appropriate for probabilities.[7]

Exercise: $\mu = \dfrac{e^{\beta x}}{e^{\beta x} + 1}$. Determine μ for $\beta x = -2$, $\beta x = 0$, and $\beta x = 2$.

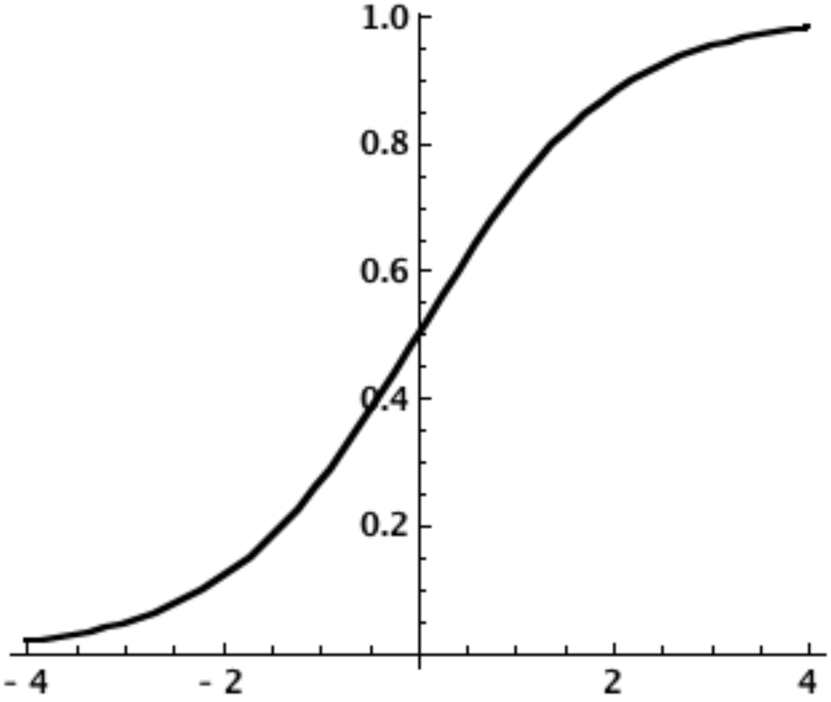
[Solution: $e^{-2} / (e^{-2} + 1) = 0.119$. $e^0 / (e^0 + 1) = 0.5$. $e^2 / (e^2 + 1) = 0.881$.
Comment: These all make sense as probabilities.]

[7] Note that $F(x) = e^x / (e^x + 1)$, $-\infty < x < \infty$ is the logistic function.

Here is a graph of the logit link function, $\ln(\frac{x}{1 - x})$, for 0 < x < 1, with range -∞ to ∞:



Here is a graph of the inverse of the logit link function, the logistic function: $e^x / (e^x + 1)$, $-\infty < x < \infty$, with range 0 to 1:



It is common to pick the form of the variable X, to be a member of an exponential family.
In that case, there are corresponding "canonical link functions".[8]

Canonical Link Functions:

Distribution Form	Canonical Link Function
Normal	Identity
Poisson	Log: ln(µ)
Gamma	Reciprocal: 1/µ
Binomial	Logit: ln[µ/(1-µ)]
Inverse Gaussian	$1/\mu^2$

Using the canonical link function makes the estimate from the GLM unbiased.

The Normal Distribution is used for example in ordinary linear regression.
The Poisson Distribution could be used to model claim frequencies or claim counts.
The Gamma Distribution could be used to model claim severities.[9]
The Binomial Distribution could be used to model probability of policy renewal.[10]

[8] While these choices result in some nice mathematical properties, they are not required.

[9] If used to model claim severities, one could use the log link function, ln(µ).

[10] The use of the logit link function with the Binomial or special case Bernoulli is the idea behind logistic regression.

Structure of Generalized Linear Models:

One can state the assumptions of a Generalized Linear Model as:

1. Random component: Each component of Y, the target variable is independent and is from one of the exponential family of distributions.[11]

2. Systematic component:
The p explanatory variables are combined to give the linear predictor X β.

3. Link function: The relationship between the random and systematic components is specified via a link function, g, that is differentiable and monotonic such that:

$$\mathbf{E[Y] = \mu = g^{-1}(X\ \beta). \Leftrightarrow X\ \beta = g(\mu).}$$

The target variable, also called the dependent variable,Y, is the thing being modeled; it may be: frequency, severity, pure premiums, loss ratios, or something like the probability of policy renewal.

The predictor variables, also called response variables or independent variables, x's, the things being used as inputs to the model, can be things like: age, gender, amount of insurance, etc.

The linear predictor has an intercept β_0 plus p slopes: $\eta = \beta_0 + \beta_1 x_1 + ... + \beta_p x_p$.
$\eta = g(\mu)$.

Several different models may be fit to the same data, with one or more of the above features differing. Then the models would be compared using the output diagnostics, in order to determine the best model to use for the purpose.[12]

[11] Y is a vector; "each component" of Y refers to the elements of that vector.
[12] Similar diagnostics are available as for a multiple linear regression.

A One Dimensional Example of Generalized Linear Models:[13]

Let us assume a set of three observations: (1, 1), (2, 2), (3, 9).

The predictor variable x takes on the values 1, 2, and 3 for the observations.[14]
The target variable Y takes on the values 1, 2 and 9 for the observations.

In a generalized linear model, Y will have some distributional form. The mean of the distribution will vary with x. However, any other parameters will be constant.

For now let us assume the identity link function, $g(\mu) = \mu$, so that $\mu = \Sigma\beta_i x_i = \beta_0 + \beta_1$ x.[15]
Thus for now we are fitting a straight line. In general, the identity link function leads to a linear model.

Assume that Y is Poisson, with mean μ.[16]
$\mu = \beta_0 + \beta_1$ x.

For the Poisson Distribution as per Loss Models, $f(y) = e^{-\lambda} \lambda^y / y!$.
$\ln f(y) = -\lambda + y\ln(\lambda) - \ln(y!) = -\mu + y\ln(\mu) - \ln(y!)$.

The loglikelihood is the sum of the contributions from the three observations:
$-(\beta_0 + \beta_1) - (\beta_0 + 2\beta_1) - (\beta_0 + 3\beta_1) + \ln(\beta_0 + \beta_1) + 2\ln(\beta_0 + 2\beta_1) + 9\ln(\beta_0 + 3\beta_1)$
$- \ln(1) - \ln(2) - \ln(9!)$.

To maximize the loglikelihood, we set its partial derivatives equal to zero.
Setting the partial derivative with respect to β_0 equal to zero:
$0 = -3 + 1/(\beta_0 + \beta_1) + 2/(\beta_0 + 2\beta_1) + 9/(\beta_0 + 3\beta_1)$.
Setting the partial derivative with respect to β_1 equal to zero:
$0 = -6 + 1/(\beta_0 + \beta_1) + 4/(\beta_0 + 2\beta_1) + 27/(\beta_0 + 3\beta_1)$.

Solving these two equations in two unknowns: $\beta_0 = -12/5 = -2.4$ and $\beta_1 = 16/5 = 3.2$.[17]
$\mu = -2.4 + 3.2x$. For x = 1, $\mu = 0.8$. For x = 2, $\mu = 4.0$. For x = 3, $\mu = 7.2$.[18]

[13] I do not expect you to have to go into this level of detail on your exam.
See page 15 of "A Practitioners Guide to Generalized Linear Models," by Duncan Anderson, Sholom Feldblum, Claudine Modlin, Dora Schirmacher, Ernesto Schirmacher and Neeza Thandi, in the 2004 CAS Discussion Paper Program, not on the syllabus of this exam.

[14] It is not clear in this example whether x can take on values other than 1, 2 and 3. These may be the only possible values, or they might be the three values for which we happen to have had an observation. In practical applications, when x is discrete, we would expect to have many observations for each value of x.

[15] I have treated x_0 as the constant 1 and x_1 as the predictor variable x.

[16] In the case of a Poisson, there are no additional parameters beyond the mean.

[17] I used a computer to solve these two equations. One can confirm that these values satisfy these equations.

[18] This differs from what would be obtained if one assumed Y was Normal rather than Poisson.

This model should be interpreted as follows. For a given value of x, Y is Poisson Distributed with mean = -2.4 + 3.2x. For example, for x = 3, the mean = 7.2. However, due to random fluctuation, for x = 3 we will observe values of Y varying around the expected value of 7.2.[19] If we make a very large number of observations of individuals with x = 3, then we expect to observe a Poisson Distribution of outcomes with mean 7.2.

As discussed, another important decision is the choice of the link function.
In this example, let us maintain the assumption of a Poisson Distribution, but instead of the identity link function let us use the log link function.

$\ln(\mu) = \Sigma\beta_i x_i = \beta_0 + \beta_1 x. \Rightarrow \mu = \exp[\Sigma\beta_i x_i] = \exp[\beta_0 + \beta_1 x].$

$f(y) = e^{-\lambda} \lambda^y / y!.$
$\ln f(y) = -\lambda + y\ln(\lambda) - \ln(y!) = -\mu + y\ln(\mu) - \ln(y!) = -\exp[\beta_0 + \beta_1 x] + y(\beta_0 + \beta_1 x) - \ln(y!).$

The loglikelihood is the sum of the contributions from the three observations:
$-\exp[\beta_0 + \beta_1] - \exp[\beta_0 + 2\beta_1] - \exp[\beta_0 + 3\beta_1] + \beta_0 + \beta_1 + 2(\beta_0 + 2\beta_1) + 9(\beta_0 + 3\beta_1)$
$- \ln(1) - \ln(2) - \ln(9!).$

To maximize the loglikelihood, we set its partial derivatives equal to zero.
Setting the partial derivative with respect to β_0 equal to zero:
$0 = -\exp[\beta_0 + \beta_1] - \exp[\beta_0 + 2\beta_1] - \exp[\beta_0 + 3\beta_1] + 12.$

Setting the partial derivative with respect to β_1 equal to zero:
$0 = -\exp[\beta_0 + \beta_1] - 2\exp[\beta_0 + 2\beta_1] - 3\exp[\beta_0 + 3\beta_1] + 32.$

Thus we have two equations in two unknowns:
$\exp[\beta_0 + \beta_1]\{1 + \exp[\beta_1] + \exp[2\beta_1]\} = 12.$
$\exp[\beta_0 + \beta_1]\{1 + 2\exp[\beta_1] + 3\exp[2\beta_1]\} = 32.$

Dividing the second equation by the first equation:

$$\frac{1 + 2\exp[\beta_1] + 3\exp[2\beta_1]}{1 + \exp[\beta_1] + \exp[2\beta_1]} = 8/3.$$

$\Rightarrow \exp[2\beta_1] - 2\exp[\beta_1] - 5 = 0.$

[19] For the Poisson Distribution, the variance is equal to the mean.

Letting v = exp[β_1], this equation is: $v^2 - 2v - 5 = 0$, with positive solution $v = 1 + \sqrt{6} = 3.4495$.

exp[β_1] = 3.4495. ⇒ $\beta_1 = 1.238$.

⇒ exp[β_0] = 12/{exp[β_1] + exp[2β_1] + exp[3β_1]} = 12/{3.4495 + 3.4495^2 + 3.4495^3} = 0.2128.

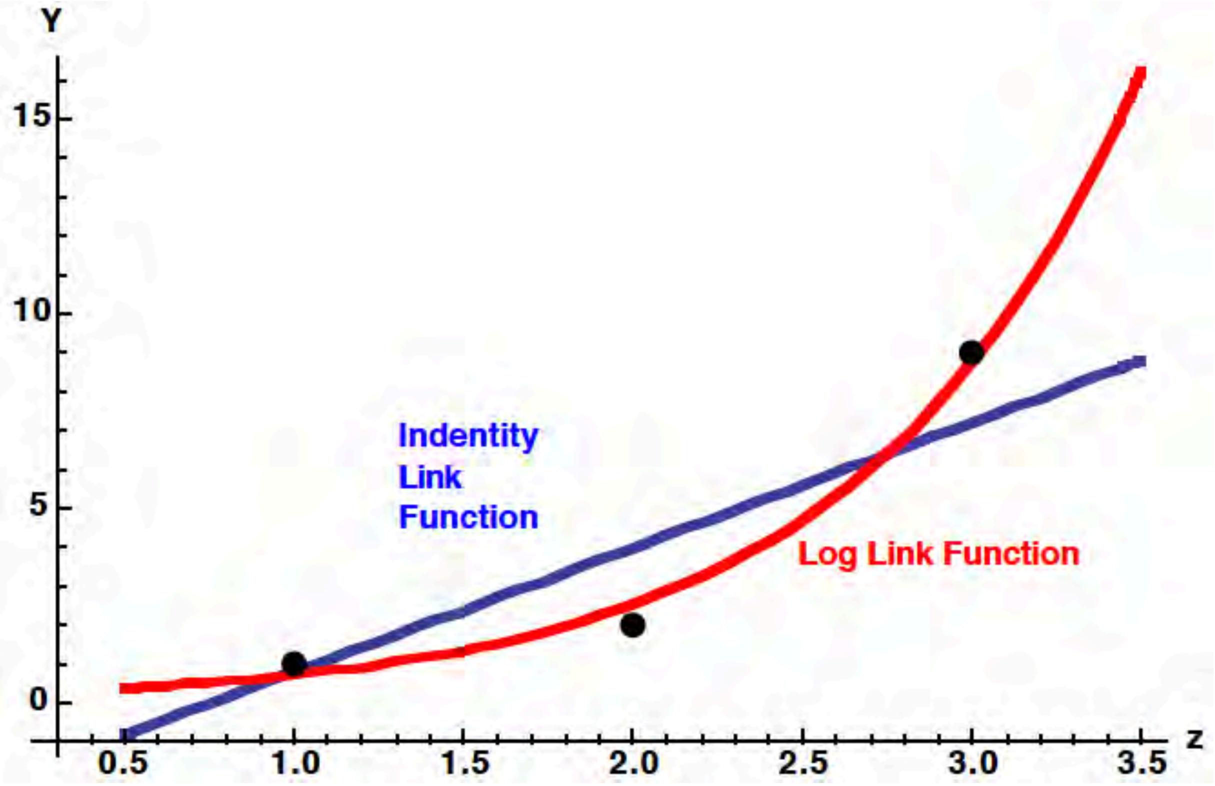
⇒ $\beta_0 = -1.547$.

μ = exp[$\beta_0 + \beta_1 x$] = exp[β_0] exp[β_1]^x = (0.2128)(3.4495^x).

For x = 1, μ = 0.734. For x = 2, μ = 2.532. For x = 3, μ = 8.735.
This differs from the result obtained previously when using the identity link function:

x	Observed	Poisson, Identity Link	Poisson, Log Link Function
1	1	0.8	0.734
2	2	4.0	2.532
3	9	7.2	8.735

Here is the same information in the form of a graph, with the data shown as dots:



In general, the choice of a link function makes a difference.
Using the log link function we got an exponential model rather than a linear model. With more explanatory variables, the log link function gives a multiplicative rather than an additive model.

Exponential Families:

Linear Exponential Families include:
Bernoulli, Binomial (m fixed), Poisson, Geometric, Negative Binomial (r fixed), Exponential, Gamma (α fixed), Normal (σ fixed), Inverse Gaussian (θ fixed), and the Tweedie Distribution.

Confusingly, when working on GLMs, “Exponential Family” means “Linear Exponential Family.”[20] This is how the syllabus reading refers to them, and thus from now on I will do the same.

Exponential Families have two parameters, μ the mean, and ϕ the dispersion parameter. The dispersion parameter is related to the variance. In a GLM ϕ is fixed across the observations and is treated as a nuisance parameter, in the same way that σ is treated in multiple regression.

It turns out that the relationship between the mean and variance uniquely identifies which linear exponential family we have.
Var[Y] = ϕ V(μ), where the form of V(μ) depends on which exponential family we have.

If the variance does not depend on the mean, then we have a Normal Distribution.
If the variance is proportional to the square of the mean, then we have a Gamma Distribution.
If the variance is proportional to the cube of the mean, then we have a Inverse Gaussian Distribution.
If the variance is proportional to the mean and we have a discrete distribution, then we have a Poisson Distribution.

For the Gamma Distribution, $f(y) = (y/\theta)^\alpha \exp[-y/\theta] / (y\ \Gamma[\alpha])$. $E[Y] = \alpha\theta$. $Var[Y] = \alpha\theta^2$.
If used in a GLM, then we are assuming that we have a Gamma Distribution with α fixed.
Then, Variance = $\alpha\theta^2 = (\alpha\theta)^2 / \alpha = (\text{mean})^2 / \alpha$.
Thus for the Gamma Distribution (with α fixed) the variance is proportional to the square of the mean.
For the Gamma Distribution: $V(\mu) = \mu^2$ and $\phi = 1/\alpha$.

For the following members of the exponential family of distributions, where μ is their mean, their variance is proportional to μ^p:
- **Normal distribution, p = 0.**
- **Poisson distribution, p = 1.**
- **Gamma distribution, p = 2.**
- **Tweedie distribution, 1 < p < 2.**
- **Inverse Gaussian distribution, p = 3.**

[20] Linear Exponential families are defined via the form of their density; however, this definition is not on the syllabus of this exam.

The syllabus reading gives a list of V(μ) for different exponential families.[21]
The syllabus reading does not go into detail on how to relate the parameterization of exponential families using μ and φ to that which you may already be familiar from for example Loss Models. However, in order to make things a little more concrete here is a table.

Distribution	μ	ϕ	$V(\mu)$
Normal	μ	σ^2	1
Poisson	λ	1	μ
Gamma	$\alpha\theta$	$1/\alpha$	μ^2
Inverse Gaussian	μ	$1/\theta$	μ^3
Negative Binomial	β/κ	1	$\mu(1 + \kappa\mu)$
Binomial	mq	1	$\mu (1 - \mu/m)$
Tweedie			μ^p

As discussed subsequently, for the overdispersed Poisson $\phi > 1$.

Gamma as per Loss Models, with mean = $\alpha\theta$ and variance = $\alpha\theta^2$, with α fixed.

Inverse Gaussian as per Loss Models, with mean = μ and variance = μ^3/θ, with θ fixed.

For the Negative Binomial, $\kappa = 1/r$, fixed. κ is called the overdispersion parameter.
As per Loss Models, the Negative Binomial has mean = r β and variance = r β $(1+\beta)$.[22]

Binomial, as per Loss Models with m fixed, with mean = mq and variance = mq(1-q)[23].
m = 1 is a Bernoulli. Goldburd, Khare, and Tevet give V(μ) for the case where m = 1.

The Tweedie Distribution will be discussed subsequently.

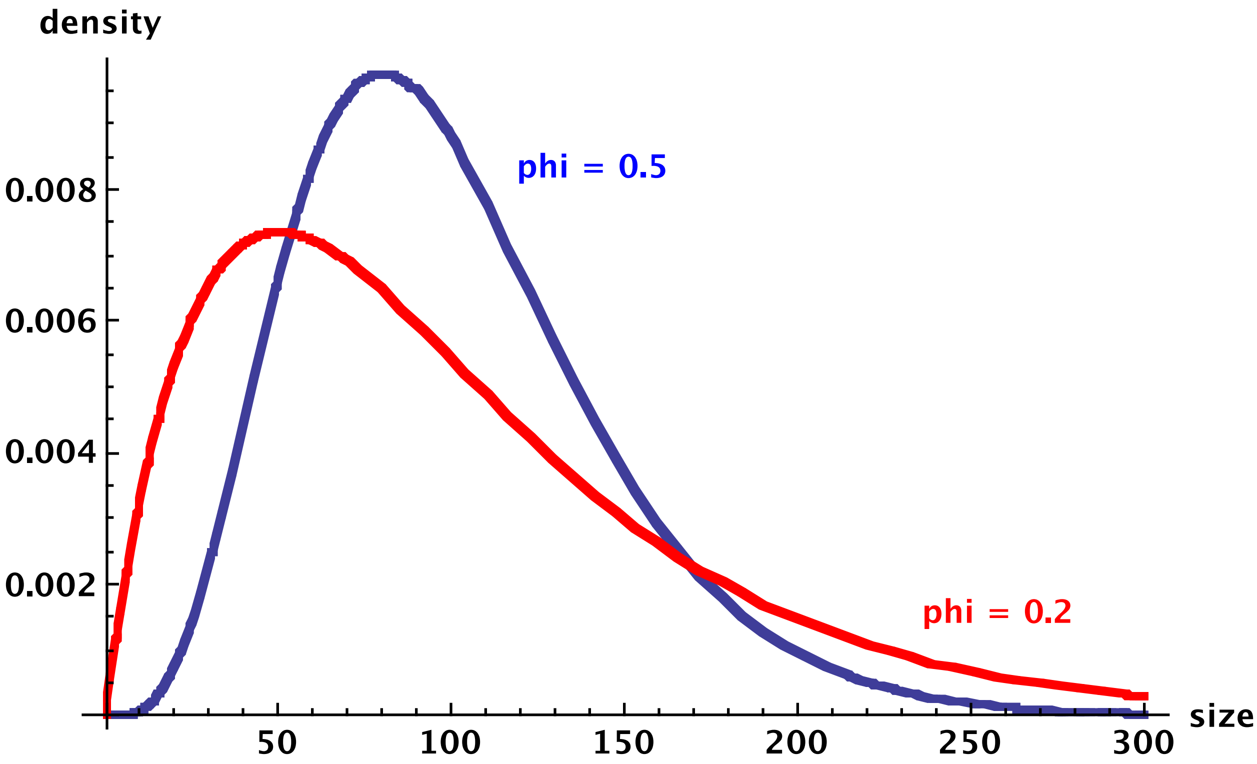
[21] See Table 1 in Generalized Linear Models for Insurance Rating.
[22] For the Negative Binomial Distribution, the dispersion parameter φ is restricted to be 1.
[23] One could introduce overdispersion via a Beta-Binomial Distribution, not on the syllabus of this exam. See for example Loss Models.

Gamma Distribution:

$f(x) = (x/\theta)^a \exp[-x/\theta] / (x\, \Gamma[\alpha])$, $x > 0$.[24]
Mean = $\alpha\theta$.
Variance = $\alpha\theta^2$. CV = $1/\sqrt{\alpha}$.
$\phi = 1/\alpha$.
$V(\mu) = \mu^2$.

The Gamma Distribution is commonly used to model severity.

Here are graphs of the densities of Gamma Distributions with $\mu = 100$ and $\phi = 1/5$ or $1/2$:[25]



The Gamma Distribution has support from 0 to infinity. The Gamma Distribution is right-skewed (has positive skewness), with a sharp peak and a long tail to the right.

Exercise: Determine the variance for a Gamma Distribution with $\mu = 20$ and $\phi = 1/4$.
[Solution: Variance = $\phi\, V(\mu) = \phi\, \mu^2 = (1/4)(20^2) = 100$.
Comment: The coefficient of variation is: $\sqrt{100}\,/20 = 1/2 = \sqrt{\phi}$.]

[24] Parameterized as per Loss Models, not on the syllabus of this exam.
I do not expect you to need to know the density.
[25] The first has $\alpha = 5$ and $\theta = 20$, while the second has $\alpha = 2$ and $\theta = 50$.

Inverse Gaussian Distribution:[26]

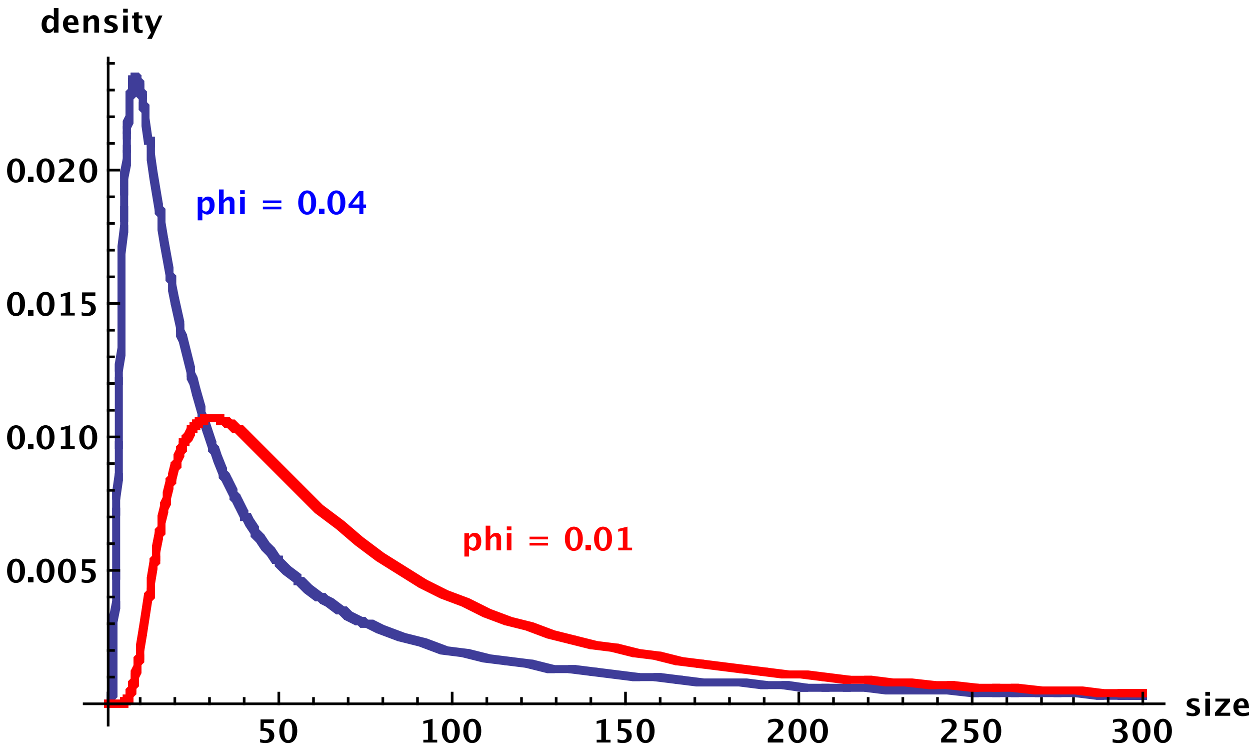
As per Loss Models: $f(x) = \sqrt{\frac{\theta}{2\pi}} \frac{\exp\left[-\frac{\theta\left(\frac{x}{\mu} - 1\right)^2}{2x}\right]}{x^{1.5}}$, $x > 0$.[27]

Mean = μ.
Variance = μ^3 / θ.
$\phi = 1/\theta$.
$V(\mu) = \mu^3$.

The Inverse Gaussian Distribution can be used to model severity. The Inverse Gaussian Distribution is appropriate when the severity has a larger skewness than for a Gamma.

Exercise: Determine the variance for an Inverse Gamma Distribution with $\mu = 20$ and $\phi = 1/5$.
[Solution: Variance = $\phi\ V(\mu) = \phi\ \mu^3 = (1/5)(20^3) = 1600$.]

Graphs of the densities of Inverse Gaussian Distributions with $\mu = 100$ and $\phi = 0.04$ or 0.01:[28]



[26] While the Gaussian (normal) describes a Brownian Motion's level at a fixed time, the Inverse Gaussian describes the distribution of the time a Brownian Motion with positive drift takes to reach a fixed positive level.
The cumulant generating function is the natural log of the moment generating function.
The cumulant generating function of an Inverse Gaussian is the inverse function of that of a Gaussian (Normal).

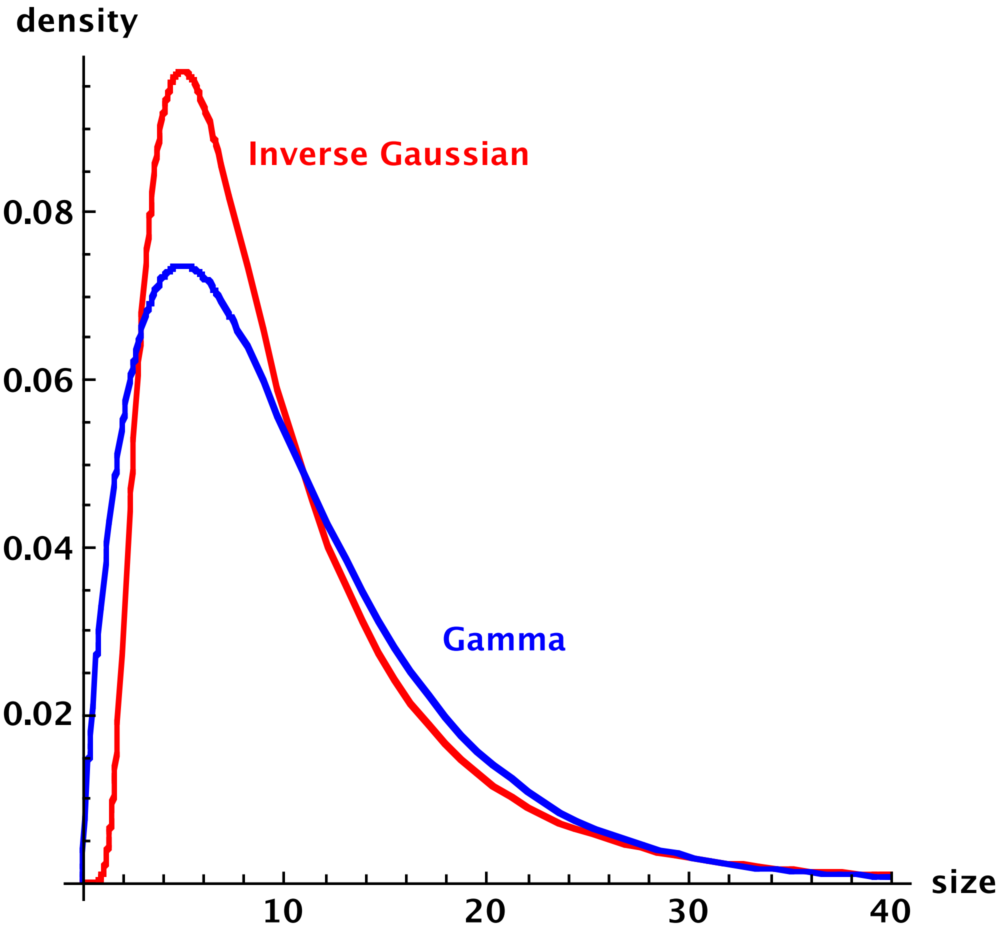
[27] I do not expect you to need to know the density.

[28] The first has $\theta = 25$, while the second has $\theta = 100$.

For the Gamma the variance is proportional to the square of the mean,
while for the Inverse Gaussian the variance is proportional to the cube of the mean.
The Inverse Gaussian and Gamma are similar, but the Inverse Gaussian has larger skewness and a higher peak.[29]

For example, a Gamma Distribution with μ = 10 and α = 2 has mean = 10, and variance = $10^2/2$ = 50. An Inverse Gaussian Distribution with μ = 10 and ϕ = 20 has mean = 10, and variance = $10^3/20$ = 50. Thus these two distributions have the same mean and variance.

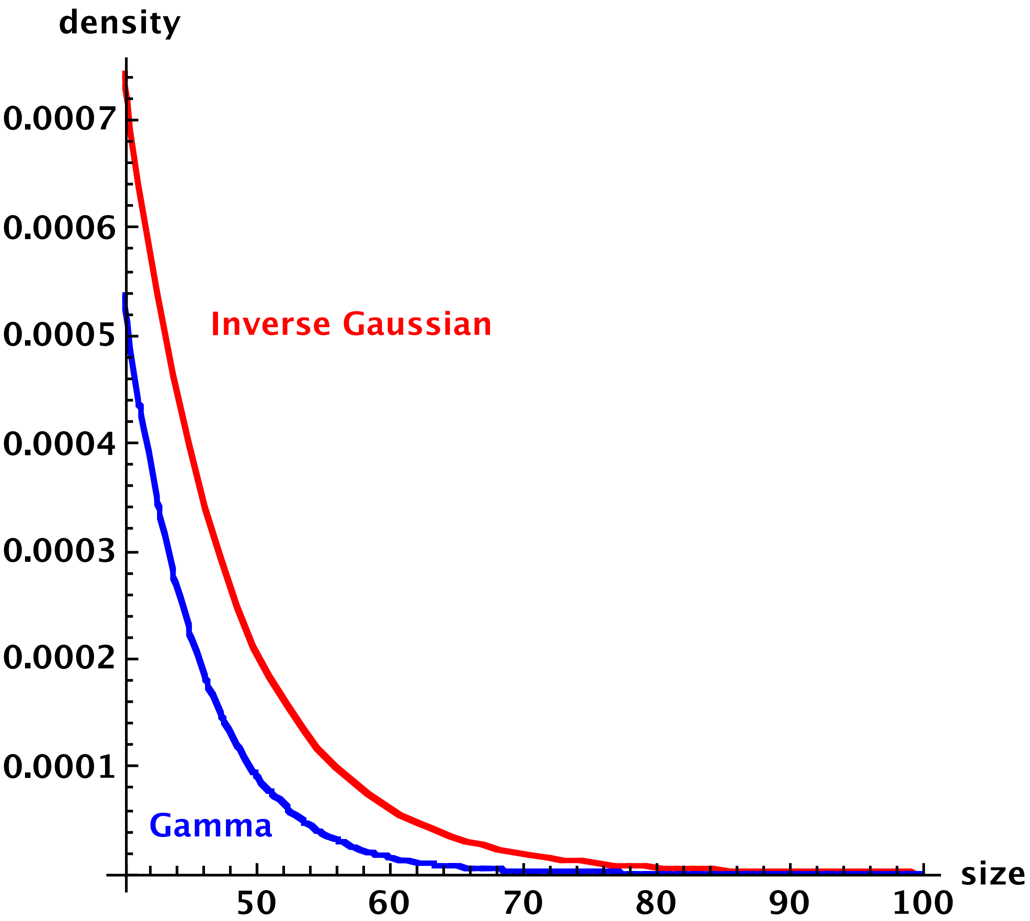
Here is a graph comparing these two densities:



The Inverse Gaussian Distribution has a higher peak than the Gamma Distribution.

[29] The skewness for the Gamma distribution is always twice times the coefficient of variation.
The skewness for the Inverse Gaussian distribution is always three times the coefficient of variation.

The Inverse Gaussian has more probability in the extreme righthand tail. With the aid of a computer, for this Gamma Distribution the survival function at 40 is S(40) = 0.30%, while for this Inverse Gaussian Distribution, S(40) = 0.58%.



A Two Dimensional Example of Generalized Linear Models:[30]

Let us assume we have two types of drivers, male and female, and two territories, urban and rural. Then there are a total of four combinations of gender and territory.
We assume an equal number of claims in each of the four combinations.

Let us assume that we have the following observed severities:

		Urban	Rural
Male		800	500
Female		400	200

Let us assume the following generalized linear model:
Gamma Function
Reciprocal link function[31]

Define male and rural as the base level, which introduces a constant term.
Then the constant, β_0, applies to all observations.
Let X_1 = 1 if female and 0 if male.[32]
Let X_2 = 1 if urban and 0 if rural.[33]

$$1/\mu = \Sigma\beta_i x_i = \beta_0 + \beta_1 x_1 + \beta_2 x_2\ .\ \Rightarrow \mu = \frac{1}{\beta_0 + \beta_1 x_1 + \beta_2 x_2}\ .$$

Therefore, the modeled means are:

		Urban	Rural
Male		$1/(\beta0+\beta2)$	$1/\beta0$
Female		$1/(\beta0+\beta1+\beta2)$	$1/(\beta0+\beta1)$

For the Gamma Distribution as per <u>Loss Models</u>, $f(y) = (y/\theta)^a \exp[-y/\theta] / (y\ \Gamma[\alpha])$.

$\ln f(y) = (\alpha-1)\ln(y) - y/\theta - \alpha\ln(\theta) - \ln[\Gamma[\alpha]] = (\alpha-1)\ln(y) - y/(\mu/\alpha) - \alpha\ln(\mu/\alpha) - \ln[\Gamma[\alpha]]$

$= (\alpha-1)\ln(y) - \alpha y/\mu - \alpha\ln(\mu) + \alpha\ln(a) - \ln[\Gamma[\alpha]]$

$= (\alpha-1)\ln(y) - \alpha y(\beta_0 + \beta_1 x_1 + \beta_2 x_2) + \alpha\ln(\beta_0 + \beta_1 x_1 + \beta_2 x_2) + \alpha\ln(\alpha) - \ln[\Gamma[\alpha]]$.

[30] I do <u>not</u> expect you to have to go into this level of detail on your exam.
See page 24 and Appendix F of "A Practitioners Guide to Generalized Linear Models," by Anderson, et. al.

[31] One could instead use the log link function, and obtain somewhat different results.

[32] Since we have taken male as the base level, the covariate has to involve not male.

[33] Since we have taken rural as the base level, the covariate has to involve not rural.

The loglikelihood is the sum of the contributions from the four observations:
$(\alpha-1)\{\ln(800) + \ln(400) + \ln(500) + \ln(200)\}$
$- \alpha\{800(\beta_0 + \beta_2) + 400(\beta_0 + \beta_1 + \beta_2) + 500\beta_0 + 200(\beta_0 + \beta_1)\}$
$+ \alpha\{\ln(\beta_0 + \beta_2) + \ln(\beta_0 + \beta_1 + \beta_2) + \ln(\beta_0) + \ln(\beta_0 + \beta_1)\} + 4\alpha\ln(\alpha) - 4\ln[\Gamma(\alpha)]$.

To maximize the loglikelihood, we set its partial derivatives equal to zero.
Setting the partial derivative with respect to β_0 equal to zero:
$0 = -\alpha(800 + 400 + 500 + 200) + \alpha\{1/(\beta_0 + \beta_2) + 1/(\beta_0 + \beta_1 + \beta_2) + 1/\beta_0 + 1/(\beta_0 + \beta_1)\}$.
$\Rightarrow 1/(\beta_0 + \beta_2) + 1/(\beta_0 + \beta_1 + \beta_2) + 1/\beta_0 + 1/(\beta_0 + \beta_1) = 1900$.
Setting the partial derivative with respect to β_1 equal to zero:
$0 = -\alpha(400 + 200) + \alpha\{1/(\beta_0 + \beta_1 + \beta_2) + 1/(\beta_0 + \beta_1)\}$. $\Rightarrow 1/(\beta_0 + \beta_1 + \beta_2) + 1/(\beta_0 + \beta_1) = 600$.
Setting the partial derivative with respect to β_2 equal to zero:
$0 = -\alpha(800 + 400) + \alpha\{1/(\beta_0 + \beta_2) + 1/(\beta_0 + \beta_1 + \beta_2)\}$. $\Rightarrow 1/(\beta_0 + \beta_2) + 1/(\beta_0 + \beta_1 + \beta_2) = 1200$.

Solving these three equations in three unknowns:[34]
$\beta_0 = 0.00223811$, $\beta_1 = 0.00171142$, and $\beta_2 = -0.00106605$.

$$\mu = \frac{1}{0.00223811 + 0.00171142x_1 - 0.00106605x_2}.$$

For Male and Urban: $x_1 = 0$, $x_2 = 1$, and $\mu = 1 / (0.00223811 - 0.00106605) = 853.20$.
For Female and Urban: $x_1 = 1$, $x_2 = 1$,
and $\mu = 1 / (0.00223811 + 0.00171142 - 0.00106605) = 346.80$.
For Male and Rural: $x_1 = 0$, $x_2 = 0$, and $\mu = 1/0.00223811 = 446.81$.
For Female and Rural: $x_1 = 1$, $x_2 = 0$, and $\mu = 1/(0.00223811 + 0.00171142) = 253.20$.

The fitted severities by cell are:[35]

		Urban	Rural	Average
Male		853.20	446.81	650.01
Female		346.80	253.20	300.00
Average		600.00	350.01	475.00

[34] I used a computer to solve these three equations.
There is no need to solve for α in order to calculate the fitted pure premiums by cell.
However, using a computer, the maximum likelihood alpha is 45.6.

[35] The averages were computed assuming the same number of claims by cell.

This compares to the observed severities by cell:

		Urban	Rural	Average
Male		800	500	650
Female		400	200	300
Average		600	350	475

Notice how the averages for male, female, urban, and rural are equal for the fitted and observed. The overall experience of each class and territory has been reproduced by the model.

In general, the estimates will be in balance as they were here, when one uses the canonical link function; the canonical link function for the Gamma is the reciprocal link function.[36]

Exercise:
For the Urban territory, what the relativity of male compared to female indicated by the GLM?
[Solution: 853.20/346.80 = 2.460.]

Exercise:
For the Rural territory, what the relativity of male compared to female indicated by the GLM?
[Solution: 446.81/253.20 = 1.765.]

The relativities are different in the different territories. In general for a particular GLM, the relativities for one predictor variable can depend on the level(s) of the other predictor variable(s).

Here we have used the reciprocal link function. If instead the log link function had been used, the model would have been multiplicative, and the indicated multiplicative relativities would not have depended on territory. If instead the identity link function had been used, the model would have been additive, and the indicated additive relativities would not have depended on territory.

We could instead change the definitions of the covariates, and have a model without an intercept:
x_1 = 1 if male.
x_2 = 1 if female.
x_3 = 1 if urban and x_3 = 0 if rural.

Then $1/\mu = \sum\beta_i x_i = \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3$. $\Rightarrow \mu = \dfrac{1}{\beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3}$.

[36] See “A Systematic Relationship Between Minimum Bias and Generalized Linear Models”, by Stephen J. Mildenhall, PCAS 1999, <u>not</u> on the syllabus.

Therefore, the modeled means are:

		Urban	Rural
Male		$1/(\beta 1+\beta 3)$	$1/\beta 1$
Female		$1/(\beta 2+\beta 3)$	$1/\beta 2$

For the Gamma Distribution as per Loss Models, $f(y) = (y/\theta)^a \exp[-y/\theta] / (y\ \Gamma[\alpha])$.

$\ln f(y) = (\alpha-1)\ln(y) - y/\theta - \alpha\ln(\theta) - \ln[\Gamma[\alpha]] = (\alpha-1)\ln(y) - y/(\mu/\alpha) - \alpha\ln(\mu/\alpha) - \ln[\Gamma[\alpha]]$

$= (\alpha-1)\ln(y) - \alpha y/\mu - \alpha\ln(\mu) + \alpha\ln(a) - \ln[\Gamma[\alpha]]$

$= (\alpha-1)\ln(y) - \alpha y\ (\beta_1x_1 + \beta_2x_2 + \beta_3x_3) + \alpha\ln(\beta_1x_1 + \beta_2x_2 + \beta_3x_3) + \alpha\ln(\alpha) - \ln[\Gamma[\alpha]]$.

The loglikelihood is the sum of the contributions from the four observations:

$(\alpha-1)\{\ln(800) + \ln(400) + \ln(500) + \ln(200)\}$

$- \alpha\{800(\beta_1 + \beta_3) + 400(\beta_2 + \beta_3) + 500\beta_1 + 200\beta_2\}$

$+ \alpha\{\ln(\beta_1 + \beta_3) + \ln(\beta_2 + \beta_3) + \ln(\beta_1) + \ln(\beta_2)\} + 4\alpha\ln(\alpha) - 4\ln[\Gamma[\alpha]]$.

To maximize the loglikelihood, we set its partial derivatives equal to zero.
Setting the partial derivative with respect to β_1 equal to zero:

$0 = -\alpha(800 + 500) + \alpha\{1/(\beta_1 + \beta_3) + 1/\beta_1\}. \Rightarrow 1/(\beta_1 + \beta_3) + 1/\beta_1 = 1300.$

Setting the partial derivative with respect to β_2 equal to zero:

$0 = -\alpha(400 + 200) + \alpha\{1/(\beta_2 + \beta_3) + 1/\beta_2\}. \Rightarrow 1/(\beta_2 + \beta_3) + 1/\beta_2 = 600.$

Setting the partial derivative with respect to β_3 equal to zero:

$0 = -\alpha(800 + 400) + \alpha\{1/(\beta_1 + \beta_3) + 1/(\beta_2 + \beta_3)\}. \Rightarrow 1/(\beta_1 + \beta_3) + 1/(\beta_2 + \beta_3) = 1200.$

Solving these three equations in three unknowns:[37]

$\beta_1 = 0.00223811$, $\beta_2 = 0.00394952$, and $\beta_3 = -0.00106605$.

$$\mu = \frac{1}{0.00223811x_1 + 0.00394952x_2 - 0.00106605x_3}.$$

For Male and Urban: $x_1 = 1$, $x_2 = 0$, $x_3 = 1$, and $\mu = 1 / (0.00223811 - 0.00106605) = 853.20$.

For Female and Urban: $x_1 = 0$, $x_2 = 1$, $x_3 = 1$, and $\mu = 1 / (0.00394952 - 0.00106605) = 346.80$.

For Male and Rural: $x_1 = 1$, $x_2 = 0$, $x_3 = 0$, and $\mu = 1/0.00223811 = 446.81$.

For Female and Rural: $x_1 = 0$, $x_2 = 1$, $x_3 = 0$, and $\mu = 1/0.00394952 = 253.20$.

The modeled means are the same as in the other version of the model with a base level.

[37] I used a computer to solve these three equations.

Instead fit an Inverse Gaussian with the inverse square link function to this same data.[38] [39]

For the Inverse Gaussian: $f(x) = \sqrt{\frac{\theta}{2\pi}} \; \frac{\exp\left[-\frac{\theta\left(\frac{x}{\mu} - 1\right)^2}{2x}\right]}{x^{1.5}}$, mean = μ, variance = μ^3 / θ.

Ignoring terms that do not involve μ,

$$\ln f(x) = -\frac{\theta\left(\frac{x}{\mu} - 1\right)^2}{2x} = -\frac{\theta}{2x}\left(\frac{x^2}{\mu^2} - 2\frac{x}{\mu} + 1\right) = -\frac{\theta x}{2\mu^2} + \frac{\theta}{\mu} - \frac{\theta}{2x}.$$

Use $x_1 = 1$ if male.
$x_2 = 1$ if female.
$x_3 = 1$ if urban and $x_3 = 0$ if rural.

Using the squared reciprocal link function: $1/\mu^2 = \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3$.
Thus ignoring terms that do not include μ, the loglikelihood is:

$$\frac{-\theta}{2}\{800(\beta_1 + \beta_3) + 500(\beta_1) + 400(\beta_2 + \beta_3) + 200(\beta_2)\} + \theta\{\sqrt{\beta_1 + \beta_3} + \sqrt{\beta_1} + + \sqrt{\beta_2 + \beta_3}\}.$$

Setting the partial derivative with respect to β_1 equal to zero:

$$0 = \frac{-\theta}{2}\{800 + 500\} + \frac{\theta}{2}\{1/\sqrt{\beta_1 + \beta_3} + 1/\sqrt{\beta_1}\}. \Rightarrow 1300 = 1/\sqrt{\beta_2 + \beta_3} + 1/\sqrt{\beta_2}.$$

Setting the partial derivative with respect to β_2 equal to zero:

$$0 = \frac{-\theta}{2}\{400 + 200\} + \frac{\theta}{2}\{1/\sqrt{\beta_2 + \beta_3} + 1/\sqrt{\beta_2}\}. \Rightarrow 600 = 1/\sqrt{\beta_2 + \beta_3} + 1/\sqrt{\beta_2}.$$

Setting the partial derivative with respect to β_3 equal to zero:

$$0 = \frac{-\theta}{2}\{800 + 400\} + \frac{\theta}{2}\{1/\sqrt{\beta_1 + \beta_3} + 1/\sqrt{\beta_2 + \beta_3}\}. \Rightarrow 1200 = 1/\sqrt{\beta_1 + \beta_3} + 1/\sqrt{\beta_2 + \beta_3}.$$

Solving these three equations in three unknowns:[40]
$\beta_1 = 0.0000054693$, $\beta_2 = 0.0000134722$, and $\beta_3 = -0.00000415544$.
$1/\mu^2 = 0.0000054693x_1 + 0.0000134722x_2 - 0.00000415544x_3$.

[38] Again assuming equal claims per cell.

[39] While the inverse square is the canonical link function for the Inverse Gaussian, one could use a different link function.

[40] I used a computer to solve these three equations.
There is no need to solve for θ in order to calculate the fitted pure premiums by cell.

For Male and Urban: $x_1 = 1$, $x_2 = 0$, $x_3 = 1$, and

$\mu = 1 / \sqrt{0.0000054693 - 0.00000415544} = 872.42$.

For Female and Urban: $x_1 = 0$, $x_2 = 1$, $x_3 = 1$, and

$\mu = 1 / \sqrt{0.0000134722 - 0.00000415544} = 327.62$.

For Male and Rural: $x_1 = 1$, $x_2 = 0$, $x_3 = 0$, and $\mu = 1 / \sqrt{0.0000054693} = 427.60$.

For Female and Rural: $x_1 = 0$, $x_2 = 1$, $x_3 = 0$, and $\mu = 1 / \sqrt{0.0000134722} = 272.45$.

The fitted severities by cell differ from the previous model and are as follows:[41]

		Urban	Rural	Average
Male		872.42	427.60	650.01
Female		327.62	272.45	300.04
Average		600.02	350.03	475.02

This compares to the observed severities by cell:

		Urban	Rural	Average
Male		800	500	650
Female		400	200	300
Average		600	350	475

Notice how subject to rounding, again the averages for male, female, urban, and rural are equal for the fitted and observed. The overall experience of each class and territory has been reproduced by the model.

In general, the estimates will be in balance as they were here, when one uses the canonical link function; the canonical link function for the Inverse Gaussian is the inverse square link function.[42]
When the weights differ by cell, this balance involves weighted averages.

[41] The averages were computed assuming the same number of claims by cell.

[42] See “A Systematic Relationship Between Minimum Bias and Generalized Linear Models,” by Stephen Mildenhall, PCAS 1999, not on the syllabus.

Design Matrix:

As is the case for multiple regression, it is common in GLMs to work with a design matrix.
Each row of the design matrix corresponds to one observation in the data.[43]
Each column of the design matrix corresponds to a covariate in the model.
If there is an intercept or constant term in the model, then the first column refers to it; **the first column of the design matrix will then consist of all ones**.

A one dimensional example, with one covariate plus an intercept, was discussed previously:
Three observations: (1, 1), (2, 2), (3, 9).
$Y = \beta_0 + \beta_1 X$.

Then the design matrix is: $\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}$.

Since the intercept applies to each observation, the first column is all ones.
The second column contains the observed values of the only covariate X.

Note that the design matrix depends on the observations and the definitions of the covariates.
The design matrix does not depend on the link function or the distributional form of the errors.

The response vector would contain the observed values of Y: $\begin{pmatrix} 1 \\ 2 \\ 9 \end{pmatrix}$.

The vector of parameters is: $\begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix}$.

If one used the identity link function, then this model can be written as: $E[Y] = X\,\beta$,
where X is the design matrix and β is the vector of parameters.
If instead one used the log link function, then this model can be written as: $E[Y] = \exp[X\,\beta]$.

In general, with a link function g, a GLM can be written as: $\mathbf{E[Y] = g^{-1}[X\,\beta]}$.

With more covariates, things get a little more complicated. There is not a unique way to define the covariates. The important thing is to have the design matrix be consistent with the chosen definitions of the covariates.

[43] When we have more than one exposure or claim in a cell, a row may correspond to several observations grouped.

A two dimensional model was previously discussed:

		Urban	Rural
Male		800	500
Female		400	200

Usually on your exam, one would define a base level, which introduces a constant term.
For example, as before we could define male/rural as the base level.[44]
Then the constant, β_0, would apply to all observations.
Let X_1 = 1 if female and 0 if male.[45]
Let X_2 = 1 if urban and 0 if rural.[46]

Then with link function g, the GLM is: $g(E[Y]) = \beta_0 + \beta_1 X_1 + \beta_2 X_2$.

If we order the observations as follows, then the design matrix is:

$$\begin{pmatrix} \text{Male/Urban} \\ \text{Male/Rural} \\ \text{Female/Urban} \\ \text{Female/Rural} \end{pmatrix} \Leftrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

The first column of ones corresponds to the constant term which applies to all observations.
The first row of the design matrix corresponds to male/urban: $X_1 = 0$, $X_2 = 1$.
The second row corresponds to male/rural: $X_1 = 0$, $X_2 = 0$.
The third row corresponds to female/urban: $X_1 = 1$, $X_2 = 1$.
The last row corresponds to female/rural: $X_1 = 1$, $X_2 = 0$.

[44] One could define any of the four combinations as the base level.
[45] Since male is the base level, the covariate has to involve not male.
[46] Since rural is the base level, the covariate has to involve not rural.

As stated previously, on your exam the model is likely to be defined with a base level.
Nevertheless, one could instead define:
X_1 = 1 if male. (0 if female)
X_2 = 1 if female. (0 if male)
X_3 = 1 if urban and 0 if rural.
Then with link function g, the GLM is: $g(E[Y]) = \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3$.

Then if we order the observations as before, then the design matrix is:[47]

$$\begin{pmatrix} \text{Male/Urban} \\ \text{Male/Rural} \\ \text{Female/Urban} \\ \text{Female/Rural} \end{pmatrix} \Leftrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

The first row corresponds to male/urban: $X_1 = 1$, $X_2 = 0$, and $X_3 = 1$.
The second row corresponds to male/rural: $X_1 = 1$, $X_2 = 0$, and $X_3 = 0$.
The third row corresponds to female/urban: $X_1 = 0$, $X_2 = 1$, and $X_3 = 1$.
The last row corresponds to female/rural: $X_1 = 0$, $X_2 = 1$, and $X_3 = 0$.

The response vector would contain the observed values of Y,
in the same order as the rows of the design matrix:

$$\begin{pmatrix} \text{Male/Urban} \\ \text{Male/Rural} \\ \text{Female/Urban} \\ \text{Female/Rural} \end{pmatrix} \Leftrightarrow \begin{pmatrix} 800 \\ 500 \\ 400 \\ 200 \end{pmatrix}.$$

The vector of parameters is: $\begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}$.

[47] One can put the observations in any order, as long as one is consistent throughout.

This definition of covariates is not unique. For example instead define:
X_1 = 1 if urban. (0 if rural)
X_2 = 1 if rural. (0 if urban)
X_3 = 1 if female and 0 if male.

Exercise: For these definitions, what are the design matrix and the response vector?
[Solution: If we order the observations as before, then the design matrix is:

$$\begin{pmatrix} \text{Male/Urban} \\ \text{Male/Rural} \\ \text{Female/Urban} \\ \text{Female/Rural} \end{pmatrix} \Leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

The response vector would contain the observed values of Y,
in the same order as the rows of the design matrix:

$$\begin{pmatrix} \text{Male/Urban} \\ \text{Male/Rural} \\ \text{Female/Urban} \\ \text{Female/Rural} \end{pmatrix} \Leftrightarrow \begin{pmatrix} 800 \\ 500 \\ 400 \\ 200 \end{pmatrix}.$$

Comment: While the design matrix is different than before, this version of the model is just as valid as the previous ones, as long as everything is handled consistently.
The first row of the design matrix corresponds to male/urban: $X_1 = 1$, $X_2 = 0$, and $X_3 = 0$.
The second row corresponds to male/rural: $X_1 = 0$, $X_2 = 1$, and $X_3 = 0$.
The third row corresponds to female/urban: $X_1 = 1$, $X_2 = 0$, and $X_3 = 1$.
The last row corresponds to female/rural: $X_1 = 0$, $X_2 = 1$, and $X_3 = 1$.]

Generalized Linear Models, An Example of Adding Dimensions:

Assume we have a one-dimensional model with two territories: Urban and Rural.
While there are several different ways to set up this model, let us define:
Urban is the base level, β_0 is the intercept, $X_1 = 1$ if Rural.

Let us now add another dimension, gender: Male or Female.

We can either let Female/Urban be the base level and $X_2 = 1$ if Male,
or let Male/Urban be the base level and $X_2 = 1$ if Female.
In either case, we add only one more variable to the model we had for one dimension.

We could now add another dimension such as age: Young, Senior, Other. Regardless of which model we had for two dimensions, we would add two more variables to include age. Age has three levels, and in order to add it to our model we need to add 3 - 1 = 2 variables to the model.

Assume our model for two dimensions had:
Female/Rural as the base level, β_0 is the intercept, with $X_1 = 1$ if Urban, $X_2 = 1$ if Male.
Then for example we could take:
Female/Rural/Other as the base level, β_0 is the intercept, with $X_3 = 1$ if Young and $X_4 = 1$ if Senior.

If the model has a base level and corresponding constant term, then each categorical variable introduces a number of covariates equal to the number of its levels minus 1.

In this example, the number of covariates is: (constant term) + (2-1) + (2-1) + (3-1) = 5.

In practical applications, it is important to choose the base level of each category to be one with lots of data. If the chosen base level has little data, then the standard errors of the coefficients will be larger than if one had chosen a base level with lots of data.[48]

[48] See Figure 2 in Generalized Linear Models for Insurance Rating. Even when the base level has lots of data, the standard errors of coefficients corresponding to levels with little data will be wider than those levels with more data.

For example, assume instead our model for two dimensions had:
No base level with X_1 = 1 if Urban, X_2 = 1 if Rural, and X_3 = 1 if Male.
Then for example we could take: X_4 = 1 if Young and X_5 = 1 if Senior.

Without a base level and corresponding constant term, then one and only one of the categorical variables has a number of covariates equal to the number of its levels.
Each of the other categorical variables introduces a number of covariates equal to the number of its levels minus one.

For this example, without a base level, territory has a number of covariates equal to its number of levels, while gender and age each have a number of covariates equal to their number of levels minus one. The total number of covariates is: 2 + (2-1) + (3-1) = 5, the same as before.

Design Matrices, Continuous Variables:

We have looked at discrete categorical variables such as territory. GLMs can also use continuous variables such as amount of insurance and time living at current residence.[49] With continuous variables, determining the design matrix is somewhat different than it is with discrete variables.

Let us assume we are modeling pure premiums for homeowners and observe five policies:

Policy	Amount of Insurance ($000)	Time at Residence	Pure Premium ($000)
1	100	3	0
2	130	11	30
3	180	0	0
4	250	7	80
5	400	16	0

If X_1 = Amount of Insurance and X_2 = Time at Residence,
then the design matrix and response vector are:

$$X = \begin{pmatrix} 100 & 3 \\ 130 & 11 \\ 180 & 0 \\ 250 & 7 \\ 400 & 16 \end{pmatrix} \quad Y = \begin{pmatrix} 0 \\ 30 \\ 0 \\ 80 \\ 0 \end{pmatrix}.$$

The GLM is: g(E[Y]) = βX.

[49] In some cases, the model will perform better if such continuous variables are grouped into several categories.

Poisson Distribution:

$f(x) = e^{-\lambda} \lambda^x / x!$, $x = 0, 1, 2, ...$
Mean = λ.
Variance = λ.
$\phi = 1$.
$V(\mu) = \mu$.

The Poisson Distribution is commonly used to model frequency.

Overdispersion:

$\text{Var}[Y_i] = \phi \, E[Y_i]$. Since for the Poisson $\phi = 1$, the variance is equal to the mean.

When the variance is greater than the mean, one could use a Negative Binomial Distribution, which has a variance greater than its mean.[50]

We can instead use an overdispersed Poisson with $\phi > 1$.
$\text{Var}[Y_i] = \phi \, E[Y_i]$. For $\phi > 1$, variance is greater than the mean.
While this does not correspond to the likelihood of any exponential family, otherwise the GLM mathematics works.[51] [52]

Using an overdispersed Poisson (ODP), we get the same estimated betas as for the usual Poisson regression.[53]
However, the standard errors of all of the estimated parameters are multiplied by $\sqrt{\phi}$. [54]

Although not mentioned in the syllabus readings, the usual estimator of the dispersion parameter ϕ is: $\hat{\phi} = \frac{1}{n - p} \sum_{i=1}^{n} \frac{(y_i - \mu_i)^2}{\mu_i}$.

[50] One way the variance can be greater than the mean is if frequency is Poisson for each insured, but the means of the Poissons vary between insureds. If the Poisson means follow a Gamma Distribution, then the mixed distribution is a Negative Binomial Distribution.
[51] This is called using a quasi-likelihood, although the syllabus reading dos not use that term.
[52] Often using a Negative Binomial Distribution or an overdispersed Poisson approach to fit a GLM will produce similar results.
[53] This is the same reason we can fit the betas in a Normal regression without fitting σ.
[54] The variance of the estimated parameter is multiplied by ϕ.

Negative Binomial Distribution:

$$f(x) = \frac{\Gamma(x+r)}{x!\,\Gamma(r)} \frac{\beta^x}{(1+\beta)^{x+r}} = \frac{\Gamma(x+1/\kappa)}{x!\,\Gamma(1/\kappa)} \frac{(\kappa\mu)^x}{(1+\kappa\mu)^{x+r}},\ x = 0, 1, 2, ...$$

Mean = $r\beta = \beta/\kappa$.

Variance = $r\beta(1+\beta) = (\beta/\kappa)\,(1+\beta)$.

$\phi = 1$.

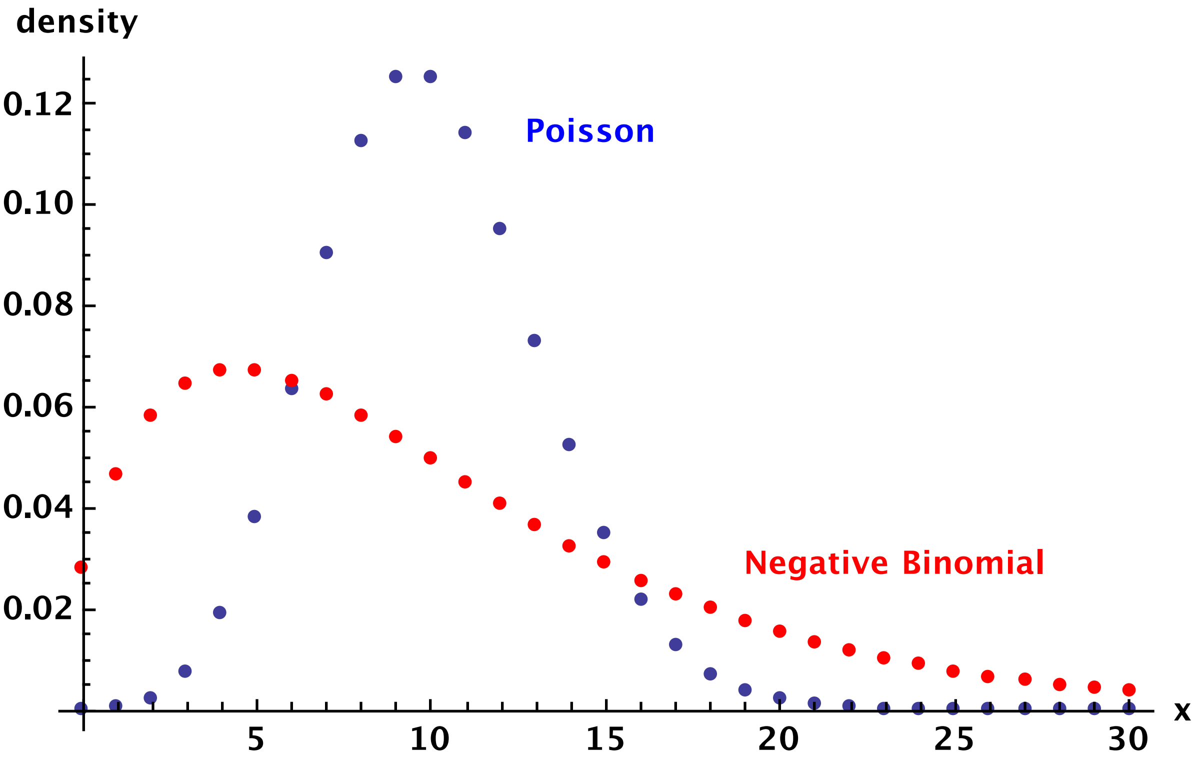
$V(\mu) = \mu(1 + \kappa\mu)$.

$\kappa = 1/r$ is called the overdispersion parameter.
As κ approaches zero while keeping the mean constant, the Negative Binomial Distribution approaches a Poisson Distribution.[55]

The Negative Binomial Distribution has its variance greater than its mean.
One way a Negative Binomial Distribution arises is as a Gamma mixture of Poisson Distributions.

The Negative Binomial Distribution is used to model frequency.

Here is a graph comparing a the densities of a Poisson with mean 5,
and a Negative Binomial with mean 5 and $\kappa = 1/2$ ($r = 2$):



[55] This is mathematically equivalent to letting β approach zero while keeping the mean constant.

One Dimensional Poisson Example with Exposures:

Exposures are a measure of how much insurance protection has been provided. Car years are an example. If one insures three cars each for two years, that is 6 car years of exposure.

Assume the same three observations: (1, 1), (2, 2), (3, 9).
However, let us assume 2, 3, and 4 exposures respectively.

Let us again fit a GLM using a Poisson with a log link function.
$\lambda_i = \exp[\beta_0 + x\beta_1]$.

We assume that Y_i is Poisson, with mean $n_i\ \lambda_i$,
where n_i is the number of exposures for observation i.
For example, the third observation is Poisson with mean: $4\ \exp[\beta_0 + 3\beta_1]$.

For the Poisson Distribution, $\ln f(y) = -\lambda + y\ln(\lambda) - \ln(y!)$.
Thus the contribution to the loglikelihood from the third observation is:
$-4\ \exp[\beta_0 + 3\beta_1] + 9\ \{\ln 4 + (\beta_0 + 2\beta_1)\} - \ln[9!]$.

The loglikelihood is the sum of the contributions from the three observations:
$-2\ \exp[\beta_0 + \beta_1] - 3\ \exp[\beta_0 + 2\beta_1] - 4\ \exp[\beta_0 + 3\beta_1] + (\beta_0 + \beta_1) + 2(\beta_0 + 2\beta_1) + 9(\beta_0 + 3\beta_1)$
$+ \ln[2] + 2\ \ln[3] + 9\ \ln[4] - \ln(1) - \ln(2) - \ln(9!)$.

To maximize the loglikelihood, we set its partial derivatives equal to zero.
Setting the partial derivative with respect to β_0 equal to zero:
$0 = -2\ \exp[\beta_0 + \beta_1] - 3\ \exp[\beta_0 + 2\beta_1] - 4\ \exp[\beta_0 + 3\beta_1] + 12$.
Setting the partial derivative with respect to β_1 equal to zero:
$0 = -2\ \exp[\beta_0 + \beta_1] - 6\ \exp[\beta_0 + 2\beta_1]) - 12\ \exp[\beta_0 + 3\beta_1] + 32$.

Solving these two equations in two unknowns: $\beta_0 = -1.97234$ and $\beta_1 = 0.91629$.[56]
$\mu_i = n_i\ \exp[-1.97234 + 0.91629\ x_i]$.

For x = 1, $\mu = 2\ \exp[-1.97234 + 0.91629] = 0.696$.
For x = 2, $\mu = 3\ \exp[-1.97234 + (2)(0.91629)] = 2.609$.
For x = 3, $\mu = 4\ \exp[-1.97234 + (3)(0.91629)] = 8.696$.

[56] I used a computer to solve these two equations. One can confirm that these values satisfy these equations.

<u>Offsets, Poisson Model with Log Link Function</u>:

When fitting a Poisson Distribution with a log link function, it is common to state the model with an offset term which is ln[exposures].
Offset terms are used to adjust for group size or differing time periods of observation.

With the log link function: $\lambda_i = \exp[\eta_i]$. We assume that Y_i is Poisson, with mean $n_i \lambda_i$, where n_i is the number of exposures for observation i.
$\mu_i = n_i \lambda_i = n_i \exp[\eta_i]$. $\Leftrightarrow$ $\mathbf{\ln[\mu_i] = \ln[n_i] + \eta_i}$. [57]

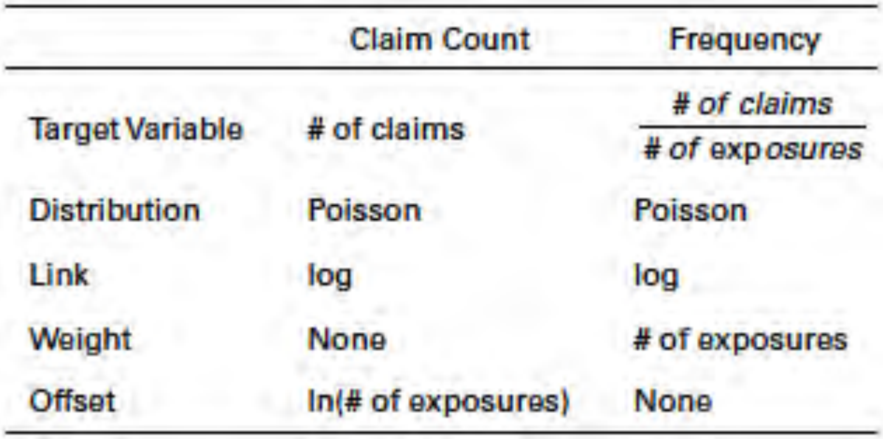
Thus we have rewritten the usual equation relating the mean to the linear predictor, $\eta = X\beta$, with an additional term, **$\ln[n_i]$ which is called the offset. Note that the offset involves a vector of known amounts, the number of exposures corresponding to each observation**.

Computer software to fit GLMs will have an option to include an offset term.

In the previous example: $\ln[\mu_i] = \ln[n_i] + \beta_0 + \beta_1 x_i$. $\Leftrightarrow$ $\mu_i = n_i \exp[\beta_0 + \beta_1 x_i]$.
Thus the use of an offset term will produce an equivalent model and the same result as obtained previously taking into account exposures.

One can show that "**a claim count model that includes exposure as an offset is exactly equivalent to a frequency model that includes exposure as a weight (but not as an offset)**—that is, they will yield the same predictions, relativity factors and standard errors."[58]

Since the number of exposures are known quantities, if one predicts the number of claims one also predicts the claim frequency and vice versa. For the Poisson and a log link function, one can either use an offset of ln(exposures) in a claim count model, or one can use exposures as weights in a claim frequency model. The following table summarizes this equivalence:[59]



	Claim Count	Frequency
Target Variable	# of claims	$\frac{\textit{\# of claims}}{\textit{\# of exposures}}$
Distribution	Poisson	Poisson
Link	log	log
Weight	None	# of exposures
Offset	ln(# of exposures)	None

[57] In other words, the expected number of claims is proportional to exposures, such as car years.
[58] "While this equivalence holds true for the Poisson (or overdispersed Poisson) distribution, it does not work for the negative binomial distribution since the two approaches may yield different estimates of the negative binomial parameter κ."
[59] Taken from Section 2.6 of <u>Generalized Linear Models for Insurance Rating</u>.

Offsets, When Updating Only Part of the Rating Plan:[60]

“When updating deductible factors, it is frequently desirable to calculate them using traditional loss elimination-based techniques, while the GLM is used for factors other than deductible.”

Assume for example, one is updating other parts of the rating algorithm, but is leaving the deductible credits the same.[61] The current deductibles and credits are a follows:

$500	Base
$1000	8% credit
$2500	14% credit

Then in a GLM for pure premium using a log link function:

$$\mu = \exp[X\beta]\ f_D,$$

where $X\beta$ is the linear predictor (not taking into account deductible),
and f_D is the appropriate deductible factor of: 1, 0.92, or 0.86.

$\ln[\mu] = X\beta + \ln[f_D] = X\beta + \text{offset}$.

This is mathematical the same as the use of an offset in the case of a Poisson frequency. However, there the offset was ln[exposures] while here the offset is ln[1 - deductible credit].

If an observation is from a policy with a $500 deductible, then the offset is ln[1] = 0.
If an observation is from a policy with a $1000 deductible, then the offset is ln[1 - 0.08] = -0.0834.
If an observation is from a policy with a $2500 deductible, then the offset is ln[1 - 0.14] = -0.1508.

The expected pure premium for a policy with a $2500 deductible is lower than that of a similar policy with a $500 deductible. If the mix of deductibles varies by the other classification variables, then we know that completely ignoring deductibles would lead to distorted estimates of the effects of the other classification variables. The use of the offset term takes into account deductible; however, we are assuming the effects of deductibles are known based on the current credits and that there is no (significant) interaction of effects between deductible amount and other classification variables.

In general, **an offset factor is a vector of known amounts which adjusts for known effects not otherwise included in the GLM**.

“It is recommended that factors for coverage options—deductible factors, ILFs, peril group factors and the like—be estimated outside the GLM, using traditional actuarial loss elimination techniques. The resulting factors should then be included in the GLM as an offset.”

[60] See Section 2.6 of Generalized Linear Models for Insurance Rating.

[61] Either you will update them at some later date, or the deductible credits will be determined by some technique other than by using a GLM.

As another example, one could take the current territories and territory relativities as givens, and include an offset term in a GLM of ln[territory relativity].

"Territories are not a good fit for the GLM framework. ... Since there are usually many complicated relationships between territory and other variables, your GLM should still consider territory. This is accomplished by including territory in your model as an offset."

Offsets, General Mathematics:

The offset is added to the linear component: $g(\mu_i) = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + ...+ \beta_p x_{ip}$ + offset.
While I have discussed offsets in the context of the log link function, offsets can be used with other link functions.

For example, an actuary is planning to add a predictor to a model that estimates the probability of a policy having a claim. The actuary has decided to offset all of the current model variables before fitting the new variable. Given the following:

- The current model is a logit link binomial GLM (logistic regression).
- The current fitted probability without the new variable is 5% for an individual.

The logit link function is defined as $g(\mu) = \ln\left(\frac{\mu}{1 - \mu}\right)$.

Thus the offset for this individual is: $\ln\left(\frac{5\%}{1 - 5\%}\right) = -2.944$.

In a similar manner one would calculate an offset for each individual.
Where x is the new variable, one would now fit a model: $g(\mu_i) = \beta_0 + \beta_1 x_i + \text{offset}_i$.[62]

Assume the fitted model was: $\beta_0 = -0.5$ and $\beta_1 = 0.1$.
Assume the same individual as before has a value of the new predictor of x = 8.
Then: $\ln\left(\frac{\mu_i}{1 - \mu_i}\right) = -0.5 + (0.1)(8) - 2.944 = -2.644$.

The new fitted probability of a claim for this individual is: $\mu_i = \frac{\exp[-2.644]}{1 + \exp[-2.644]} = 6.6\%$.

Exercise: Another individual has a current fitted probability of 8%, and x = 2.
Calculate the revised fitted probability of having a claim for this individual.
[Solution: Offset = $\ln\left(\frac{8\%}{1 - 8\%}\right) = -2.442$. $-0.5 + (0.1)(2) - 2.442 = -2.742$.

$\frac{\exp[-2.742]}{1 + \exp[-2.742]} = 6.1\%$.

Comment: See 8, 11/18, Q.7.]

[62] The betas are not the same as shown previously. We would continue to use the same logit link function.

Prior Weights:[63]

When observing numbers of claims, the volume of data is numbers of exposures. When observing sizes of claims, the volume of data is numbers of claims.[64] When a given observation is based on more data we give it more weight.

Let us return to the example with two types of drivers, male and female, and two territories, urban and rural. Before we assumed an equal number of claims in each of the four combinations.
Instead let us assume that the Urban/Male combination has twice the volume of the others; in other words Urban/Male has twice as many claims as each of other the other combinations.

Let us assume that we have the same observed average severities:

		Urban	Rural
Male		800	500
Female		400	200

Let us again assume the following generalized linear model:
Gamma Function
Reciprocal link function
x_1 = 1 if male.
x_2 = 1 if female.
x_3 = 1 if urban and x_3 = 0 if rural.

Then $1/\mu = \sum\beta_i x_i = \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3$. $\Rightarrow \mu = \dfrac{1}{\beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3}$.

Therefore, the modeled means are:

		Urban	Rural
Male		$1/(\beta1+\beta3)$	$1/\beta1$
Female		$1/(\beta2+\beta3)$	1/b2

For the Gamma Distribution as per Loss Models, $f(y) = (y/\theta)^a \exp[-y/\theta] / (y\ \Gamma[\alpha])$.

$$\ln f(y) = (\alpha-1)\ln(y) - y/\theta - \alpha\ln(\theta) - \ln[\Gamma[\alpha]] = (\alpha-1)\ln(y) - y/(\mu/\alpha) - \alpha\ln(\mu/\alpha) - \ln[\Gamma[\alpha]]$$
$$= (\alpha-1)\ln(y) - \alpha y/\mu - \alpha\ln(\mu) + \alpha\ln(a) - \ln[\Gamma[\alpha]]$$
$$= (\alpha-1)\ln(y) - \alpha y(\beta_0 + \beta_1 x_1 + \beta_2 x_2) + \alpha\ln(\beta_0 + \beta_1 x_1 + \beta_2 x_2) + \alpha\ln(\alpha) - \ln[\Gamma[\alpha]].$$

[63] See Section 2.5 of Generalized Linear Models for Insurance Rating.

[64] In Buhlmann Credibility, N is number of exposures when estimating frequency or pure premiums, but N is number of claims when estimating severity.

Since it now has twice the number of claims, we multiply the contribution from Urban/Male by two.

The loglikelihood is the sum of the contributions from the four combinations:
$(\alpha\text{-}1)\{2\ \ln(800) + \ln(400) + \ln(500) + \ln(200)\}$
$- \alpha\{(2)(800)(\beta_1 + \beta_3) + 400(\beta_2 + \beta_3) + 500\beta_1 + 200\beta_2\}$
$+ \alpha\{2\ln(\beta_1 + \beta_3) + \ln(\beta_2 + \beta_3) + \ln(\beta_1) + \ln(\beta_2)\}\} + 5\alpha\ln(\alpha) - 5\ l\ \ln[\Gamma[\alpha]]$.

To maximize the loglikelihood, we set its partial derivatives equal to zero.
Setting the partial derivative with respect to β_1 equal to zero:
$0 = -\alpha(1600 + 500) + \alpha\{2/(\beta_1 + \beta_3) + 1/\beta_1\}. \Rightarrow 2/(\beta_1 + \beta_3) + 1/\beta_1 = 2100.$
Setting the partial derivative with respect to β_2 equal to zero:
$0 = -\alpha(400 + 200) + \alpha\{1/(\beta_2 + \beta_3) + 1/\beta_2\}. \Rightarrow 1/(\beta_2 + \beta_3) + 1/b_2 = 600.$
Setting the partial derivative with respect to β_3 equal to zero:
$0 = -\alpha(1600 + 400) + \alpha\{2/(\beta_1 + \beta_3) + 1/(\beta_2 + \beta_3)\}. \Rightarrow 2/(\beta_1 + \beta_3) + 1/(\beta_2 + \beta_3) = 2000.$

Solving these three equations in three unknowns:[65]
$\beta_1 = 0.00224451$, $\beta_2 = 0.00392976$, and $\beta_3 = -0.00103566$.
$\mu = 1 / (0.00224451x_1 + 0.00392976x_2 - 0.00103566x_3)$.

For Male and Urban: $x_1 = 1$, $x_2 = 0$, $x_3 = 1$, and $\mu = 1 / (0.00224451 - 0.00103566) = 827.23$.
For Female and Urban: $x_1 = 0$, $x_2 = 1$, $x_3 = 1$, and $\mu = 1 / (0.00392976 - 0.00103566) = 345.53$.
For Male and Rural: $x_1 = 1$, $x_2 = 0$, $x_3 = 0$, and $\mu = 1/0.00224451 = 445.53$.
For Female and Rural: $x_1 = 0$, $x_2 = 1$, $x_3 = 0$, and $\mu = 1/0.00392976 = 254.47$.

The fitted severities by cell are:

		Urban	Rural
Male		827.23	445.30
Female		345.53	254.47

Which differ from those obtained previously when we had equal weights.

[65] I used a computer to solve these three equations.
There is no need to solve for α in order to calculate the fitted pure premiums by cell.

Let us examine what I did in a little more detail.

My contribution to the loglikelihood for Male/Urban was:
$2 \ln f(800) = 2\{(\alpha-1)\ln(800) - 800/\theta - \alpha\ln(\theta) - \ln[\Gamma(\alpha)]\}$
$= 2 \{(\alpha-1)\ln(800) - 800\alpha/\mu - \alpha\ln(\mu/\alpha) - \ln[\Gamma(\alpha)]\}$.
This is the same as assuming two claims each of size 800 were observed.

If instead, we had two claims, one of size 600 and one of size 1000, averaging to the same 800, then the contribution to the loglikelihood for Male/Urban would be:
$\ln f(600) + \ln f(1000) =$
$(\alpha-1)\ln(600) - 600\alpha/\mu - \alpha\ln(\mu/\alpha) - \ln[\Gamma(\alpha)]] + (\alpha-1)\ln(1000) - 10000\alpha/\mu - \alpha\ln(\mu/\alpha) - \ln[\Gamma(\alpha)]]$
$= (\alpha-1) \{\ln(600) + \ln(1000)\} - 1600\alpha/m - 2\alpha\ln(\mu/\alpha) - 2 \ln[\Gamma(\alpha)]]$.

This differs from before by some constant times $\alpha - 1$. However, this does not affect the fitted maximum likelihood parameters; when we take a partial derivative with respect to β_i these terms will drop out.

If we only use the fact that Urban/Male has two claims summing to 1600, then we can use the fact that the sum of two identically distributed Gammas has twice the alpha.[66] The mean will also be twice as big, so that $\theta = \mu/\alpha$ would remain the same. Thus the contribution to the loglikelihood for Male/Urban would be the log density of this Gamma with 2α at 1600:
$(2\alpha-1)\ln(1600) - 1600\alpha/\mu - 2\alpha\ln(\mu/\alpha) - \ln[\Gamma(\alpha)]]$.

Again, this differs from before by terms that involve constants and alpha. However, this does not affect the fitted maximum likelihood parameters; when we take a partial derivative with respect to β_i these terms will drop out.

The members of exponential families each have this nice property that the maximum likelihood fit only depends on the average and not the individual values.[67]

In general, **when modeling severity, let the weights ω_i be the number of claims**.

So for example, if an observation is the average size of 10 claims, then the variance will be 1/10 of that for an observation of the size of a single claim.

For example, for the Poisson, $f(x) = \lambda^x e^{-\lambda} / x!$. $\ln f(x) = x \ln(\lambda) - \lambda - \ln(x!)$.
If we have two (independent) exposures each with mean frequency x, then we can multiply the contribution to the loglikelihood by two: $2x \ln(\lambda) - 2\lambda - 2 \ln(x!)$.

[66] The other exponential families share the property that when one adds up independent, identical copies one gets anther member of the same family.
[67] The mean is a sufficient statistic.

If we have two (independent) exposures each with Poissons with mean λ, then the number of claims is Poisson with mean 2λ.

Then with a sum of 2x, and an average frequency of x, the log density is:
$2x \ln(2\lambda) - 2\lambda - \ln(2x!) = 2x \ln(\lambda) - 2\lambda - 2x \ln(2) - \ln(2x!)$.

Except for constants and terms involving x, this is the same loglikelihood as before.
Thus we would get the same maximum likelihood fit.
Thus when modeling claim frequencies, one can weight by the number of exposures.

When modeling claim frequency or pure premiums, let the weights be exposures.

When a weight is specified, the assumed variance for (the mean of) observation i is inversely proportional to the weight:[68] $\mathbf{Var[Y_i] = \phi\ V[\mu_i] / \omega_i}$.

[68] This is our usual assumption that the variance of an average is inversely proportional to the number of items being averaged.

A Three Dimensional Example of a GLM:[69]

Here is a three dimensional example for private passenger automobile insurance claim frequency, with: age of driver, territory, and vehicle class.[70] It is a multiplicative model, in other words a GLM with a log link function.[71]

There are 9 levels for driver age, 8 territories, and 5 classes of vehicle. An intercept term is used. Therefore, since each of the three factors is a categorical variable, each has one less parameter than its number of levels. In addition to the intercept term, there are 8 driver age parameters, 7 territory parameters, and 4 vehicle class parameters.

Choose age group 40-49, territory C, and vehicle class A, as the base levels.[72] [73]
Let β_1 correspond to the intercept term, and assign the other parameters as follows:

Age of driver			Territory			Vehicle class	
Factor level	Parameter		Factor level	Parameter		Factor level	Parameter
17-21	β_2		A	β_{10}		A	
22-24	β_3		B	β_{11}		B	β_{17}
25-29	β_4		C			C	β_{18}
30-34	β_5		D	β_{12}		D	β_{19}
35-39	β_6		E	β_{13}		E	β_{20}
40-49			F	β_{14}			
50-59	β_7		G	β_{15}			
60-69	β_8		H	β_{16}			
70+	β_9						

The total number of cells is: (9)(8)(5) = 360.
So the design matrix would have 360 rows, assuming that there are no cells lacking data.

[69] See pages 31 to 32 of "A Practitioner's Guide to Generalized Linear Models," by Duncan Anderson; Sholom Feldblum; Claudine Modlin; Doris Schirmacher; Ernesto Schirmacher; and Neeza Thandi, (Third Edition), CAS Study Note, February 2007. Not on the syllabus of this exam.

[70] Presumably, there would be another GLM fit to severity.

[71] We are not told what distributional form is assumed, but it is probably Poisson.
We are not given any details of the fitting or any diagnostics.

[72] One could make another set of choices and should get the same fitted frequencies.

[73] The standard errors of the fitted parameters are smaller if one chooses as the base level the one with the most exposures.

For example, the first row of the design matrix is probably for age 17-21, Territory A, and Class A, with ones in column 1, 2, and 10:[74]
1 1 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0.

The last row of the design matrix is probably for age 70+, Territory H, and Class E, with ones in column 1, 9, 16, and 20:
1 0 0 0 0 0 0 0 1 0 0 0 0 0 0 1 0 0 0 1.

Exercise: For age 35-39, Territory F, and Class C, what does the corresponding row of the design matrix look like?
[Solution: Ones in columns 1, 6, 14, and 18:
1 0 0 0 0 1 0 0 0 0 0 0 0 1 0 0 0 1 0 0.]

The fitted parameters are an intercept term of 0.1412 and:[75] [76]

Age of driver			Territory			Vehicle class	
Factor level	Multiplier		Factor level	Multiplier		Factor level	Multiplier
17-21	1.6477		A	0.9407		A	1.0000
22-24	1.5228		B	0.9567		B	0.9595
25-29	1.5408		C	1.0000		C	1.0325
30-34	1.2465		D	0.9505		D	0.9764
35-39	1.2273		E	1.0975		E	1.1002
40-49	1.0000		F	1.1295			
50-59	0.8244		G	1.1451			
60-69	0.9871		H	1.4529			
70+	0.9466						

The estimated frequency for a 40-49 year old driver, from Territory C and Vehicle Class A, is 0.1412; the estimate for the base levels is the intercept term.[77]
For example, a 22-24 year old driver, from Territory G and Vehicle Class D would have an estimated frequency of: (1.5228)(1.1451)(0.9764)(0.1412) = 0.2404.

Exercise: What is the estimated frequency for a 30-34 year old driver, from Territory B and Vehicle Class E?
[Solution: (1.2465)(0.9567)(1.1002)(0.1412) = 0.1853.]

[74] How you arrange the rows of the design matrix does not affect the result, as long as everything is done consistently.

[75] For example, the fitted value of β_2 is ln(1.6447).
The multipliers for the base levels are one by definition.

[76] This is presumably illustrative rather than the output of a GLM fit in a practical application.

[77] In order to estimate the overall average frequency, one would need the distribution of exposures by cell.

Tweedie Distribution:[78]

Another (linear) exponential family is the Tweedie Distribution.
The Tweedie Distribution has mean μ and its **variance is proportional to μ^p, for $1 < p < 2$**.[79] [80]

The Tweedie Distribution is used to model pure premiums (losses divided by exposures) or loss ratios; there is a point mass of probability at zero corresponding to no loss.
The Tweedie Distribution is mathematically a special case of a Compound Poisson Distribution.

When the Tweedie is used in GLMs, p and ϕ are constant across all observations.

When using the Tweedie distribution, it turns out that an increase in pure premium is made up of both an increase in frequency and an increase in severity.[81] Even if this assumption does not hold in an given application, the Tweedie GLM can still produce very useful and well fitting models of pure premium.

Details of the Tweedie Distribution:

It is a Poisson frequency with a Gamma severity, with parameters of the Poisson and Gamma:[82]

$$\lambda = \frac{\mu^{2-p}}{\phi\ (2 - p)},\ \alpha = \frac{2 - p}{p - 1},\ \text{and}\ \theta = \phi\ (p-1)\ \mu^{p-1}.$$

Exercise: Verify the mean and variance of the Tweedie as a Compound Poisson.

[Solution: Mean = $\lambda\alpha\theta = \frac{\mu^{2-p}}{\phi\ (2 - p)}\ \frac{2 - p}{p - 1}\ \phi\ (p-1)\ \mu^{p-1} = \mu$.

Variance = λ (2nd moment of Gamma) = $\lambda\ \alpha(\alpha+1)\theta^2$

$$= \frac{\mu^{2-p}}{\phi\ (2 - p)}\ \frac{2 - p}{p - 1}\ \frac{1}{p - 1}\ \{\phi\ (p-1)\ \mu^{p-1}\}^2 = \phi\ \mu^p.]$$

Exercise: What is the point mass at zero of the Tweedie as a Compound Poisson.
[Solution: This corresponds to the Poisson in the Compound Poisson being zero.
This has probability $e^{-\lambda}$.]

[78] See Section 2.7.3 of Generalized Linear Models for Insurance Rating.

[79] For insurance modeling, p is typically between 1.5 and 1.8.
In some software packages, one can specify the Tweedie distribution, which will in turn cause the software package to find the best value for the power parameter, p, when solving for the parameters (betas) in the linear equation.

[80] While the syllabus reading puts this restriction on p, mathematically p can be any nonnegative value other than $0 < p < 1$.

[81] This is far from obvious. Why this is the case is discussed subsequently.

[82] For use in a GLM, ϕ and α (and thus p) are fixed for all observations.

$\alpha = \frac{2 - p}{p - 1}$. $\Rightarrow p = \frac{\alpha+2}{\alpha+1}$.

As alpha, the shape parameter of the Gamma, approaches infinity, p approaches 1, and the Tweedie approaches a Poisson. For p near one, the CV of the Gamma is small, and most of the randomness is due to the Poisson frequency.[83] As alpha approaches zero, p approaches 2, and the Tweedie approaches a Gamma.

Several of the other exponential family distributions are in fact special cases of Tweedie, dependent on the value of p:

- A Tweedie with $p = 0$ is a Normal distribution.
- A Tweedie with $p = 1$ and $\phi = 1$ is a Poisson distribution.[84]
- A Tweedie with $p = 2$ is a Gamma distribution.
- A Tweedie with $p = 3$ is an inverse Gaussian distribution.

The mean of the Tweedie is: $\mu = \lambda\alpha\theta$. Also it turns out that: $\phi = \frac{\lambda^{1-p} (\alpha\theta)^{2-p}}{2 - p}$.

For a Compound Poisson with Gamma severity, we have Prob[X = 0] = $e^{-\lambda}$, and for $x > 0$:[85]

$$f(x) = \sum_{n=1}^{\infty} \frac{e^{-\lambda}\lambda^n}{n!} \frac{e^{-x/\theta} x^{n\alpha-1}}{\Gamma[n\alpha]\,\theta^{n\alpha}} = \exp[-x/\theta - \lambda] \sum_{n=1}^{\infty} \frac{(\lambda / \theta^{\alpha})^n x^{n\alpha-1}}{n!\,\Gamma[n\alpha]}.$$

$\theta^{\alpha} = \phi^{(2-p)/(p-1)} (p-1)^{(2-p)/(p-1)} \mu^{2-p}$. We had: $\lambda = \frac{\mu^{2-p}}{\phi\,(2 - p)}$. Thus λ/θ^{α} does <u>not</u> depend on μ.

Thus the above sum does <u>not</u> depend on μ.

For a given GLM using the Tweedie, ϕ and $1 < p < 2$ are fixed. $\Rightarrow \alpha = \frac{2 - p}{p - 1}$ is fixed.

If μ increases, then $\lambda = \frac{\mu^{2-p}}{\phi\,(2 - p)}$ and $\theta = \phi\,(p-1)\,\mu^{p-1}$ each also increase.

Thus if the mean increases, then both mean frequency = λ, and mean severity = $\alpha\theta$ increase.

[83] For the Gamma Distribution, the coefficient of variation is $1/\sqrt{\alpha}$.

[84] If $p = 1$ and $\phi \neq 1$, then the Tweedie is an overdispersed Poisson distribution.

[85] Using the fact that the sum of n independent, identically distributed Gammas is another Gamma Distribution with parameters $n\alpha$ and θ.

An Example of a Tweedie Distribution:

Exercise: Take $\mu = 10$, $p = 1.5$, and $\phi = 4$.
Determine the parameters of the Poisson and Gamma.

[Solution: $\lambda = \frac{\mu^{2-p}}{\phi\ (2 - p)} = \frac{10^{0.5}}{(4)\ (2 - 1.5)} = 1.581$. $\alpha = \frac{2 - p}{p - 1} = (2 - 1.5) / (1.5 - 1) = 1$.

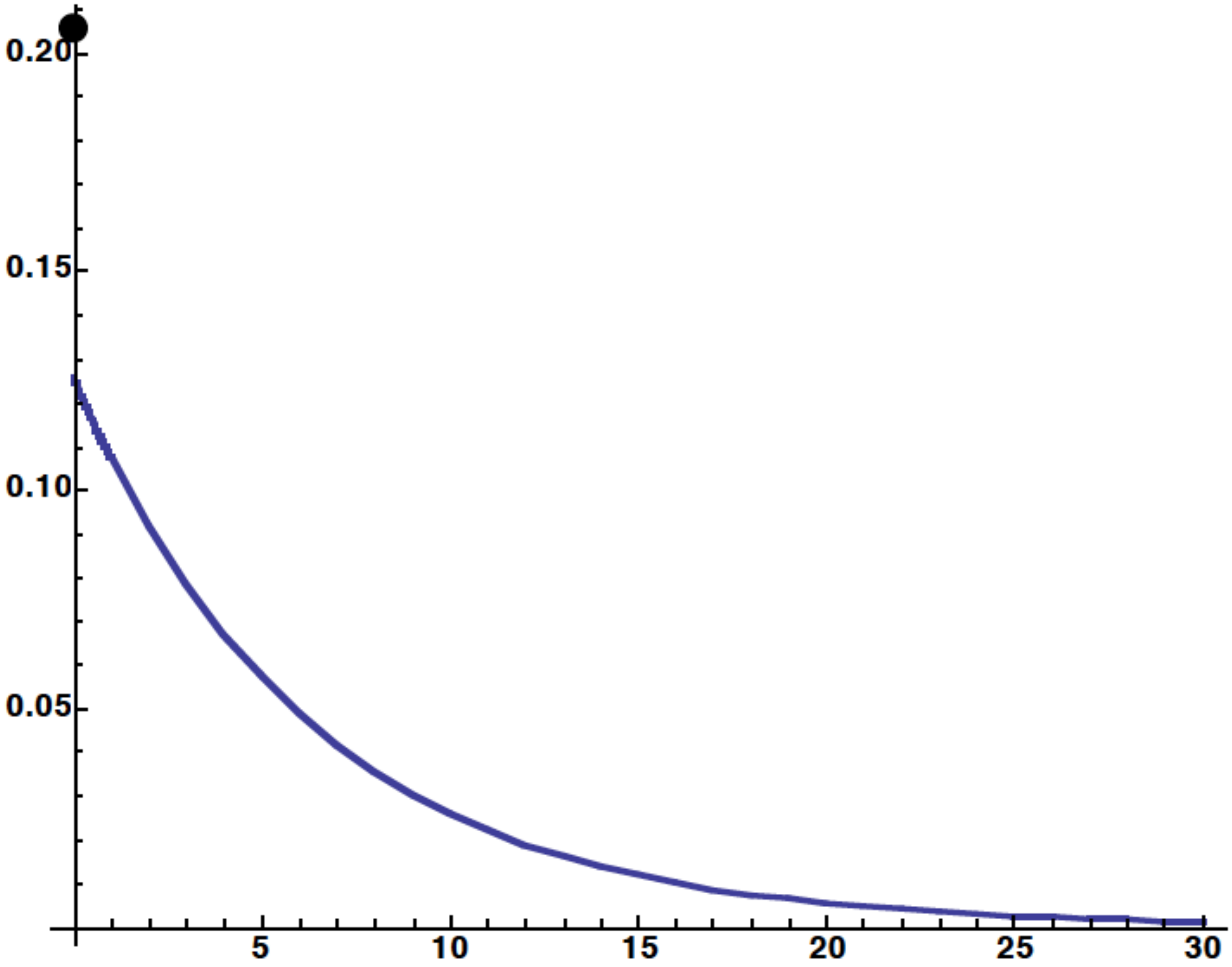
$\theta = \phi\ (p\text{-}1)\ \mu^{p-1} = (4)(1.5 - 1)\ 10^{(1.5-1)} = 6.325$.

Comment: The severity piece of the Compound Poisson is an Exponential with mean 6.325.
The mean of the Compound Poisson is: (1.581) (6.325) = 10 = μ.
The variance of the Compound Poisson is:
(mean of Poisson) (second moment of the Exponential) = $(1.581)\ \{(2)(6.325^2)\} = 126.5 = (4)(10^{1.5}) = \phi\ \mu^p$.]

The density at zero of the Poisson is: $e^{-1.581} = 20.58\%$.
Thus there is a point mass of probability of 20.58% at zero.

Using a computer, this Tweedie has density at one of 0.1072.
This Tweedie has density at ten of 0.0258.
This Tweedie has density at twenty five of 0.0024.

Here is graph of the density of this Tweedie Distribution, including the point mass of probability 20.58% at zero:[86] [87]



[86] For example, at 0.01 the density of the Tweedie is 0.1254.

[87] See Figure 5 in Generalized Linear Models for Insurance Rating.

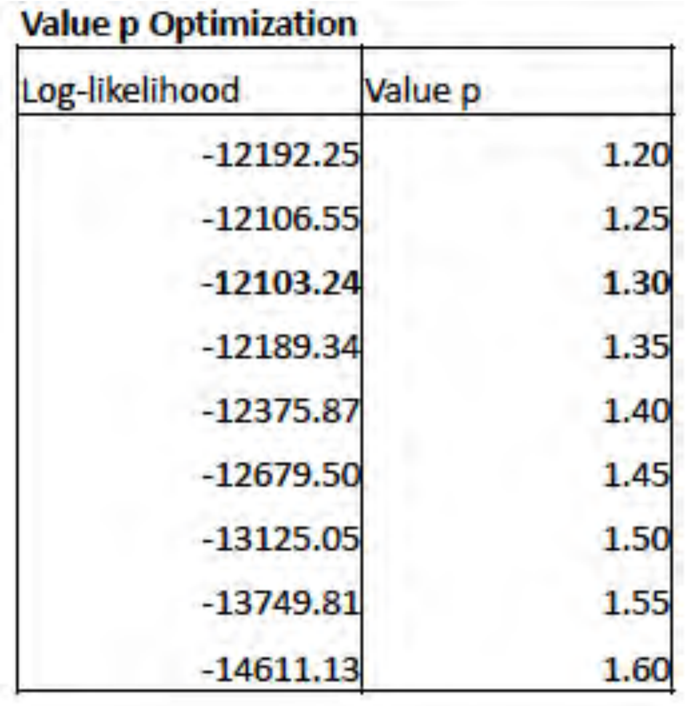
Determination of the p parameter of the Tweedie Distribution:

In order to use the Tweedie Distribution, one needs to determine the p parameter.
There are number of different ways to do so:

- Some model-fitting software packages provide the functionality to estimate p as part of the model-fitting process.[88]
- Several candidate values of p (1 < p < 2) can be considered and tested with the goal of optimizing a statistical measure such as loglikelihood or using cross-validation.
- Simply judgmentally select some value that makes sense.[89]

This last may be the most practical in many situations, as the fine-tuning of p is unlikely to have a very material effect on the model estimates.

Here is an example of the second approach of determining the value of p for a GLM using the Tweedie Distribution:[90]



Value p Optimization

Log-likelihood	Value p
-12192.25	1.20
-12106.55	1.25
-12103.24	1.30
-12189.34	1.35
-12375.87	1.40
-12679.50	1.45
-13125.05	1.50
-13749.81	1.55
-14611.13	1.60

For a sequence of values of the parameter p in the Tweedie model, we compute the loglikelihood of the fitted model. In this case, the loglikelihoods show a smooth inverse U-shape. One would then select the value of p that corresponds to the maximum loglikelihood, in this case p = 1.30.

[88] Using this option may increase the computation time considerably, particularly for larger datasets.
[89] Common choices being 1.6, 1.67, or 1.7.
[90] “Loss Cost Modeling vs. Frequency and Severity Modeling,” by Jun Yan, 2017 CAS Spring Meeting.

Standard Errors and Confidence Intervals for Fitted Parameters:

A standard error is the standard deviation of an estimated coefficient.[91] Computer software for fitting GLMs will output the fitted coefficients and the corresponding standard errors.[92]

For GLMs for large samples, the Maximum Likelihood estimator is approximately multivariate Normal and asymptotically unbiased. Thus in GLM output, it is common to graph the fitted parameters and also bands plus or minus two standard errors.[93]

For example we might have fitted coefficients of:

$\hat{\beta}_0 = 223$, $\hat{\beta}_1 = 1.95$, and $\hat{\beta}_2 = -1.07$.

With corresponding standard errors of: 30.3, 0.607, and 0.632.

An approximate 95% confidence interval for β_0 is:
223 ± (1.960)(30.3) = (164, 282).

95% confidence interval for β_i is: $\hat{\beta}_i$ ± 1.96 (standard error of β_i).

Exercise: Determine an approximate 95% confidence interval for β_1,
[Solution: 1.95 ± (1.960)(0.607) = (0.76, 3.14).]

Exercise: Determine an approximate 95% confidence interval for β_2,
[Solution: -1.07 ± (1.960)(0.632) = (-2.31, 0.17).]

A standard error of 30.3 for $\hat{\beta}_0$ can be thought of as follows: if one simulated similar sized data sets many times and fit GLMs, the estimated intercepts would have a variance of 30.3^2.
A smaller standard error gives us more confidence in the estimate of the corresponding coefficient.

Larger data sets will produce smaller standard errors than otherwise smaller data sets;
the standard errors go down approximately as the square root of the sample size.
The larger the estimated dispersion parameter ϕ, the more randomness there is in the data, and thus the larger the standard error; the standard error goes up as $\sqrt{\phi}$.

[91] This is similar to the standard error of a regression.

[92] Most software will also output the covariance matrix. The variances are along the diagonal.
The standard errors are the square roots of the variances.

[93] See Figure 2 in Generalized Linear Models for Insurance Rating.

One can perform hypothesis tests. For example, we can test $\beta_1 = 0$ versus $\beta_1 \neq 0$.
The probability value of this two-sided test is: $2\{1 - \Phi[1.95/0.607]\} = 2\{1 - \Phi[3.21]\} = 0.1\%$.[94]

p-value = Prob[test statistic takes on a value equal to its calculated value or
a value less in agreement with H_0 (in the direction of H_1) | H_0].

For a p-value sufficiently small, we can reject the null hypothesis in favor of the alternative hypothesis that the slope is non-zero. In this case, with a p-value of 0.1% we reject the hypothesis that $\beta_1 = 0$.

Exercise: Test $\beta_2 = 0$ versus $\beta_2 \neq 0$.
[Solution: p-value = $2\,\Phi[-1.07/0.632] = 2\,\Phi[-1.69] = 9.1\%$.
Therefore, we reject the null hypothesis at 10% and do not reject the null hypothesis at 5%.
Comment: Since zero was not in the 95% confidence interval for b_2,
we reject the null hypothesis at 5%.
Note that “not reject” is the correct statistical language, although actuaries sometimes say “accept”.]

At the 10% significance level we can reject the hypothesis that $\beta_2 = 0$. However, at the 5% significance level there is insufficient evidence to reject the hypothesis that $\beta_2 = 0$.

We can perform two-sided tests: $\beta_2 = 0$ versus $\beta_2 \neq 0$.
We can also perform one-sided tests: $\beta_2 = 0$ versus $\beta_2 > 0$, or $\beta_2 = 0$ versus $\beta_2 < 0$.

[94] A table of the Normal Distribution will not be attached to your exam.

Using a p-value of 5%:

“A common statistical rule of thumb is to reject the null hypothesis where the p-value is 0.05 or lower. However, while this value may seem small, note that it allows for a 1-in-20 chance of a variable being accepted as significant when it is not. Since in a typical insurance modeling project we are testing many variables, this threshold may be too high to protect against the possibility of spurious effects making it into the model.”[95]

For example, if we are testing the potential usefulness of 60 possible predictor variables, then if we use a p-value of 5%, even if none of the variables actually predict the outcome, on average three of these 60 variables will be selected as significant.

I performed a simulation experiment. I simulated 500 random observations from each of 60 independent normally distributed predictor variables. Then I simulated 500 observations from a normally distributed response variable.[96]

However, the response variable was independent of the predictor variables. In other words, none of the 60 predictor variables was actually useful for predicting the response variable. Then I fit a multiple regression to this data.[97]

The p-values of the 60 fitted slopes, were from smallest to largest:
0.005, 0.009, 0.020, 0.095, 0.109, 0.121, 0.148, 0.159, 0.177, 0.181, 0.196, 0.206, 0.253, 0.275, 0.331, 0.333, 0.387, 0.421, 0.423, 0.455, 0.494, 0.495, 0.495, 0.513, 0.521, 0.522, 0.545, 0.549, 0.562, 0.591, 0.593, 0.610, 0.614, 0.618, 0.629, 0.637, 0.645, 0.649, 0.653, 0.676, 0.684, 0.707, 0.707, 0.758, 0.778, 0.778, 0.790, 0.806, 0.825, 0.861, 0.886, 0.894, 0.894, 0.916, 0.941, 0.952, 0.980, 0.982, 0.987, 0.993.

We note that even though none of the 60 potential predictor variables is useful, three of the slopes are significant at the 5% level.[98] This illustrates the difficulty of relying on p-values when one starts with a large number of potential predictor variables. In such situations, it is very important to test any selected model on a separate holdout data set, as has been discussed.[99]

[95] Quoted from Section 2.3.2 of Generalized Linear Models for Insurance Rating.
[96] A Normal Distribution was used for simplicity.
[97] This is a special case of a GLM, with a Normal response using the identity link function.
[98] While we would expect (5%)(60) = 3 significant slopes, the fact that in this simulation it is exactly three is a coincidence.
[99] One can instead use k-fold validation, as discussed previously.

Log Link Function and Continuous Variables:[100]

As will be discussed, taking the log of continuous variables provides more variety of behaviors.
⇒ One is more likely to find a behavior that fits your data.

Assume we are using the log link function.
For example: $\mu = \exp[\beta_0 + \beta_1 x_1 + \beta_2 x_2]$.

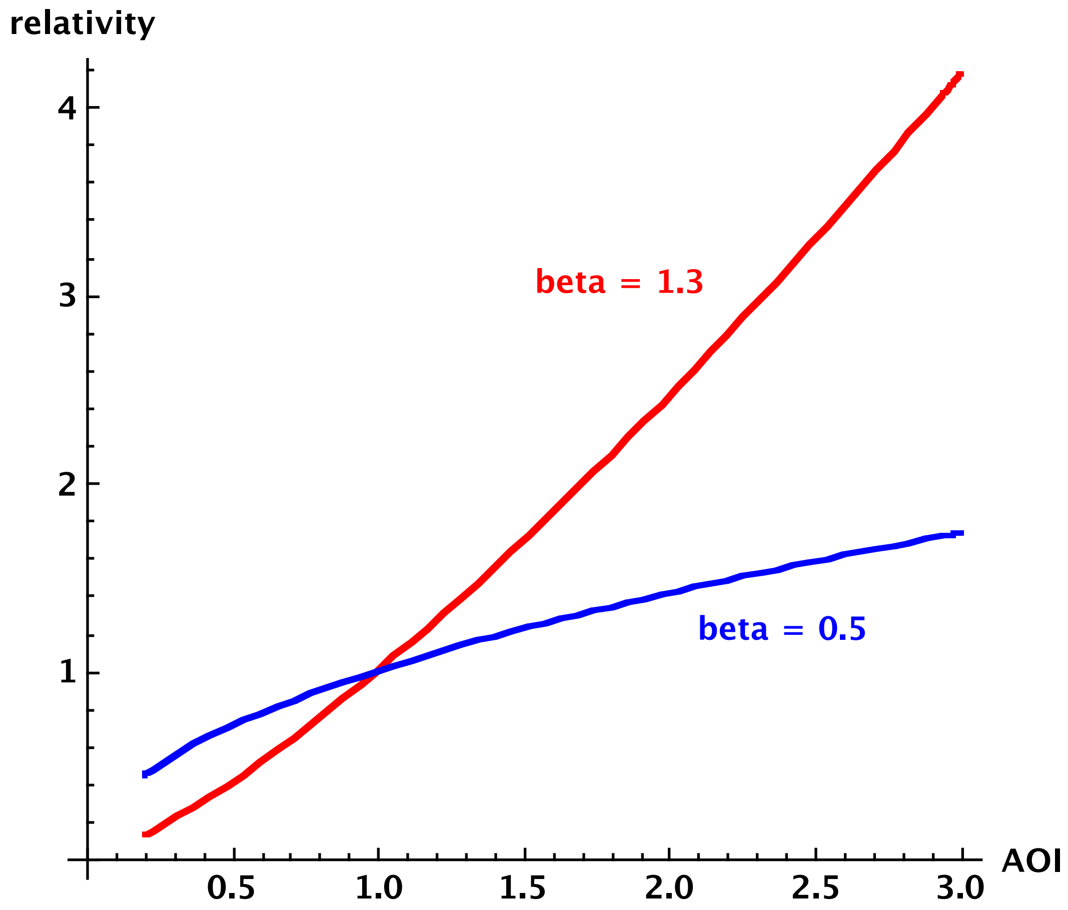
Then $\mu = \exp[\beta_0 + \beta_2 x_2] \exp[\beta_1]^{x_1}$.
Thus the multiplicative relativity for x_1 is $\exp[\beta_1]^{x_1}$.

Assume x_1 is a continuous variable such as amount of insurance.[101]
For example, if $\beta_1 = 0.5$, then $\exp[\beta_1]^{x_1} = 1.649^{AOI}$.
If instead $\beta_1 = 1.1$, then $\exp[\beta_1]^{x_1} = 3.004^{AOI}$.
Both of these curves have the same form, exponential growth: c^x, where c is some constant.

What if instead of using x_1 as the predictor variable, we used $\ln[x_1]$?
$\mu = \exp[\beta_0 + \beta_1 \ln[x_1] + \beta_2 x_2] = \exp[\beta_0 + \beta_2 x_2]\, {x_1}^{\beta_1}$.
Now the multiplicative relativity for amount of insurance is AOI^{β_1}.

For example, if $\beta_1 = 0.5$, then the multiplicative relativity is $AOI^{0.5}$.
If instead $\beta_1 = 1.3$, then the multiplicative relativity is $AOI^{1.3}$.
These are significantly different behaviors.[102]

[100] See Section 2.4.1 of Generalized Linear Models for Insurance Rating.
[101] We have not grouped the variable into levels.
[102] See Figure 1 in Generalized Linear Models for Insurance Rating.



This variety of behaviors makes it more likely to find a model that fits the data.
⇒ **The authors recommend that when using the log link function in a GLM, you log your continuous predictor variables**.[103]

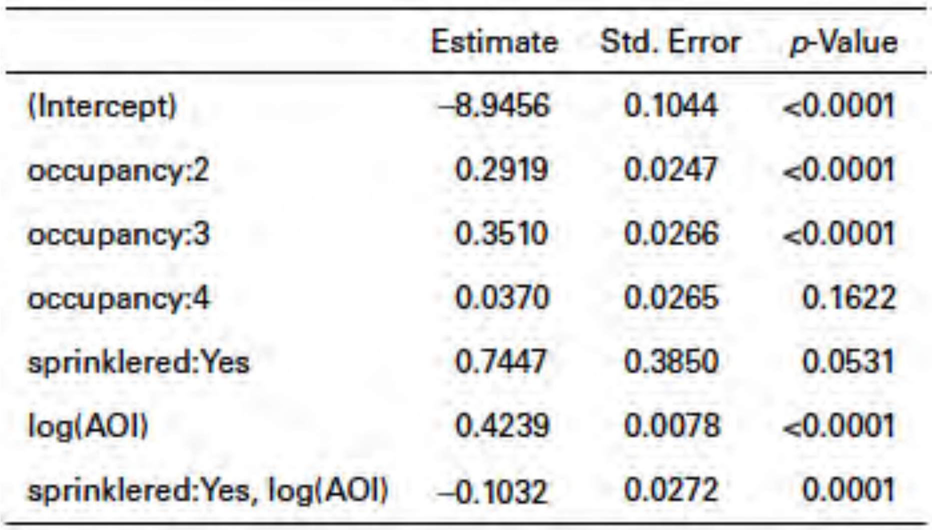
“Note that this suggestion is not due to any statistical law, but rather it is a rule of thumb specific to the context of insurance modeling, and is based on our a priori expectation as to the relationship between losses and the continuous predictors typically found in insurance models.”

“For some variables, logging may not be feasible or practical. For example, variables that contain negative or zero values cannot be logged without a prior transformation. Also, for artificial continuous variables (such as credit scores) we may not have any a priori expectation as to whether the natural form or the logged form would better capture the loss response.”

Usually the model will be easier to interpret if for example we used ln[AOI / 200,000], rather than ln[AOI]. While this will be easier to interpret, it produces a mathematically equivalent model to using ln[AOI].

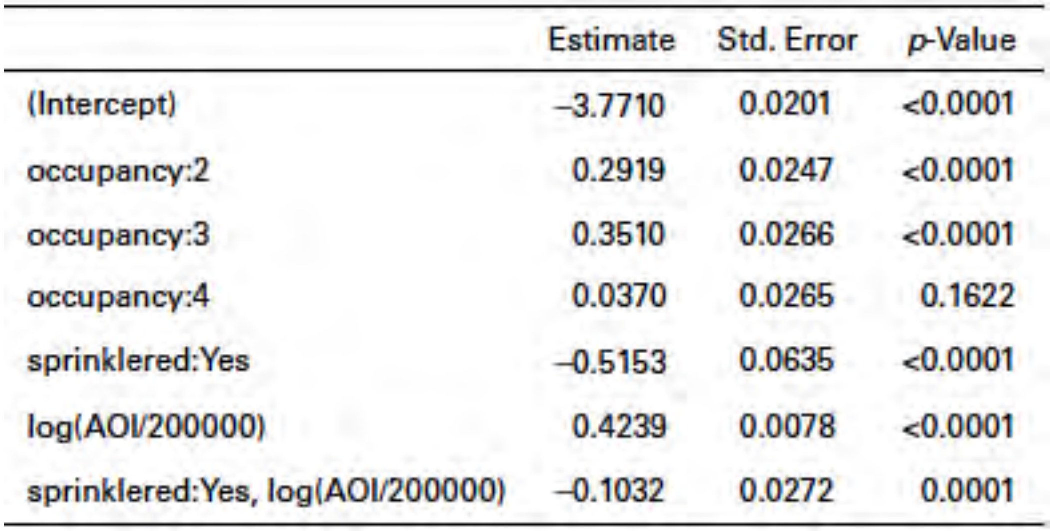
[103] “This allows the scale of the predictors to match the scale of the entity they are linearly predicting, which in the case of a log link is the log of the mean of the outcome.”
This is an empirical question. There will be cases where not taking the log of a continuous predictor variable will result in a GLM that better fits the data; for example, this may be the case when the continuous predictor is year.

The syllabus reading has an example of a commercial building claims frequency model. When ln[AOI] was used the output was:[104]



	Estimate	Std. Error	*p*-Value
(Intercept)	–8.9456	0.1044	<0.0001
occupancy:2	0.2919	0.0247	<0.0001
occupancy:3	0.3510	0.0266	<0.0001
occupancy:4	0.0370	0.0265	0.1622
sprinklered:Yes	0.7447	0.3850	0.0531
log(AOI)	0.4239	0.0078	<0.0001
sprinklered:Yes, log(AOI)	–0.1032	0.0272	0.0001

When instead ln[AOI/200,000] was used the output was:[105]



	Estimate	Std. Error	*p*-Value
(Intercept)	–3.7710	0.0201	<0.0001
occupancy:2	0.2919	0.0247	<0.0001
occupancy:3	0.3510	0.0266	<0.0001
occupancy:4	0.0370	0.0265	0.1622
sprinklered:Yes	–0.5153	0.0635	<0.0001
log(AOI/200000)	0.4239	0.0078	<0.0001
sprinklered:Yes, log(AOI/200000)	–0.1032	0.0272	0.0001

Most of the fitted coefficients stay the same. However, both the intercept and the coefficient of sprinklered have changed.

Originally the sprinklered coefficient was positive and now it is negative. In the first model, it appears that having sprinklers would lead to a higher claims frequency, which does not make intuitive sense. However, we need to also take into account the interaction term.

In the first model, for example, for AOI = 50,000, having sprinklers adds to the linear predictor: 0.7447 + (-0.1032)ln[50,000] = -0.3719. Thus, sprinklered has a lower predicted frequency.

[104] See Table 11 in Generalized Linear Models for Insurance Rating.
[105] See Table 12 in Generalized Linear Models for Insurance Rating.

In the first model, this addition to the linear predictor would be zero for:
0 = 07447 + (-0.1032) ln[AOI]. ⇒ AOI = 1361.

In this example, almost all, if not all insured buildings have amounts of insurance larger than 1361. Thus in both models, for actual amounts of insurance for insured buildings, having sprinklers reduces the predicted frequency.

Exercise: An insured building has is occupancy = 2, has AOI = 80,000, and is sprinklered.
For each of the two models, determine the predicted frequency.
[Solution: For the first model:
exp[-8.9456 + 0.2919 + 0.7447 + 0.4239 ln[80,000] - 0.1032 ln[80,000]] = exp[-4.288] = 1.37%.
For the second model:
exp[-3.7710 + 0.2919 - 0.5153 + 0.4239 ln[80,000/200,000] - 0.1032 ln[80,000/200,000]]
= exp[-4.288] = 1.37%.
Comment: The two models give the same result.
Note that: -8.9456 + 0.4239 ln[200,000] ≅ -3.7710.
0.7447 - 0.1032 ln[200,000] ≅ -0.5153.
Thus one can infer what the new intercept and sprinklered coefficient must be after centering.]

The authors recommend that when using the log link function in a GLM, prior to logging a continuous predictor variables you divide by the base level of that continuous variable; in other words, center your continuous variables at their base level.

Centering has the following advantages:[106]

- If all continuous variables are divided by their base values prior to being logged and included in the model, then the intercept term after exponentiating yields the indicated frequency at the base case when all variables are at their base levels. This is both more intuitive and easier to interpret.
- When terms are not centered, you can have unintuitive results. In the given example, the sprinkler coefficient is positive which can appear to indicate a higher frequency for sprinklered buildings than for non-sprinklered buildings. (However, when taking into account the interaction term, this is not true for values of log(AOI) for insured buildings.) This would not happen if AOI had been centered at its base level; the coefficients are more intuitive to understand when variables are centered.
- In this example, with the AOI predictor in this form, the sprinklered coefficient has a more natural interpretation: it is the (log) sprinklered relativity for a risk with the base AOI.

[106] See Section 5.6.2 of Generalized Linear Models for Insurance Rating.

Logistic Regression:[107]

A variable can be categorical; there are a discrete number of categories, but the labels attached to them may have no significance. Variables can be binary; this is a special case of categorical variable with only two categories, which can be thought of as either 0 or 1. Examples include: whether a policyholder renews its policy, whether a newly opened claim will exceed $10,000, whether a newly opened claim will lead to a subrogation opportunity, whether a newly opened claim is fraudulent, etc.

When the response variable is binary we use the Bernoulli Distribution, the Binomial with m = 1. In that case, the probability of the event is μ and the probability of not having the event is $1-\mu$. The ratio **$\mu/(1-\mu)$ is called the odds**.

Exercise: If the probability of an event is 80%, what are the odds?
[Solution: 80% / (1 - 80%) = 4.
Comment: The event is 4 times as likely to occur as not occur.]

The most common link function to use in this case is the logit, the log of the odds:[108]
$g(m) = \ln[\mu/(1-\mu)]$. $\Leftrightarrow \mu = \exp[x'b] / \{1 + \exp[x'b]\}$.

One can group similar observations in which case one has a Binomial with parameters m_i and q_i, where m_i is the number of observations in the given group.

A GLM with the Bernoulli or Binomial Distribution using the logit link function is called a Logistic Regression.

Example of Logistic Regression:[109]

Fit a logistic regressions to data on whether or not a vehicle had a claim.
If x is the vehicle value in units of $10,000, the model is:
$\ln[\mu/(1-\mu)] = \beta_0 + \beta_1 x + \beta_2 x^2$, with $\hat{\beta}_0 = -2.893$, $\hat{\beta}_1 = 0.220$, $\hat{\beta}_2 = -0.026$.

For a vehicle worth $30,000, $x\beta = -2.893 + (0.220)(3) + (-0.026)(3^2) = -2.467$.
Thus the expected probability of a claim for a vehicle worth $30,000 is:
$e^{-2.467} / (1 + e^{-2.467}) = 7.8\%$.

Exercise: Determine the expected probability of a claim for a vehicle worth $70,000.
[Solution: $x\beta = -2.893 + (0.220)(7) + (-0.026)(7^2) = -2.627$. $e^{-2.627} / (1 + e^{-2.627}) = 6.7\%$.]

[107] See Section 2.8 of Generalized Linear Models for Insurance Rating.
[108] The logit is the canonical link function for the Binomial Distribution, including the special case the Bernoulli.
[109] See Section 7.3 of Generalized Linear Models for Insurance Data, by de Jong and Heller, not on the syllabus.

One can also fit a model, using instead a categorical version of vehicle value such as 6 groups: less than 25,000, 25K to 50K, 50K to 75K, 75K to 100K, 100K to 125K, more than 125,000. With the first group as the base level, the fitted model had:
$\hat{\beta}_0$ = -2.648, $\hat{\beta}_1$ = 0.174, $\hat{\beta}_2$ = 0.102, $\hat{\beta}_3$ = -0.571, $\hat{\beta}_4$ = -0.397, $\hat{\beta}_5$ = -0.818.

Thus a vehicle of value less than $25,000 has an expected probability of a claim of:
exp[-2.648] / (1 + exp[-2.648]) = 6.61%.

A vehicle of value $25,000 to $50,000 has an expected probability of a claim of:
exp[-2.648 + 0.174] / (1 + exp[-2.648 + 0.174]) = 7.77%.

A vehicle of value greater than $125,000 has an expected probability of a claim of:
exp[-2.648 - 0.818] / (1 + exp[-2.648 - 0.818]) = 3.03%.

The odds for a vehicle of value less than $25,000, the base level is:
6.61%/(1 - 6.61%) = 0.0708 = exp[-2.648] = exp[β_0].

The odds for a vehicle of value 25,000 to $50,000 is:
7.77%/(1 - 7.77%) = 0.0842 = exp[-2.648]exp[0.174] = exp[β_0]exp[β_1].

Thus the odds for the second level are those for the first base level times exp[β_1]. The odds for the second level are higher than those for the base level by a factor of exp[0.174] = 1.190. The odds for the last level are lower than those for the base level by a factor of exp[-0.818] = 0.441.

Grouping Data:

When one has binary variables, one can group the data into the possible combinations.
For example, with vehicle insurance data using driver's age (6 groups), area (6 territories), vehicle body (13 types), and vehicle value (6 groups), there are (6)(6)(13)(6) = 2808 cells.
Only some of these cells contain data.

For example, assume that driver age group 1, Area A, Hatchback, of value less than $25,000 in value has 554 polices with 47 claims.
We would take this as a random draw from a Binomial with m = 554.
In general, for a cell with n_i policies, we would assume the number of claims follows B(n_i, q_i).

We get the same fitted parameters and standard errors using either individual or grouped data, although the test statistics will differ.

Correlation Among Predictors:[110]

When the correlation between two predictor variables is large (in absolute value), the GLM will be unstable. The standard errors of the corresponding coefficients can be large and small changes in the data can produce large changes in the coefficients.

For example, years of education of the father and mother are likely to be highly positively correlated.
Including both in a model may produce problems.[111]

Software may not catch the presence of highly correlated variables and try to fit the model anyway. Due to the extreme correlation, the model will be highly unstable; the fitting procedure may fail to converge, and even if the model run is successful the estimated coefficients will be nonsensical.

When you start with a very long list of possible predictors to use in a GLM, it is common for some pairs of predictors to be highly correlated. Thus one should check the correlations of pairs of proposed predictor variables with each other.

If potential problems are found, one can:
1. Remove one or more predictors from the model.[112]
2. Use techniques that combine predictors in order to reduce the dimension, such as Principal Component Analysis and Factor Analysis.[113]

"Determining accurate estimates of relativities in the presence of correlated rating variables is a primary strength of GLMs versus univariate analyses; unlike univariate methods, the GLM will be able to sort out each variable's unique effect on the outcome, as distinct from the effect of any other variable that may correlate with it, thereby ensuring that no information is double-counted."

[110] See Section 2.9 of Generalized Linear Models for Insurance Rating.
[111] One could instead include an average of these two variables.
[112] While simple, this may lead us to lose valuable information.
[113] You are not responsible for any details.
I discuss Principal Component Analysis in my section on the paper by Robertson.
When a set of variables are highly correlated, either positively or negatively, the first principal component or the first two principal components capture most of the variation in the original variables.
The first principal component is a linear combination of the original variables.

Multicollinearity:

Multicollinearity is a similar situation which also leads to potential problems.
Multicollinearity occurs when two or more predictors in a model are strongly predictive of another one of the predicator variables.[114]

As discussed, we are concerned when pairs of variables are highly correlated. However, even in situations where pairs of variables are not highly correlated, problems can occur when looking at three or more predictor variables in combination.

For example, an insurer uses among others the following policyholder characteristics: age, years of education, and income. The first two characteristics would help to predict the final characteristic. Depending on how close this relationship was for this insurer's data, this could create a problem with the output of a GLM due to multicollinearity.

A high degree of multicollinearity, usually leads to unreliable estimates of the parameters. The estimation equations are ill-conditioned.

A useful statistic for detecting multicollinearity is the variance inflation factor (VIF).
If one or more of the VIFs is large, that is an indication of multicollinearity.
A common statistical rule of thumb is that a VIF greater than 10 is considered high, indicating possible problems from multicollinearity.

You will not be asked to compute VIF.[115] [116] Most software packages give VIF as an output.

Aliasing:

Where two predictors are perfectly correlated, they are said to be aliased, and the GLM will not have a unique solution. Equivalently, aliasing can be defined as a linear dependency among the columns of the design matrix X.

Intrinsic aliasing is a linear dependency between covariates due to the definition.

For example, if you have only three territories, then knowing an insured is not in territory one or territory two, implies they are in territory three. Such intrinsic aliasing is common with categorical variables; every insured must be in one and only one of the categories.

[114] Let X be the design matrix and X' be its transpose. In the case of regression, this is often described as X'X being an ill-conditioned matrix; one can also say the data is ill-conditioned.
In this case, the determinant of X'X will be very small.

[115] "The VIF for any predictor is a measure of how much the (squared) standard error for the predictor is increased due to the presence of collinearity with other predictors. It is determined by running a linear model for each of the predictors using all the other predictors as inputs, and measuring the predictive power of those models."

[116] In the case of regression, regress the i^{th} independent variable against all of the other independent variables, and let R_i^2 be the coefficient of determination of this regression.
Then the Variance Inflation Factor is: $VIF_i = 1/(1 - R_i^2)$.

Initially we have three covariates for the three territories and corresponding coefficients: β_1, β_2, and β_3. Ignoring any other factors, the linear predictor is: $\eta = X_1\ \beta_1 + X_2\ \beta_2 + X_3\ \beta_3$.
However, $X_1 + X_2 + X_3 = 1$, so we can eliminate any one of three variables from the model.
For example, $\eta = X_1\ \beta_1 + X_2\ \beta_2 + (1 - X_1 - X_2)\ \beta_3 = X_1\ (\beta_1 - \beta_3) + X_2\ (\beta_2 - \beta_3) + \beta_3$.
Thus one can eliminate X_3 from the model, and include an intercept term if it does not already exist.

The fitted values will be the same regardless of which level is eliminated.
Selecting as the base level for each factor the one with the most exposure is helpful, since this minimizes the standard errors associated with other parameter estimates.

Exercise: Age of driver has only three levels: Youth, Adult, and Senior.
Demonstrate how aliasing can be used to exclude a level from the age variable.
[Solution: We have that $1 = X_{\text{youth}} + X_{\text{adult}} + X_{\text{senior}}$, and thus $X_{\text{adult}} = 1 - X_{\text{youth}} - X_{\text{senior}}$.
Therefore, we can eliminate β_{adult} from the model and include an intercept term if it does not already exist.
Comment: One could have eliminated any of the levels.
The adult level, which has the most exposures, would be a good choice for a base level.
The intercept term would now corresponds to the adult base level; there is no separate parameter for adult.
We would still have a parameter for Youth and a parameter for Senior.]

In general, **when we have a categorical variable with N levels, the model should have N-1 parameters in addition to an intercept term**. The chosen base level, which is often the one with the most exposures, is associated with the intercept term and will not have a separate associated parameter.

As another example of intrinsic aliasing, age of vehicle would alias with model year, since if you know one you can determine the other.

Extrinsic aliasing is a linear dependency between covariates that arises due to the particular values in the observed data rather than inherent properties of the covariates themselves.[117]

For example, if all sports cars in a data base just happen to be red cars and vice-versa.

Most software will detect aliasing and automatically drop one of those predictors from the model.

[117] Goldburd, Khare, and Tevet, do not distinguish between intrinsic and extrinsic aliasing.

Limitations of GLMs:[118]

1. GLMs assign full credibility to the data.[119]
2. GLMs assume that the randomness of outcomes are uncorrelated.[120]

As has been discussed on the exam on Basic Ratemaking, when estimating classification relativities by older techniques, an actuary uses credibility. The estimated relativities of classes with less data are given less than full weight.

However, using GLMs the estimated relativities are given full weight.

In fact, for a GLM with just one categorical predictor variable, the estimates will just be the observed average for each level. An actuary would not use the observed average for a small class (or the ratio of its observed average to the observed average for the base level) as a reasonable estimate of the future.

It should be noted, that for a class with little data, the standard errors of the fitted coefficient will be large. Thus we may not reject a value of zero for the coefficient of that small class. In a multiplicative model this would imply a relativity of one. Alternately, we could combine the small class with another class. However, neither of these alternatives is as flexible as giving the observed relativity for this small class some positive weight less than one.

In a regression, we assume that the random components, in other words the errors, ε_i, are uncorrelated.[121] Similarly, in a GLM we assume that the random components are uncorrelated.[122] [123]
This assumption can be violated.

For example, the data set may include several years of data from a single policyholder, which appear as separate records. The outcomes of a single policyholder are correlated.
Another example, in the case of wind losses, the outcomes for policyholders in the same area will be correlated.[124]

If there are large correlations of random components, then the GLM would pick up too much random noise, and produce sub-optimal predictions and overoptimistic measures of statistical significance.

[118] See Section 2.10 of Generalized Linear Models for Insurance Rating.
[119] Section 10 of Generalized Linear Models for Insurance Rating t, not on the syllabus, discusses two ways to incorporate something similar to credibility: generalized linear mixed models and elastic net GLMs.
[120] Goldburd, Khare, and Tevet, mention two methods that account for correlation in the data: generalized linear mixed model, and generalized estimating equations.
[121] This assumption is often violated when dealing with time series.
[122] We assume that the systematic components are correlated.
For example, drivers in the same class and territory are assumed to have similar expected pure premiums.
[123] The random component is the portion of the outcomes driven by causes not in our model.
[124] I am thinking about wind losses from other than catastrophes; catastrophes would not be modeled using GLMs.

The Model-Building Process:[125]

The authors discuss how actuaries build models; much of the material is not specific to GLMs. They give a list of steps or components:[126]

- Setting of objectives and goals
- Communicating with key stakeholders
- Collecting and processing the necessary data for the analysis
- Conducting exploratory data analysis
- Specifying the form of the predictive model
- Evaluating the model output
- Validating the model
- Translating the model results into a product
- Maintaining the model
- Rebuilding the model

Setting Goals and Objectives:

- Determine the goals.
- Determine appropriate data to collect.
- Determine the time frame.
- What are key risks and how can they be mitigated?
- Who will work on the project; do they have the necessary knowledge and expertise?

Communication with Key Stakeholders:

- Legal and regulatory compliance
- Information Technology (IT) Department
- Underwriters
- Agents

Collecting and Processing Data:[127]

- Time-consuming.
- Data is messy.
- Often an iterative process.
- **The data should also be split into at least two subsets, so that the model can be tested on data that was not used to build it**.
- Formulate a strategy for validating the model.

Any analysis performed by an actuary is no better than the quality of the data that goes into that analysis![128]

[125] See Section 3 of Generalized Linear Models for Insurance Rating.

[126] As always, such lists are somewhat arbitrary. Many actuaries do not require such lists to do their jobs. Another possible step is to read the literature to see what has been done in similar situations in the past.

[127] For more detail, see Section 4 of Generalized Linear Models for Insurance Rating.

[128] Garbage in, garbage out.

Conducting Exploratory Data Analysis (EDA):

Spend some time to better understand the nature of the data and the relationships between the target and explanatory variables. Helpful EDA plots include:

- Plotting each response variable versus the target variable to see what (if any) relationship exists. For continuous variables, such plots may help inform decisions on variable transformations.
- Plotting continuous response variables versus each other, to see the correlation between them.[129]

Specifying Model Form:[130]

- What type of predictive model works best?
- What is the target variable, and which response variables should be included?
- Should transformations be applied to the target variable or to any of the response variables?
- Which link function should be used?

Evaluating Model Output:[131]

- Assessing the overall fit of the model.
- Identifying areas in which the model fit can be improved.
- Analyzing the significance of each predictor variable, and removing or transforming variables accordingly.
- Comparing the lift of a newly constructed model over the existing model or rating structure.

Model Validation:[132]

- Assessing fit with plots of actual vs. predicted on holdout data.
- Measuring lift.
- For Logistic Regression, use Receiver Operating Characteristic (ROC) Curves.

Translating the Model into a Product:

For GLMs, often the desired result is a rating plan.

- The product should be clear and understandable.
- Are there other rating factors included in the rating plan that were not part of the GLM?
  Then it is important to understand the potential relationship between these additional variable(s) and other variables that were included in the model.
  Judgmental adjustments may be needed.

[129] Recall that a high correlation, either positive or negative, between pairs of predictor variables may lead to problems with the fitted GLM.

[130] For more detail, see Section 5 of Generalized Linear Models for Insurance Rating.

[131] For more detail, see Section 6 and 7 of Generalized Linear Models for Insurance Rating.

[132] For more detail, see Section 7 of Generalized Linear Models for Insurance Rating.

Maintaining and Rebuilding the Model:

Models should be periodically rebuilt in order to maximize their predictive accuracy, but in the interim it may be beneficial to merely refresh the existing model using newer data. In other words, more frequently one would update the classification relativities without updating the rating algorithm or classification definitions. Less frequently, one would do a more complete update, investigating changing the classification definitions, the predictor variables used, and/or the rating algorithm.

In a somewhat different context, perhaps every 2 years one would update ELPPFs using the latest data but the existing grouping of classifications into hazard groups. Perhaps every 10 or 15 years one would update the grouping of classifications into hazard groups.[133]

Data Preparation and Considerations:[134]

Much of this is not unique to GLMs.
Data preparation is time consuming.[135]
Correcting one data error might help you discover another.

- Combining Policy and Claim Data.
- Modifying the Data.
- Splitting the Data.

Ratemaking Data:

Data is used by actuaries for many purposes including ratemaking.
For classification and territory ratemaking, more detailed data on exposures, premiums, losses, and ALAE is used, broken down by class and territory.
Ratemaking data is usually aggregated into calendar years, accidents years, and/or policy years.

[133] See the syllabus reading by Robertson.

[134] See Section 4 of Generalized Linear Models for Insurance Rating.

[135] At a large insurer, much of this work would have been routinely done by someone other than the actuary working on a specific GLM project. The actuary is responsible for determining whether it is reasonable to rely on the data supplied by others. See for example, Actuarial Standard of Practice 23 on Data Quality, not on the syllabus.

Combining Policy and Claim Data:

An insurer's data is often contained in a policy data base with exposures and premiums, and a separate claims data base with losses and alae.[136] These data bases have to be combined in a manner useful to the actuary.

Issues discussed by the authors:

- Are there timing considerations with respect to the way these databases are updated that might render some of the data unusable?
- Is there a unique key that can be used to match the two databases to each other in such a way that each claim record has exactly one matching policy record?
- What level of detail should the data sets be aggregated to before merging?
- Are there fields that can be safely discarded?
- Are there fields that should be in the database but aren't?[137]

Finding and Correcting Errors in the Data:[138]

Any dataset of sufficient size is likely to have errors.

- Check for duplicate records.
- Check categorical fields against available documentation.
- Check numerical fields for unreasonable values.[139]
- Decide how to handle each error or missing value that is discovered.

[136] See for example Chapter 3 of "Basic Ratemaking" by Werner and Modlin, on the syllabus of Exam 5.

[137] In which case, the actuary may initiate the process to start collecting this additional information. There are many pieces of information currently collected by insurers and rating bureaus that were not collected 50 years ago.

[138] When I worked at a rating bureau, a good percentage of my time was spent on this. We developed many systematic ways to detect errors. More than one group of people would be looking at the data from somewhat different points of view. Large errors were easy to find, but smaller errors required more diligence to find. Unfortunately, one can never find all of the errors.

[139] For example, an insurer reported to the rating bureau that an employer had as much payroll as the entire state. This error was quickly spotted and when pointed out to the insurer was quickly corrected.

Splitting the Data into Subsets:[140]

For modeling purposes one should split the data into either two or three parts.
This can be done either at random or based on time, for example policy year.

The simpler approach is to **split the data into a training set and test (holdout) set.**[141]
For example, the training set could be 2/3 of the data while the test set is the remaining 1/3.

One develops the model on the training set. Then once one has come up with a final model or a few candidates for a final model, **one would test performance on the test set of data, which was not used in developing the model**.[142]

The model was developed to fit well to the training set. In doing so, we are concerned that the model may be picking up peculiarities of the training set. If the model does a good job of predicting for the test set, which was not used in developing the model, then it is likely to also work well at predicting the future.[143]

Reasons to split the data into a training set and a test set:
● Attempting to test the performance of any model on the same set of data on which the model was built will produce overoptimistic results. The model-fitting process optimizes the parameters to best fit the data used to train it. Using the training data to compare our model to a model built on different data would give our model an unfair advantage.
● As we increase the complexity of the model, the fit to the training data will always get better. Thus the performance on the training data can not be used to compare models of different complexity. On the other hand, for data the model fitting process has not seen, eventually increased complexity will worsen the performance of the model.[144] Thus the performance on the test data can be used to compare models of different complexity.

The split of data can be performed either by randomly allocating records between the training and test sets, or by splitting on the basis of a time variable.[145] The latter approach has the advantage in that the model validation is performed "out of time" as well as out of sample, giving us a more accurate view into how the model will perform on unseen years.

[140] See Section 4.3 of Generalized Linear Models for Insurance Rating.
[141] This is done in the syllabus reading by Couret and Venter.
[142] Such testing will be discussed subsequently.
[143] We are interested in how the GLM will perform at predicting the response variable on some future set of data rather than on the set of past data with which we are currently working.
[144] See Figure 7 in Generalized Linear Models for Insurance Rating.
[145] One could split by month or by calendar/accident year.
As in Couret and Venter one could select either the even or odd years of data as the training set and the other as the test set, in order to be neutral with respect to trend and maturity.

“Out-of-time validation is especially important when modeling perils driven by common events that affect multiple policyholders at once. An example of this is the wind peril, for which a single storm will cause many incurred losses in the same area. If random sampling is used for the split, losses related to the same event will be present in both sets of data, and so the test set will not be true unseen data, since the model has already seen those events in the training set. This will result in overoptimistic validation results. Choosing a test set that covers different time periods than the training set will minimize such overlap and allow for better measures of how the model will perform on the completely unknown future.”[146]

The actuary should wait as long as possible in the process to use the test set. Once you start comparing to the test set, if you go back and change the form of the model, the usefulness of the test set for further comparisons has been diminished.

Thus sometimes, one uses the more complicated approach of splitting the data in three subsets: **a training set, validation set, and test (holdout) set**.[147]
For example, the split might be 40%, 30%, 30%.

As before, one develops the model on the training set. Then once one has come up with a good model or several good models, one would test performance on the validation set of data, which was not used in developing the model(s). If any changes in the form of the model are indicated, one goes back and works again with the training set. This iteration continues until the actuary is satisfied.

Then one would test performance on the test set of data, which was not used so far.

In either the simpler or more complicated case, **once a final form of the model has been decided upon, one should go back and use all of the available data to fit the parameters of the GLM**.

[146] Quoting from Section 4.3.1 of Generalized Linear Models for Insurance Rating.
See 8 11/17, Q.4c.

[147] Hopefully the total amount of data available is big enough to allow this.

Underfitting and Overfitting:

A model may be either overfit or underfit. Think of fitting a polynomial to 20 points. A straight line with no intercept, in other words a model with one parameter, will probably not do a good job of fitting the points. A fitted 19th degree polynomial, in other words a model with 20 parameters, will pass through all of the points.

However, actuaries are using a model to predict the behavior in the future. The one parameter model will probably not do a very good job, since it ignored some of the information in the data. It is underfit. The 20 parameter model will not do a good job of predicting, since it picked up all of the random fluctuation (noise) in the data. It is overfit.

A model should be made as simple as possible, but not simpler.

Underfit. ⇔ Too few Parameters. ⇔ Does not use enough of the useful information.
⇔ Does not capture enough of the signal.

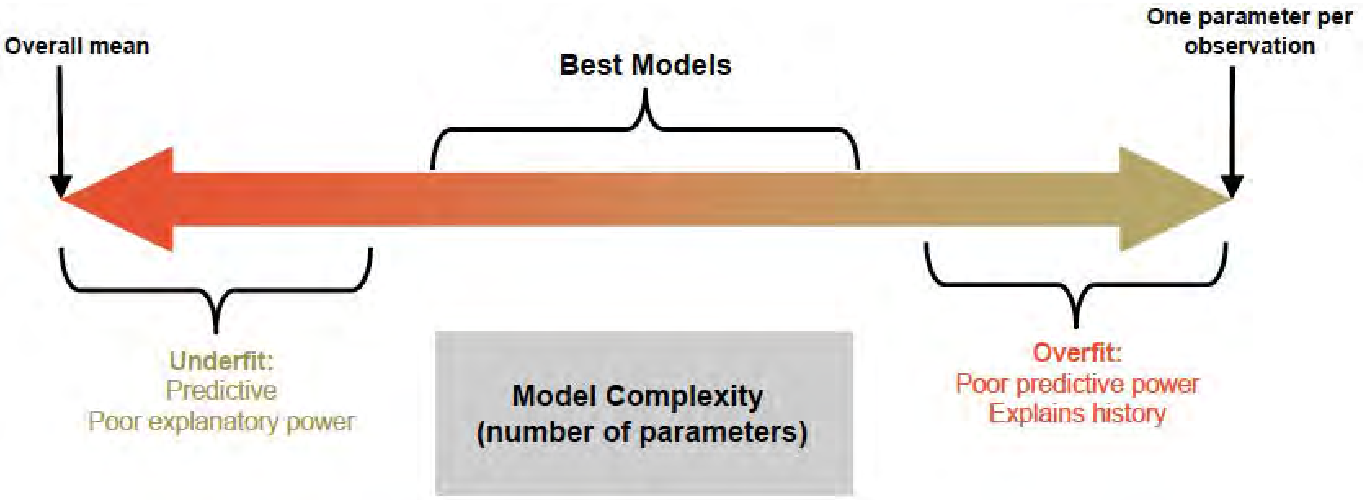
Overfit. ⇔ Too many Parameters. ⇔ Reflects too much of the noise.

We wish to avoid both underfitting and overfitting a model.

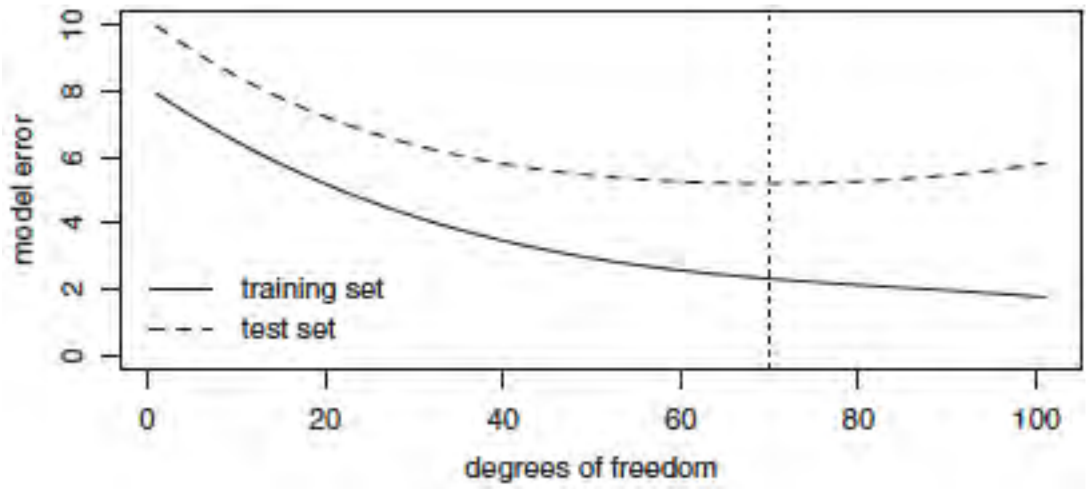
Think of fitting loss distributions. We would not use the most complicated model possible.[148]
We would only add parameters to the extent they were statistically significant.[149]
In a particular situation, it might be that an Exponential Distribution (one parameter) is an underfit model, a Transformed Gamma Distribution (3 parameters) is an overfit model, while a Gamma Distribution (2 parameters) is just right.

[148] Recall that a mixture of two or more distributions can have a lot of parameters.
[149] Think of the Likelihood Ratio Test, AIC, or BIC (the Schwarz Bayesian Criterion).

In order to produce a sensible model that explains recent historical experience and is likely to be predictive of future experience, one needs to avoid both too little and too much complexity:[150]



Each added parameter adds a degree of freedom to the model. This can be due to the addition of a new predictor variable, the addition of a polynomial term, the addition of an interaction term, etc. Each added degree of freedom makes the model more complex.
Our goal in modeling is to find the right balance where we pick up as much of the signal as possible with minimal noise. This is illustrated in Figure 7 of the syllabus reading:



As we add more parameters, we get a model that fits the training set better. However, when we compare such a model fitted to the training data to the test data, there is a point past which added parameters <u>reduce</u> the fit to the test data. The right balance is indicated by the vertical dotted line, at about 70 degrees of freedom in this case.[151]

[150] Taken from "GLM II, Basic Modeling Strategy", presented by Lenard Shuichi Llaguno, FCAS, at the 2012 CAS Ratemaking and Product Management Seminar.
[151] Here the authors use degrees of freedom to refer to the number of parameters in the fitted model. In for example the F-test, many authors instead define the degrees of freedom as number of observations minus number of fitted parameters for the fitted model.

Cross Validation:[152] [153]

Cross Validation is another technique for data splitting, although it is often of limited usefulness for actuarial work.

Split the data into for example 10 groups. Each group is called a fold. For each fold:
- Train the model using the other folds.[154]
- Test the model using the given fold.

Cross validation has the advantage of using all of the data (at some point) to estimate the mean squared error, rather than only using the portion of the data in the holdout set to do so. Thus cross validation should produce a better estimate of the MSE.

In the case of 10-fold cross validation, fit model form A on the data for the first 9 folds. Then compute the mean squared error (MSE) of this fitted model used to make predictions to the data in the remaining tenth fold.

Now fit the same model form A on the data for the folds other than the ninth. Then compute the mean squared error (MSE) of this fitted model used to make predictions to the data in the remaining ninth fold.

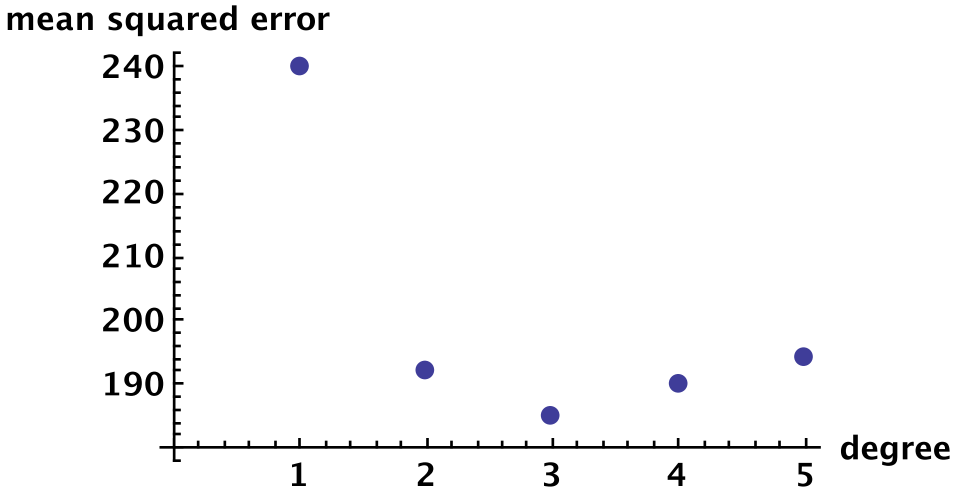
We would continue in this manner and then average these ten mean squared errors. This would be the estimated test MSE for model form A.

We could then determine the MSE of several other model forms, B, C, etc., in a similar manner.[155] The form of model with the lowest test MSE would be best.

For example, we might compare polynomial models with different number of powers of a predictor variable.

[152] See Section 4.3.4 of Generalized Linear Models for Insurance Rating.
[153] See also, An Introduction to Statistical Learning with Applications in R, by James, Witten, Hastie, and Tibshirani, not on the syllabus of this exam. They also discuss how to apply cross validation to other modeling techniques such as ridge regression and the lasso.
[154] According to the authors, this training procedure has to include all of the steps of the model building, including the variable selection and transformation; these steps usually include significant amounts of actuarial judgement.
[155] For example, Model A and Model B might use different sets of predictor variables.

One has fit similar GLMs on a set of data, where one of the predictors enters using polynomials of different degrees. The test MSEs were estimated using ten-fold cross-validation:



The model using the third degree model seems to perform best.[156]

Limitations of Cross-Validation for Actuarial Work:

Cross-validation can be useful for deciding how many polynomial terms to include.[157] However, cross validation is often of limited usefulness for most insurance modeling applications.

The actuary usually applies a great deal of care and judgment in selecting the variables to be included in the model. If using cross validation, this actuarial judgement should be applied separately to each of the data sets created by leaving out one fold. This is not really practical. Thus, using cross validation in place of a holdout set is only really appropriate where a purely automated variable selection process is used.[158]

For most actuarial modeling, the use of a holdout set is preferred to the use of cross validation. The final model valuation should always be done using a distinct set of data held out until the end.

[156] Due to the data being assigned to the 10 folds at random, if one performed cross validation again, one would get somewhat different estimates of the test MSEs. Therefore, in practical applications one would perform cross validation several times and compare the results.

[157] This is an example of evaluating a "tuning parameter" of the model.

[158] This is the opinion of the authors of the syllabus reading, who have plenty of experience using GLMs for actuarial modeling.

An Example of k-Fold Cross-Validation:[159]

Eight observations of three independent variables and one dependent variable:

X_1	X_2	X_3	Y
-2	1	-4	6
1	-1	0	8
3	4	4	33
6	-4	8	14
11	0	12	40
15	8	16	118
17	-8	20	2
20	-6	24	61

I will perform 4-fold cross-validation, so that each fold contains 8/4 = 2 observations.
We need to divide the original data into 4 random subsets; the estimated test MSE will depend to some extent on this random subdivision. My four folds will be: (1, 7), (2, 4), (3, 5), (6, 8).

If we leave out the first and seventh observations, and fit a regression model to the remaining six observations, the fitted parameters are:
$\hat{\beta}_0 = 3.78881$, $\hat{\beta}_1 = 5.10444$, $\hat{\beta}_2 = 5.17811$, $\hat{\beta}_3 = -0.621247$.
We now plug into this fitted model the values of the predictors for the first observation:
$(5.10444)(-2) + (5.17811)(1) + (-0.621247)(-4) = 1.24303$.
We now plug into this fitted model the values of the predictors for the seventh observation:

$\hat{Y} = 3.78881 + (5.10444)(17) + (5.17811)(-8) + (-0.621247)(20) = 36.7145$.
The mean squared difference between the observed values and these predicted values is:
$\text{MSE}_1 = \{(6 - 1.24303)^2 + (36.7145 - 2)^2\} / 2 = 613.863$.

Similarly, we would now instead leave out the 2nd and 4th observations.
We continue in this manner, and the four mean squared errors are:
613.863, 231.863, 697.906, 1458.9.
The average of these four values is the 4-fold cross-validation estimate of the test MSE:
750.633.

I used R to perform this same process five separate times and the estimated test MSEs were:[160]
449.2197, 1249.365, 616.1268, 680.8828, 754.928.
With only 8 observations, we see considerable variation in these estimates.

[159] Solely in order to give a simple concrete example; you are not responsible for any details.
[160] Using the R function cv.glm. Each time a different set of random folds is used.

Selection of Model Form:[161]

“Selecting the form of a predictive model is an iterative process, and is often more of an art than a science.”

Important decisions on the form of a GLM include:

- Choosing the target variable.
- Choosing a distribution for the target variable.
- Choosing the predictor variables.
- Whether to apply transformations to the predictor variables.
- Grouping categorical variables.
- Whether to include interactions.

[161] See Section 5 of Generalized Linear Models for Insurance Rating.

Frequency/Severity versus Pure Premium:[162]

An actuary could build two separate models: one for frequency and one for severity.[163] Alternately the actuary could build a single model for pure premium. If there is time, an actuary could do both of these different approaches and compare the results.

Advantages of the frequency/severity approach over pure premium modeling:
● Provides the actuary with more insight.
● Each of frequency and severity is more stable than pure premium.[164]

Disadvantages of pure premium modeling versus the frequency/severity approach:
● Some interesting effects may go unnoticed.
● Pure premium modeling can lead to underfitting or overfitting.
● The Tweedie distribution used to model pure premium contains the implicit assumption that an increase in pure premiums is made up of an increase in both frequency and severity.[165]

For example, urban driving tends to lead to a higher frequency of accidents (per mile driven) than rural driving. However, urban driving tends to lead to a lower severity of accidents than rural driving.
These two separate effects could be masked in a pure premium model. In any case, with just a pure premium model, the actuary would not get this interesting and perhaps important insight.

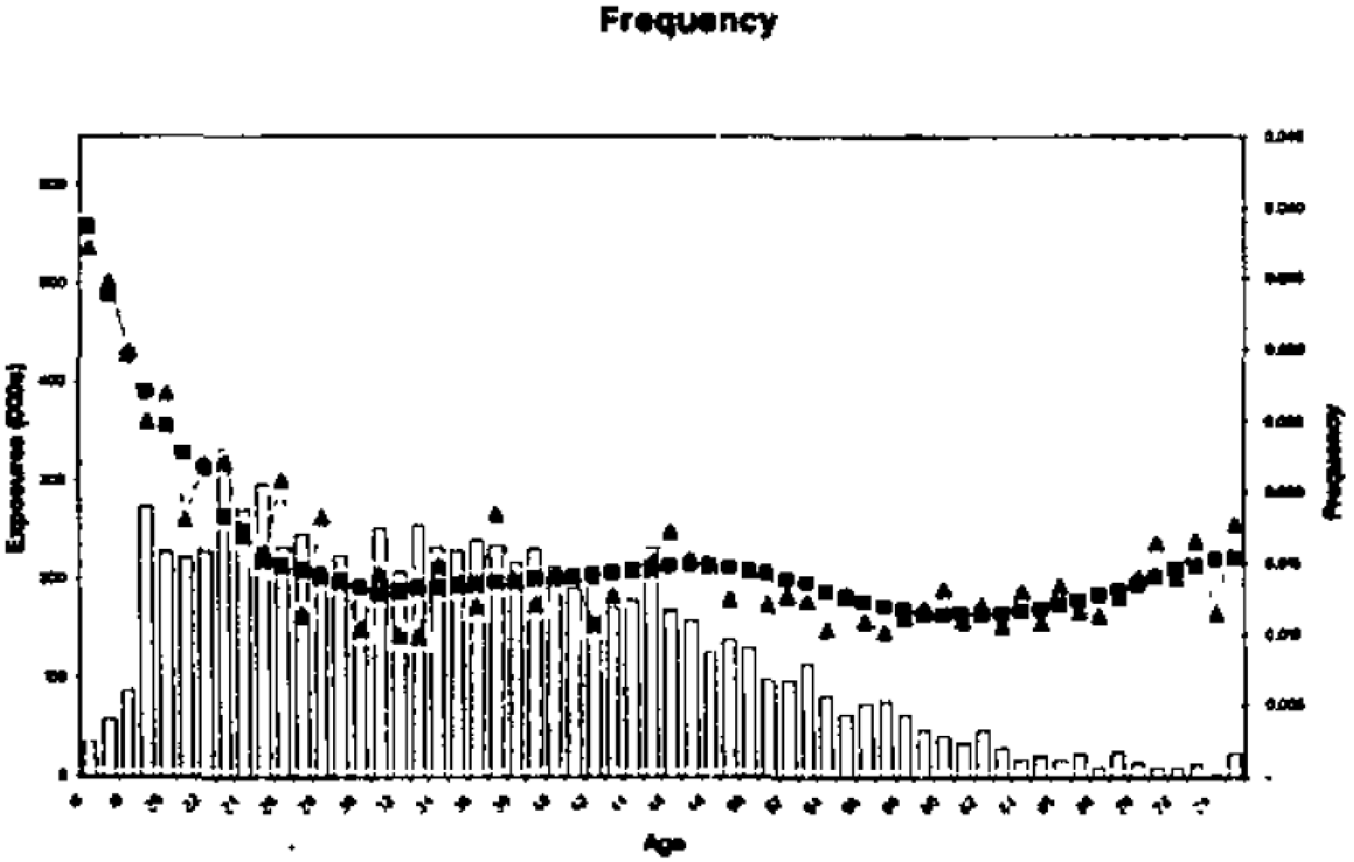
While territory would show up as significant in a frequency model, when testing it in a pure premium model the high variance in severity may overwhelm this effect, rendering the territory statistically insignificant.[166] Thus, a useful predictive variable will be excluded from the model, leading to underfitting.

Assume that a predictor variable has a significant effect on frequency and no effect on severity. If that variable is included in a pure premium model, then the fitted GLM will pick up any effect of severity in the training data even if it is just noise. The corresponding parameter will be overfit.

For frequency and severity, a priori expected patterns help the actuary to produce a better model. To the extent that the historical pattern is erratic, the actuary will be able to use appropriate techniques and knowledge about insurance to build a model that captures the signal in the data.

[162] See Section 5.1.1 of Generalized Linear Models for Insurance Rating.
[163] If the log link function is used for both, then the pure premium (multiplicative) relativities will be the product of the separate frequency and severity relativities.
[164] Recall that the standard for full credibility for pure premium is the sum of those for frequency and severity.
[165] The authors assume that one would use the Tweedie Distribution to model pure premiums.
[166] While this could happen in general, in the example I have chosen it is unlikely to do so.

For example, when modeling auto collision frequency, the actuary may expect the frequency by age to decrease from youthful to adult and increase again for the most mature drivers.[167] The following figure compares the historical frequencies (triangles) and modeled frequencies (squares) by age.[168]



The modeled frequencies follow the general pattern expected.

[167] These are frequencies per car year. Most senior citizens have higher expected frequencies per mile driven. However, their average number of miles driven per year is lower.
[168] Figure 5, from "GLM Basic Modeling: Avoiding Common Pitfalls," by Geoff Werner and Serhat Guven, CAS Forum Winter 2007, not on the syllabus.

Modeling Loss Ratios:

If the goal of the project is to identify deficiencies in the existing rating plan, loss ratio may be an appropriate target variable for the GLM.[169] However, there are disadvantages to modeling loss ratios rather than pure premiums or frequency/severity.

Theoretical and practical disadvantages to loss ratio modeling:[170]

- One needs to put premiums on-level at a granular level; difficult and time consuming.
  One has to put on the current rate level individual policies;
  overall on-level factors will not do.
- There is no generally accepted error distribution.[171]
- Difficult to distinguish noise from pattern, compared to modeling frequency/severity.
- If changes are made to the rates, then models cannot be reused from the last review.

[169] See Section 5.1 of Generalized Linear Models for Insurance Rating.

[170] Taken from "GLM II: Basic Modeling Strategy, " by Claudine Modlin,
CAS Predictive Modeling Seminar, October 2008.

[171] However, as discussed previously, one could use the Tweedie Distribution.

Policies with Multiple Coverages and Perils:[172]

A Businessowners package policy includes building, business personal property, and liability coverage.[173] Each of those coverages should be modeled separately.

In addition, one may models each peril individually.[174] For the Businessowners building model, one may wish to create separate models for: fire and lightning, wind and hail, and all other perils.[175]

One way to combine separate models by peril in order to get a model for all perils:

1. Use the separate models by peril to generate predictions of expected loss due to each peril for some set of exposure data.[176]
2. Add the peril predictions together to form a combined loss cost for each record.
3. Run a model on that data, using the combined loss cost calculated in Step 2 as the target, and the union of all the individual model predictors as the predictor variables.

Transforming the Target Variable:[177]

Sometimes it is useful to transform the target variable. Among the possible transformations:

- Cap large losses for purposes of modeling pure premium or severity.[178]
- Remove catastrophe losses.
- Losses may need to be developed.[179]
- Losses and/or exposures may need to be trended.
- Premium may need to be put on level.[180]

Year could be included in the model, which should pick up any effects on the target variable related to time, such as trend, loss development, and rate changes.

[172] See Section 5.1.2 of Generalized Linear Models for Insurance Rating.

[173] Similar ideas would apply to Homeowners Insurance.

[174] Or group of perils.

[175] Wind and hail should be divided between catastrophe and non-catastrophes; with catastrophes modeled separately as discussed in the syllabus reading by Grossi and Kunreuther.

[176] The data used for this procedure should reflect the expected mix going forward, and so using only the most recent year of exposures may be ideal. Since the target data fed into this new model is extremely stable, this procedure doesn't require a whole lot of data.

[177] See Section 5.1.3 of Generalized Linear Models for Insurance Rating,
which discusses familiar things done in ratemaking.

[178] Ideally the level chosen for the cap should capture most of the signal and eliminate most of the noise.
This is similar in concept to choosing a reasonable accident limit to use in an Experience Rating Plan.

[179] Either to ultimate or to a common level of maturity.
For a severity model, the development factor should reflect only expected future development on known claims.
Since larger claims take on average longer to report, this may not address the whole issue.
For some lines of insurance, one may be better off not using more recent but less mature data in the model.
For a pure premium or loss ratio model, the development factor should include the effect of pure IBNR claims as well.

[180] Premium would be used in a loss ratio model.

Choosing the Distribution for the Target Variable:[181] [182]

If modeling claim frequency, the distribution is likely to be either Poisson or Negative Binomial.[183]

If modeling a binary response, then the Bernoulli or Binomial Distributions are used.

If modeling claim severity, common choices for the distribution are Gamma and Inverse Gaussian.

If modeling pure premiums, the Tweedie Distribution is a common choice.

Selection of Predictor Variables:[184]

Sometimes the actuary is just updating the parameters a model using newer data. Other times, the actuary will do a full review of all aspects of a model, including which predictor variables to include.

One would like a predictor variable to have a statistical significant effect on the target variable. Statistical tests can be performed. One would like a small probability value for the null hypothesis that the corresponding parameter is zero.

There is no magic cutoff, although a p-value of 5% or less is often used.[185] However, if the p-value is 5%, that means that there is 1/20 chance we are including a predictor variable in the model when we should not. If there is large set of possible predictor variables that are tested for inclusion in the model, this can lead to problems.[186]

In addition to statistical significance, the actuary must take into account practical considerations.[187] For example:
- Will it be cost effective?
- Actuarial standards of practice.
- Regulatory and legal requirements.
- Can the IT (Information Technology) department easily implement the change?

[181] See Section 5.2 of Generalized Linear Models for Insurance Rating.
[182] Analysis of the deviance residuals, to be discussed subsequently, can help the actuary to choose.
[183] Recall that one can also use an overdispersed Poisson.
[184] See Section 5.3 of Generalized Linear Models for Insurance Rating.
[185] For a further discussion of p-values see the following subsection, the ASA statement not on the syllabus.
[186] There are automated variable selection algorithms, which are not on the syllabus.
[187] See ASOP 12: Risk Classification.

ASA Statement on Statistical Significance and P-values:[188] [189]

Introduction

Increased quantification of scientific research and a proliferation of large, complex datasets in recent years have expanded the scope of applications of statistical methods. This has created new avenues for scientific progress, but it also brings concerns about conclusions drawn from research data. The validity of scientific conclusions, including their reproducibility, depends on more than the statistical methods themselves. Appropriately chosen techniques, properly conducted analyses and correct interpretation of statistical results also play a key role in ensuring that conclusions are sound and that uncertainty surrounding them is represented properly.

Underpinning many published scientific conclusions is the concept of "statistical significance," typically assessed with an index called the p-value. While the p-value can be a useful statistical measure, it is commonly misused and misinterpreted. This has led to some scientific journals discouraging the use of p-values, and some scientists and statisticians recommending their abandonment, with some arguments essentially unchanged since p-values were first introduced.

In this context, the American Statistical Association (ASA) believes that the scientific community could benefit from a formal statement clarifying several widely agreed upon principles underlying the proper use and interpretation of the p-value. The issues touched on here affect not only research, but research funding, journal practices, career advancement, scientific education, public policy, journalism, and law. This statement does not seek to resolve all the issues relating to sound statistical practice, nor to settle foundational controversies. Rather, the statement articulates in non-technical terms a few select principles that could improve the conduct or interpretation of quantitative science, according to widespread consensus in the statistical community.

What is a p-value?

Informally, a p-value is the probability under a specified statistical model that a statistical summary of the data (for example, the sample mean difference between two compared groups) would be equal to or more extreme than its observed value.

[188] February 5, 2016.
Edited by Ronald L. Wasserstein, Executive Director
on behalf of the American Statistical Association Board of Directors

[189] Not on the syllabus.

Principles

1. P-values can indicate how incompatible the data are with a specified statistical model.

A p-value provides one approach to summarizing the incompatibility between a particular set of data and a proposed model for the data. The most common context is a model, constructed under a set of assumptions, together with a so-called “null hypothesis.” Often the null hypothesis postulates the absence of an effect, such as no difference between two groups, or the absence of a relationship between a factor and an outcome. The smaller the p-value, the greater the statistical incompatibility of the data with the null hypothesis, if the underlying assumptions used to calculate the p-value hold. This incompatibility can be interpreted as casting doubt on or providing evidence against the null hypothesis or the underlying assumptions.

2. P-values do not measure the probability that the studied hypothesis is true, or the probability that the data were produced by random chance alone.

Researchers often wish to turn a p-value into a statement about the truth of a null hypothesis, or about the probability that random chance produced the observed data. The p-value is neither. It is a statement about data in relation to a specified hypothetical explanation, and is not a statement about the explanation itself.

3. Scientific conclusions and business or policy decisions should not be based only on whether a p-value passes a specific threshold.

Practices that reduce data analysis or scientific inference to mechanical “bright-line” rules (such as “$p < 0.05$”) for justifying scientific claims or conclusions can lead to erroneous beliefs and poor decision-making. A conclusion does not immediately become “true” on one side of the divide and “false” on the other. Researchers should bring many contextual factors into play to derive scientific inferences, including the design of a study, the quality of the measurements, the external evidence for the phenomenon under study, and the validity of assumptions that underlie the data analysis. Pragmatic considerations often require binary, “yes-no” decisions, but this does not mean that p-values alone can ensure that a decision is correct or incorrect. The widespread use of “statistical significance” (generally interpreted as “$p \le 0.05$”) as a license for making a claim of a scientific finding (or implied truth) leads to considerable distortion of the scientific process.

4. Proper inference requires full reporting and transparency.

P-values and related analyses should not be reported selectively. Conducting multiple analyses of the data and reporting only those with certain p-values (typically those passing a significance threshold) renders the reported p-values essentially uninterpretable. Cherry-picking promising findings, also known by such terms as data dredging, significance chasing, significance questing, selective inference and “p-hacking,” leads to a spurious excess of statistically significant results in the published literature and should be vigorously avoided. One need not formally carry out multiple statistical tests for this problem to arise: Whenever a researcher chooses what to present based on statistical results, valid interpretation of those results is severely compromised if the reader is not informed of the choice and its basis. Researchers should disclose the number of hypotheses explored during the study, all data collection decisions, all statistical analyses conducted and all p-values computed. Valid scientific conclusions based on p-values and related statistics cannot be drawn without at least knowing how many and which analyses were conducted, and how those analyses (including p-values) were selected for reporting.

5. A p-value, or statistical significance, does not measure the size of an effect or the importance of a result.

Statistical significance is not equivalent to scientific, human, or economic significance. Smaller p-values do not necessarily imply the presence of larger or more important effects, and larger p-values do not imply a lack of importance or even lack of effect. Any effect, no matter how tiny, can produce a small p-value if the sample size or measurement precision is high enough, and large effects may produce unimpressive p-values if the sample size is small or measurements are imprecise. Similarly, identical estimated effects will have different p-values if the precision of the estimates differs.

6. By itself, a p-value does not provide a good measure of evidence regarding a model or hypothesis.

Researchers should recognize that a p-value without context or other evidence provides limited information. For example, a p-value near 0.05 taken by itself offers only weak evidence against the null hypothesis. Likewise, a relatively large p-value does not imply evidence in favor of the null hypothesis; many other hypotheses may be equally or more consistent with the observed data. For these reasons, data analysis should not end with the calculation of a p-value when other approaches are appropriate and feasible.

Other approaches

In view of the prevalent misuses of and misconceptions concerning p-values, some statisticians prefer to supplement or even replace p-values with other approaches. These include methods that emphasize estimation over testing, such as confidence, credibility, or prediction intervals; Bayesian methods; alternative measures of evidence, such as likelihood ratios or Bayes Factors; and other approaches such as decision-theoretic modeling and false discovery rates. All these measures and approaches rely on further assumptions, but they may more directly address the size of an effect (and its associated uncertainty) or whether the hypothesis is correct.

Conclusion

Good statistical practice, as an essential component of good scientific practice, emphasizes principles of good study design and conduct, a variety of numerical and graphical summaries of data, understanding of the phenomenon under study, interpretation of results in context, complete reporting and proper logical and quantitative understanding of what data summaries mean. No single index should substitute for scientific reasoning.

Transformation of Predictor Variables:[190]

In many cases, a variable will need to be transformed in some way such that the resulting GLM is a better fit to the data. We have already discussed how with a log link function it often make sense to take the log of a continuous variable.

Regardless of whether the original variable has been logged or not, it is crucial to test the assumption of linearity and make adjustments where appropriate.[191]
Partial Residual Plots are one way for the actuary to detect such non-linear effects.

Partial Residual Plots:[192]

Concentrate on one of the explanatory variables X_j.

Then the partial residuals are: r_i = (ordinary residual) $g'(\mu_i) + x_{ij}\ \hat{\beta}_j$. [193]

In a Partial Residual Plot, we plot the partial residuals versus the variable of interest.

If there seems to be curvature rather than linearity in the plot, that would indicate a departure from linearity between the explanatory variable of interest and $g(\mu)$, adjusting for the effects of the other independent variables.

For a log link, $g'(\mu) = 1/\mu$, so that:

$$r_i = \frac{y_i - \mu_i}{\mu_i} + \hat{\beta}_j\ x_{ij}.$$

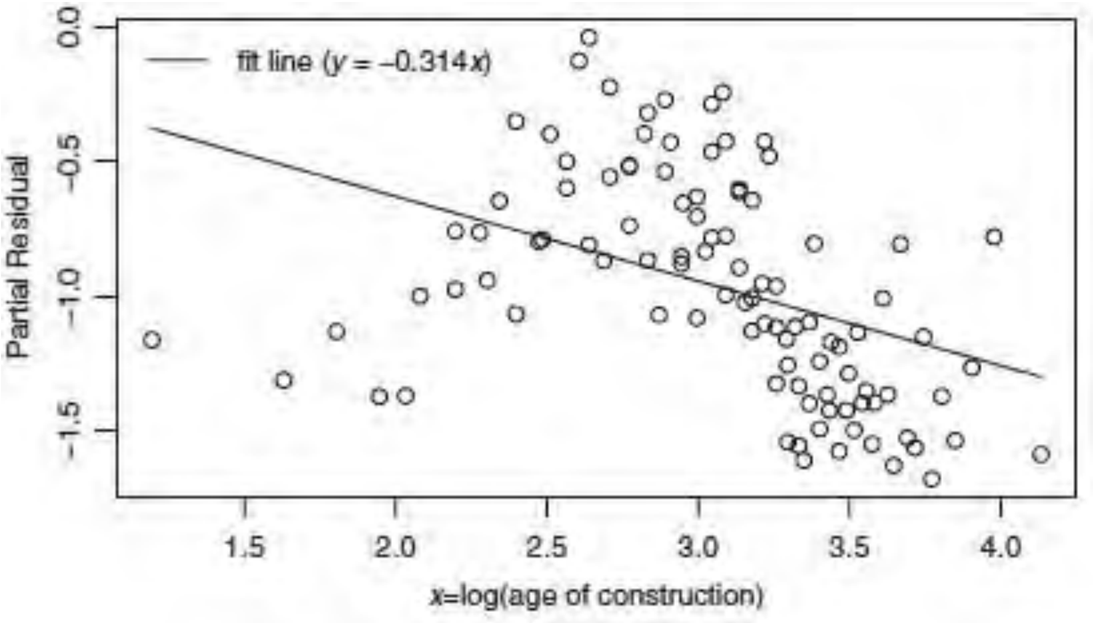
[190] See Section 5.4 of Generalized Linear Models for Insurance Rating

[191] The form of a GLM assumes that $g(\mu) = \beta x$, with the linear predictor on the righthand side of the equation.

[192] See Section 5.4.1 of Generalized Linear Models for Insurance Rating

[193] For the identity link ratio, this matches the definition of the partial residual for multiple regressions; the second term removes the effect of the j^{th} variable on the prediction of the i^{th} response. leaving the partial residual.

For example, assume a GLM where the fitted coefficient on ln[age of building] is -0.314. Assume the following graph of the partial residuals:[194]



The linear estimate of the GLM, -0.314*x,* is superimposed over the plot of the partial residuals. The points are missing the line in a systematic way, indicating that this model can be improved. The model is overpredicting for risks where log building age is less than 2.5, underpredicts between 2.5 and 3.25, and once again overpredicts for older buildings.

As will be discussed subsequently, one can add polynomial terms and examine the resulting graphs of partial residuals.

Binning Continuous Predictors:[195]

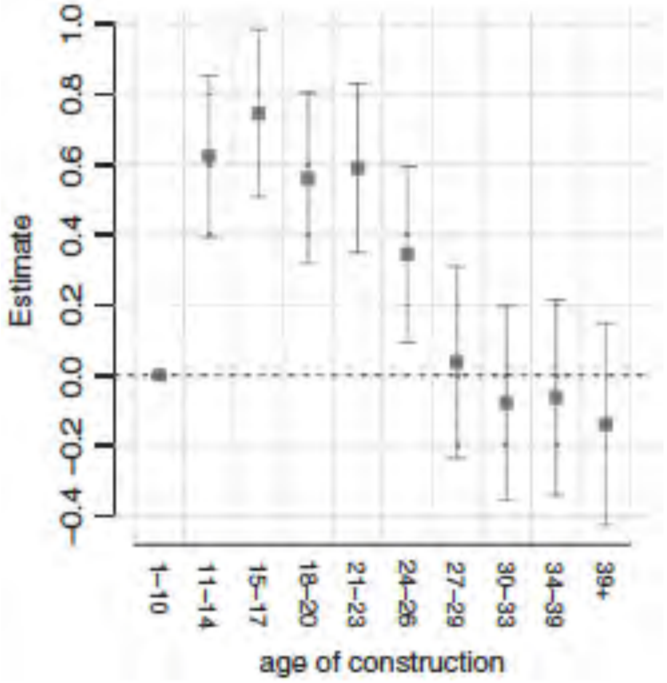
If there is nonlinearity, one possible fix for a continuous variable is to group it into intervals.

For example, rather than treat age of construction as a continuous variable, one can group it into several categories. We have converted a continuous variable into a categorical variable.

For their example, the authors group age of construction into ten bins.[196]

[194] Figure 8 taken from Generalized Linear Models for Insurance Rating, by Goldburd, Khare and Tevet.
[195] See Section 5.4.2 of Generalized Linear Models for Insurance Rating.
[196] The bins were chosen so that they each have roughly the same amount of data.
While having bins with roughly equal amounts of data has advantages, it is not a necessity.

Figure 9 in the syllabus reading shows the resulting model:[197]



"The model picked up a shape similar to that seen in the points of the partial residual plot. Average severity rises for buildings older than ten years, reaching a peak at the 15-to-17 year range, then gradually declining."

Disadvantages of binning (grouping) continuous variables:
1. Adds parameters to the model.[198]
2. Continuity in the estimates is not guaranteed.
There is no guarantee that the pattern among intervals makes sense.[199] [200]
3. Variation within intervals is ignored.
For example, it may be that the relativity for age of construction less than 5 years may be significantly different than that for 6 to 10 years. However, if we use an interval consisting of less than 10, our model can not pick up any such difference.
4. There may not be enough data in each bin to be credible.
5. There could be non-intuitive results, such as reversals.

[197] The error bars correspond to plus or minus two standard errors around the estimate.
[198] By the principle of parsimony, we wish to avoid adding unnecessary parameters to the model.
[199] For example, in the previous graph, the estimate for 21-23 years does not follow the general pattern.
[200] One may be able to alleviate this problem by applying some smoothing process to the estimates from the model. Alternately, one could group together two or more intervals.

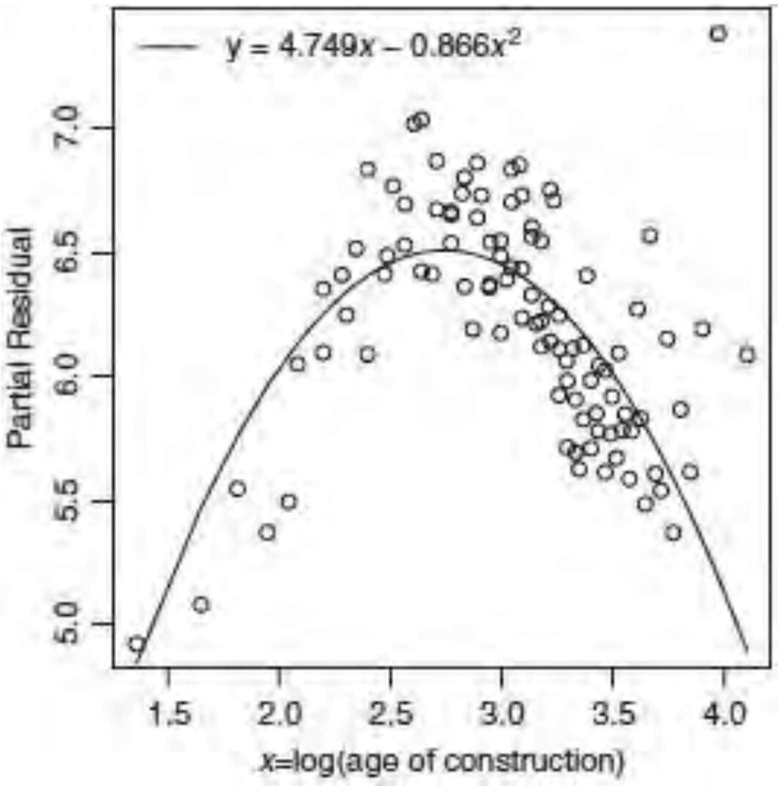
Adding Polynomial Terms:[201]

Rather than a model that uses $\beta_0 + \beta_1 x_1 + \beta_2 x_1 + ...$,

one can use $\beta_0 + \beta_1 x_1 + \beta_2 x_1^2 + ...$, or $\beta_0 + \beta_1 x_1 + \beta_2 x_1^2 + \beta_3 x_1^3 + ...$

The more polynomial terms that are included, the more flexibility, at the cost of greater complexity.

The authors added the square of the logged building age to their model. Here is the resulting plot of partial residuals with the curve formed by both building age terms superimposed:[202] [203]

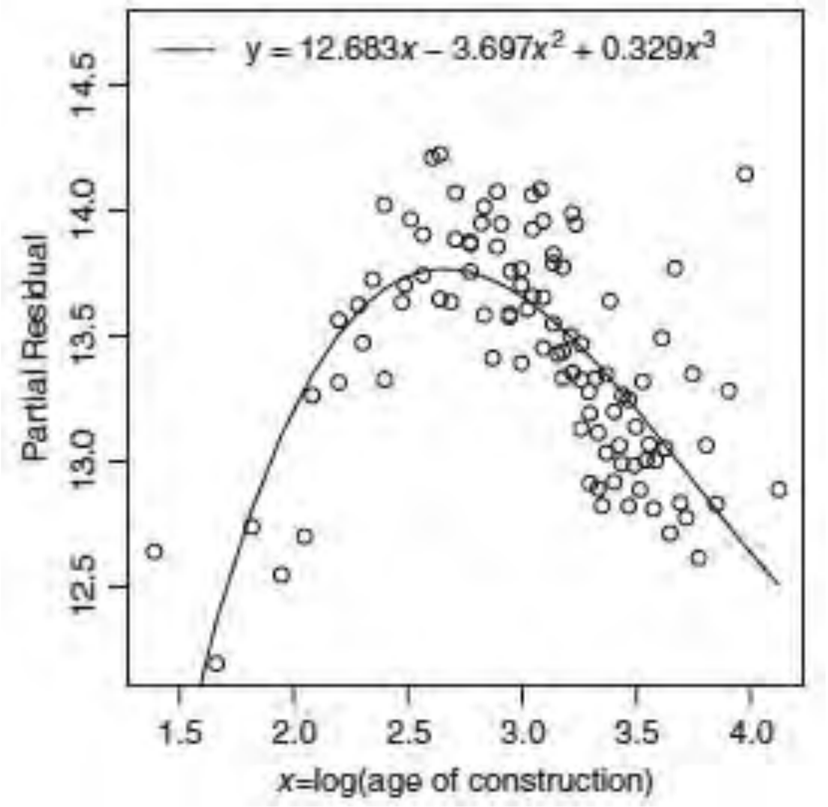


[201] See Section 5.4.3 of Generalized Linear Models for Insurance Rating.

[202] See the lefthand panel in Figure 10 in Generalized Linear Models for Insurance Rating. T

[203] The definition of partial residuals was extended to include all terms related to the variable being evaluated; i.e., the $\beta_j x_{ij}$'s for all polynomial terms are added back to the working residual rather than the single $\beta_j x_{ij}$ term.

Then the authors added the cube of the logged building age to their model. Here is the resulting plot of partial residuals with the curve formed by both building age terms superimposed:[204] [205]



"This perhaps yields a better fit, as the points seem to indicate that the declining severity as building age increases does taper off toward the higher end of the scale."

Unfortunately, it is hard to interpret these models that include powers of the logged building age.

"**One potential downside to using polynomials is the loss of interpretability**. From the coefficients alone it is often very difficult to discern the shape of the curve; to understand the model's indicated relationship of the predictor to the target variable it may be necessary to graph the polynomial function."

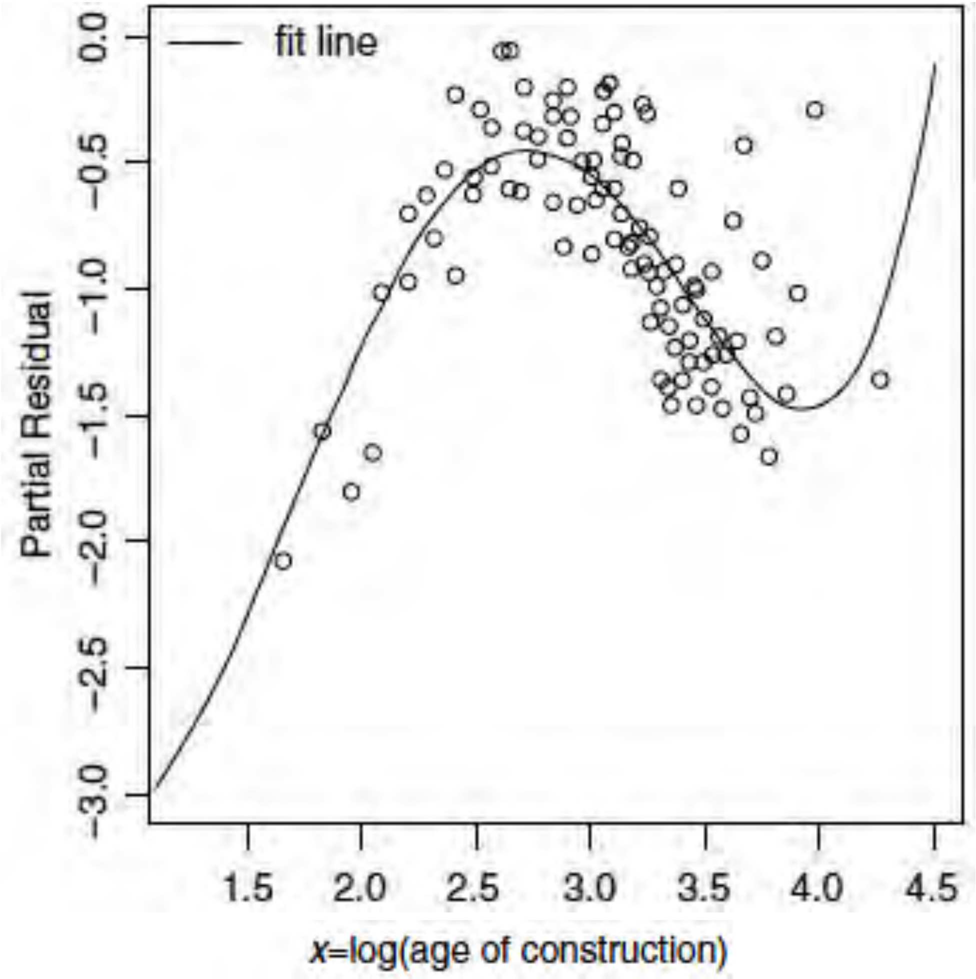
"Another drawback is that **polynomial functions have a tendency to behave erratically at the edges of the data, particularly for higher-order polynomials**."[206]

[204] See the righthand panel of Figure 10 in Generalized Linear Models for Insurance Rating.

[205] The definition of partial residuals was extended to include all terms related to the variable being evaluated; i.e., the $\beta_j x_{ij}$'s for all polynomial terms are added back to the working residual rather than the single $\beta_j x_{ij}$ term.

[206] Splines can suffer from the same problem. This can be solved by constraining the function to be linear at the edges; this what is done for natural cubic splines, to be discussed subsequently,

Continuing the age of construction example, here is the partial residual plot that would result if we were instead to use five polynomial terms:[207].



"The fitted curve veers sharply upward near the upper bound of the data, and would most likely generate unreasonably high predictions for ages of construction higher than typical."

Drawbacks of using polynomials:
1. Loss of interpretability
2. Tendency to behave erratically at the edges of the data

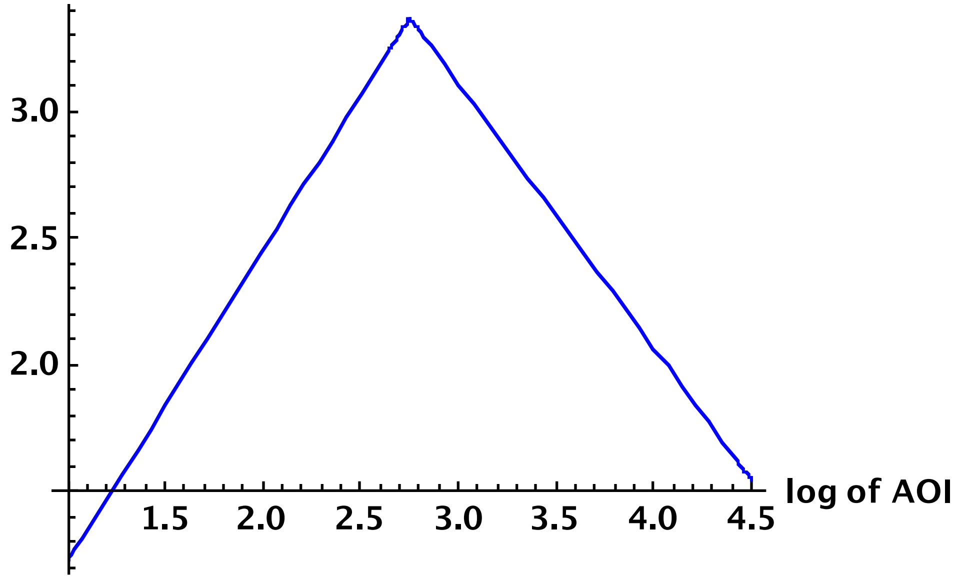
[207] Figure 11 in Generalized Linear Models for Insurance Rating.

Using Piecewise Linear Functions:[208]

Let X_+ be X if X ≥ 0 and 0 if X < 0.
Then a **hinge function** is: max[0, X - c)] = $(X - c)_+$, for some constant c.
The constant c would be called the breakpoint.
Hinge functions can be used to create piecewise linear functions which can be used in GLMs.

For example, let X = ln[AOI]. Then a usual linear estimator is: β_0 - 0.314 x + ...
Using instead a hinge function: $\beta_0 + 1.225\ x - 2.269\ (x - 2.75)_+ + ...$[209]

Here is a graph of the broken line that results from including the hinge function:

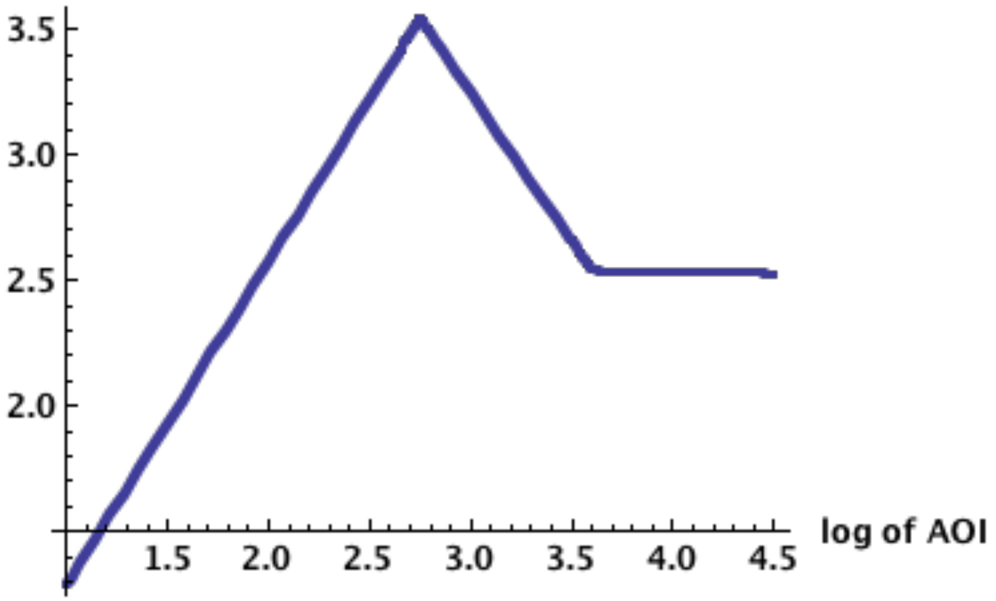


For ln[AOI] < 2.75, we have slope 1.225, while for ln[AOI] > 2.75 we have a slope of:
1.225 - 2.269 = -1.044.

Instead we can use two hinge functions:
$\beta_0 + 1.289\ x - 2.472\ (x - 2.75)_+ + 1.170\ (x - 3.60)_+ + ...$[210]

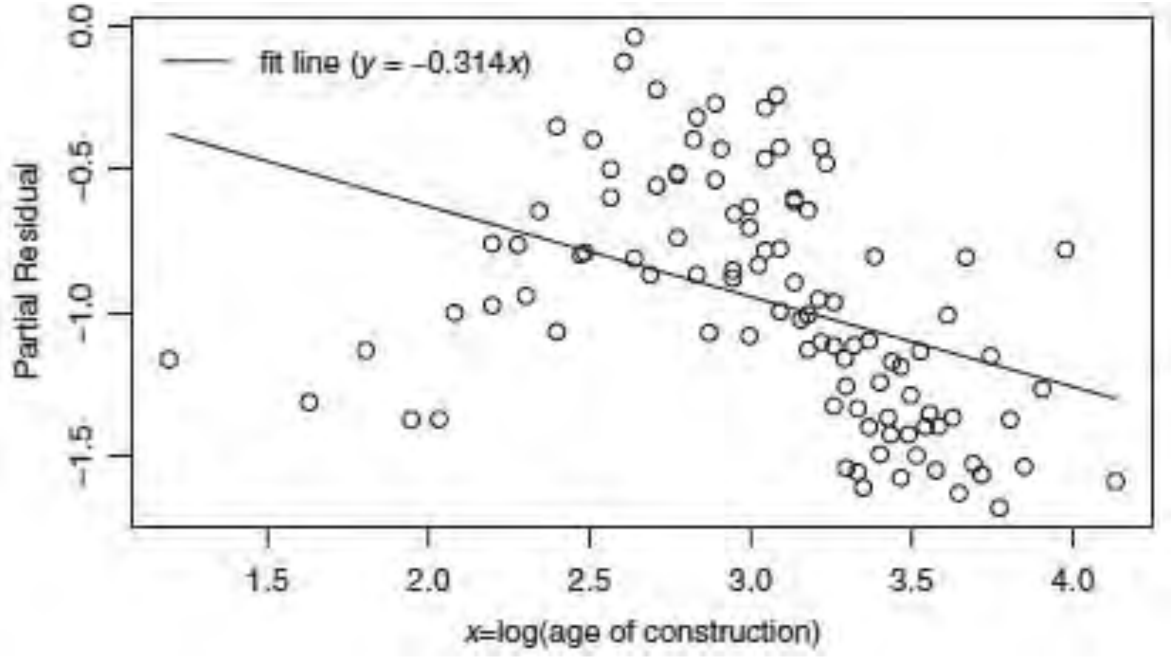
[208] See Section 5.4.4 of Generalized Linear Models for Insurance Rating.
[209] See Table 6 in Generalized Linear Models for Insurance Rating, "adding a breakpoint at 2.75."
[210] See Table 7 in Generalized Linear Models for Insurance Rating, "adding an additional breakpoint at 3.6."

Here is a graph of the broken line that results from including two hinge functions:



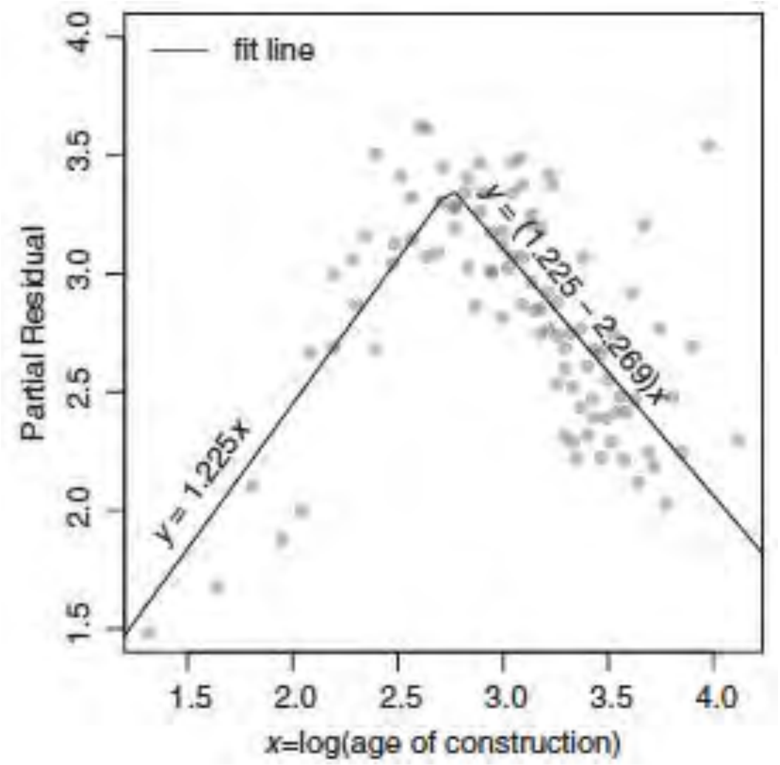
For ln[AOI] < 2.75,we have slope 1.289,
for 3.60 > ln[AOI] > 2.75 the slope is: 1.289 - 2.472 = -1.183,
while for 3.60 > ln[AOI] > 3.60 the slope is: 1.289 - 2.472 + 1.170 = -0.013.

Here is a graph of the partial residuals for the straight line:[211]



[211] See Figure 8 in Generalized Linear Models for Insurance Rating.

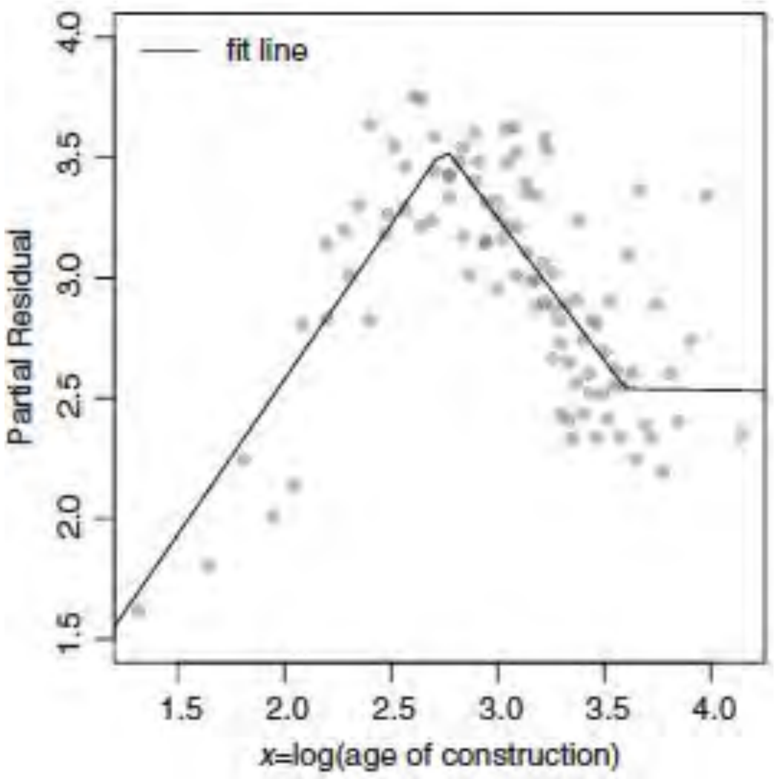
Here is a graph of the partial residuals for the broken line that results from using one hinge function:[212]



The model using the broken line does a better job of fitting the authors' data than the model that uses the straight line.

[212] See Figure 12 in G Generalized Linear Models for Insurance Rating.

Here is a graph of the partial residuals for the broken line using two hinge functions:[213]



The model using two hinge functions may do a somewhat better job of fitting the authors' data than the model that uses one hinge function. With limited data it is hard to tell.[214]

Hinge functions provide more flexibility at the cost of greater complexity.

The breakpoints must be selected by the user.
"Generally, break points are initially guesstimated by visual inspection of the partial residual plot, and they may be further refined by adjusting them to improve some measure of model fit. However, the GLM provides no mechanism for estimating them automatically."[215]

At each breakpoint, the hinge function is not smooth; its first derivative is not continuous. While the function is continuous, it abruptly changes direction at each breakpoint. This potential downside is not shared by natural cubic splines, to be discussed subsequently.

Drawbacks of using piecewise linear functions:
1. The need to select the breakpoints.
2. Lack of smoothness.

[213] See Figure 12 in Generalized Linear Models for Insurance Rating.

[214] "As this leveling-off effect comports with our intuition, we may decide to keep the third hinge function term in the model."

[215] MARS is a variant of the GLM, which among other things, fits non-linear curves using hinge function, and does it in an automated fashion with no need for tweaking by the user.

A Cherry Tree Example:

We are given the height, diameter, and volume of 31 black cherry trees:[216]

Diameters are: 83, 86, 88, 105, 107, 108, 110, 110, 111, 112, 113, 114, 114, 117, 120, 129, 129, 133, 137, 138, 140, 142, 145, 160, 163, 173, 175, 179, 180, 180, 206.

Heights are: 70, 65, 63, 72, 81, 83, 66, 75, 80, 75, 79, 76, 76, 69, 75, 74, 85, 86, 71, 64, 78, 80, 74, 72, 77, 81, 82, 80, 80, 80, 87.

Volumes are: 103, 103, 102, 164, 188, 197, 156, 182, 226, 199, 242, 210, 214, 213, 191, 222, 338, 274, 257, 249, 345, 317, 363, 383, 426, 554, 557, 583, 515, 510, 770.

I took X_1 = ln[diameter], X_2 = ln[height], and Y = volume.
A GLM was fit using a Gamma Distribution and a log link function.

The fitted parameters were: $\hat{\beta}_0 = -8.94859$, $\hat{\beta}_1 = 1.98041$, $\hat{\beta}_2 = 1.13288$.

$\hat{y}$ = exp[-8.94859 + 1.9804 ln[diameter] + 1.13288 ln[height]]

= 0.00012992 diameter$^{1.9804}$ height$^{1.13288}$.

The covariance matrix is: $\begin{pmatrix} 0.556725 & 0.00760542 & -0.13715 \\ 0.00760542 & 0.00545975 & -0.00788943 \\ -0.13715 & -0.00788943 & 0.0405552 \end{pmatrix}$.

Exercise: Based on geometry, it would make sense for $\beta_1 = 2$. Test whether $\beta_1 = 2$.

[Solution: $(1.98041 - 2) / \sqrt{0.00545975} = -0.265$. p-value is: $2\,\Phi[-0.265] = 79.1\%$.

Comment: We do not reject the null hypothesis that $\beta_1 = 2$.]

Exercise: Based on geometry, it would make sense for $\beta_2 = 1$. Test whether $\beta_2 = 1$.

[Solution: $(1.13288 - 1) / \sqrt{0.0405552} = 0.660$. p-value is: $2\,\{1- \Phi[0.660]\} = 50.9\%$.

Comment: We do not reject the null hypothesis that $\beta_2 = 1$.]

[216] The diameter is measured at 4'6" above the ground.
Data from a study by Ryan, Joiner, and Ryan.

The first predicted volume is: exp[-8.94859 + 1.9804 ln[83] + 1.13288 ln[70]] = 101.04.
Thus the first residual is: 103 - 101.04 = 1.96.

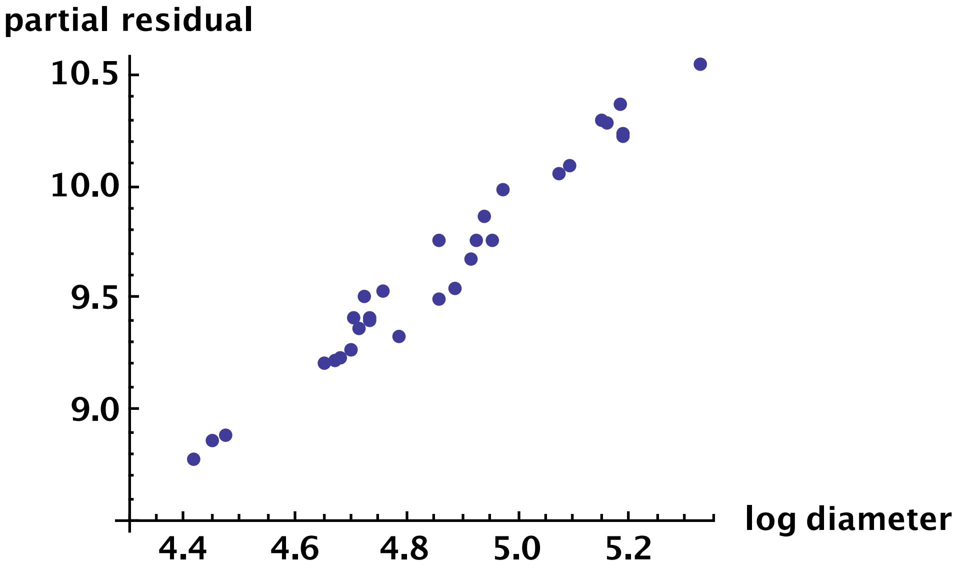
The residuals are: 1.96, 3.32, 1.31, -2.19, -9.14, -9.43, -9.12, -8.85, 16.96, 1.22, 28.50, 2.06, 6.06, 16.79, -35.74, -35.71, 36.48, -50.59, -20.02, -0.86, 23.33, -23.46, 38.15, 0.29, -2.43, 43.48, 27.42, 44.45, -29.52, -34.52, -12.21.

For this example, g(μ) = ln(μ). Thus g'(μ) = 1/μ. ⇒ $r_i = \frac{y_i - \mu_i}{\mu_i} + \hat{\beta}_j x_{ij}$.

Thus for ln[diameter], the partial residuals are: $(y_i - \hat{y}_i) / \hat{y}_i$ + ln[(diameter)$_i$] 1.98041.

The first partial residual is: (103 - 101.04)/101.04 + ln[83](1.980401) = 8.77.

The partial residuals for the ln[diameter] are: 8.77, 8.85, 8.88, 9.2, 9.21, 9.23, 9.25, 9.26, 9.41, 9.35, 9.50, 9.39, 9.41, 9.52, 9.32, 9.49, 9.75, 9.53, 9.67, 9.75, 9.86, 9.75, 9.97, 10.05, 10.08, 10.29, 10.28, 10.36, 10.23, 10.22, 10.54.
Here is a graph of these partial residuals versus ln[diameter]:



A departure from linearity is not evident.[217]

[217] If there seems to be curvature rather than linearity in the plot, that would indicate a departure from linearity between the independent variable of interest and g(μ), adjusting for the effects of the other independent variables.

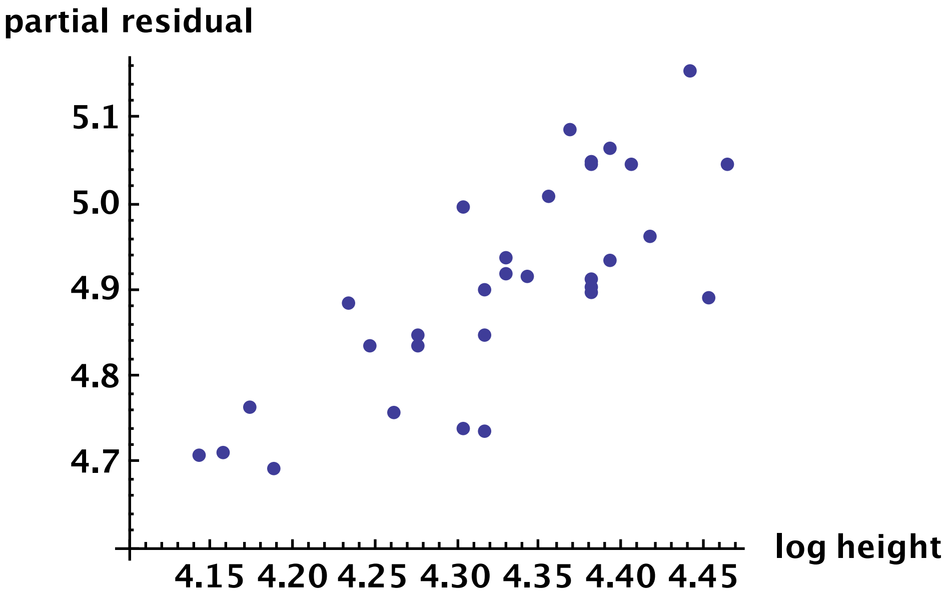
Exercise: For ln[height] what is the first partial residual?

[Solution: The partial residuals are: $(y_i - \hat{y}_i) / \hat{y}_i + \ln[(\text{height})_i]$ 1.13288.

The first partial residual is: (103 - 101.04)/101.04 + ln[70] (1.13288) = 4.83.

The partial residuals for the ln[height] are: 4.83, 4.76, 4.71, 4.83, 4.93, 4.96, 4.69, 4.84, 5.05, 4.90, 5.08, 4.92, 4.94, 4.88, 4.73, 4.74, 5.15, 4.89, 4.76, 4.71, 5.01, 4.90, 4.99, 4.85, 4.92, 5.06, 5.04, 5.05, 4.91, 4.90, 5.04.

Here is a graph of these partial residuals versus ln[height]:



A departure from linearity is not evident.

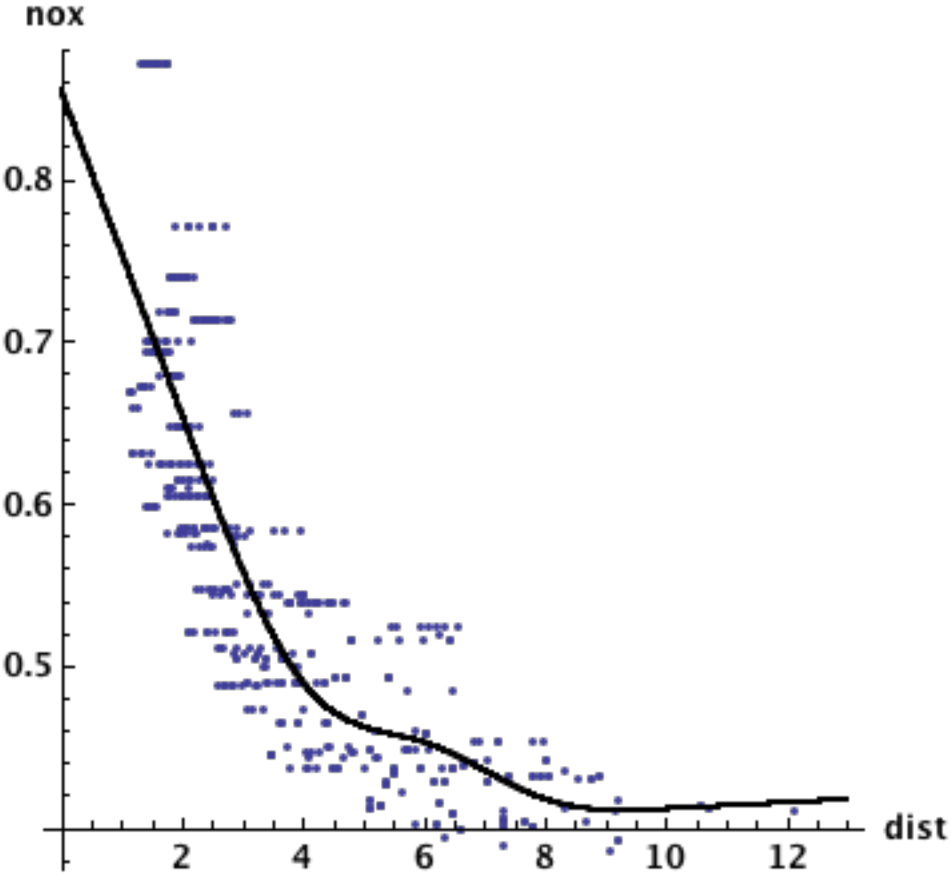
Natural Cubic Splines:[218] [219]

Another way to handle non-linear effects is to use Regression Splines.
An important special case are Natural Cubic Splines.

One has to choose breakpoints, called knots.[220] The spline will be continuous at these knots.
In between each of the knots, a cubic spline follows a cubic polynomial.
Below the first knot and above the last knot, a natural cubic spline is linear.

“As with polynomial functions, natural cubic splines do not lend themselves to easy interpretation based on the model coefficients alone, but rather require graphical plotting to understand the modeled effect.”

An example of a natural cubic spline, with 5 knots at 2, 4, 6, 8, and 10, fit to some data:[221]



[218] See Section 5.4.5 of Generalized Linear Model for Insurance Rating.
[219] Little detail is given. For more detail, see An Introduction to Statistical Learning with Applications in R by Gareth James, Daniela Witten, Trevor Hastie, Robert Tibshirani, on Exam MAS-1.
[220] One has to choose how many knots to use and where to place them.
[221] The fitted spline relates nitrogen oxides concentration (in parts per 10 million) to the weighted mean of the distances to five Boston employment centers.

Some of characteristics of natural cubic splines:[222]

- The first and second derivatives of the fitted curve function are continuous at the breakpoints (knots).[223] [224]

- The fits at the edges of the data (before the first selected breakpoint and after the last) are restricted to be linear.[225] [226]

- The use of breakpoints makes it more suitable than regular polynomial functions for modeling more complex effect responses, such as those with multiple rises and falls.

[222] As listed in Generalized Linear Model for Insurance Rating.

[223] Also, the spline is continuous at each of the breakpoints (knots).

[224] In a practical sense this means that the curve will appear fully “smooth” with no visible breaks in the pattern.

[225] This curtails the potential for the kind of erratic edge behavior, exhibited for example by regular polynomial functions.

[226] This linearity at the the edges is what distinguishes a natural cubic spline from a cubic spline.

An Example of Fitting a Natural Cubic Spline:[227]

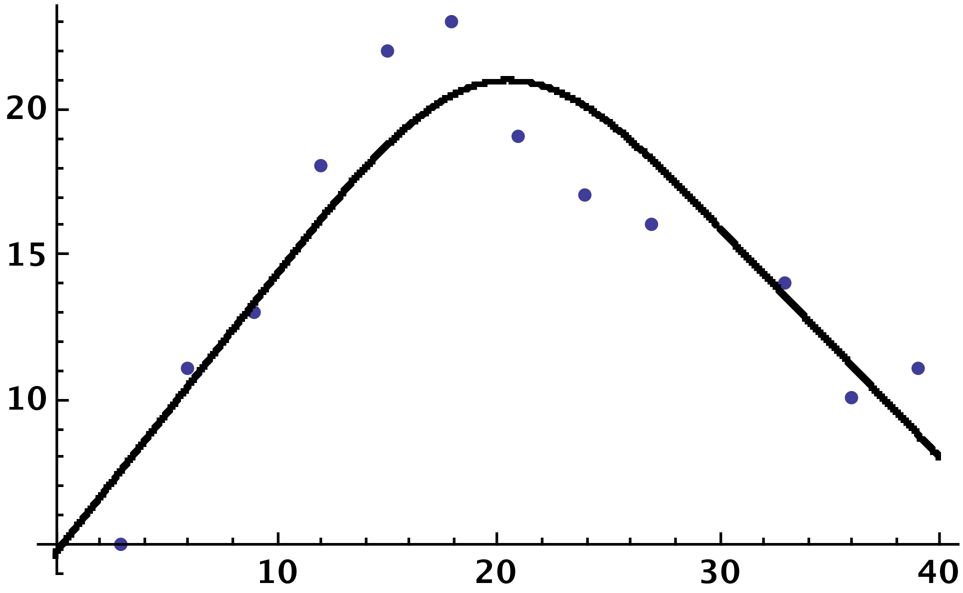
Start with 12 observations: {3, 5}, {6, 11}, {9, 13}, {12, 18}, {15, 22}, {18, 23}, {21,19}, {24, 17}, {27, 16}, {33, 14}, {36, 10}, {39, 11}.

Use three knots at 10, 20 and 30.[228] A natural cubic spline can be written as:[229]

$$Y = \beta_0 + \beta_1 X + \beta_2 \{\frac{(X-10)_+^3 - (X-30)_+^3}{30 - 10} - \frac{(X-20)_+^3 - (X-30)_+^3}{30 - 20}\}.$$

Using a computer, we minimize the sum of squared errors. The fitted betas are:[230]
β_0 = 4.578601, β_1 = 0.964608, β_2 = -0.058667.

Here is the data and the fitted natural cubic spline with knots at 10, 20 and 30:[231]



[227] Not on the syllabus of this exam. Some people will benefit from seeing a concrete example.
[228] One has to choose how many knots to use and where to place them.
[229] You are not responsible for this representation of a natural cubic spline. The sub-plus means that if what is inside is negative then we make it zero, while otherwise we leave what is inside alone.
[230] The sum of squared errors is 52.24.
[231] At each of the knots, the cubic spline is continuous, and has continuous first and second derivatives. Below 10 and above 30, the natural cubic spline is linear.

It is interesting to see what happens if we use more knots.
For example, let us take four knots at 8, 16, 24, and 32.

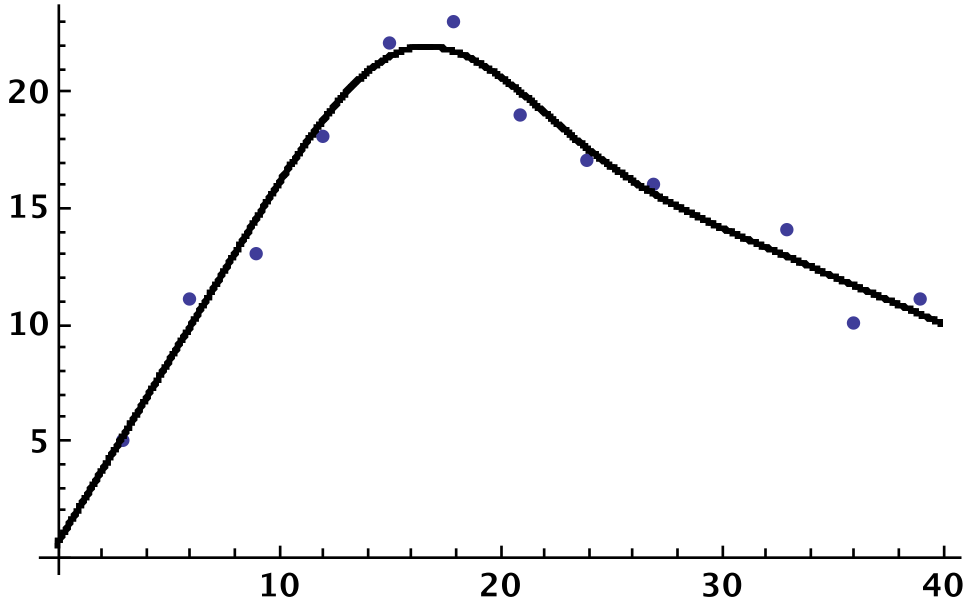
Then one can write a natural cubic spline as:

$$Y = \beta_0 + \beta_1 X + \beta_2 \{\frac{(X-8)_+^3 - (X-32)_+^3}{32 - 8} - \frac{(X-24)_+^3 - (X-32)_+^3}{32 - 24}\}$$

$$+ \beta_3 \{\frac{(X-16)_+^3 - (X-32)_+^3}{32 - 16} - \frac{(X-24)_+^3 - (X-32)_+^3}{32 - 24}\}.$$

Using a computer, we minimize the sum of squared errors. The fitted betas are:[232]
$\beta_0 = 0.412248$, $\beta_1 = 1.559231$, $\beta_2 = -0.165939$, $\beta_3 = 0.249697$.

Here is the data and the fitted natural cubic spline with knots at 8, 16, 24, and 32:[233]



This natural cubic spline with four knots seems to do a better job than the natural cubic spline with only three knots.

[232] The sum of squared errors is 11.68.
[233] At each of the knots, the cubic spline is continuous, and has continuous first and second derivatives. Below 8 and above 32, the natural cubic spline is linear.

Grouping Categorical Variables:[234]

Some predictor variables are ordinal; they are discrete with several categories with a natural order. Sometimes it is useful for modeling purposes to group such predictor variables into fewer categories.[235] This is particularly useful when there are many categories.[236]

For example, workers compensation claims are categorized as: medical only, temporary total, minor permanent partial, major permanent partial, permanent total, and fatal. For some purposes it might be useful to group the first three categories into nonserious and the last three categories into serious.

One can start with a model without grouping. Statistical tests can determine whether the coefficients of adjacent levels are significantly different. Then one can group adjacent levels with similar fitted coefficients. Now run a new model using these groupings, and iterate the procedure. One needs to balance the competing priorities of: predictive power, parsimony, and avoiding overfitting.

Interactions:[237]

If x_1 and x_2 are predictor variables, then we can include an interaction term: x_1x_2.
Then the model would be: $g(\mu) = \beta_0 + \beta_1x_1 + \beta_2x_2 + \beta_3x_1x_2 + ...$
This provides more flexibility at the cost of complexity.[238]

For example let x_1 be gender and x_2 be age. Then if we include an interaction term the effect of age depends on gender, and the effect of gender depends on age.

The syllabus reading gives an example with building occupancy class and sprinkler status.[239] Models are fit both with and without an interaction term.[240] The model with interactions is:

$$\mu = (\text{mean for base}) \exp[0.2303\ x_1 + 0.4588\ x_2 + 0.0701\ x_3 - 0.2895\ x_4 - 0.2847 x_1 x_4 - 0.0244\ x_2 x_4 - 0.2622\ x_3 x_4],$$

where $x_1 = 1$ if occupancy class 2, $x_2 = 1$ if occupancy class 3, $x_3 = 1$ if occupancy class 4, $x_4 = 1$ if sprinklered, and occupancy class 1 without sprinklers is the base.

[234] See Section 5.5 in Generalized Linear Models for Insurance Rating.
[235] This is analogous to Robertson grouping classes into Hazard Groups.
[236] The syllabus reading uses the example of driver age, which can be thought of as either continuous or discrete. In the case of age, there may not be any clear breakpoints to use for grouping; actuarial judgement may be needed.
[237] See Section 5.6 in Generalized Linear Models for Insurance Rating.
[238] One would only include the interaction term if its coefficient were significantly different from zero.
[239] This is a commercial building claims frequency model using a Poisson with a log link function.
[240] See Tables 8 and 9 in Generalized Linear Models for Insurance Rating.
While two of the interaction terms are significantly different from zero, the remaining one is not.
They show an intercept which only makes sense if there are other predictor variables in the model.

For a non-sprinklered building in occupancy class 2, the multiplicative relativity to the base is: exp[0.2303] = 1.259.

For a sprinklered building in occupancy class 2, the multiplicative relativity to the base is: exp[0.2303 - 0.2895 - 0.2847] = 0.709.

Exercise: For a non-sprinklered building in occupancy class 4, determine the multiplicative relativity to the base.
[Solution: exp[0.0701] = 1.073.]

Exercise: For a sprinklered building in occupancy class 4, determine the multiplicative relativity to the base.
[Solution: exp[0.0701 - 0.2895 - 0.2622] = 0.618.
Comment: For occupancy class 4, the effect of sprinklers is small, while for occupancy class 2, the effect of sprinklers is large.]

The syllabus reading also shows another fitted model, with occupancy class, sprinklered, ln[AOI/200,000], plus an interaction term between sprinklered and ln[AOI/200,000]:[241]

$$\mu = \text{(mean for base)} \exp[0.2919\ x_1 + 0.3510\ x_2 + 0.0370\ x_3 - 0.5153\ x_4 + 0.4239\ x_5 - 0.1032\ x_4 x_5],$$

where x_1 = 1 if occupancy class 2, x_2 = 1 if occupancy class 3, x_3 = 1 if occupancy class 4,
x_4 = 1 if sprinklered, x_5 = ln[AOI/200,000],
and AOI = 200,000 in occupancy class 1 without sprinklers is the base.

For a non-sprinklered building in occupancy class 2 with AOI = 500,000, the multiplicative relativity to the base is: exp[0.2919 + 0.4239 ln[2.5]] = 1.975.

Exercise: For a sprinklered building in occupancy class 2 with AOI = 500,000, determine the multiplicative relativity to the base.
[Solution: exp[0.2919 - 0.5153 + 0.4239 ln[2.5] - 0.1032 ln[2.5]] = 1.073.]

For range of sizes of AOI for buildings that are insured, the expected frequency increases at a slower rate with AOI for sprinklered buildings than for non-sprinklered buildings.

[241] See Table 12 in Generalized Linear Models for Insurance Rating. They show an intercept of -3.771, which implies an expected frequency for the base level of: exp[-3.771] = 2.3%.

Loglikelihood:[242]

The loglikelihood is the sum of the contributions of the ln[density] at each of the observations. All other things being equal, a larger loglikelihood indicates a better fit. However, the principle of parsimony means that we should not add additional parameters to a model unless it significantly increases the loglikelihood.

The saturated model has as many parameters as the number of observations.
Each fitted value equals the observed value.
The saturated model has the largest possible likelihood, of models of a given form.
The null model has only one parameter, the intercept.
The null model has the smallest possible likelihood, of models of a given form.[243]

[242] See Section 6.1.1 in Generalized Linear Models for Insurance Rating.
They do not give any details on the form of the scaled deviance for the different distributions.
See for example, An Introduction to Generalized Linear Models by Dobson and Barnett.

[243] In the context of GLMs we would be comparing models with the same distributional form and link function, that have been fit to the same data.

Deviance:[244]

The scaled deviance is twice the difference between the maximum loglikelihood for the saturated model (with as many parameters as data points) and the maximum loglikelihood for the model of interest.

D* = Scaled Deviance
= 2 {(loglikelihood for the saturated model) - (loglikelihood for the fitted model)}.

The smaller the scaled deviance, the better the fit of the GLM to the data.[245]

Maximizing the loglikelihood is equivalent to minimizing the scaled deviance.

By definition, the scaled deviance of the saturated model is zero. Even though the saturated model fits the data perfectly we would not use it to predict the future, since the saturated model is overfit; the saturated model picks up too much of the randomness in the data (called the noise).

The null model (with only an intercept) has the largest possible scaled deviance while the saturated model has the smallest possible scaled deviance of zero. The scaled deviance of a fitted model will lie between those two extremes.

We will be comparing the deviance of models with the same distributional form, and same link function, that have been fit to the same data.[246]

D = unscaled deviance = (scaled deviance) (dispersion parameter) = D* ϕ.[247]

The unscaled deviance is independent of the dispersion parameter and thereby useful for comparing models with different estimates of the dispersion parameter. [248]

[244] See Section 6.1.2 in Generalized Linear Models for Insurance Rating.

[245] Subsequently we will discuss how to test whether an improvement in scaled deviance is statistically significant.

[246] If a variable has missing values for some records, the default behavior of most model fitting software is to toss out those records when fitting the model. In that case, the resulting measures of fit are no longer comparable, since the second model was fit with fewer records than the first.

[247] Confusingly, while some sources use the same terminology as the syllabus reading, some other sources reverse the labels,

[248] For cases where the dispersion parameter is one, such as for a Poisson or Negative Binomial Distribution, this is not an advantage. For other cases, such as a Gamma or Normal Distribution, this is an advantage.

Nested Models and the F-Test:[249] [250]

We can use the F-Test to compare two nested models, in other words when one model is a special case of the other. The bigger (more complex) model always has a smaller (better) unscaled deviance than the smaller (simpler) model. The question is whether the unscaled deviance of the bigger model is significantly better than that of the smaller model (special case).

Assume that we have two nested models.
Then the test statistic (asymptotically) follows an F-Distribution with numbers of degrees of freedom equal to: ν_1 = the difference in number of parameters, and
ν_2 = number of observations minus number of fitted parameters for the bigger model.[251]

The test statistic is: $\dfrac{(D_S - D_B) / (\text{number of added parameters})}{\hat{\phi}_B} \sim F_{df_S - df_B,\, df_B}$.

D_S = unscaled deviance for the smaller (simpler) model.
D_B = unscaled deviance for the bigger (more complex) model.
df_S = number of degrees of freedom for the smaller (simpler) model.
= number of observations minus number of fitted parameters for the simpler model.
df_B = number of degrees of freedom for the bigger (more complex) model
= number of observations minus number of fitted parameters for the more complex model.
number of added parameters = $df_S - df_B$.
$\hat{\phi}_B$ = estimated dispersion parameter for the bigger (more complex) model.[252] [253] [254]

[249] See Section 6.2.1 in Generalized Linear Models for Insurance Rating.

[250] This F-Test is analogous to that used to test slopes in multiple regression.

[251] A Table of the F-Distribution is not attached to your exam, although they could give some values in a question. An F-Distribution is the ratio of two independent Chi-Square Distributions, with each Chi-Square divided by its number of degrees of freedom. ν_1 = the number of degrees of freedom of the Chi-Square in the numerator.
ν_2 = the number of degrees of freedom of the Chi-Square in the denominator.
If $\nu_1 = 1$, then the F-Distribution is related to the t-distribution.
Prob[F-Distribution with 1 and n degrees of freedom > c^2] =
Prob[absolute value of t-distribution with n degrees of freedom > c].
Thus if the difference in the number of parameters is one, then this test reduces to a t-test.

[252] The syllabus reading does not discuss how to estimate the dispersion parameter. One way to estimate the dispersion parameter in a model is as the ratio of the unscaled deviance to the number of degrees of freedom of the model.

[253] There is no requirement that the estimated dispersion parameters of the two models be equal.

[254] For cases where the dispersion parameter is one, such as for a Poisson or Negative Binomial Distribution, an actuary would normally use instead the likelihood ratio test, not discussed in the syllabus reading.
See "A Practitioners Guide to Generalized Linear Models," by Duncan Anderson, Sholom Feldblum, Claudine Modlin, Dora Schirmacher, Ernesto Schirmacher and Neeza Thandi, in the 2004 CAS Discussion Paper Program.

If the F-Statistic is sufficiently big, then reject the null hypothesis that the data is from the smaller model in favor of the alternate hypothesis that the data is from the bigger model.[255]

Exercise: A GLM using a Gamma Distribution has been fit for modeling expenditures upon admission to a hospital. There are 150 observations. It uses 25 variables.
It uses 4 categories of self-rated physical health: poor, fair, good, and very good.
The unscaled deviance is 35.1.
An otherwise similar GLM excluding self-rated physical health has an unscaled deviance of 38.4. The estimated dispersion parameter for the more complex model is 0.3.
Discuss how you would determine whether physical health is a useful variable for this model.
[Solution: The more complex model has 25 variables, and 150 - 25 = 125 degrees of freedom.
In order to incorporate physical health, avoiding aliasing, we need 4 - 1 = 3 variables.
Thus the simpler model has 22 variables, and 150 - 22 = 128 degrees of freedom.
The difference in degrees of freedom is: 128 - 125 = 3 = number of additional variables.

Test statistic is: $\dfrac{D_S - D_B}{(\text{number of added parameters})\ \hat{\phi}_B} = \dfrac{38.4 - 35.1}{(3)\ (0.3)} = 3.67.$

We compare the test statistic to an F-distribution with 3 and 125 degrees of freedom.
The null hypothesis is to use the simpler model, the one without physical health
The alternate hypothesis is to use the more complex model.
We reject the null hypothesis if the test statistic is sufficiently big.
Comment: The syllabus reading gives a similar example.
It may be helpful to briefly review the F-Test in Statistics, covered on an earlier exam.]

Using a computer, the p-value (probability-value) of this test is 1.4%.[256]
Thus at a 2.5% significance level we would reject the simpler model in favor of the more complex model. At a 1% significance level we would not reject the simpler model.

If we had used a 2.5% significance level, we would have decided to use physical health.
We had used four levels of physical health: poor, fair, good, and very good.
The next step would be to see how many of these levels are useful. For example, does it significantly improve model performance to separate good from very good?

[255] The F-Distribution with ν_1 and ν_2 > 2 degrees of freedom has a mean of $\nu_2/(\nu_2 - 1)$. For ν_2 large this mean is approximately 1. We reject the null hypothesis if the F-Statistic is significantly greater than 1.

[256] The 2.5% critical value is 3.222, while the 1% critical value is 3.942.
In other words, for the F-Distribution with 3 and 125 degrees of freedom, the survival function at 3.222 is 2.5%.

$$F = \frac{D_S - D_B}{(\text{number of added parameters})\ \hat{\phi}_B}.$$

$\hat{\phi}_B$ in the denominator is the estimate of the dispersion parameter for the bigger model.

It turns out that $\hat{\phi}_B$ is a good estimate of the amount by which we can expect unscaled deviance to go down for each new parameter added to the model, if the new parameter adds no predictive power. Thus

$\hat{\phi}_B$ (number of added parameters)

= expected drop in unscaled deviance when there is no added predictive value.

Thus for the added complexity to add predictive value to the model, it must reduce unscaled deviance by significantly more than $\hat{\phi}_B$ (number of added parameters), the denominator of the F-statistic.

Thus in the absence of added predictive value, the F-statistic has an expected value of approximately 1. If the F-statistic is significantly greater than 1, we may conclude that the added variables do indeed improve the model.

Statistical theory says that the F-statistic follows an F distribution. Thus we can perform an F-test, as in the previous example, to determine whether the F-Statistic is significantly bigger than one. If the p-value of the F-test is sufficiently small, we may conclude that the parameters added to the model are a significant improvement; in other words, one would use the bigger rather than the smaller model.

AIC and BIC:[257]

AIC and BIC are each methods of comparing models.
In each case, a smaller value is better.
These penalized measures of fit are particularly useful for comparing models that are not nested.

The Akaike Information Criterion (AIC) is used to compare a bunch of models all fit via maximum likelihood to the same data.[258] The model with the smallest AIC is preferred. For a particular model:
AIC = (-2) (maximum loglikelihood) + (number of parameters)(2).

The number of parameters fitted via maximum likelihood are the betas (slopes plus if applicable an intercept).[259]

Since the scaled deviance =
(2) (saturated max. loglikelihood - maximum loglikelihood for model),
we can compare between the models: Scaled Deviance + (number of parameters)(2).[260]

Assume for example, assume we have three Generalized Linear Models fit to the same data:

Model #	Number of Parameters	Scaled Deviance	Scaled Deviance + (number of parameters)(2)
1	4	888.7	896.7
2	5	886.2	896.2
3	6	884.4	896.4

We prefer Model #2, since it has the smallest AIC.[261]

The Bayesian Information Criterion (BIC) can also be used to compare a bunch of models all fit via maximum likelihood to the same data.[262] The model with the smallest BIC is preferred.
For a particular model:
BIC = (-2) (max. loglikelihood) + (number of parameters) ln(number of data points).[263]

[257] See Section 6.2.2 in Generalized Linear Models for Insurance Rating.
[258] Thus AIC can be applied to Generalized Linear Models.
[259] If a dispersion parameter is fit via maximum likelihood, then the number of parameters in the above formula for AIC is one more. However, if one is using AIC to compare models, it does not matter, as long as one is consistent, since the only difference is to add the same constant to each AIC.
[260] The maximum likelihood for the saturated model is the same in each case.
[261] In each case, the AIC is:
Scaled Deviance + (number of parameters)(2) - (2)(loglikelihood for the saturated model).
[262] Thus BIC can be applied to Generalized Linear Models.
[263] The GLM monograph uses ln and log interchangeably to both mean the natural log.

Since the scaled deviance = (2) (saturated max. loglikelihood - maximum likelihood for model),
we can compare between the models:
Scaled Deviance + (number of parameters) ln(number of data points).[264]

Assume that we have three Generalized Linear Models fit to the same data set of size 20:

Model #	Number of Parameters	Scaled Deviance	Scaled Deviance + (number parameters) ln(20)
1	4	888.7	900.7
2	5	886.2	901.2
3	6	884.4	902.4

We prefer Model #1, since it has the smallest BIC.[265]
We note that in this case, using AIC or BIC would result in different conclusions.

BIC is mathematically equivalent to the Schwarz Bayesian Criterion.[266] Using the Schwarz Bayesian Criterion, one adjusts the loglikelihoods by subtracting in each case the penalty:
(number of fitted parameters) ln(number of data points) / 2.
One then compares these penalized loglikelihoods directly; larger is better.
For a model, when BIC is smaller this penalized loglikelihood is bigger and vice-versa.

“As most insurance models are fit on very large datasets, the penalty for additional parameters imposed by BIC tends to be much larger than the penalty for additional parameters imposed by AIC. In practical terms, the authors have found that **AIC tends to produce more reasonable results**. **Relying too heavily on BIC may result in the exclusion of predictive variables from your model**.”

[264] The maximum likelihood for the saturated model is the same in each case.
[265] In each case, the BIC is:
Scaled Deviance + (number of parameters)ln[20] - (2)(loglikelihood for the saturated model).
[266] See for example Loss Models, not on the syllabus of this exam.

A Communicable Disease Example:[267]

Assume we have the following reported occurrences of a communicable disease in two areas:

Number in Area A	Number in Area B	Month
8	9	2
8	12	4
10	9	6
11	14	8
14	15	10
17	19	12
13	20	14
15	21	16
17	25	18
15	23	20

Let X_1 = 0 if Region A and 1 if Region B.
Let X_2 = ln[month].
Fit a GLM with a Poisson using a log link function.
$\mu = \text{Exp}[\beta_0 + \beta_1 X_1 + \beta_2 X_2]$.

The fitted parameters are: β_0 = 1.54894, β_1 = 0.265964, β_2 = 0.435105.

The covariance matrix is: $\begin{pmatrix} 0.0618301 & -0.00781226 & -0.0226385 \\ -0.00781226 & 0.0138001 & -6.28837 \times 10^{-18} \\ -0.0226385 & -6.28837 \times 10^{-18} & 0.00948766 \end{pmatrix}$.

Therefore, approximate 95% confidence intervals for the parameters are:
$1.54894 \pm 1.960 \sqrt{0.0618301} = (1.06, 2.04)$,

$0.265964 \pm 1.960 \sqrt{0.0138001} = (0.04, 0.50)$,

$0.435105 \pm 1.960 \sqrt{0.00948766} = (0.24, 0.63)$.

The loglikelihood is: -47.0892.
The Scaled Deviance is: 4.45650.

[267] Adapted from Section 18.4 of Applied Regression Analysis by Draper and Smith, not on the syllabus.

In order to test whether $\beta_1 = 0$, the test statistic is:

$\hat{\beta}_1$ / StdDev[$\hat{\beta}_1$] = 0.265964 / $\sqrt{0.0138001}$ = 2.264.

The probability value of a two-sided test is: 2{1 - Φ[2.264]} = 2.4%.[268]

Exercise: Test whether $\beta_2 = 0$.

[Solution: $\hat{\beta}_2$ / StdDev[$\hat{\beta}_2$] = 0.435105 / $\sqrt{0.00948766}$ = 4.467.

The probability value of a two-sided test is: 2{1 - Φ[4.467]} = 0%.

Comment: Using a computer, the p-value is 8 x 10^{-6}.]

Exercise: Test whether $\beta_0 = 2$.

[Solution: ($\hat{\beta}_0$ - 2) / StdDev[$\hat{\beta}_0$] = (1.54894 - 2) / $\sqrt{0.0618301}$ = -1.814.

The probability value of a two-sided test is: 2 Φ[-1.814] = 7.0%.]

Now fit an otherwise similar GLM ignoring region, in other words without the dummy variable X_1.
The fitted parameters are: β_0 = 1.69074, β_2 = 0.435105.

The covariance matrix is: $\begin{pmatrix} 0.0574127 & -0.0226404 \\ -0.0226404 & 0.00948839 \end{pmatrix}$.

Therefore, approximate 95% confidence intervals for the parameters are:
β_0: 1.69074 ± 1.960 $\sqrt{0.0574127}$ = (1.22, 2.16),
β_2: 0.435105 ± 1.960 $\sqrt{0.00948839}$ = (0.24, 0.63).

The loglikelihood is: -49.6747.
The Scaled Deviance is: 9.62755.

For the model including region, the loglikelihood is -47.0892.
There are 20 data points and this model has 3 fitted betas.
AIC = (-2)(-47.0892) + (3)(2) = 100.178.
BIC = (-2)(-47.0892) + 3 ln(20) = 103.166.

For the simpler model excluding region, the loglikelihood is -49.6747.
This model has only 2 fitted betas.
AIC = (-2)(-49.6747) + (2)(2) = 103.349.
BIC = (-2)(-49.6747) + 2 ln(20) = 105.341.

[268] There is not a Normal Distribution Table attached to your exam.

The first more complicated model has the smaller AIC and thus is preferred on this basis.
The more complicated model has the smaller BIC and thus is also preferred on this basis.

The first model has a Scaled Deviance of 4.45650, while the second simpler model has a Scaled Deviance of 9.62755. Equivalently, we can use these rather than using AIC or BIC directly.

For the first model, Scaled Deviance + (number of parameters)(2)
= 4.45650 + (3)(2) = 10.45650.
For the second model, Scaled Deviance + (number of parameters)(2) = 9.62755 + (2)(2) = 13.62755.
Since 10.45650 < 13.62755, the first more complicated model is preferred on this basis.[269]

For the first model, Scaled Deviance + (number of parameters) ln(sample size)
= 4.45650 + 3 ln(20) = 13.444.
For the second model, Scaled Deviance + (number of parameters) ln(sample size)
= 9.62755 + 2 ln(20) = 15.619.
Since 113.444 < 15.619, the first more complicated model is also preferred on this basis.[270]

Deviance Residuals:[271]

The (ordinary) residuals are the difference between the observed and fitted values.
Other types of residuals are useful when working with GLMs, including
Deviance Residuals.[272] [273] Deviance Residuals provide a more general quantification of the conformity of a case to the model specification.

Deviance Residuals are based on the form of the unscaled deviance for the particular distribution. Since the syllabus reading does not discuss these forms, you are not responsible for them on this exam.

The square of the deviance residual is the corresponding term in the sum that is the unscaled deviance.

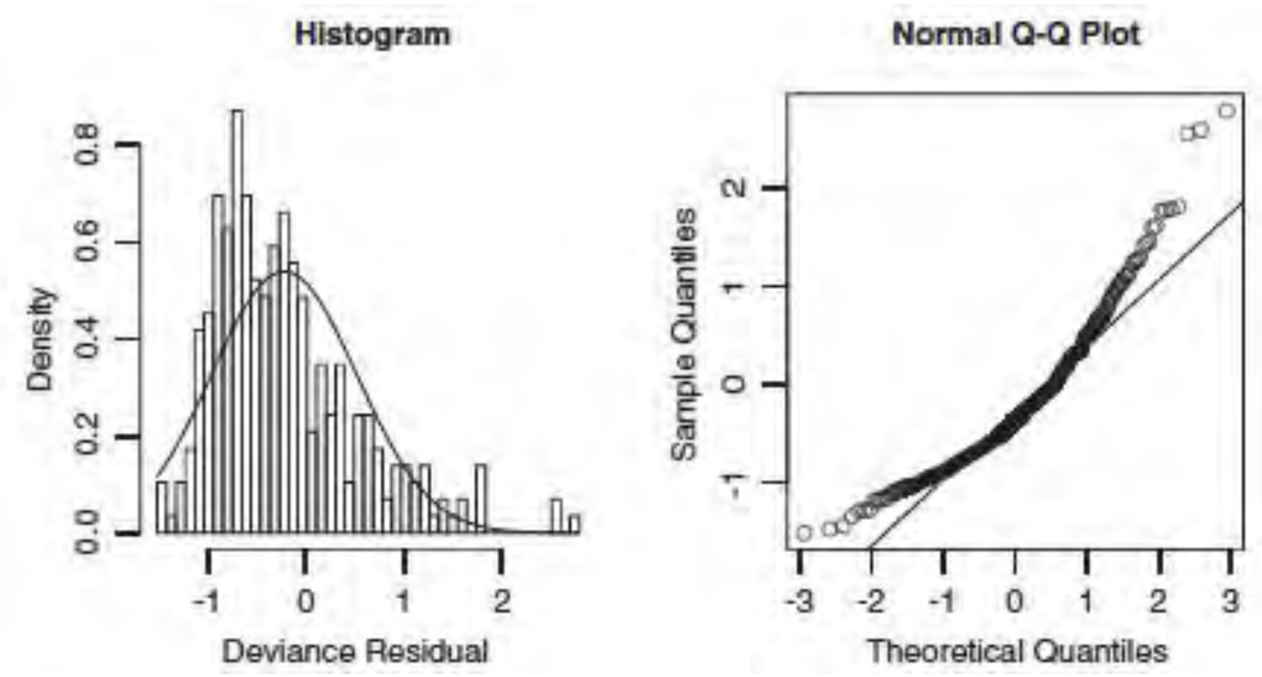
We take the sign of the deviance residual as the same as that of the (ordinary) residual $y_i - \hat{\mu}_i$.

[269] This is equivalent to comparing AICs.
[270] This is equivalent to comparing BICs.
[271] See Section 6.3.1 in Generalized Linear Models for Insurance Rating.
[272] Working Residuals will be discussed subsequently.
[273] Pearson Residuals and Anscombe Residuals are also used, but these are not on the syllabus.
See for example Generalized Linear Models by McCullagh and Nelder,
Generalized Linear Models for Insurance Data by de Jong and Heller,
and An Introduction to Generalized Linear Models by Dobson and Barnett.

"We can think of **the deviance residual as the residual adjusted for the shape of the assumed GLM distribution**, **such that its distribution will be approximately Normal** if the assumed GLM distribution is correct."

If the fitted model is appropriate, then we expect:
- **The deviance residuals should follow no predictable pattern**.[274]
- **The deviance residuals should be Normally distributed, with constant variance**.[275]

The syllabus reading shows an example of how to determine whether the deviance residuals are Normal. In the first case, a model was fit with a Gamma Distribution:[276]

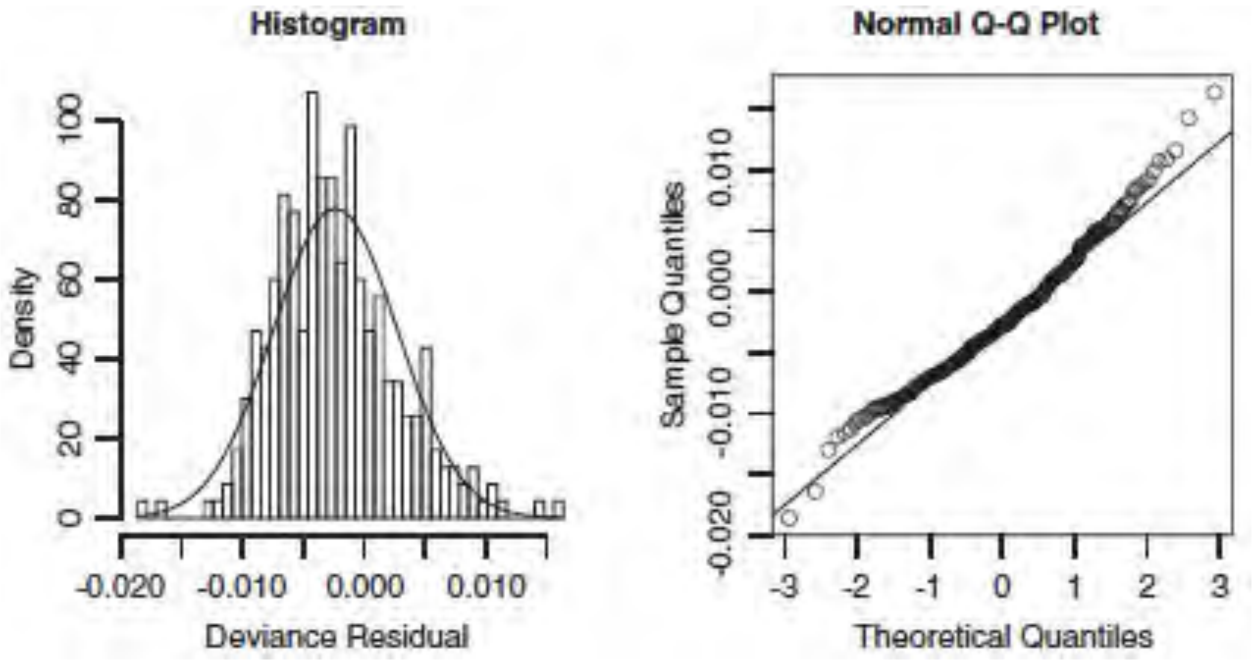


In the histogram, the deviance residuals do not seem close to the best fit Normal.[277]
In the Normal Q-Q plot, the deviance residuals are not near the comparison straight line.[278]
We conclude that the deviance residuals are not Normal and therefore the Gamma Distribution is probably not a good choice to model this data.

In the histogram, the deviance residuals are skewed to the right. Thus an Inverse Gaussian Distribution with greater skewness than a Gamma Distribution, might be better for modeling this data.

[274] If we discover a pattern in the deviance residuals then we can probably improve our model to pick this pattern up.
[275] The property of constant variance is called homoscedasticity.
Homoscedasticity is more closely followed for standardized deviance residuals, not on the syllabus.
If the model is correct, standardized residuals should (approximately) follow a Standard Normal Distribution.
See an An Introduction to Generalized Linear Models by Dobson and Barnett.
[276] See Figure 16 in Generalized Linear Models for Insurance Rating.
[277] See for example Loss Models, not on the syllabus of this exam.
[278] See for example Loss Models, not on the syllabus of this exam.

Here is similar graphs for a model that was fit with an Inverse Gaussian Distribution:[279]



In the histogram, the deviance residuals are much closer to the best fit Normal than before. In the Normal Q-Q plot, the deviance residuals are much nearer to the comparison straight line than before.

We conclude that the deviance residuals are closer to Normal, and therefore the Inverse Gaussian Distribution is probably a better choice to model this data than the Gamma Distribution.

Deviance Residuals for Discrete Distributions:

For discrete distributions such as Poisson or Negative Binomial, or distributions that have a point mass such as the Tweedie, the deviance residuals will likely not follow a Normal Distribution.[280] This makes deviance residuals less useful for assessing the appropriateness of such distributions, when each record is for a single risk.[281]

Fortunately, for data sets where one record may represent the average frequency for a large number of risks, deviance residuals are more useful than when each record is for a single risk.[282]

[279] See Figure 17 in Generalized Linear Models for Insurance Rating.

[280] This is because the deviance residuals do not adjust for the discreteness; the large numbers of records having the same target values cause the residuals to be clustered together in tight groups.

[281] One possible solution is to use randomized quantile residuals, which add random jitter to the discrete points so that they wind up more smoothly spread over the distribution.

[282] The target variable will take on a larger number of distinct values, effectively smoothing out the resulting distribution causing it to lose much of its discrete property and approach a continuous distribution.

Review, Histograms:

A histogram is an approximate graph of the probability density function.
First we need to group the data into intervals.

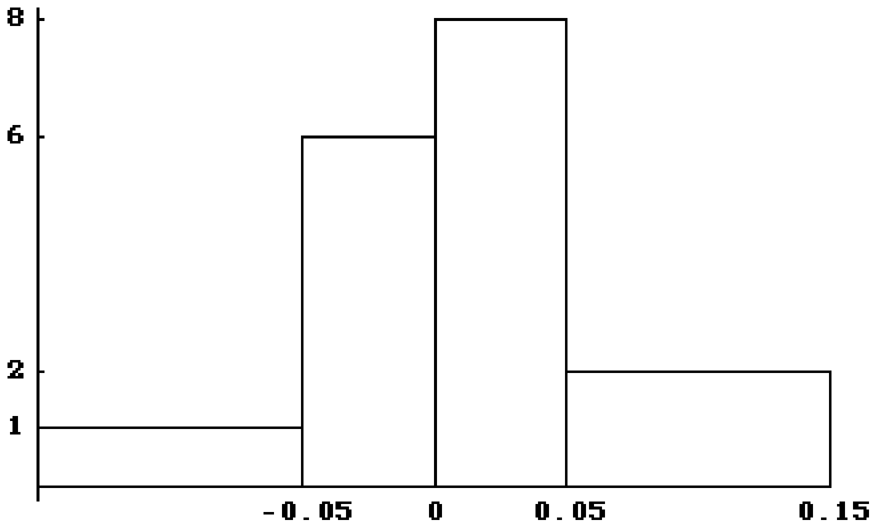
The height of each rectangle = $\dfrac{\text{\# values in the interval}}{\text{(total \# values) (width of interval)}}$.

For example, let us assume we observe 100 values and group them into four intervals:
Number that are between -0.15 and -0.05: 10
Number between -0.05 and 0: 30
Number between 0 and 0.05: 40
Number between 0.05 and 0.15: 20

The first interval has width 0.1. The probability in the first interval is: 10/100.
We want the area of the first rectangle to be equal to the probability in the first interval.
(0.1)(height) = 10/100. ⇒ Height = (10/100) / (0.1) = 1.

Similarly, the height of the second rectangle is: (30/100) / (0.05) = 6.
The height of the third rectangle is: (40/100) / (0.05) = 8.
The height of the fourth rectangle is: (20/100) / (0.10) = 2.

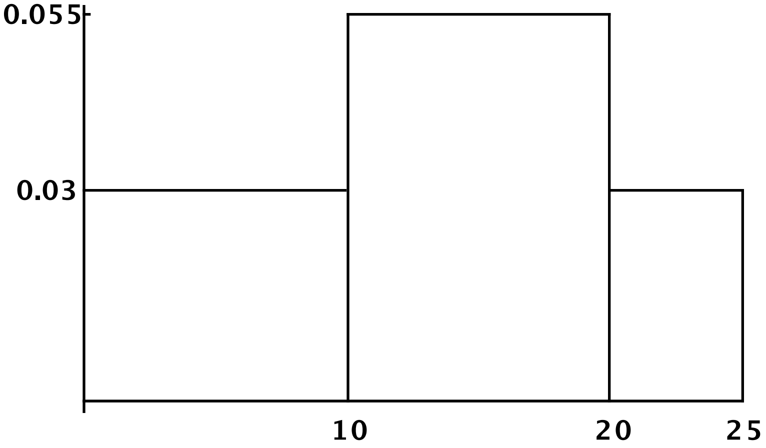
The histogram of these 100 values:



The sum of the areas of the rectangles is: (0.1)(1) + (0.05)(6) + (0.05)(8) + (0.1)(2) = 1.
In general the area under a histogram should sum to one, just as for the graph of a probability density function.

Exercise: Draw a histogram of the following grouped data: 0 -10: 6, 10-20: 11, 20-25: 3.

[Solution: The heights are: $\frac{6}{(20)(10)}$ = 0.03, $\frac{11}{(20)(10)}$ = 0.055, and $\frac{3}{(20)(5)}$ = 0.03.



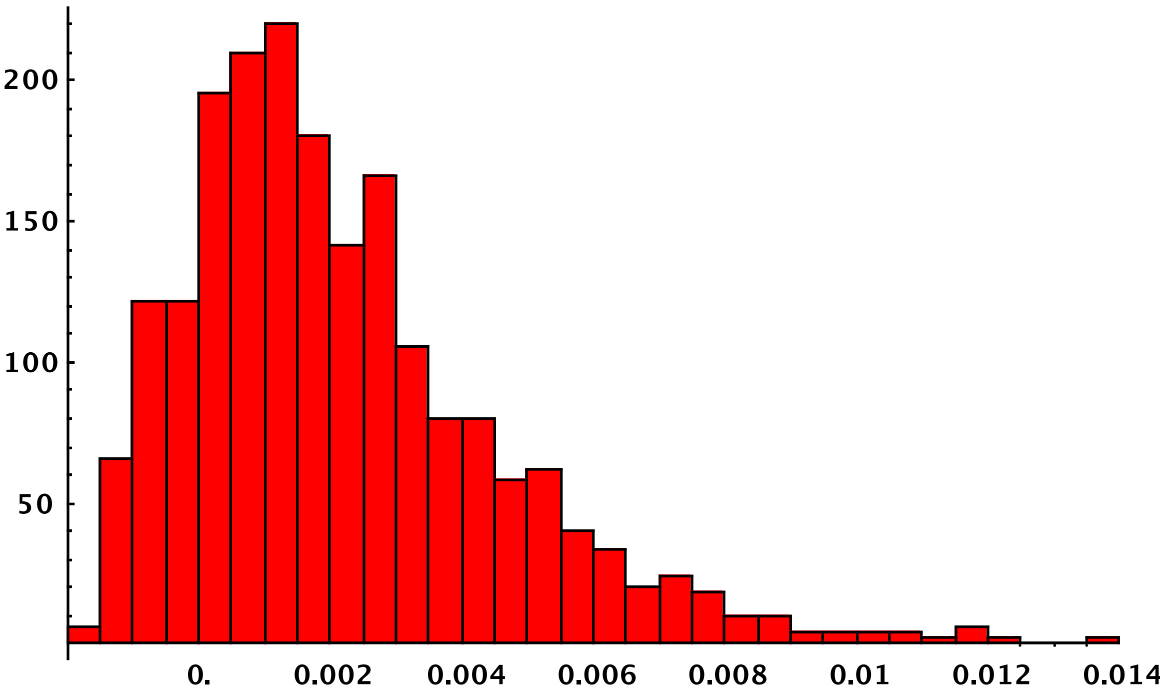
Comment: The sum of the areas of the rectangles is: (10)(0.03) + (10)(0.055) + (5)(0.03) = 1. With more data, we would get a better idea of the probability density function from which this data was drawn.]

Creating a histogram and comparing the histogram to a graph of a Normal Distribution is one way to determine whether the items of interest appear to be Normally distributed.

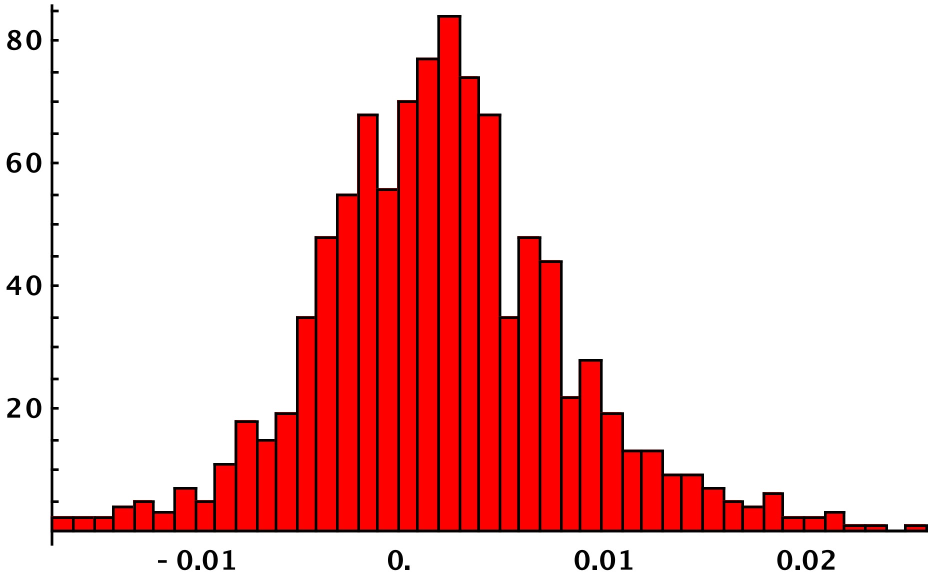
First we would want the histogram to look roughly symmetric, since the Normal Distribution is symmetric around its mean.[283]

[283] If the values are from a Normal Distribution, then one would expect the skewness of the observed values to be close to zero. In addition, since a Normal Distribution has a kurtosis of 3, if the values are from a Normal Distribution, then one would expect the kurtosis of the observed values to be close to 3.

The following histogram is not symmetric, and thus not likely to be a sample from a Normal Distribution:[284]

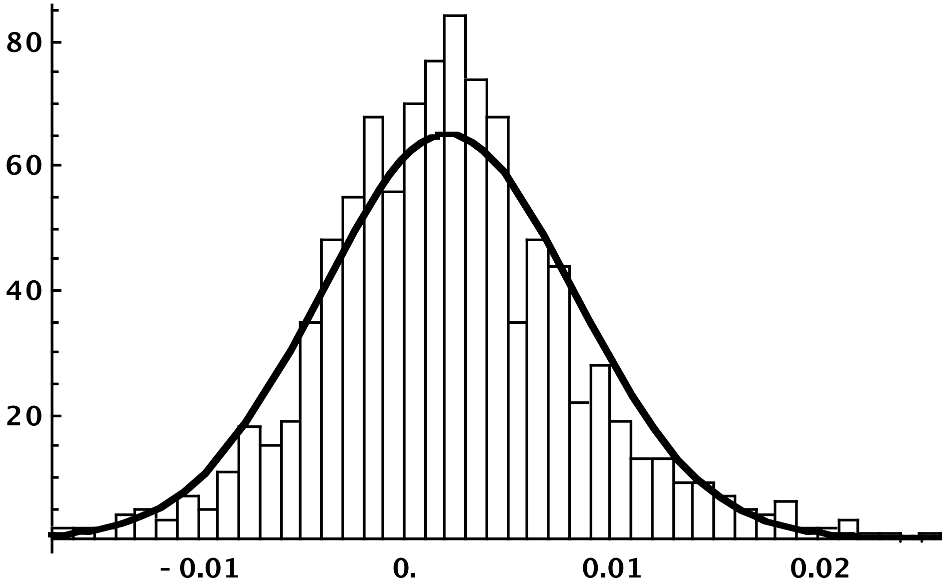


The following histogram looks approximately symmetric:



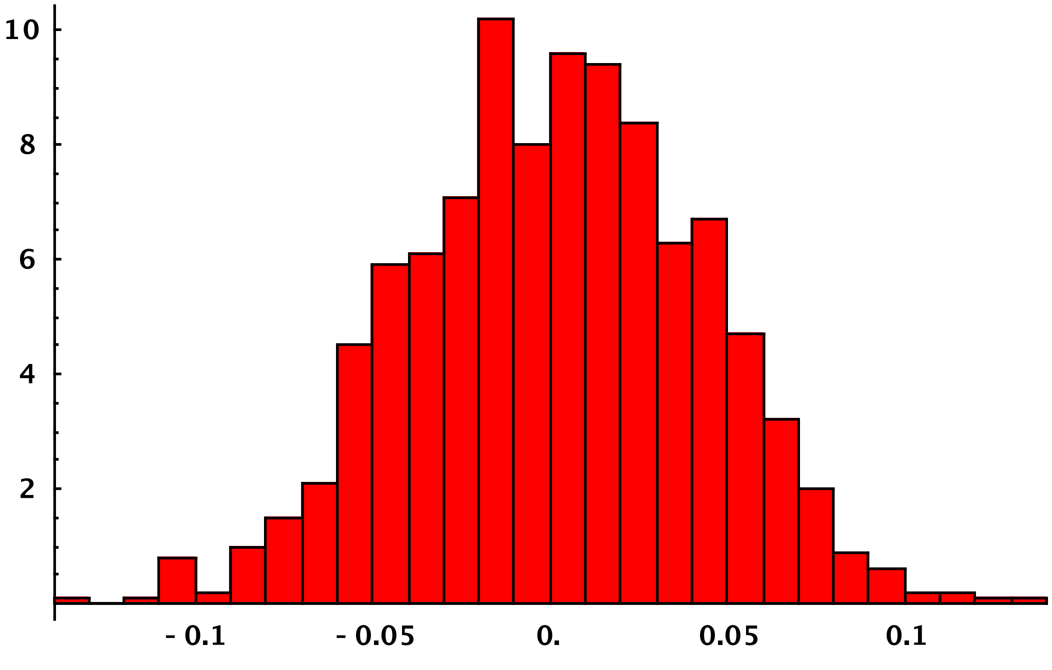
[284] *This histogram was based on 1000 data points simulated from a shifted Gamma Distribution.*

However, one can superimpose upon it a Normal Distribution with parameters $\mu = \bar{X}$ and σ = sample variance:



The histogram of the data seems to be more highly peaked than the Normal and may have heavier tails.[285] *This data has a larger kurtosis than a Normal; the graph displays leptokurtosis.*[286]

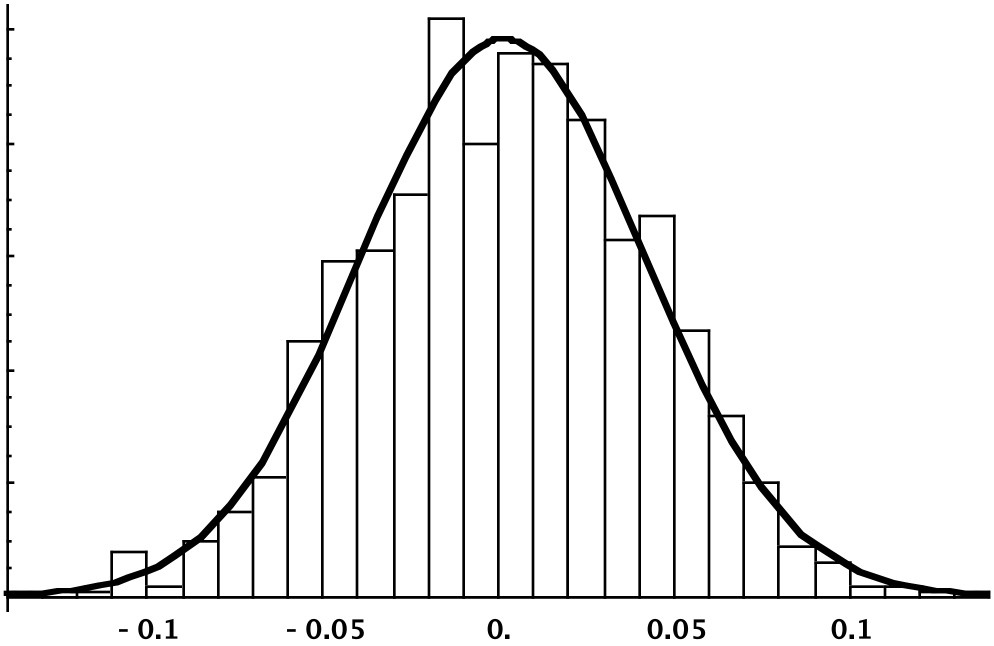
The following histogram, is based on a random sample of size 1000 from a Normal Distribution:



[285] Heavier tails means more probability in both the lefthand and righthand tails.

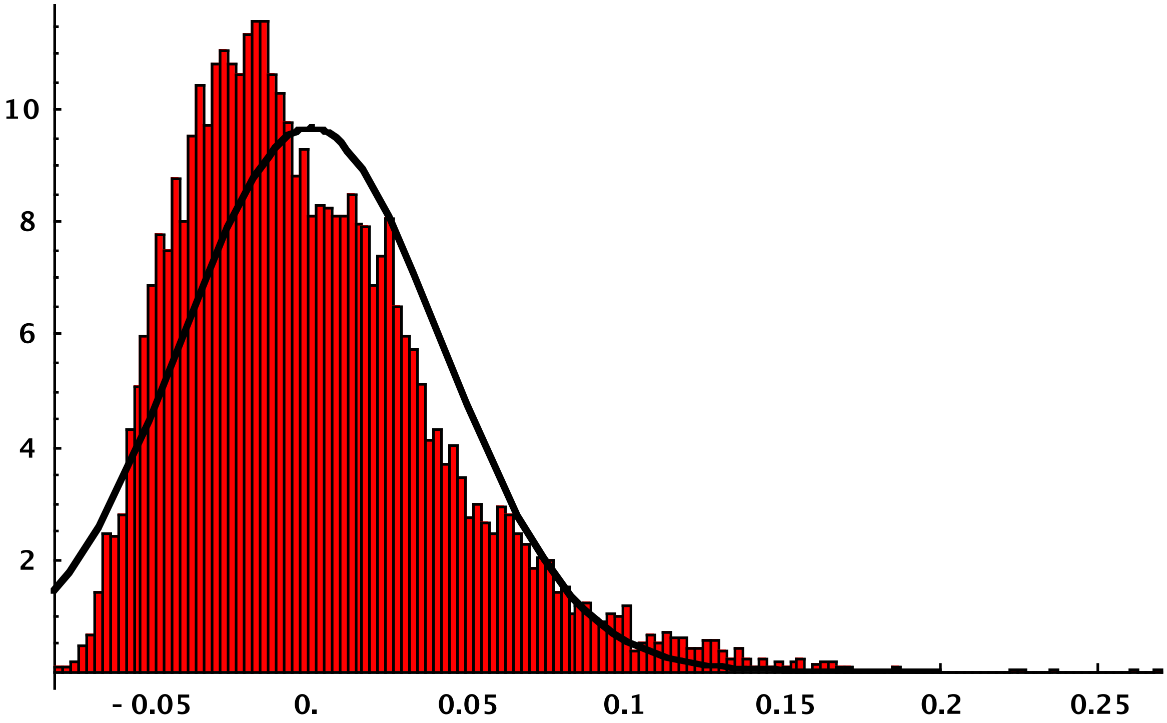
[286] *Kurtosis = 4th central moment / square of the variance. All Normal Distributions have a kurtosis of 3, so one would want the kurtosis of the data to also be close to 3. For the data that generated this histogram the kurtosis is 3.85, indicating somewhat heavier tails than a Normal Distribution.*

I superimposed upon the above histogram a Normal Distribution, with parameters $\mu = \bar{X}$ and σ = sample variance:



As with any finite sample, while the match between the data and a fitted Normal Distribution seems reasonable, it is far from perfect.

Next I simulated 10,000 random draws from a Gamma Distribution (with $\alpha = 4$), and then subtracted a constant.[287] I then compared a histogram of the data to the probability density function of a Normal Distribution with parameters based on the sample mean and sample variance of the data:



The curve of the Normal Distribution is a poor match to the data represented by the histogram.[288]
Even if we did not know the data was simulated from another distribution, we would conclude that this data was not drawn from a Normal Distribution.

Review, Q-Q Plots:

A Q-Q plot or quantile-quantile plot is a graphical technique which can be used to either compare a data set and a distribution or compare two data sets. Q-Q plots are most commonly used as a visual test of whether data is appears to be from a Normal Distribution. These are sometimes called Normal Q-Q Plots.

The 95th percentile is also referred to as $Q_{0.95}$, the 95% quantile.
For a distribution, the quantile Q_α is such for $F(Q_\alpha) = \alpha$. In other words, $Q_\alpha = F^{-1}(\alpha)$.
For example, $Q_{0.95}$ for a Standard Normal Distribution is 1.645, since $\Phi[1.645] = 0.95$.

[287] The key idea here is that the Gamma is some distribution different than the Normal Distribution.
[288] Since this Normal Distribution has the same mean and variance as the data, we would expect it to be a good match to the data, provided the data were drawn from a Normal Distribution.

In order to see whether data is drawn from some member of a Distribution Family, which has a scale and/or location parameter, we can create a Q-Q Plot for a standard member of that family F.

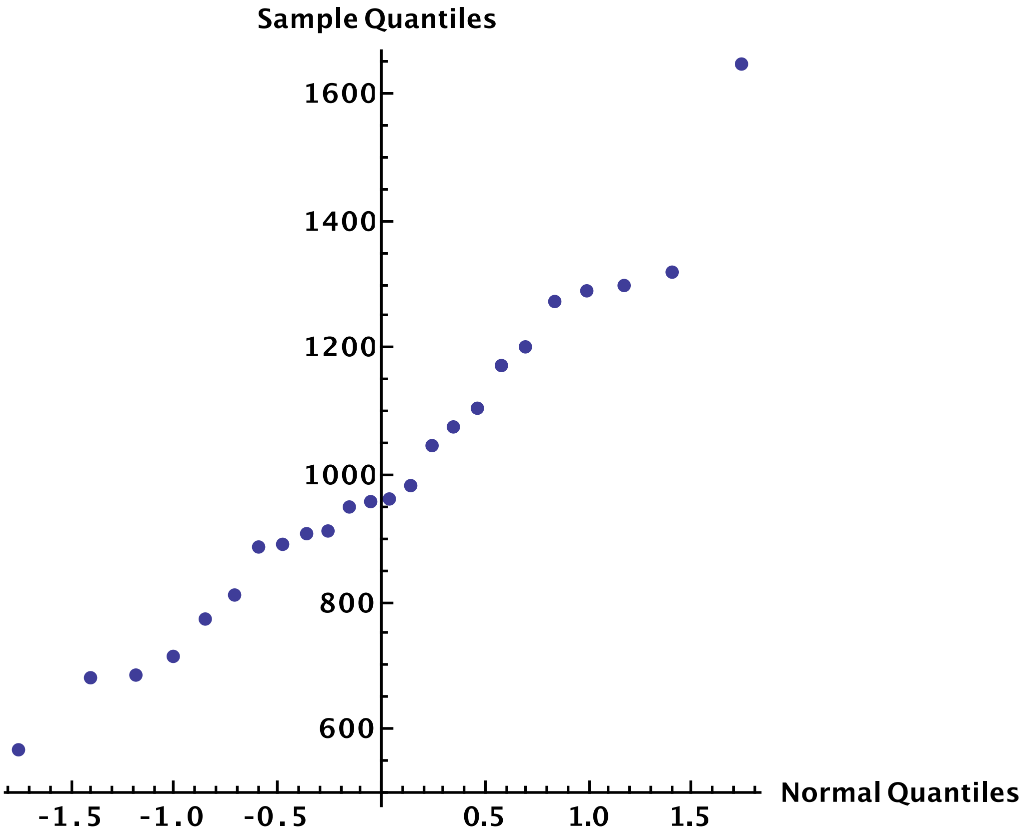
- Grade the n data points from smallest to largest.
- For i = 1 to n, plot the points: $(F^{-1}[\frac{i}{n+1})], x_{(i)})$.

If the data is drawn from the given distribution family, then we expect the plotted points to lie close to some straight line.

Take the following 24 data point arranged from smallest to largest:
565, 678, 681, 713, 769, 809, 883, 890, 906, 909, 946, 956, 961, 983, 1046, 1073, 1103, 1171, 1198, 1269, 1286, 1296, 1316, 1643.
For the Standard Normal, $Q_{1/25} = Q_{0.04} = -1.751$.
Thus the first plotted point in a Normal Q-Q Plot is: (-1.751, 565).

Exercise: What is the second plotted point?
[Solution: $Q_{2/25} = Q_{0.08} = -1.405$. Thus the second plotted point is: (-1.405, 678).]

Here is the resulting Normal Q-Q Plot:

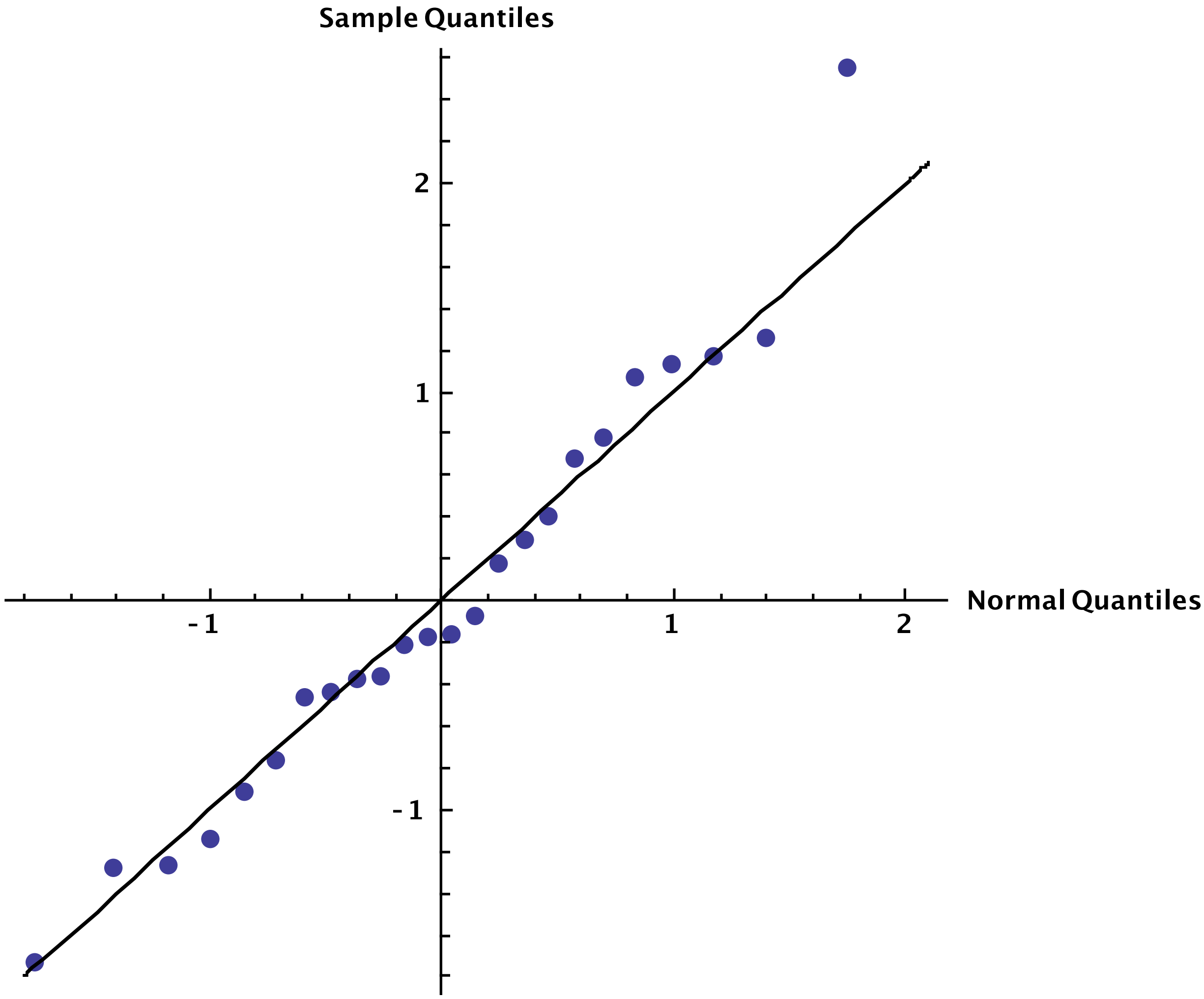


Other than the final point, the plotted points seem approximately linear, and thus this data could very well be from a single Normal Distribution.[289]

One could standardize each data point prior to constructing the Q-Q plot.
The data has a sample mean of 1002.08, and a sample variance of 63,387.9.
Thus we would subtract 1002.08 from each data point and divide by $\sqrt{63{,}387.9}$.

For example, (565 - 1002.08) / $\sqrt{63{,}387.9}$ = -1.736.

Here is the Q-Q Plot, using the standardized data, including the comparison line x = y:[290]



Again, other than the final point, the plotted points are close to the 45 degree comparison line, and thus this data could very well be from a single Normal Distribution.

[289] With small data sets it is hard to draw a definitive conclusion.
There is no specific numerical test we would apply to the Q-Q plot.

[290] Having standardized the data, when we compare to the Standard Normal Distribution, we expect the plotted points to be close to the 45 degree comparison line x = y.

Form of the Deviance Residual:[291]

The form of the deviance residual depends on the distribution and thus the form of the unscaled deviance.

Distribution	Square of the Deviance Residual
Normal	$\frac{1}{\sigma^2} \sum_{i=1}^{n} (y_i - \hat{\mu}_i)^2$
Poisson	$2 \sum_{i=1}^{n} \{y_i \ln[y_i / \hat{\lambda}_i] - (y_i - \hat{\lambda}_i)\}$
Binomial	$2 \sum_{i=1}^{n} \{y_i \ln[\frac{y_i}{\hat{y}_i}] + (m_i - y_i) \ln[\frac{m_i - y_i}{m_i - \hat{y}_i}]\}$
Gamma	$2 \alpha \sum_{i=1}^{n} \{-\ln[y_i / \hat{y}_i] + (y_i - \hat{y}_i) / \hat{y}_i\}$
Inverse Gaussian	$\theta \sum_{i=1}^{n} \frac{(y_i - \hat{y}_i)^2}{\hat{y}_i^2 \; y_i}$
Negative Binomial	$2 \sum_{i=1}^{n} \{y_i \ln[y_i / \hat{y}_i] - (y_i + r) \ln[\frac{y_i + r}{\hat{y}_i + r}]\}$

Exercise: For a GLM using a Gamma Distribution, the first observed value is 800 with corresponding fitted value of 853.20. The maximum likelihood fitted parameter $\alpha = 45.6$
What is the corresponding deviance residual?

[Solution: $d_1^2 = (2)(45.6) \{-\ln[800/853.20] + (800 - 853.20)/853.20\} = 0.1850$.

Since 800 - 853.20 is negative, we take the deviance residual as negative.

$d_1 = -\sqrt{0.1850} = -0.430$.

Comment: This is for the two-dimensional example I discussed previously, using a reciprocal link function.]

[291] Not on the syllabus of this exam.

Communicable Disease Example Continued:

For the Poisson Distribution, the unscaled deviance is:

$$D = 2 \sum_{i=1}^{n} \{y_i \ln[y_i / \hat{\lambda}_i] - (y_i - \hat{\lambda}_i)\}.$$

Then the square of the deviance residual is the corresponding term in the above sum:

$d_i^2 = 2 \{y_i \ln[y_i / \hat{\lambda}_i] - (y_i - \hat{\lambda}_i)\}$.

For example, for the Communicable Disease Example which uses a Poisson Distribution, the first observed count is 8 with corresponding fitted value 6.3632.

Thus $d_1^2 = 2 \{8 \ln[8 / 6.3632] - (8 - 6.3632)\} = 0.3889$.

Since the first ordinary residual is positive, $d_1 = \sqrt{0.3889} = 0.6236$.

Exercise: For this example, the third observed count is 10 with corresponding fitted value 10.263.

Determine the corresponding deviance residual.

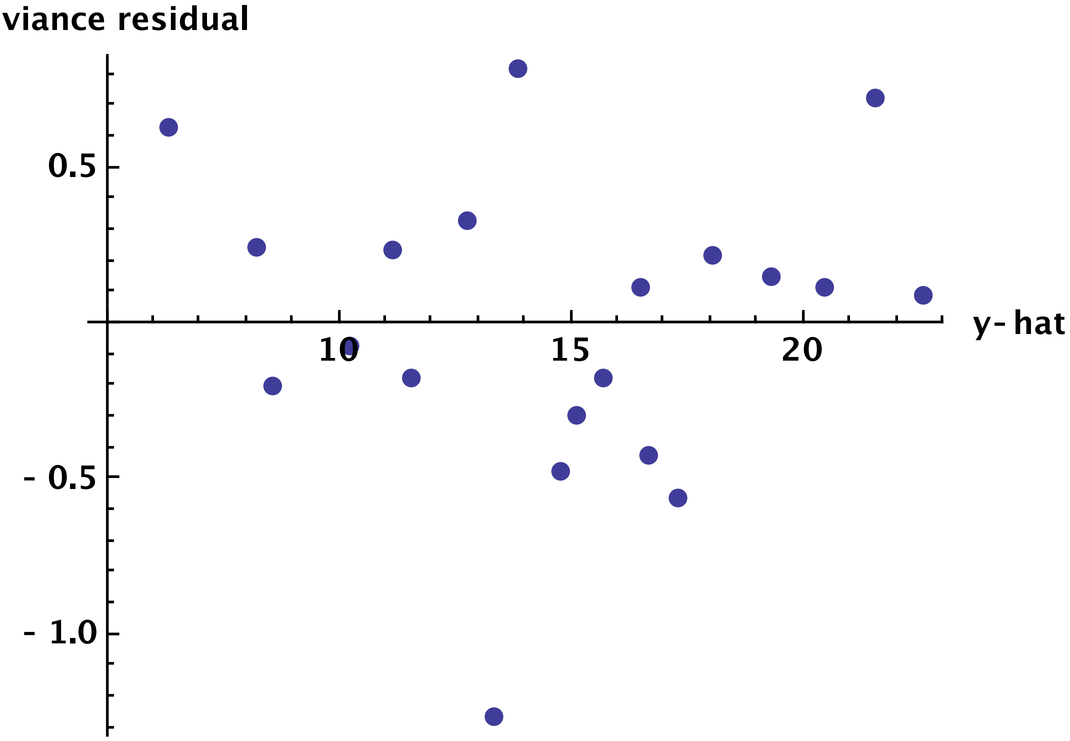
[Solution: $d_3^2 = 2 \{10 \ln[10 / 10.263] - (10 - 10.263)\} = 0.006798$.

Since $10 - 10.263 < 0$, we take the deviance residual as negative.

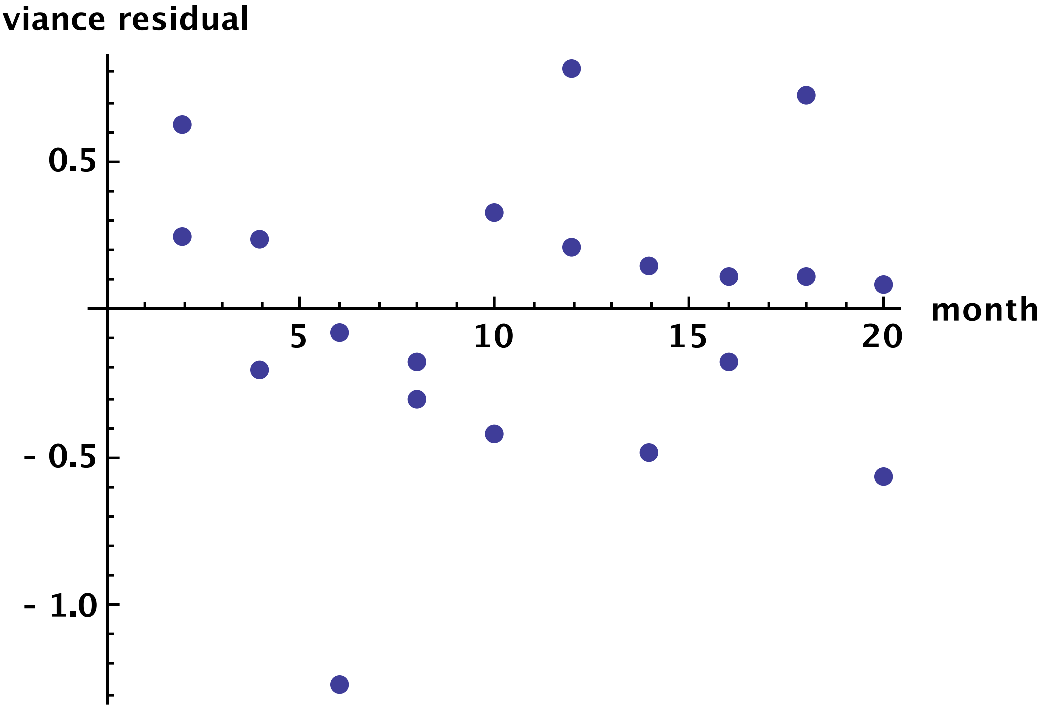
$d_3 = -\sqrt{0.006798} = -0.0824$.]

For this example, the deviance residuals are: 0.6237, -0.2081, -0.0824, -0.1869, 0.3254, 0.8099, -0.4876, -0.1845, 0.1094, -0.5728, 0.2390, 0.2289, -1.2763, -0.3058, -0.4288, 0.2090, 0.1448, 0.1061, 0.7144, 0.0819.

Here is a graph of the deviance residuals versus the fitted values:

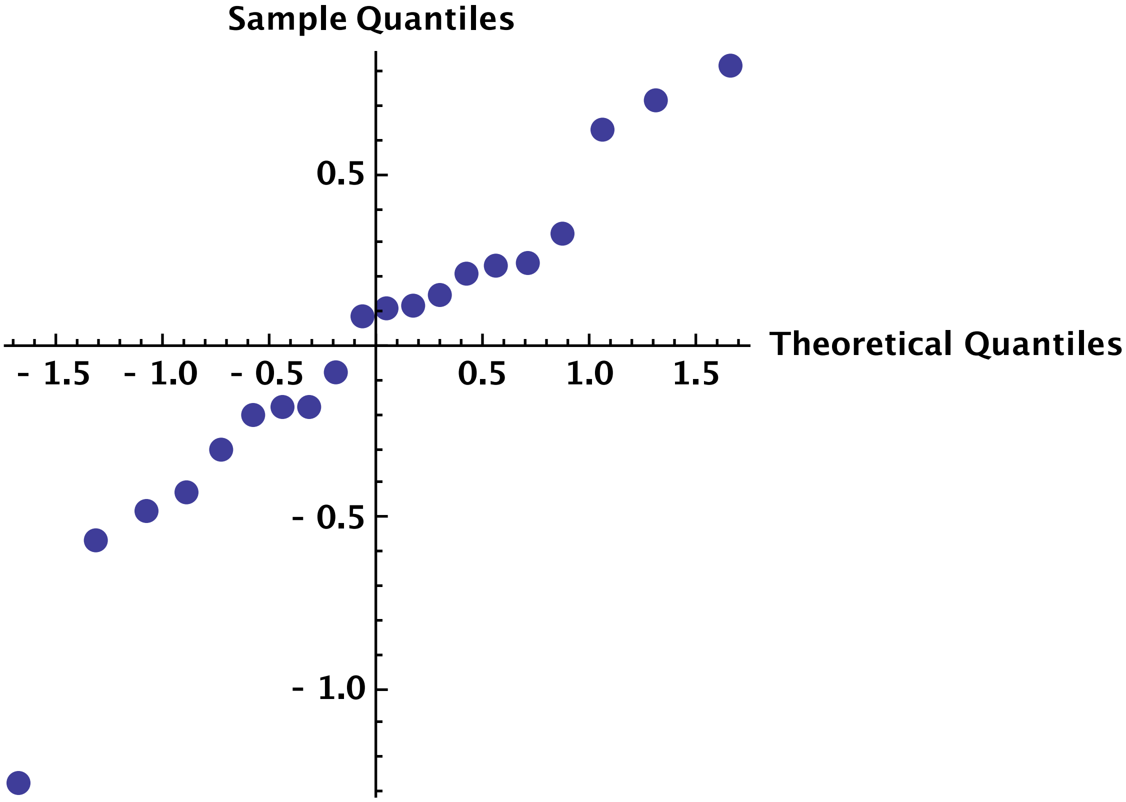


Here is a graph of the deviance residuals versus month:



In neither case do I observe an obvious pattern.

Here is a Normal Q-Q plot of the deviance residuals:



Other than the first point, the points seem to be approximately along a straight line, thus this data could be from a Normal. However, there is too little data to make a definite conclusion.[292]

[292] It turns out the standardized residuals do not seem to follow a Standard Normal. It turns out that for a Gamma Distribution rather than a Poisson Distribution, the standardized residuals seem to follow a Standard Normal. Thus a Gamma Distribution seems to be a better model for this data.

Working Residuals:[293]

Working residuals are another useful type of residual, which can be used to analyze the appropriateness of a fitted GLM.[294] [295]

The form of the deviance residuals depends on the distributional form used in the model. The form of the working residuals instead depends on the link function used in the model.

Working Residual is:

$$\mathbf{wr_i = (y_i - \mu_i)\, g'(\mu_i)}.$$

Recall that the partial residual was defined as: r_i = (ordinary residual) $g'(\mu_i) + x_{ij}\,\beta_j$.
Thus the partial residual is the working residual with effect of the j^{th} predictor removed.

Exercise: What is the form of the working residual for the identity link function?
[Solution: $g(\mu) = \mu$. ⇒ $g'(\mu) = 1$. ⇒ $wr_i = y_i - \mu_i$ = ordinary residual.

Comment: Thus the working residuals can be thought of as a generalization of the ordinary residuals used for example in linear regression.]

Exercise: What is the form of the working residual for the log link function?
[Solution: $g(\mu) = \ln(\mu)$. ⇒ $g'(\mu) = 1/\mu$. ⇒ $wr_i = (y_i - \mu_i)/\mu_i$.]

Exercise: What is the form of the working residual for the logit link function?
[Solution: $g(\mu) = \ln\left(\frac{\mu}{1 - \mu}\right)$. ⇒ $g'(\mu) = \frac{1}{\left(\frac{\mu}{1-\mu}\right)} \frac{(1 - \mu) - (\mu)(-1)}{(1 - \mu)^2} = \frac{1}{\mu\,(1 - \mu)}$. ⇒ $wr_i = \frac{y_i - \mu_i}{\mu_i\,(1 - \mu_i)}$.]

Link Function	Working Residual
Identity	$y_i - \mu_i$
Log	$(y_i - \mu_i)/\mu_i$
Logit	$\frac{y_i - \mu_i}{\mu_i\,(1 - \mu_i)}$

[293] See Section 6.3.2 of Generalized Linear Models for Insurance Rating.
[294] Working residuals are available in the statistical language R, as well as the computer language Mathematica.
[295] "Most implementations of GLM fit the model using the Iteratively Reweighted Least Squares (IRLS) algorithm, the details of which are beyond the scope of this monograph. Working residuals are quantities that are used by the IRLS algorithm during the fitting process."

Exercise: A GLM has been fit. The fifth response is 0.8, and the corresponding prediction is 0.6.
Determine the fifth working residual for each of the following cases:
Identity Link Function, Log Link Function, and Logit Link Function.
[Solution: For the Identity Link Function: 0.8 - 0.6 = 0.2.
For the Log Link Function: (0.8 - 0.6)/0.6 = 0.333.
For the Logit Link Function: (0.8 - 0.6)/ {(0.6)(0.4)} = 0.833.]

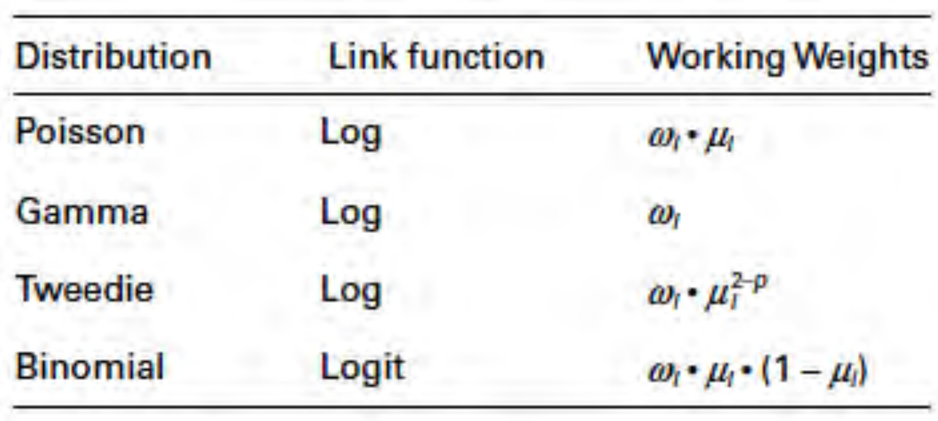
As has been discussed, graphing residuals is useful. However, most insurance models have thousands or even millions of observations, making such graphs much less useful.[296]

Therefore, **it can very useful to <u>bin</u> the working residuals**.
One groups together similar values on the x-axis (of the intended plot), and then takes a weighted average of the corresponding working residuals.[297] "Binning the residuals aggregates away the volume and skewness of individual residuals, and allows us to focus on the signal."

In order to take this weighted average of working residuals within each bin,
one uses working weights:

$$ww_i = \frac{\omega_i}{V(\mu_i)\, g'(\mu_i)^2}.$$

The working weight depends on the weights assigned in the model to each observation as well as the link function and the distributional form. Here is the form for some examples:[298] [299]



Distribution	Link function	Working Weights
Poisson	Log	$\omega_i \cdot \mu_i$
Gamma	Log	ω_i
Tweedie	Log	$\omega_i \cdot \mu_i^{2-p}$
Binomial	Logit	$\omega_i \cdot \mu_i \cdot (1-\mu_i)$

For the Normal with an identity link function, in other words linear regression, V(μ) = 1 and g(μ) = μ. Therefore, in this case, the working weight is just ω, the (ordinary) weight.

[296] Also the quantity being modeled is usually highly skewed, adding to the difficulty of interpreting a graph of residuals,

[297] "The advantage of working residuals is that they can be aggregated in a way that preserves the common properties of residuals – that is, they are unbiased (i.e., have no predictable pattern in the mean) and homoscedastic (i.e.,have no pattern in the variance) for a well-fit model."
See the Appendix of <u>Generalized Linear Models for Insurance Rating</u>.

[298] Taken from the footnote at page 71 of <u>Generalized Linear Models for Insurance Rating</u>.
What is shown for the Binomial is actually for the Bernoulli special case.

[299] Personally, I would <u>not</u> memorize these forms of working weights.

Exercise: A GLM has been fit. The sixth observation was given a weight of 10.
The sixth response is 0.5, and the corresponding prediction is 0.7.
Determine the sixth working weight for each of the following cases: Poison with log link, Gamma with log link, Tweedie with p = 1.4 and log link, and Bernoulli with logit link.
[Solution: Poison with log link: (10)(0.7) = 7. Gamma with log link: 10,
Tweedie with p = 1.4 and log link: (10) $(0.7^{0.6})$ = 8.07.
Bernoulli with logit link: (10)(0.7)(0.3) = 2.1.]

Exercise: An actuary has fit a severity GLM using an Inverse Gaussian Distribution with log link function. Number of claims were used as the weights.
The actuary is creating a plot of working residuals in order to assess the model fit.
The following eight observations will be binned together.
Compute the binned working residual for this bin.

Observed	Predicted	Number of Claims
334	444	6
412	383	3
560	487	11
621	642	5
448	370	8
509	581	4
380	426	7
495	411	9

[Solution: working residual: $wr_i = (y_i - \mu_i)\ g'(\mu_i)$.
For the log link function: $g(\mu) = \ln(\mu)$. ⇒ $g'(\mu) = 1/\mu$. ⇒ $wr_i = (y_i - \mu_i)/\mu_i$.

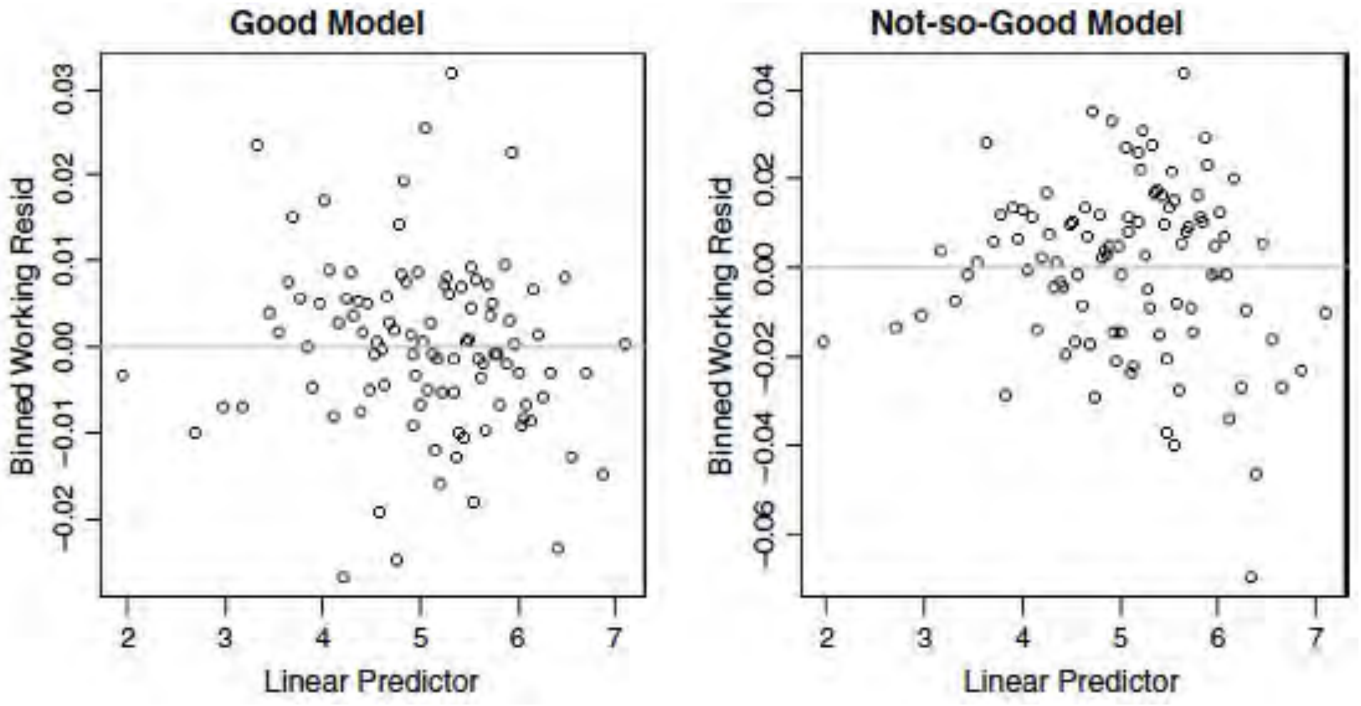
working weights: $ww_i = \dfrac{\omega_i}{V(\mu_i)\ g'(\mu_i)^2}$.

For the Inverse Gaussian Distribution: $V(\mu) = \mu^3$. ⇒ $ww_i = \omega_i / \mu_i$.

$wr_i\ ww_i = \omega_i\ (y_i - \mu_i) / \mu_i^2$.
The numerator of the weighted average is the sum of the product of the working residuals and working weights: $(6)(334 - 444)/444^2 + (3)(412 - 383)/383^2 + (11)(560 - 487)/487^2$
$+ (5)(621 - 642)/642^2 + (8)(448 - 370)/370^2 + (4)(509 - 581)/581^2 + (7)(380 - 426)/426^2$
$+ (9)(495 - 411)/411^2 = 0.006782$.
The denominator of the weighted average is the sum of the working weights:
6/444 + 3/383 + 11/487 + 5/642 + 8/370 + 4/581 + 7/426 + 9/411 = 0.1186.
The binned working residual is: 0.006782/0.1186 = 0.0572.]

It can be useful to plot working residuals versus: the Linear Predictor, Values of a Predictor Variable, or the Weight Variable. Ideally we should detect no pattern in these residual plots. Any such pattern may reveal flaws in the GLM.

Plotting the working residuals versus the value of the linear predictor, xβ, may reveal places where the model is systematically underpredicting or overpredicting.
Here are two examples of such plots:[300] [301]



In the left-hand plot, the points form an uninformative cloud with no apparent pattern, as they should for a well-fit model. In contrast, the right-hand plot displays a pattern. The dots near the middle tend to be higher, while those on either side tend to be lower. The model has a tendency to underpredict in the middle region, and to overpredict on either side.[302] [303] Thus this model is not so good.

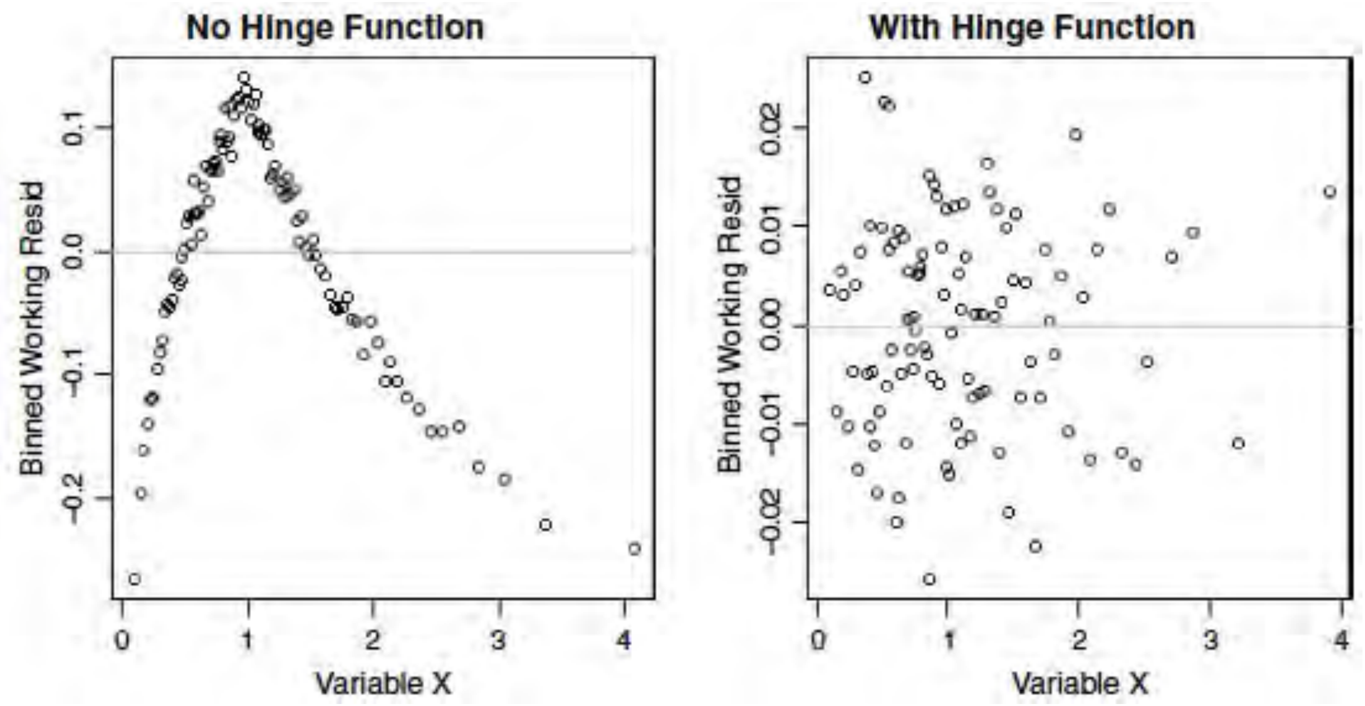
[300] See Figure 18 in Generalized Linear Models for Insurance Rating.

[301] The plotted points each represent the result of grouping observations with similar values of the linear predictor, and then taking a weighted average of their working residuals.

[302] The residual is positive when the observed is larger than the predicted; thus a positive residual corresponds to an underprediction.

[303] The cause may be made clearer with plots of residuals over the various predictors.

Plots of working residuals over each of the various predictors in the model are also useful. Here are two examples of such plots:[304] [305]



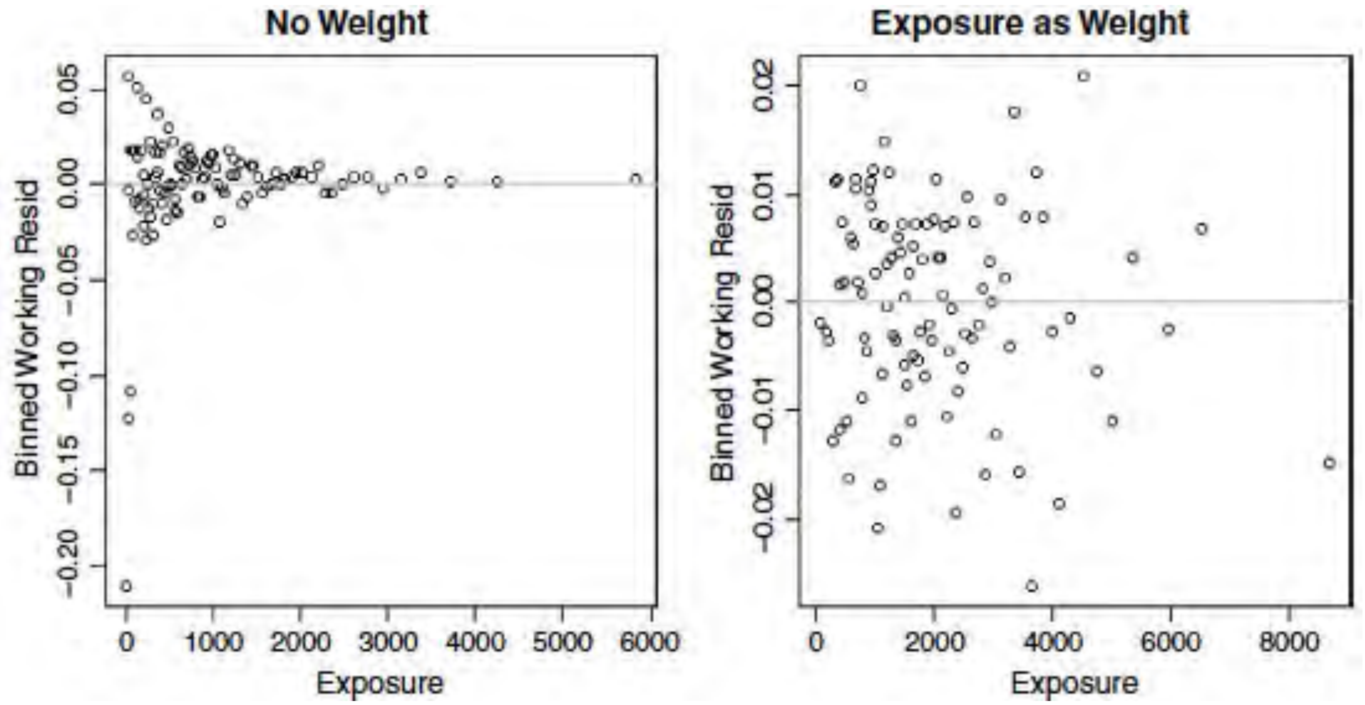
“The left-hand plot clearly reveals that Variable X has a non-linear relationship with the target variable that is not being adequately addressed. The right-hand shows the plot that results after this issue had been fixed with a hinge function.”

[304] See Figure 19 in Generalized Linear Models for Insurance Rating.

[305] The plotted points each represent the result of grouping observations with similar values of Variable X, and then taking a weighted average of their working residuals.

“A plot of residuals over the weight variable used in the model (or over a variable that could potentially be a good choice of weight in the model) may reveal information about the appropriateness of the model weight (or lack thereof).”

Here are two examples of such plots:[306] [307]



The lefthand plot did not use exposure as a weight in the model. The lower-exposure records show more variance, and the higher-exposure records show less variance, which violates our desire for homoscedasticity.[308]

Observations based on larger volume of exposure tend to be more stable. Thus we expect the pattern seen in the left-hand plot, when no weights are used. This problem can be fixed by using exposure as the weight in the model.

The righthand plot shows the result of adding exposure as the weight in the model; the expectation of lower variance with higher exposure has now been incorporated into the GLM. In the righthand plot, the working residuals form a homoscedastic cloud; as desired, there is no longer any pattern.

[306] See Figure 20 in Generalized Linear Models for Insurance Rating.
[307] The plotted points each represent the result of grouping observations with similar values of exposure, and then taking a weighted average of their working residuals.
[308] We do not want to see a pattern in the variance.

Assessing Model Stability:[309]

The actuary would like the GLM to be stable; in other words, the predictions of the model should not be overly sensitive to small changes in the data.

An observation is influential if it has a large effect on the fitted model. An outlier is an observation such that the corresponding fitted value is far from the observation.

An influential observation is such that its removal from the data set causes a significant change to our modeled results. An observation is influential when one or more of its predictor values are far from its mean and the observation is an outlier.

A common measure of influence is Cook's distance.[310] **The larger the value of Cook's distance, the more influential the observation**.[311]

The actuary should rerun the model excluding the most influential points to see their impact on the results. If this causes large changes in some of the parameter estimates, the actuary should consider for example whether to give these influential observations less weight.

Cross-validation, as discussed previously, can also be used to assess the stability of a GLM. For example, we can divide the data into ten parts. By combining these parts, we can create ten different subsets each of which contains 90% of the total data. We then fit the model to each of these ten subsets.

The results of the models fit to these different subsets of the data ideally should be similar. The amount by which these results vary is a measure of the stability of the model.

Bootstrapping via simulation can also be used to assess the stability of a GLM.[312] The original data is randomly sampled with replacement to create a new set of data of the same size. One then fits the GLM to this new set of data. By repeating this procedure many times one can estimate the distribution of the parameter estimates of the GLM; we can estimate the mean, variance, confidence intervals, etc. "Many modelers prefer bootstrapped confidence intervals to the estimated confidence intervals produced by statistical software in GLM output."

[309] See Section 6.4 of Generalized Linear Models for Insurance Rating.

[310] The syllabus reading gives no details on how Cook's Distance is calculated.
Computer software to fit GLMs will usual include Cook's Distance as one of the possible outputs.

[311] Values of Cook's Distance greater than unity may require further investigation.

[312] See An Introduction to Statistical Learning with Applications in R, by James, Witten, Hastie, and Tibshirani, not on the syllabus of this exam.

Scoring Models:[313]

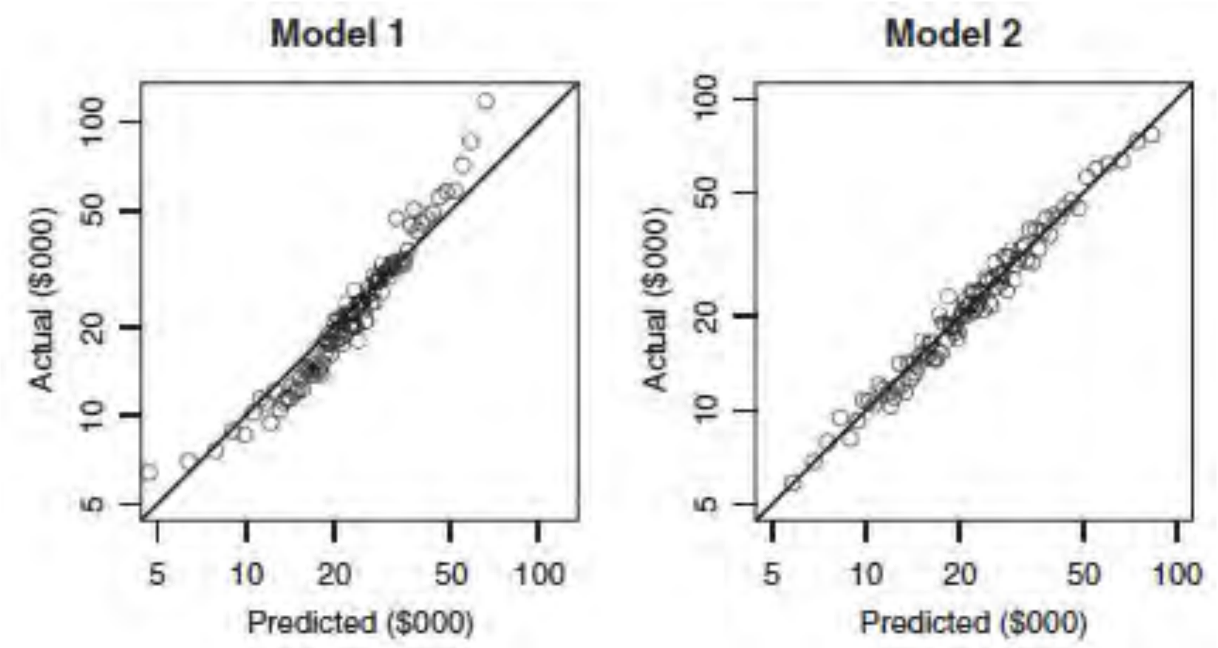
We have a rating plan or rating plans. We may not know what model if any that the plan(s) came from.[314] We wish to evaluate a rating plan or compare two rating plans.

Methods that are discussed: Plots of Actual vs. Predicted, Simple Quantile Plots,
Double Lift Charts, Loss Ratio Charts, the Gini Index, and ROC Curves.

In order for these techniques to be used, one only needs a database of historical observations plus the predictions from each of the competing models. The process of assigning predictions to individual records is called scoring.

Assessing Fit with Plots of Actual versus Predicted:[315]

Create a plot of the actual target variable (on the y-axis) versus the predicted target variable (on the x-axis) for each model. If a model fits well, then the actual and predicted target variables should follow each other closely. Here are two examples:[316]



Model 2 fits the data better than Model 1, as there is a much closer agreement between the actual and predicted target variables for Model 2 than there is for Model 1.

These plots should not use data that was used to fit or train the models.
It is common to group the data, for example into percentiles.
Often one will plot the graph on a log scale, as in the above examples.

[313] See Section 7 of Generalized Linear Models for Insurance Rating.
[314] One or more of the rating plans may be proprietary.
[315] See Section 7.1 of Generalized Linear Models for Insurance Rating.
[316] See Figure 21 of Generalized Linear Models for Insurance Rating.

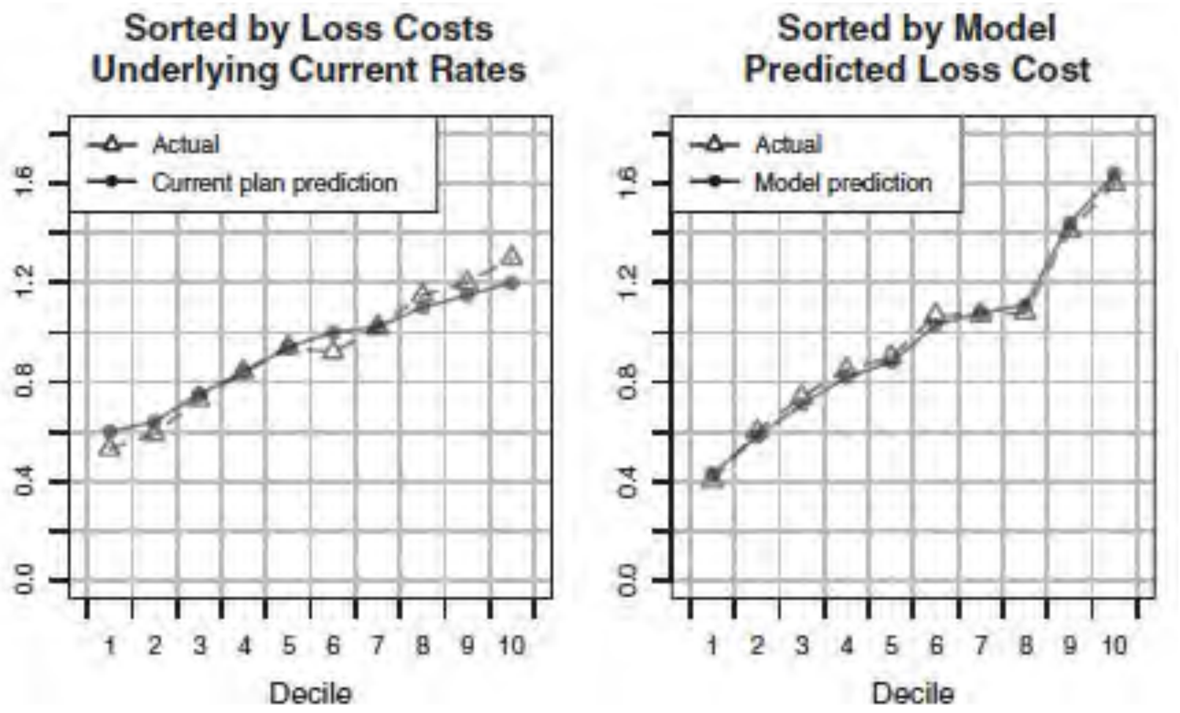
Measuring Model Lift:[317]

Lift refers to a model's ability to prevent adverse selection, **measuring the approximate "economic value" of the model**. Economic value is produced by comparative advantage in avoidance of adverse selection; thus model lift is a relative concept, comparing two or more competing models, or a model and the current plan. Lift measures a model's ability to charge each insured an actuarially fair rate, thereby minimizing the potential for adverse selection. Model lift should always be measured on holdout data, in other words not using data used to fit or build the model.

Simple Quantile Plots:[318]

To create a quantile plot of a model.
1. Sort the dataset based on the model predicted loss cost from smallest to largest.[319]
2. Group the data into quantiles with equal volumes of exposures.[320]
3. Within each group, calculate the average predicted pure premium based on the model, and the average actual pure premium.
4. Plot for each group, the actual pure premium and the predicted pure premium.

One can create separate quantile plots for two models, for example the current rating plan and a proposed rating plan and compare them:[321]

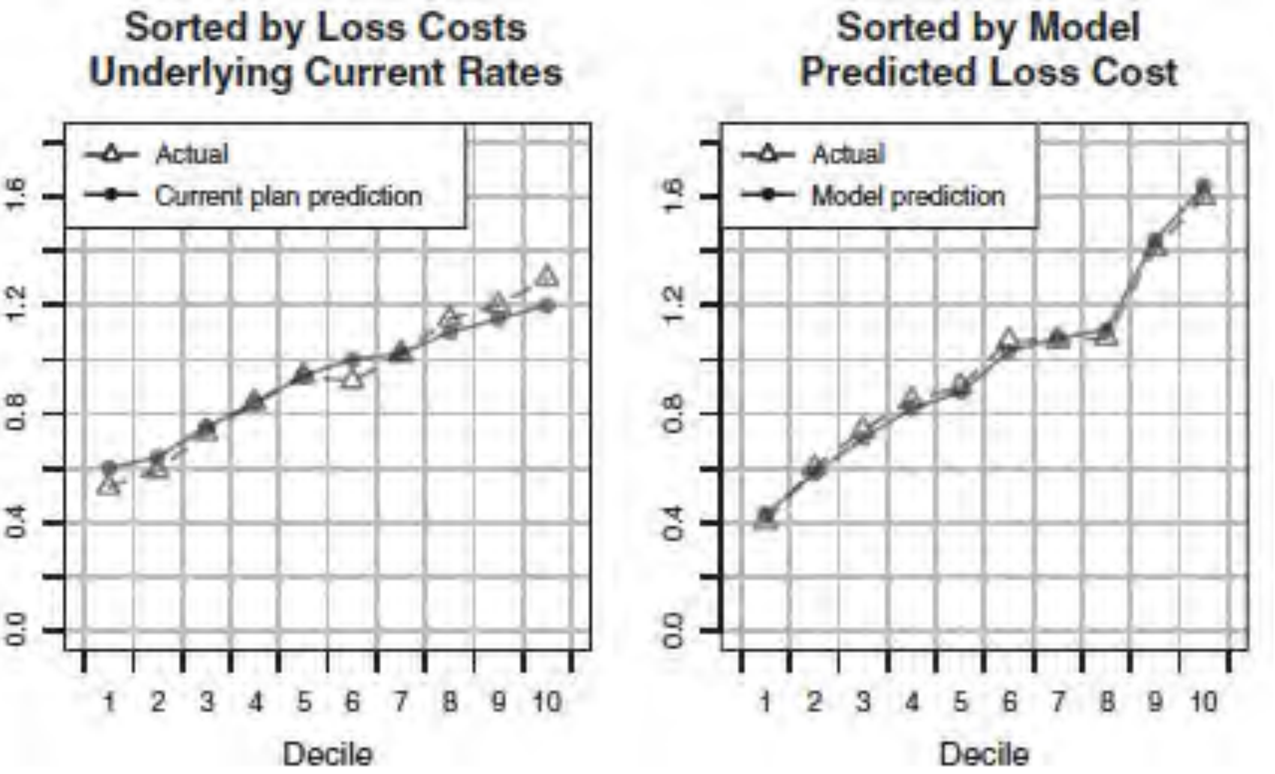


[317] See Section 7.2 of Generalized Linear Models for Insurance Rating Lift differs from goodness of fit measures.
[318] See Section 7.2.1 of Generalized Linear Models for Insurance Rating.
[319] The plots shown seem to be sorted on predicted pure premiums (losses per exposure)
The syllabus reading says "loss costs", which can mean pure premiums.
[320] For example: quintiles (5 buckets), deciles (10 buckets), or vigintiles (20 buckets).
[321] See Figure 22 in Generalized Linear Models for Insurance Rating.

To compare the models use the following 3 criteria:
1. **Predictive accuracy**.
2. **Monotonicity**. The actual pure premium should increase.[322]
3. **Vertical distance between the actuals in the first and last quantiles**.

"A large difference (also called "lift") between the actual pure premium in the quantiles with the smallest and largest predicted loss costs indicates that the model is able to maximally distinguish the best and worst risks."

The previous set of graphs can be used to compare the current and proposed model.



1. Predictive accuracy: the proposed model does a better job of predicting.
2. Monotonicity: the current plan has a reversal in the 6th decile, whereas the proposed model does better with no significant reversals.
3. Vertical distance between the first and last quantiles: The spread of actual loss costs for the current plan is 0.55 to 1.30. The spread of the proposed model is 0.40 to 1.60, which is larger and thus better.

Thus, by all three criteria, the proposed plan outperforms the current one.

[322] Although small reversals are okay.

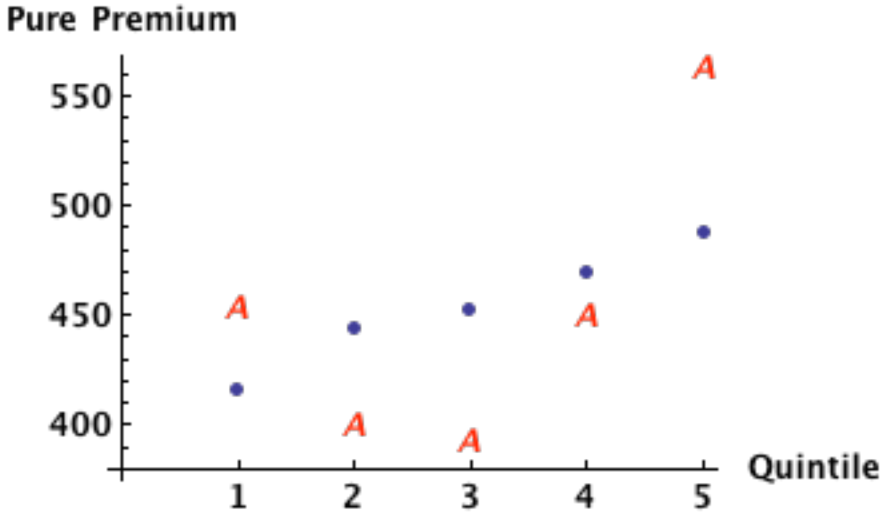
Exercise: An insurer uses a GLM for classification ratemaking.
You are given the following data on five insureds.

Insured	Actual Loss Cost	Loss Cost Predicted by the Model	Exposures
1	$45,000	$39,000	80
2	$56,000	$62,000	140
3	$72,000	$75,000	160
4	$86,000	$79,000	190
5	$98,000	$113,000	250

Construct a Simple Quantile Plot; sort the data based on predicted pure premium.
[Solution: The order of predicted pure premiums is: 4, 2, 5, 3, 1.

Insured	Actual Loss Cost	Actual Pure Premium	Model Loss Cost	Model Pure Premium	Exposures
1	$45,000	$563	$39,000	$488	80
2	$56,000	$400	$62,000	$443	140
3	$72,000	$450	$75,000	$469	160
4	$86,000	$453	$79,000	$416	190
5	$98,000	$392	$113,000	$452	250

The corresponding predicted pure premiums are: 416, 443, 452, 469, 488.
The corresponding actual pure premiums are: 453, 400, 392, 450, 563.
The Simple Quantile Plot, with the actual pure premiums shown as A and the predicted pure premiums shown as dots:



Comment: One would construct a similar Simple Quantile Plot for a proposed model, in order to compare that proposed model to the current model.]

Double Lift Charts:[323]

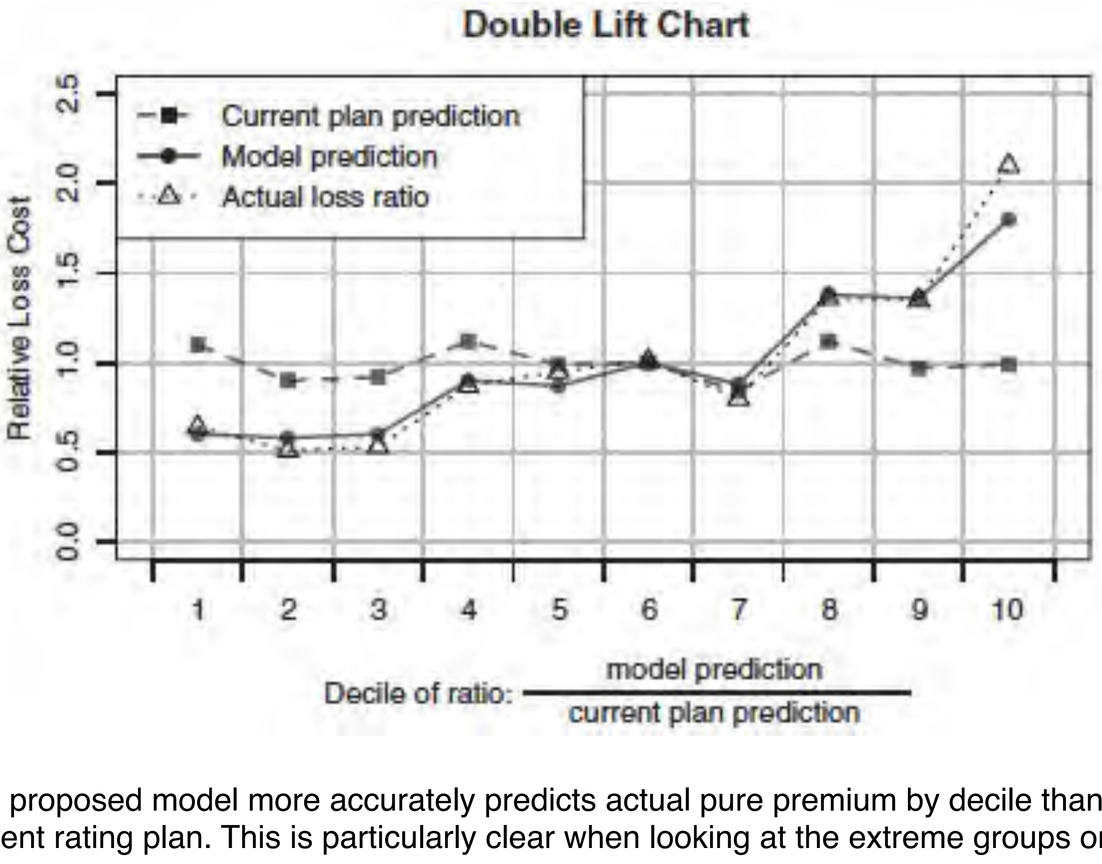
A double lift chart directly compares two models A and B.

To create a double lift chart:

1. For each observation, calculate Sort Ratio = $\frac{\text{Model A Predicted Loss Cost}}{\text{Model B Predicted Loss Cost}}$.[324]
2. Sort the dataset based on the Sort Ratio, from smallest to largest.
3. Group the data.[325]
4. For each group, calculate the pure premiums: predicted by Model A, predicted by Model B, and actual. Then divide the group average by the overall average.
5. For each group, plot the three relativities calculated in the step 4.

The first group contains those risks which Model A thinks are best relative to Model B, while the last group contains those risks which Model B thinks are best relative to Model A. The first and last groups contain those risks on which Models A and B disagree the most in percentage terms.

The "winning" model is the one that more closely matches the actual pure premiums.
Here is an example of a double lift chart, comparing a current and proposed plan:[326]



The proposed model more accurately predicts actual pure premium by decile than does the current rating plan. This is particularly clear when looking at the extreme groups on either end.

[323] See Section 7.2.2 of Generalized Linear Models for Insurance Rating.

[324] Thus a low sort ratio means that model A predicts a lower loss cost than does model B.

[325] For example into 5 buckets (quintiles) or 10 buckets (deciles).

[326] See Figure 23 in Generalized Linear Models for Insurance Rating.

Exercise: An actuary has built two generalized linear models to predict loss costs.
Output for each model are shown below:

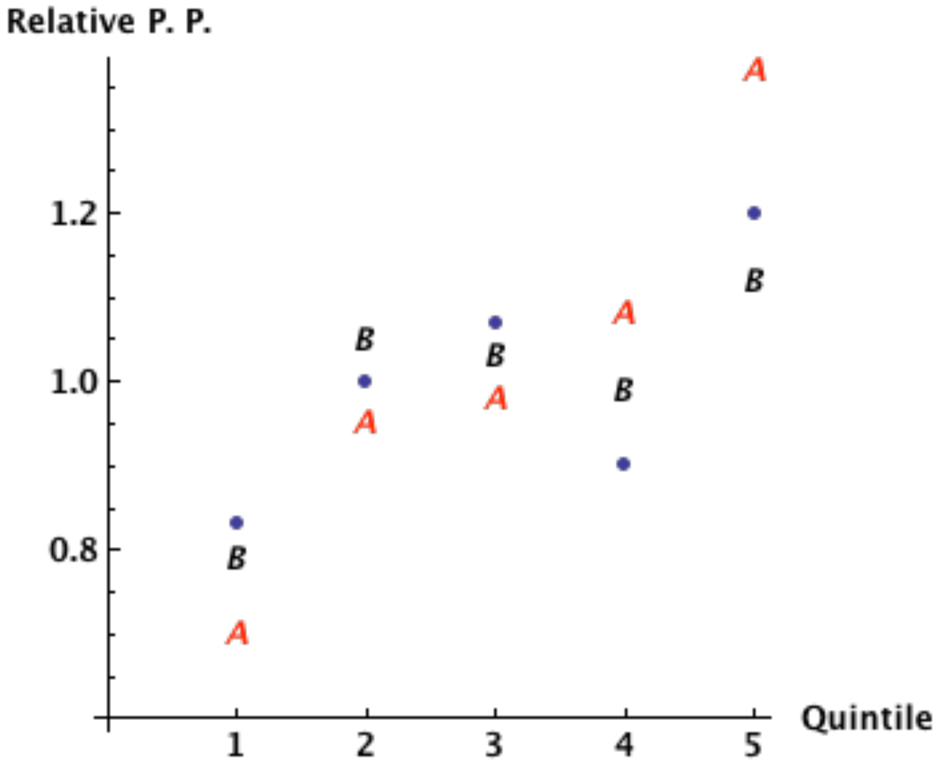
Observation	Actual Loss Cost	Model A Loss Cost	Model B Loss Cost	Exposures
1	\$41,000	\$46,000	\$38,000	150
2	\$34,000	\$28,000	\$32,000	180
3	\$43,000	\$51,000	\$47,000	210
4	\$61,000	\$55,000	\$58,000	250
5	\$68,000	\$64,000	\$71,000	300

Construct a double lift chart.

[Solution: Sort the data based on the ratio:
(Model A Predicted Pure Premium) / (Model B Predicted Premium).

Obs	Actual Loss Cost	Actual Pure Premium	Model A Loss Cost	Model A Pure Premium	Model B Loss Cost	Model B Pure Premium	Exposures	Sort Ratio
1	$41,000	$273	$46,000	$307	$38,000	$253	150	1.21
2	$34,000	$189	$28,000	$156	$32,000	$178	180	0.88
3	$43,000	$205	$51,000	$243	$47,000	$224	210	1.09
4	$61,000	$244	$55,000	$220	$58,000	$232	250	0.95
5	$68,000	$227	$64,000	$213	$71,000	$237	300	0.90
Tot.	$247,000	$227	$244,000	$224	$246,000	$226	1,090	

The sort ratios from smallest to largest give the order: 2, 5, 4, 3, 1.
In each case, we divide the individual pure premiums by the total pure premium.
The Actual P.P. relativities are: (189, 227, 244, 205, 273) / 227 = 0.83, 1.00, 1.07, 0.90, 1.20.
Model A P.P. relativities are: (156, 213, 220, 243, 307) / 224 = 0.70, 0.95, 0.98, 1.08, 1.37.
Model B P.P. relativities are: (178, 237, 232, 224, 253) / 226 = 0.79, 1.05, 1.03, 0.99, 1.12.
The double lift chart, with actual shown as dots, Model A shown as A, and Model B shown as B:



Comment: Model B more closely matches the actual pure premiums.
One would work with many more than 5 observations; I would not draw any conclusions based on such a small amount of data.]

"As an alternate representation of a double lift chart, one can plot two curves: the percent error for the model predictions and the percent error for the current loss costs, where percent error is calculated as: $\frac{\text{Predicted Loss Cost}}{\text{Actual Loss Cost}}$ - 1. In this case, the winning model is the one with the flatter line centered at y = 0, indicating that its predictions more closely match actual pure premium."

Exercise: An actuary has built two generalized linear models to predict losses.
Output for each model are shown below:

Observation	Actual Loss Cost	Model A Loss Cost	Model B Loss Cost	Exposures
1	\$41,000	\$46,000	\$38,000	150
2	\$34,000	\$28,000	\$32,000	180
3	\$43,000	\$51,000	\$47,000	210
4	\$61,000	\$55,000	\$58,000	250
5	\$68,000	\$64,000	\$71,000	300

Construct a double lift chart using the alternative of percent errors.

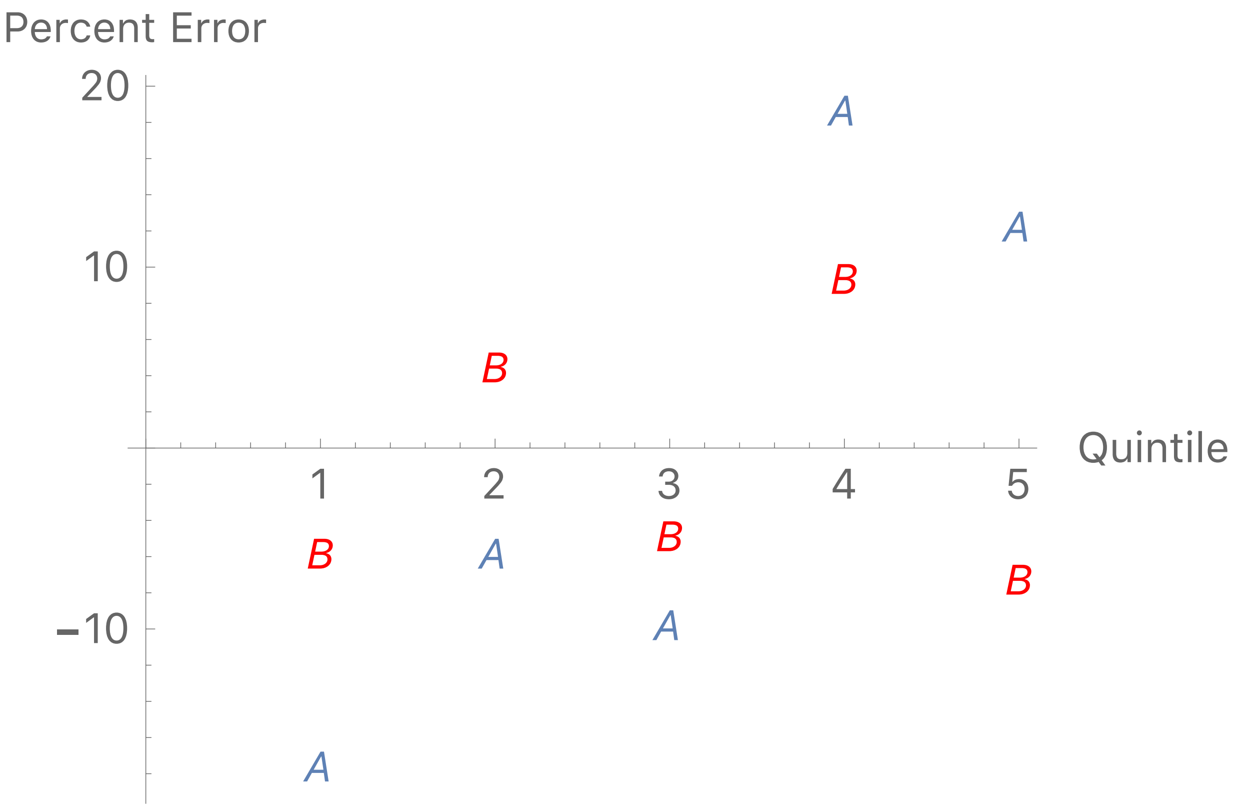
[Solution: Sort the data based on the ratio:
(Model A Predicted Loss Cost) / (Model B Predicted Loss Cost).

Obs	Actual Loss Cost	Model A Loss Cost	Model A Error	Model B Loss Cost	Model B Error	Sort Ratio
1	$41,000	$46,000	12.2%	$38,000	-7.3%	1.21
2	$34,000	$28,000	-17.6%	$32,000	-5.9%	0.88
3	$43,000	$51,000	18.6%	$47,000	9.3%	1.09
4	$61,000	$55,000	-9.8%	$58,000	-4.9%	0.95
5	$68,000	$64,000	-5.9%	$71,000	4.4%	0.90

The sort ratios from smallest to largest give the order: 2, 5, 4, 3, 1.

Then we compute the percent errors: $\frac{\text{Predicted Loss Cost}}{\text{Actual Loss Cost}}$ - 1.

For example for Model A: 46,000/41,000 - 1 = 12.2%.
The double lift chart, with Model A shown as A, and Model B shown as B:



Comment: Model B has the flatter line centered at y = 0, indicating that its predictions more closely match the actual pure premium than Model A.]

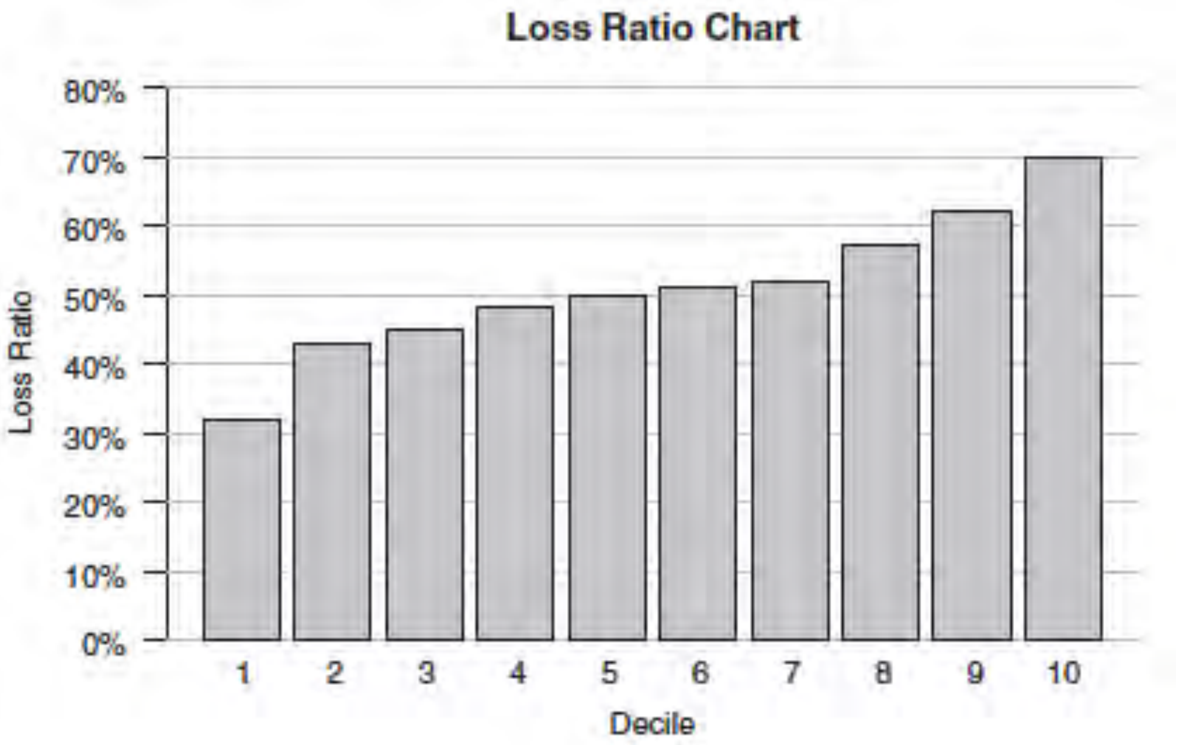
Loss Ratio Charts:[327]

A loss ratio chart is similar to a simple quantile chart, except one works with loss ratios (with respect to the premiums for the current plan) rather than pure premiums.[328]

To create a loss ratio chart:
1. Sort the dataset based on the model prediction, in other words modeled loss ratios.
2. Group the data into quantiles with equal volumes of exposures.
3. Within each group, calculate the actual loss ratio (under the current plan).[329]

If the proposed model is able to segment the data into lower and higher loss ratio buckets, then the proposed model is better than the current model.

Here is an example:[330]



The proposed model is able to segment the data into lower and higher loss ratio buckets, indicating that the proposed model is better than the current model.

"The advantage of loss ratio charts over quantile plots and double lift charts is that they are simple to understand and explain."

[327] See Section 7.2.3 of Generalized Linear Models for Insurance Rating.
[328] One should work with a test (holdout) set of data not used to develop the proposed model.
[329] The premiums used in the denominator of the loss ratios should be at present rates, reflecting the current model
[330] See Figure 24 in Generalized Linear Models for Insurance Rating.

Exercise: An insurer uses a GLM for classification ratemaking. The insurer is considering using a different GLM instead. You are given the following data on five insureds.

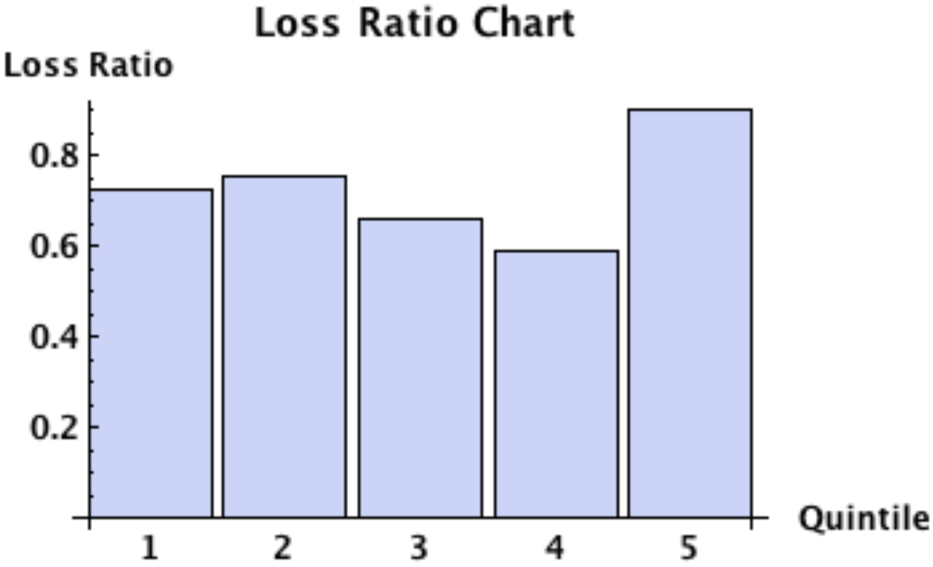
Insured	Actual Loss Cost	Loss Cost Predicted by Proposed Model	Earned Premium at Present Rates
1	13,000	17,000	22,000
2	21,000	19,000	29,000
3	25,000	27,000	38,000
4	37,000	33,000	41,000
5	34,000	31,000	45,000

Construct a Loss Ratio Chart.

[Solution: Sort the data based on the loss ratio predicted by the proposed model.

Insured	Actual Loss Cost	Actual Loss Ratio	Model Loss Cost	Model Loss Ratio	Earned Premium at Present Rates
1	13,000	59.1%	17,000	77.3%	22,000
2	21,000	72.4%	19,000	65.5%	29,000
3	25,000	65.8%	27,000	71.1%	38,000
4	37,000	90.2%	33,000	80.5%	41,000
5	34,000	75.6%	31,000	68.9%	45,000

For the proposed model, the order of predicted loss ratios is: 2, 5, 3, 1, 4.
The corresponding actual loss ratios are: 72.4%, 75.6%, 65.8%, 59.1%, 90.2%.



Comment: See 8, 11/19, Q.2a. One would work with many more than 5 observations.]

Lorenz Curves:[331]

The Lorenz Curve is used to define the Gini Index, to be discussed subsequently.

Assume that the incomes in a country follow a distribution function F(x).[332]
Then F(x) is the percentage of people with incomes less than x.

The income earned by such people is: $\int_0^x t\ f(t)\ dt = E[X \wedge x] - x\ S(x) = \int_0^x S(t)\ dt$.

The percentage of total income earned by such people is:

$$\frac{\int_0^x y\ f(y)\ dy}{E[X]} = \frac{E[X \wedge x] - x\ S(x)}{E[X]}\ .$$

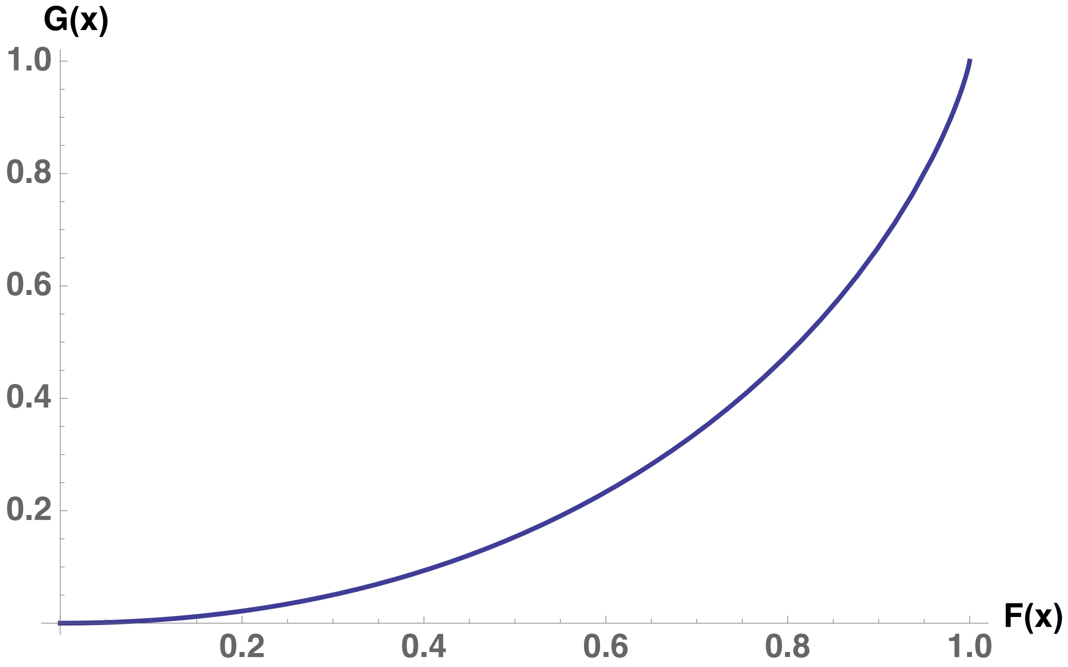
Define $G(x) = \dfrac{\int_0^x y\ f(y)\ dy}{E[X]} = \dfrac{E[X \wedge x] - x\ S(x)}{E[X]}$.[333]

For example, assume an Exponential Distribution.
Then $F(x) = 1 - e^{-x/\theta}$.
$G(x) = \dfrac{E[X \wedge x] - x\ S(x)}{E[X]} = \dfrac{\theta\ (1 - e^{-x/\theta}) - x\ e^{-x/\theta}}{\theta} = 1 - e^{-x/\theta} - (x/\theta)\ e^{-x/\theta}$.

Let $t = F(x) = 1 - e^{-x/\theta}$. Therefore, $x/\theta = -\ln(1 - t)$.[334]
Then, $G(t) = t - \{-\ln(1-t)\}\ (1-t) = t + (1-t)\ \ln(1-t)$.

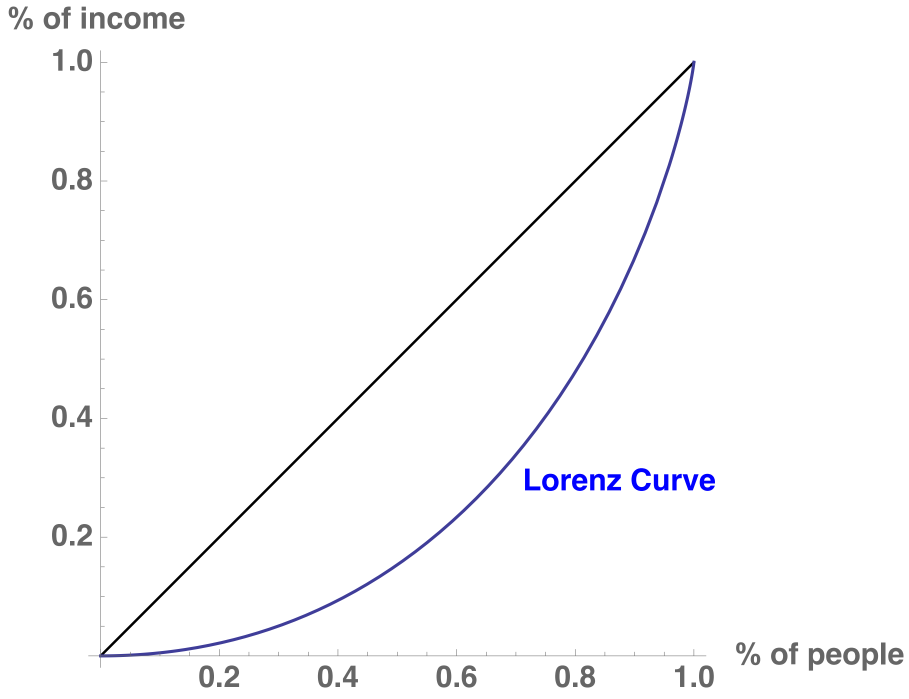
[331] You should not be responsible for any details of the mathematics of Lorenz curves.
[332] Of course, the mathematics applies regardless of what is being modeled.
The distribution of incomes is just the most common context.
[333] This is not standard notation. I have just used G to have some notation.
[334] This is just the VaR (Value at Risk) formula for the Exponential Distribution.

Then we can graph G as a function of F:



This curve is referred to as the Lorenz curve or the concentration curve.

Since F(0) = 0 = G(0) and F(∞) = 1 = G(∞), the Lorenz curve passes through the points (0, 0) and (1, 1). Usually one would also include in the graph the 45° reference line connecting (0, 0) and (1, 1), called the line of equality, as shown below:



$$G(t) = G[F(x))] = \frac{\int_0^x y\ f(y)\ dy}{E[X]}.$$

$$\frac{dG}{dt} = \frac{dG}{dx} / \frac{dF}{dx} = \frac{x\ f(x)}{E[X]} / f(x) = \frac{x}{E[X]} > 0.$$

$$\frac{d^2G}{dt^2} = \frac{1}{E[X]}\ \frac{dx}{dx} / \frac{dF}{dx} = \frac{1}{E[X]\ f(x)} > 0.$$

Thus, in the above graph, as well as in general, the Lorenz curve is increasing and concave up. The Lorenz curve is below the 45° reference line, except at the endpoints when they are equal.

The vertical distance between the Lorenz curve and the 45° comparison line is: F - G.

Thus, this vertical distance is a maximum when: $0 = \frac{dF}{dF} - \frac{dG}{dF}$.

$$\Rightarrow \frac{dG}{dF} = 1. \Rightarrow \frac{x}{E[X]} = 1. \Rightarrow x = E[X].$$

Thus the vertical distance between the Lorenz curve and the line of equality is a maximum at the mean income.

Exercise: If incomes follow an Exponential Distribution, what is this maximum vertical distance between the Lorenz curve and the line of equality?
[Solution: The maximum occurs when $x = \theta$.
$F(x) = 1 - e^{-x/\theta}$. From previously, $G(x) = 1 - e^{-x/\theta} - (x/\theta)\ e^{-x/\theta}$.
$F - G = (x/\theta)\ e^{-x/\theta}$. At $x = \theta$, this is: $e^{-1} = 0.3679$.]

Exercise: Determine the form of the Lorenz Curve, if the distribution of incomes follows a Shifted Pareto Distribution, with $\alpha > 1$.[335]

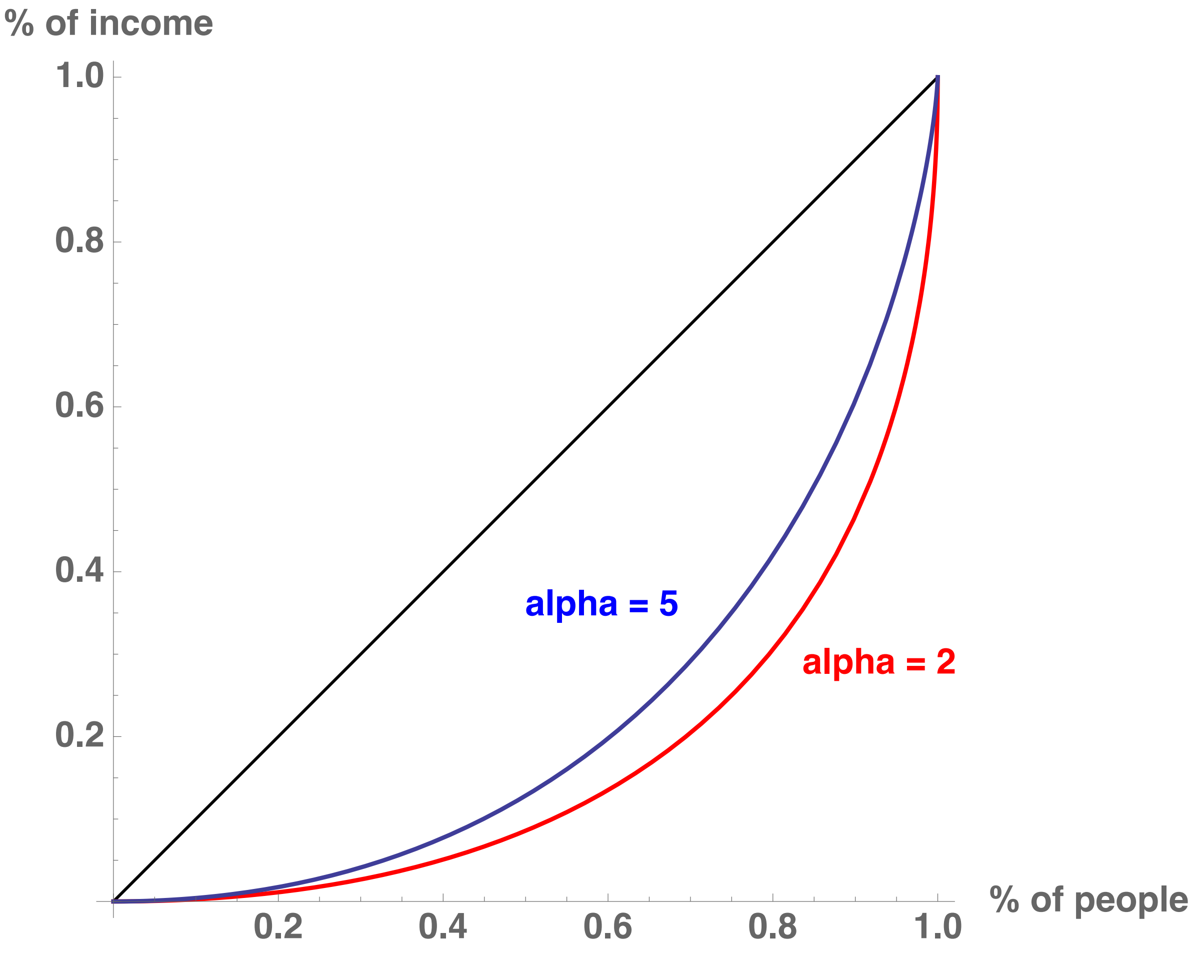
[Solution: $F(x) = 1 - \left(\frac{\theta}{\theta+x}\right)^{\alpha}$, $x > 0$. $E[X] = \frac{\theta}{\alpha - 1}$. $E[X \wedge x] = \frac{\theta}{\alpha - 1}\left\{1 - \left(\frac{\theta}{\theta+x}\right)^{\alpha-1}\right\}$.

$$G(x) = \frac{E[X \wedge x] - x\,S(x)}{E[X]} = \frac{\frac{\theta}{\alpha-1}\{1 - \left(\frac{\theta}{\theta+x}\right)^{\alpha-1}\} - x\,S(x)}{\theta / (\alpha-1)} = 1 - \left(\frac{\theta}{\theta+x}\right)^{\alpha-1} - (\alpha-1)\,\frac{x}{\theta}\,S(x).$$

Let $t = F(x) = 1 - \left(\frac{\theta}{\theta+x}\right)^{\alpha}$. $\Rightarrow \left(\frac{\theta}{\theta+x}\right)^{\alpha} = S(x) = 1 - t$. Also, $x/\theta = (1 - t)^{-1/\alpha} - 1$.[336]

Therefore, $G(t) = 1 - (1 - t)^{(\alpha-1)/\alpha} - (\alpha-1)\{(1 - t)^{-1/\alpha} - 1\}\,(1 - t) = t + \alpha - t\alpha - \alpha\,(1-t)^{1-1/\alpha}$, $0 \le t \le 1$.
Comment: $G(0) = \alpha - \alpha = 0$. $G(1) = 1 + \alpha - \alpha - 0 = 1$.]

Here is graph comparing the Lorenz curves for Shifted Pareto Distributions with $\alpha = 2$ and $\alpha = 5$:

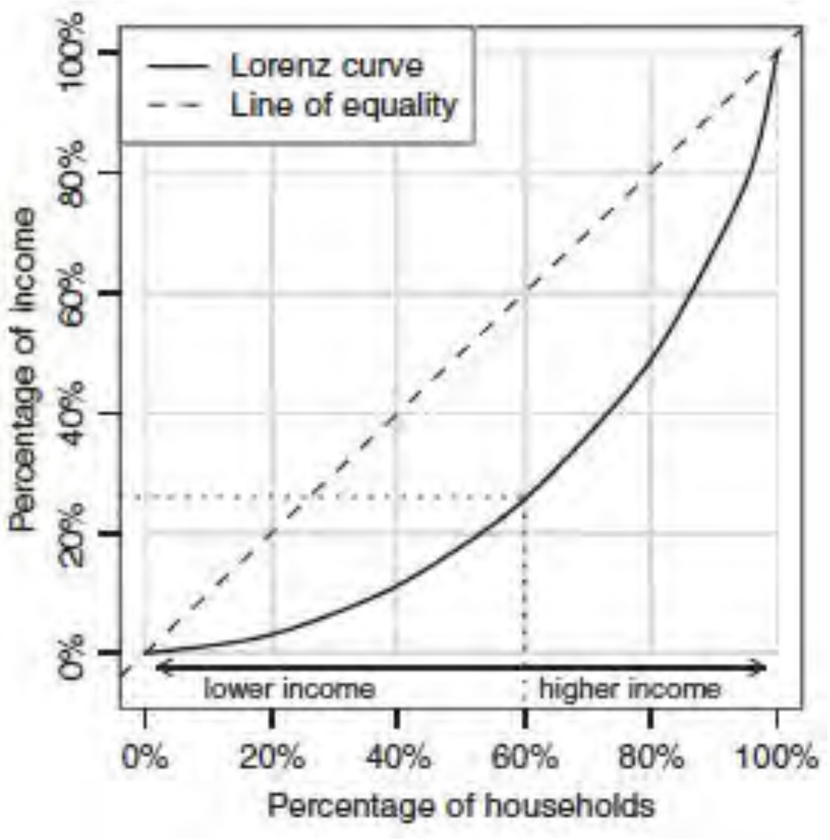


[335] What Bahnemann calls a Shifted Pareto, Loss Models simply calls a Pareto.
[336] This is just the VaR (value at risk) formula for the Shifted Pareto Distribution.

The Shifted Pareto with α = 2 has a heavier righthand tail than the Shifted Pareto with α = 5. If incomes follow a Shifted Pareto with α = 2, then there are more extremely high incomes compared to the mean, than if incomes follow a Shifted Pareto with α = 5. In other words, if α = 2, then income is more concentrated in the high income individuals than if α = 5.[337]

The Lorenz curve for α = 2 is below that for α = 5. In general, the lower curve corresponds to a higher concentration of income. In other words, a higher concentration of income corresponds to a smaller area under the Lorenz curve. Equivalently, a higher concentration of income corresponds to a larger area between the Lorenz curve and the 45° reference line.

Here is a Lorenz Curve for United States 2014 Household Income:[338]



The Gini index is calculated as twice the area between the Lorenz curve and the line of equality. In this case, the Gini index is 48.0%.

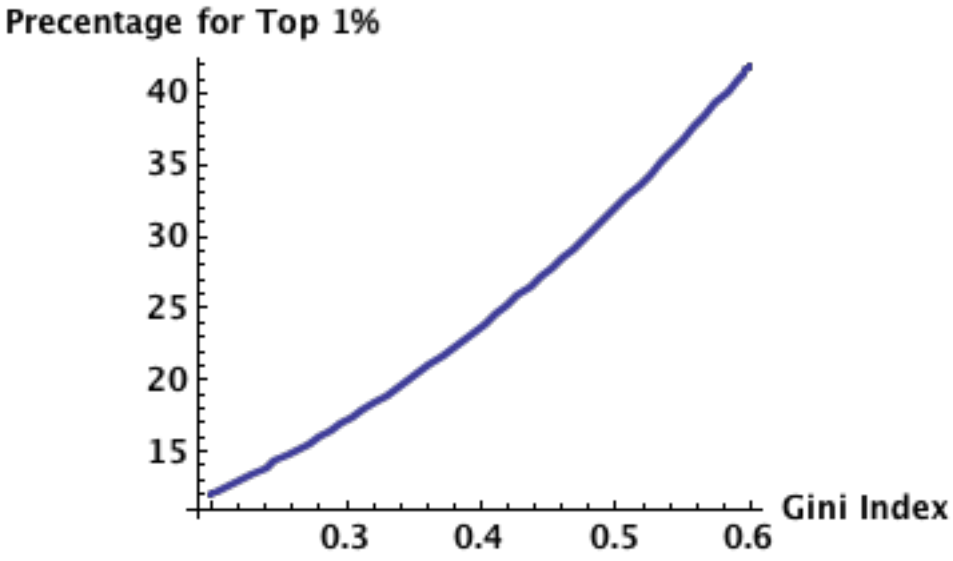
[337] An Exponential Distribution has a lighter righthand tail than either Shifted Pareto. Thus if income followed an Exponential, it would less concentrated than if it followed any Shifted Pareto.
[338] See Figure 25 of Generalized Linear Models for Insurance Rating.

The Gini Index depends on the type of distribution and its (shape) parameters. Assume for example that household incomes follow a Shifted Pareto Distribution. Then for $\alpha > 1$, it turns out that the Gini Index = $\frac{1}{2\alpha - 1}$. As alpha approaches one, the Gini Index approaches one.

It turns out, that for the Shifted Pareto, the portion of total income earned by the top p percent is as a function of the Gini Index γ: $\frac{1}{2}(1 + 1/\gamma)\ p^{\frac{1-\gamma}{1+\gamma}} + p\ (1 - 1/\gamma)/2$.

Exercise: If income follows a Shifted Pareto, for a Gini Index of 0.4, what percent of total income is earned by the top one percent?
[Solution: $(1/2)(1 + 1/0.4)\ 0.01^{0.6/1.4} + (0.01)(1 - 1/0.4)/2 = 23.6\%$.
Comment: $\alpha = (1 + 1/0.4)/2 = 1.75$.]

For the Shifted Pareto, here is a graph of the portion of total income earned by the top 1% as a function of the Gini Index:[339]



This may help to give you some intuition as the meaning of different values of the Gini Index.

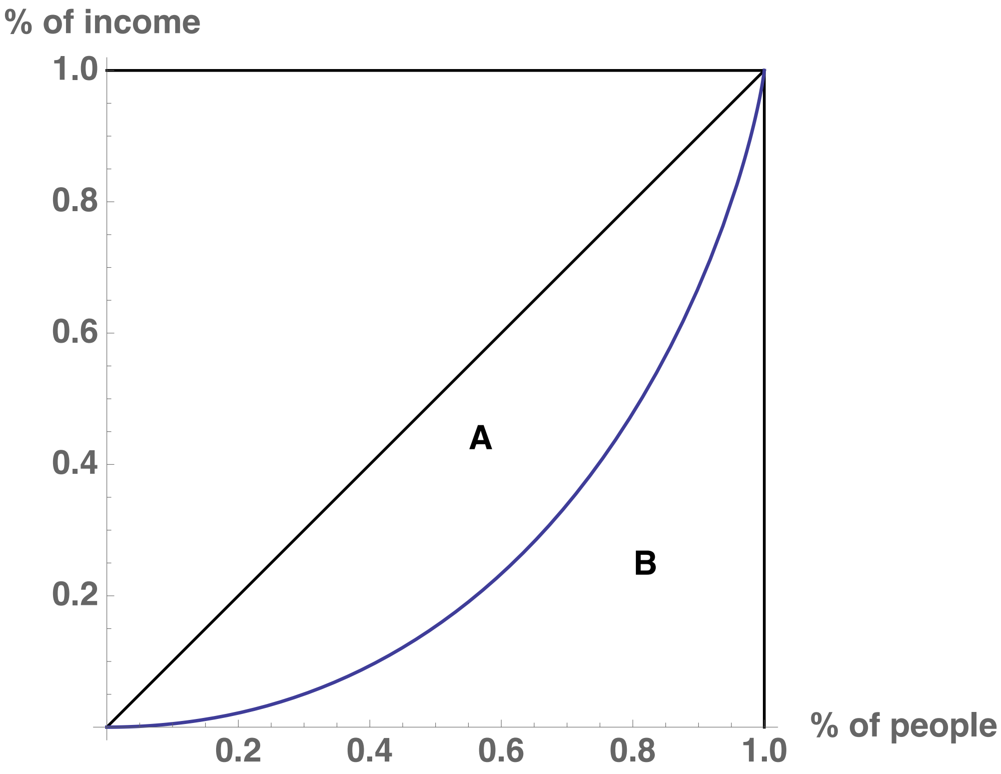
[339] Gini indexes for countries range from about 0.25 to about 0.60.

Gini Index:[340]

The Gini Index comes up for example in economics, when looking at the distribution of incomes. A subsequent section will discuss how the Gini index can be used to evaluate a rating plan.

The Gini index is a measure of inequality. For example if all of the individuals in a group have the same income, then the Gini index is zero. As incomes of the individuals in a group became more and more unequal, the Gini index would increase towards a value of 1. The Gini index has found application in many different fields of study.

As discussed, for incomes, the Lorenz curve would graph percent of people versus percent of income. This correspondence between areas on the graph of the Lorenz curve the concentration of income is the idea behind the Gini index. Let us label the areas in the graph of a Lorenz Curve:



$$\text{Gini Index} = \frac{\text{Area A}}{\text{Area A + Area B}}.$$

However, Area A + Area B add up to a triangle with area 1/2.

$$\text{Therefore, Gini Index} = \frac{\text{Area A}}{\text{Area A + Area B}} = 2\text{A}$$

= twice the area between the Lorenz Curve and the line of equality = 1 - 2B.

[340] See Section 7.2.4 of Generalized Linear Models for Insurance Rating.
Also called the Gini Coefficient or coefficient of concentration.

Gini Index for Specific Distributions:[341]

For the Exponential Distribution, the Lorenz curve was: G(t) = t + (1-t) ln(1-t).

Thus, Area B = area under Lorenz curve = $\int_0^1 t + (1\text{-}t) \ln(1\text{-}t)\ dt = 1/2 + \int_0^1 s \ln(s)\ ds$.

Applying integration by parts,

$$\int_0^1 s \ln(s)\ ds = (s^2/2) \ln(s) \Big]_{s=0}^{s=1} - \int_0^1 (s^2/2)\ (1/s)\ ds = 0 - 1/4 = -1/4.$$

Thus Area B = 1/2 - 1/4 = 1/4.

Therefore, for the Exponential Distribution, the Gini Index is: 1 - (2)(1/4) = 1/2.

For the Uniform Distribution, the Gini Index is: 1/3.

For the Shifted Pareto Distribution, the Gini Index is: $1 / (2\alpha - 1)$, $\alpha > 1$.

We note that the Uniform with the lightest righthand tail of the three has the smallest Gini index, while the Shifted Pareto with the heaviest righthand tail of the three has the largest Gini index. Among Shifted Pareto Distributions, the smaller alpha, the heavier the righthand tail, and the larger the Gini index.[342]

The more concentrated the income is among the higher earners, the larger the Gini index.

For the Classical (Single Parameter) Pareto Distribution, the Gini Index is: $1 / (2\alpha - 1)$, $\alpha > 1$.

For the LogNormal Distribution, the Gini Index is: $2\Phi[\sigma/\sqrt{2}\ ] - 1$.

For the Gamma Distribution, the Gini Index is: $1 - 2\ \beta(\alpha+1, \alpha; 1/2)$.

[341] Not on the syllabus.

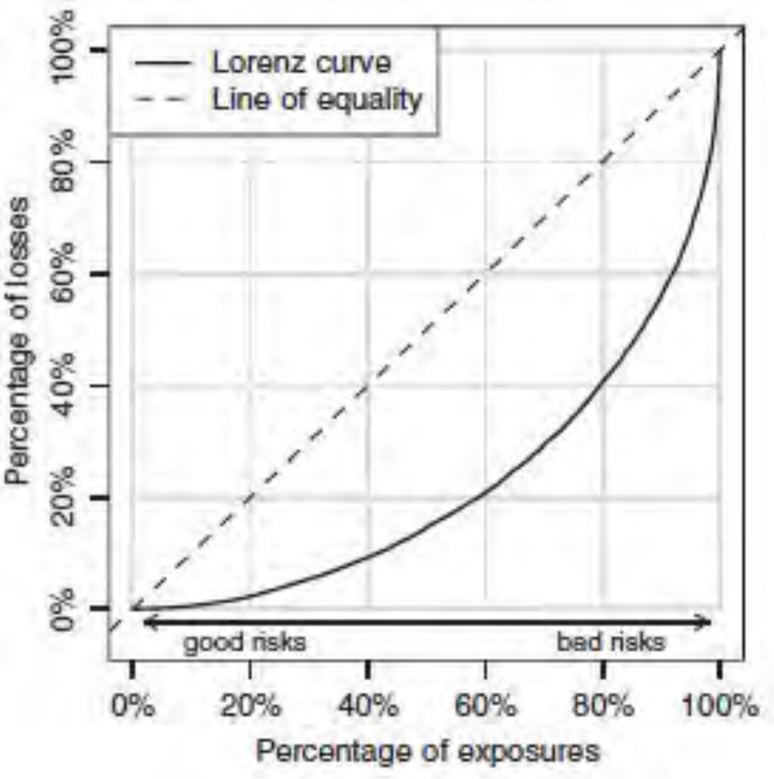
[342] As alpha approaches one, the Gini coefficient approaches one.

Gini Index and Rating Plans:[343]

The Gini index can also be used to measure the lift of an insurance rating plan by quantifying its ability to segment the population into the best and worst risks. Assume we have a rating plan. Ideally we would want the model to identify those insureds with higher expected pure premiums.

The Lorenz curve for the rating plan is determined as follows:
1. Sort the dataset based on the model predicted loss cost.[344]
2. On the x-axis, plot the cumulative percentage of exposures.
3. On the y-axis, plot the cumulative percentage of actual losses.
Draw a 45-degree line connecting (0, 0) and (1, 1), called the line of equality.

Here is an example:[345]



This model identified 60% of exposures which contribute only 20% of the total losses. **The Gini index is twice the area between the Lorenz curve and the line of equality**, in this case 56.1%. **The higher the Gini index, the better the model is at identifying risk differences**.[346]

[343] See Section 7.2.4 of Generalized Linear Models for Insurance Rating

[344] This should be done on a dataset not used to develop the rating plan.

[345] See Figure 25 of Generalized Linear Models for Insurance Rating.

[346] "Note that a Gini index does not quantify the profitability of a particular rating plan, but it does quantify the ability of the rating plan to differentiate the best and worst risks. Assuming that an insurer has pricing and/or underwriting flexibility, this will lead to increased profitability."

An Example of the Gini Index and an Insurance Rating Plan:[347]

We have four classes each with an equal number of exposures, and the result of fitting a GLM.[348]
We have already sorted the classes according to the pure premiums predicted by the GLM.[349]

Class	Predicted Pure Premium
1	100
2	200
3	300
4	400

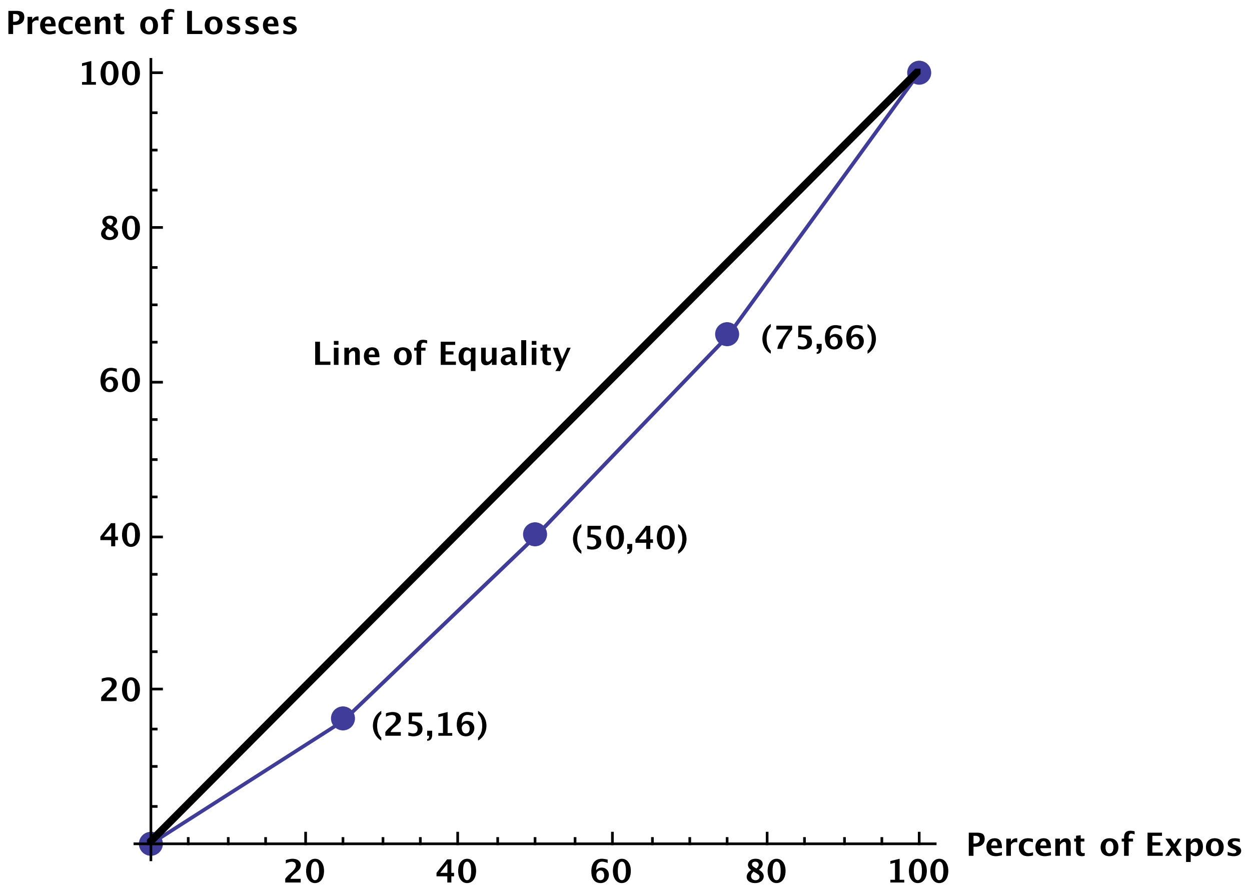
Ignoring here any misestimating of the overall rate level, the observed pure premiums would differ from the predicted pure premiums for two reasons: [350] [351]
1. Imperfection of the GLM, in other words modeling error.
2. Random fluctuation, in other words process variance.[352]

Let us assume the following Actual Pure Premiums:[353] [354]

Class	Actual P.P.	Cumulative Losses	% of Losses	% Expos
1	160	160	16%	25%
2	240	400	40%	50%
3	260	660	66%	75%
4	340	1000	100%	100%

Thus for the Lorenz curve we plot the points: (0, 0), (25, 16), (50, 40), (75, 66), (100, 100).

[347] See 8, 11/16, Q.5.
[348] I have chosen a one-dimensional example with only four levels solely for illustrative simplicity. Most GLMs would include more than one risk characteristic, and some characteristics would have more than four levels.
Also the exposures for each level would usually not all be equal.
[349] In a practical application we would have hundreds if not thousands of different cells consisting of risks with all of the same characteristics and thus the same predicted pure premium.
[350] We are using the GLM to predict class relativities rather than the overall rate level.
In some cases, the GLM output will automatically balance to the observed.
[351] Each class is not perfectly homogenous; it may be possible to refine the given classes to produce more homogeneous classes. Of course, if the classes are made too small, we would have issues with credibility.
[352] The more data in a class, the less subject to random fluctuation would be the average observed pure premium for that class.
[353] These observed pure premiums are from a dataset similar to the one to which the GLM was fit.
[354] Assuming solely for simplicity one exposure per class.



It is possible to calculate the area between the above Lorenz Curve and the Line of Equality, by dividing the area in triangles.[355] [356] This area turns out to be 0.07.[357]
Thus the Gini Index is twice that or 14%.[358]

[355] You will not be asked to do so on your exam!

[356] The six triangles I used were: {(0,0), (25,16), (25,25)}, {{25,16}, {25,25}, {50,40)},
One can calculate the area of a triangle from the length of the sides via Heron's formula, not on the syllabus.

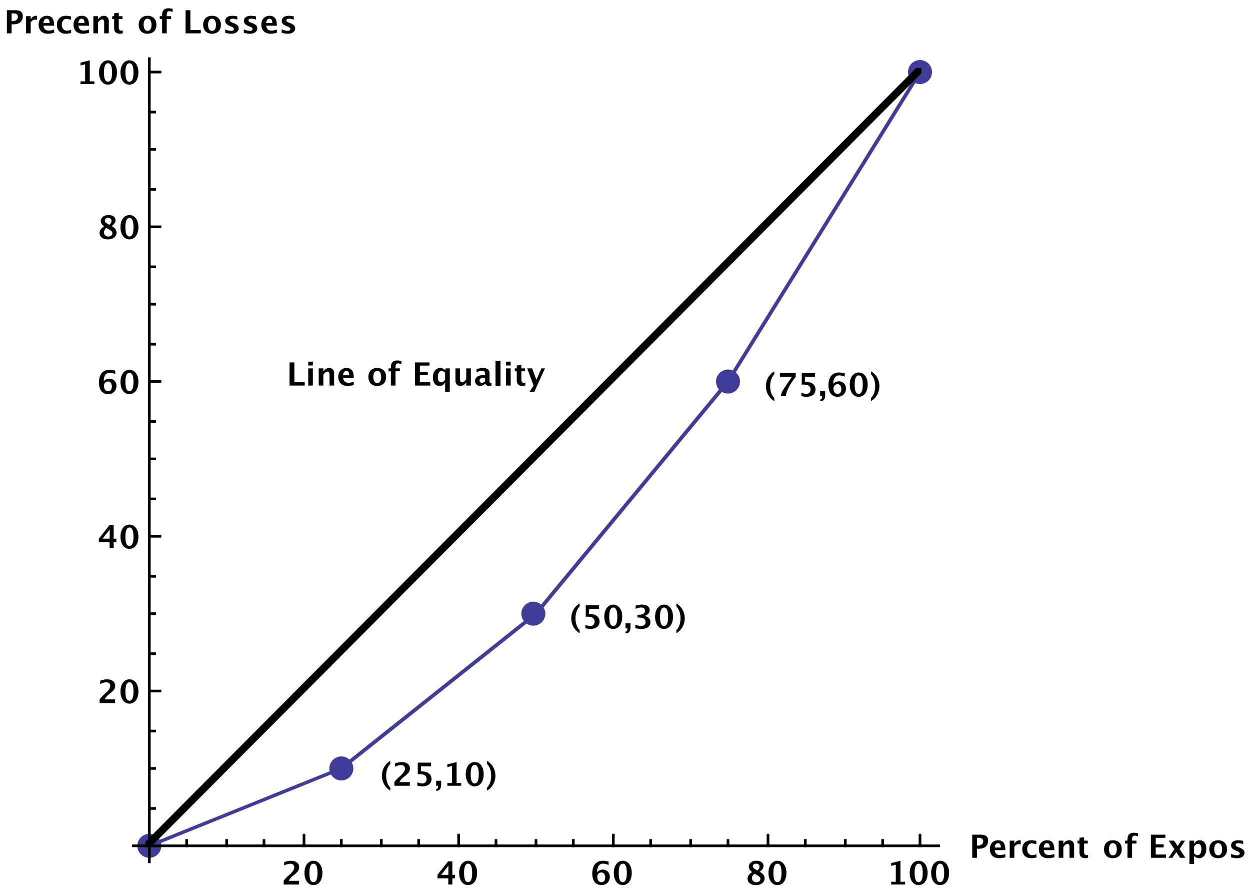
[357] Remembering that for example the value shown as 25 is actually 25% = 0.25.

[358] The higher the Gini index, the better the model is at identifying risk differences.
A more complicated model is likely to do better than this very simple class plan.

Solely for illustrative purposes, let us investigate the Gini Index if instead the actual pure premiums exactly matched the predicted pure premiums for each class.[359] [360]

Class	Actual P.P.	Cumulative Losses	% of Losses	% Expos
1	100	100	10%	25%
2	200	300	30%	50%
3	300	600	60%	75%
4	400	1000	100%	100%

Thus for the Lorenz curve we plot the points: (0, 0), (25, 10), (50, 30), (75, 60), (100, 100).



The area between the above Lorenz Curve and the Line of Equality, turns out to be 0.125. Thus the Gini Index is twice that or 25%, higher than previously.

[359] While this is will not occur in practice, this is the best possible result for this simple plan with only four classes.
[360] Assuming solely for simplicity one exposure per class.

Understanding & Validating a Model:[361]

Model Lift
How well does the model differentiate between best and worst risks?
Does the model help prevent adverse selection?
Is the model better than the current rating plan?

Simple Quantile plots:
Illustrate how well the model helps prevent adverse selection.
Double lift charts:
Compare competing models or compare new model against current rating plan.
Gini Index:
Summarizes model lift into one number.
Loss ratio charts:
Puts lift in a context most people in the insurance industry can understand.

Goodness of Fit
What kind of model statistics are available, and how do you interpret them?
What kind of residual plots should you consider, and how do you interpret them?
What are some considerations regarding actual versus predicted plots?

Internal Stability
How well does the model perform on other data?
How will the model perform over time?
How reliable are the model’s parameter estimates?

[361] “And The Winner Is…? How to Pick a Better Model,” 2015 CAS RPM Seminar, by Hernan L. Medina.

ROC Curves:[362]

Receiver Operating Characteristic (ROC) Curves can be used to compare models that use the Bernoulli or Binomial Distribution.[363]

The first step is to pick a threshold. For example, if the discrimination threshold were 8%, then we look at all cells with the fitted probability of an event > 8%, in other words $\hat{q}_i > 8\%$.[364] Then we count up the number of times there was an event when an event was predicted. For example, there might be 3740 such true positives. Assume that there 4625 total events. Then the "sensitivity" is the ratio: 3740/4625 = 0.81.

In general, **above a given threshold, the sensitivity is the portion of the time that an event was predicted by the model out of all the times there is an event** = $\frac{\text{true positives}}{\text{total times there is an event}}$. Sensitivity is the rate of true positives.[365]
All other things being equal, higher sensitivity is good.

Then we look at all cells with the fitted probability of an event ≤ 8%, in other words $\hat{q}_i \le 8\%$.
For example, there might be 54,196 such policies without an event. Assume there are a total of 63,232 policies without an event. Then the "specificity" is the ratio: 54,196/63,232 = 0.85.

Below a given threshold, the specificity is the portion of the time that an event was not predicted by the model out of all of the times these is <u>not</u> an event = $\frac{\text{true negatives}}{\text{total times there is not an event}}$.[366] All other things being equal, higher specificity is good.
Specificity is the rate of true negatives. The rate of false positives is: 1 - specificity.

For this example, for a threshold of 8%, we can display the information in a **confusion matrix**:[367]

Discrimination Threshold: 8%			
	Predicted		
Actual	Event	No Event	Total
Event	3740	884	4625
No Event	9036	54,196	63,232
Total	12,776	55,080	67,856

[362] See Section 7.3.1 in <u>Generalized Linear Models for Insurance Rating</u>.
[363] ROC analysis was originally developed during World War II for the analysis of radar images.
[364] The event could be a claim, a policy renewal, etc.
[365] If one has a model to predict the probability of a claim being fraudulent, then for a given threshold:
Sensitivity = (correct predictions of fraud) / (total number of fraudulent claims).
[366] If one has a model to predict the probability of a claim being fraudulent, then for a given threshold:
Specificity = (correct predictions of no fraud) / (total number of non-fraudulent claims).
[367] See Table 13 in <u>Generalized Linear Models for Insurance Rating</u>.

For a certain threshold, the general form of a confusion matrix:

	Predicted	
Actual	Event	No Event
Event	true positive	false negative
No Event	false positive	true negative

A confusion matrix is similar to a table from hypothesis testing,
where the null hypothesis is no event:[368]

Decision	Reject H_0	Do not reject H_0
H_1 is True	Correct	Type II Error
H_0 is True	Type I Error	Correct

The false negatives are analogous to making a Type II Error.
The false positives are analogous to making a Type I Error.

	Predicted		
Actual	Event	No Event	Total
Event	3740	884	4625
No Event	9036	54,196	63,232
Total	12,776	55,080	67,856

For the 8% threshold, the specificity was: $\frac{\text{true negatives}}{\text{total times there is not an event}} = \frac{54{,}196}{63{,}232} = 85\%.$

$$1 - \text{specificity} = \frac{\text{false positives}}{\text{total times there is not an event}} = \frac{9036}{63{,}232} = 15\%.$$

1 - specificity is analogous to:
chance of making a Type Error I = significance level of a statistical test.

For the 8% threshold, the sensitivity was: $\frac{\text{true positives}}{\text{total times there is an event}} = \frac{3740}{4625} = 81\%.$

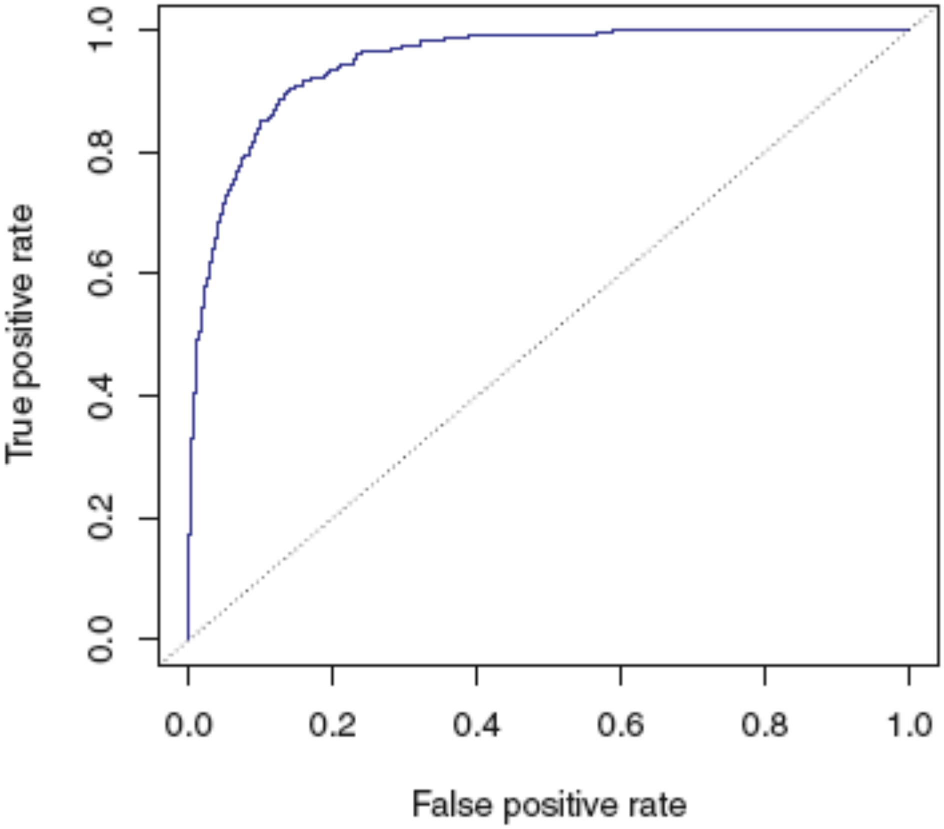
Sensitivity is analogous to: 1 - chance of making a Type Error II = power of a statistical test.

In the ROC Curve we plot the point: (1 - 0.85, 0.81) = (0.15, 0.81).

In general, **the ROC curve consists of plotting for various thresholds:**
(1 - specificity , sensitivity).
In addition, there is a 45% comparison line, the line of equality, from (0, 0) to (1, 1).

[368] While the analogy to hypothesis testing may help your understanding, it should not be tested on your exam.

Here is an example of an ROC curve,
similar to Figure 26 in Generalized Linear Models for Insurance Rating:[369]



Sensitivity = true positive rate. 1 - specificity = false positive rate.

A perfect model would be at (0, 1) in the upper lefthand corner; sensitivity = 1 and specificity = 1. The closer the model curve gets to the upper lefthand corner the better.

The comparison line (line of equality) indicates a model with sensitivity = 1 - specificity, which can be achieved by just flipping a coin to decide your prediction. Thus such models have no predictive value. The closer the model curve gets to the 45 degree comparison line (line of equality), the worse the model.

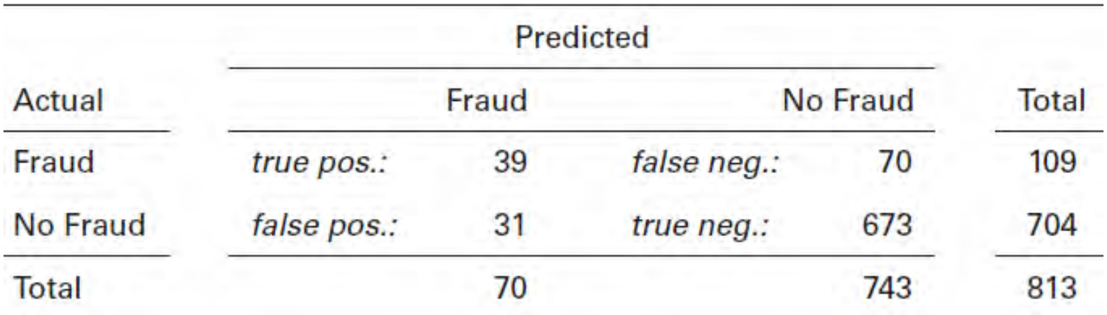
The comparison line has area 1/2 below it. The larger the area under the model curve, the better it is. The area under the above ROC curve is 0.95.

AUROC is the area under the ROC curve; the larger AUROC the better the model.[370]

[369] Figure 4.8 in An Introduction to Statistical Learning with Applications in R, by James, Witten, Hastie, & Tibshirani, not on the syllabus of this exam.

[370] The AUROC is equal to: (0.5) (normalized Gini) + 0.5, where the normalized Gini is the ratio of the model's Gini index to the Gini index of the hypothetical "perfect" model (where each record's prediction equals its actual value). Note that the prefect model will not have a Gini index of one; it's Gini index depends on the homogeneity of the risks and the randomness of the loss process.

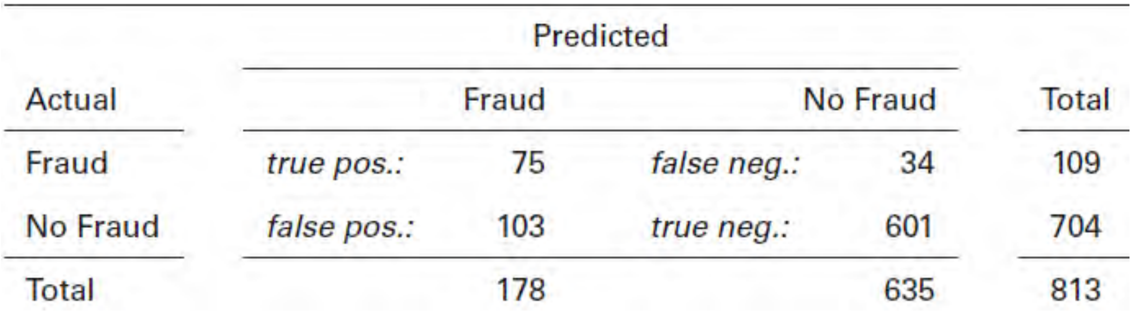
For a fraud example, the confusion matrix for a discrimination threshold of 50%:[371]



	Predicted				
Actual	Fraud		No Fraud		Total
Fraud	*true pos.:*	39	*false neg.:*	70	109
No Fraud	*false pos.:*	31	*true neg.:*	673	704
Total		70		743	813

Exercise: For a discrimination threshold of 50%, determine the sensitivity and specificity.
[Solution: Sensitivity = 39/109 = 35.8%. Specificity = 673/704 = 95.6%.
<u>Comment</u>: In the ROC Curve we plot the point: (1 - 0.956, 0.358) = (0.044, 0.358).]

For this fraud example, the confusion matrix for instead a discrimination threshold of 25%:[372]

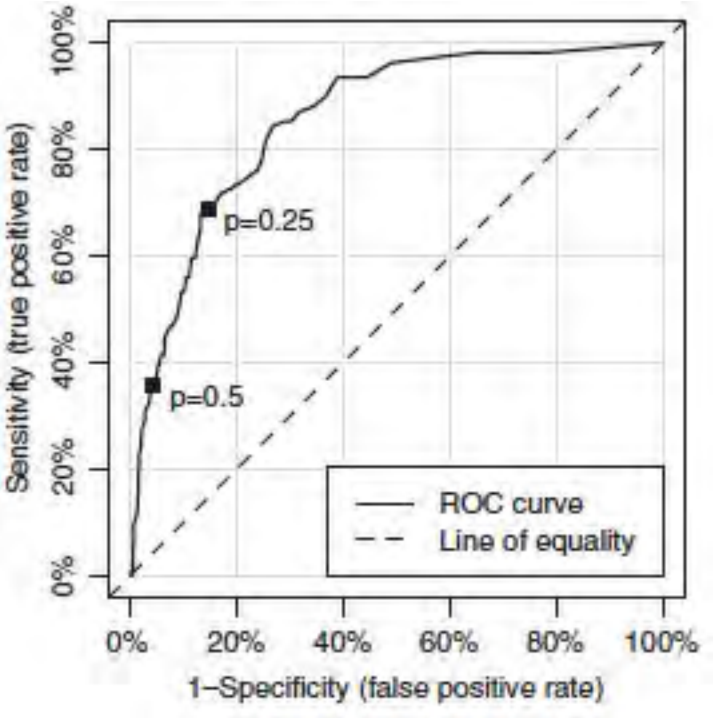


	Predicted				
Actual	Fraud		No Fraud		Total
Fraud	*true pos.:*	75	*false neg.:*	34	109
No Fraud	*false pos.:*	103	*true neg.:*	601	704
Total		178		635	813

Exercise: For a discrimination threshold of 25%, determine the sensitivity and specificity.
[Solution: Sensitivity = 75/109 = 68.8%. Specificity = 601/704 = 85.4%.
<u>Comment</u>: In the ROC Curve we plot the point: (1 - 0.854, 0.688) = (0.146, 0.688).
Lowering the threshold increased the sensitivity but decreased the specificity.]

[371] See the top of Table 13 in <u>Generalized Linear Models for Insurance Rating</u>.
[372] See the bottom of Table 13 in <u>Generalized Linear Models for Insurance Rating</u>.

For this example of modeling fraud on claims, one gets the following ROC Curve:[373]



This ROC has an area under ROC (AUROC) of 0.857.[374]

We can see how as one changes the threshold from 0.5 to 0.25, the sensitivity increases, but at the cost a lower specificity. In other words, the rate of true positives increases at the cost of also increasing the rate of false positives.

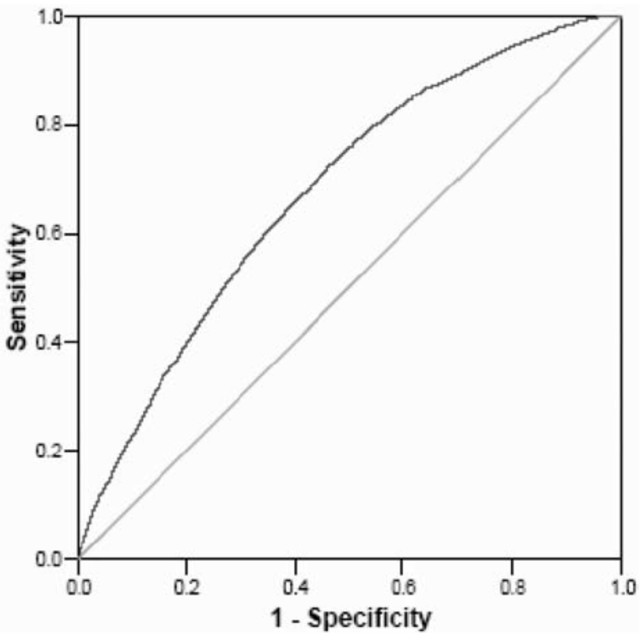
The selection of the discrimination threshold involves a trade-off: a lower threshold will result in more true positives and fewer false negatives than a higher threshold, but at the cost of more false positives and fewer true negatives.

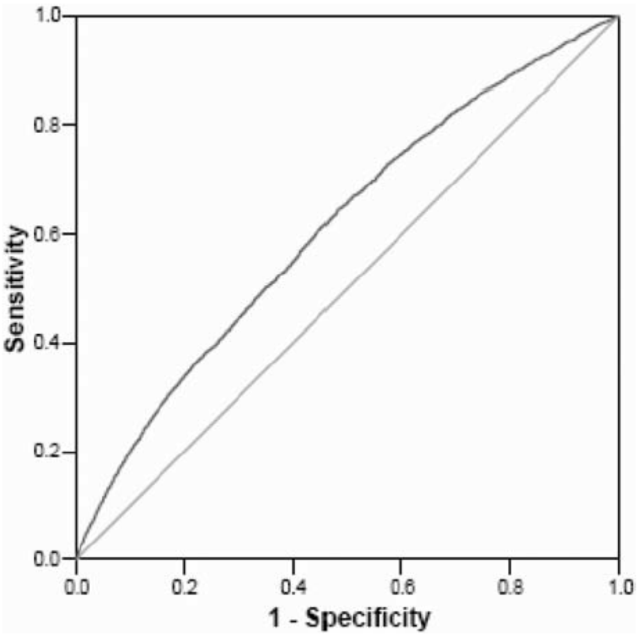
For example, let us assume an actuary has developed a GLM to predict fraudulent claims. The larger the average severity, the more worthwhile it is for the insurer to spend money to investigate cases of possible fraud. If claims are more severe, then the insurer will be more concerned about false negatives (cases where there is fraud but the modeled probability of fraud is below the threshold), than it would be about false positives (cases where there is not fraud but the modeled probability of fraud is above the threshold). Therefore, the more severe the claims, the lower the threshold that should be selected.

[373] See Figure 26 in Generalized Linear Models for Insurance Rating.
[374] The perfect model would have an AUROC of 1.

In general, the choice of an appropriate discrimination threshold involves some judgement and depends on the practical application.[375]

From a paper on detecting insurance fraud, two ROC curves for logistic models:[376]





The curve on the left has an AUROC of 0.677, while the curve on the right has an AUOC of 0.612.[377] Thus we prefer the model on the left.

[375] "Determination of the optimal threshold is typically a business decision that is out of the scope of the modeling phase."
[376] "Distinguishing the Forest from the TREES: A Comparison of Tree-Based Data Mining Methods," by Richard A. Derrig and Louise Francis, Variance, Volume 2 Issue 2.
[377] The areas between the curves and the 45 degree line are 0.177 and 0.112.

A Medical Example of ROC:[378]

Let us assume we have a medical test for a disease which results in a numerical score.
The lower the score on this test the more likely that the individual has this disease.[379]
Assume the following data:

Score on Medical Test	Number with Disease	Number without Disease
5 or less	18	1
5.1 to 7	7	17
7.1 to 9	4	36
9 or more	3	39
Total	32	93

We can pick a threshold to use with this test; if the test score is less than or equal to the chosen threshold this indicates that the individual has the disease.

For example, assume a threshold of 5. Then 18 individuals are correctly identified as diseased, and 1 is incorrectly identified as diseased. There are 18 true positives. There is one false positive. 14 individuals who are diseased are incorrectly identified as being without disease. There are 14 false negatives. 92 individuals who are not diseased are correctly identified as being without disease.

We can think of sensitivity as the rate of true positives of a medical test for a disease as a portion of positives. The rate of true positives out of all diseased is: 18/32 = 0.56.[380]

We can think of specificity as the rate of individuals that the test indicates do not have the disease out of those without the disease. The rate of negatives out of those without the disease: 92/93 = 0.99. One minus the specificity, 1%, is the rate of false positives out of those without the disease.[381]

The confusion matrix is:

Discrimination Threshold: 5			
	Predicted		
	ActualDisease	No Disease	Total
Disease	18	1	19
No Disease	14	92	106
Total	32	93	125

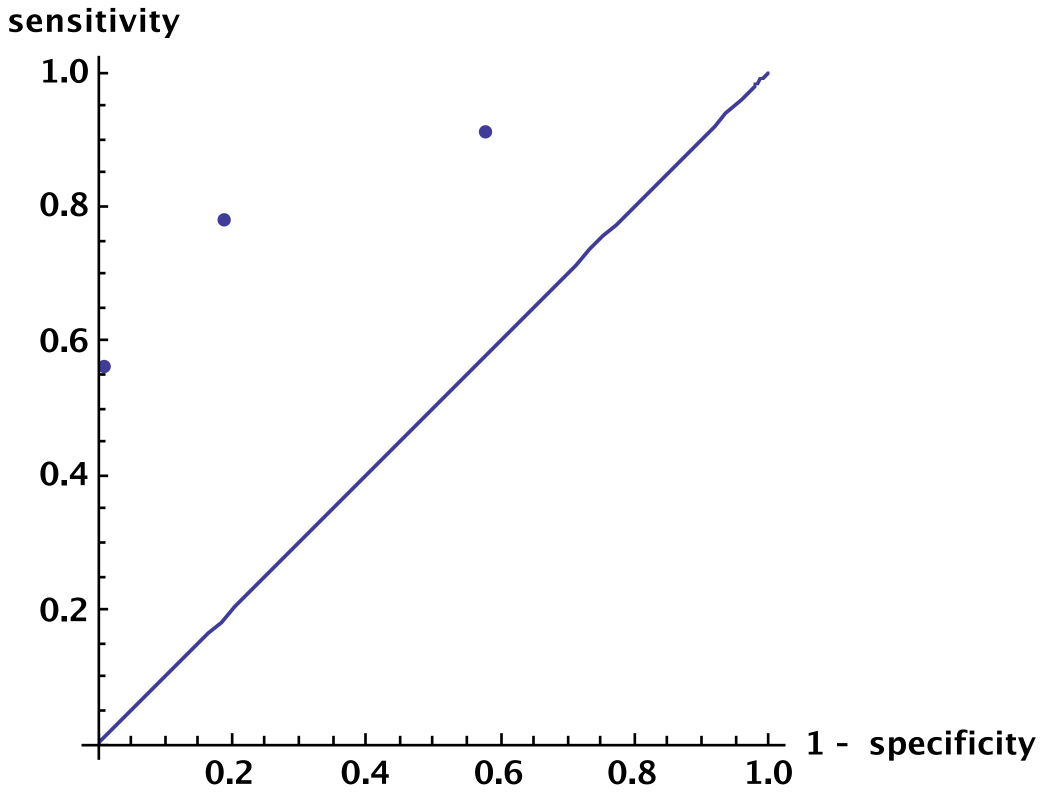
[378] http://gim.unmc.edu/dxtests/ROC1.htm
[379] While a low test score indicates the presence of the disease in this example, it could have been the reverse.
[380] Sensitivity is analogous to the probability of rejecting the null hypothesis (healthy) when it is false, which is the power of the test.
[381] One minus specificity is analogous to the probability of rejecting the null hypothesis (healthy) when it is true, which is the significance level of the test.

Exercise: What are the sensitivity and specificity if one instead uses a threshold of 7?
[Solution: 25 people have positive tests out of 32 with the disease.
⇒ sensitivity is: 25/32 = 0.78.
75 people have negative tests out of 93 who are healthy. ⇒ specificity is: 75/93 = 0.81.
Comment: With a higher threshold the sensitivity is higher but the specificity is lower.
There is a tradeoff between a high sensitivity and a high specificity.]

Exercise: What are the sensitivity and specificity if one instead uses a threshold of 9?
[Solution: 29 people have positive tests, out of 32 with the disease.
⇒ sensitivity is: 29/32 = 0.91.
39 people have negative tests out of 93 who are healthy. ⇒ specificity is: 39/93 = 0.42.]

Threshold	Sensitivity	Specificity	1 - Specificity
5	0.56	0.99	0.01
7	0.78	0.81	0.19
9	0.91	0.42	0.58

The corresponding ROC curve, where I have not connected the dots:[382]



[382] The area under the curve measures discrimination, that is, the ability of the test to correctly classify those with and without the disease.

Model Documentation:[383] [384]

Documenting your work as you go along is useful and important, when developing GLMs or doing other actuarial work.[385] This is related to but somewhat different than presenting a formal written report on your work.[386] This is related to but somewhat different than writing a paper for publication.[387]

Model documentation serves at least three purposes:[388]
- To serve as a check on your own work, and to improve your communication skills
- To facilitate the transfer of knowledge to the next owner of the model
- To comply with the demands of internal and external stakeholders

Writing down and explaining what you are doing is one way to discover mistakes, particularly when you discuss your notes with another actuary.[389] This should be an ongoing process; you should not wait until you are wrapping up the project. Documenting your work should improve your understanding of that work, as well improve your general communication skills.

Usually a model will be used more than once. For example, classification relativities may be reviewed once a year.[390] If you performed the current review, either you or someone else will be doing the next review.

If you do the next review, one year later you will be surprised at how much you do not remember or that is not longer obvious to you. You will appreciate the detailed notes that the former version of you took the previous year.

If someone else does the next review, they will appreciate your detailed notes, particularly if you are not available to explain what you did.[391]

Many people may have questions about the model: insurance regulators, internal auditors, outside auditors, and risk managers. Also many people within your organization may have questions about the model: executives, underwriters, claims adjusters, other actuaries, and information technology personnel. Good documentation will help to answer detailed questions on your work that may have been done months or years ago.

[383] See Chapter 8 of Generalized Linear Models for Insurance Rating. Added to the 2019 edition. While this chapter contains lots of good practical advice, there is not much to be tested.

[384] You might also benefit from looking at ASOP41: Actuarial Communications and ASOP56: Modeling, not on the syllabus of this exam.

[385] This is very helpful even for such a relatively simple task as updating an edition of this study guide. I have a steno book filled with detailed notes on what I have done updating various editions.

[386] Such a report may be for internal consumption, may be part of a rate filing, or may be a loss reserve opinion.

[387] A good example is "NCCI's 2007 Hazard Group Mapping," by John P. Robertson, published in Variance.

[388] Quoted from Section 8.1 of Generalized Linear Models for Insurance Rating.

[389] Personally, I find it very useful to include in my notes short numerical examples.

[390] Besides using updated data, such a review may include investigating new classification variables and/or reviewing the form of the model.

[391] Do unto others as you would have them do unto you.

To meet the needs of these stakeholders, your documentation should:[392] [393]

- Include everything needed to reproduce the model from source data to model output[394]
- Include all assumptions and justification for all decisions
- Disclose all data issues encountered and their resolution
- Discuss any reliance on external models or external stakeholders
- Discuss model performance, structure, and shortcomings
- Comply with ASOP 41 or local actuarial standards on communications[395]

As always it is preferable to have clearer computer code, and when appropriate better commented computer code.[396]

[392] Quoted from Section 8.3 of Generalized Linear Models for Insurance Rating.

[393] This is an idealized list which may not be practical in some situations.

[394] I recommend that you print out and maintain a hardcopy of all important computer programs as part of your documentation.

[395] Actuarial Standard of Practice 41: Actuarial Communications,
applies to actuaries practicing in the United States.

[396] The authors recommend that if you use the computer language R, you should use the tidyverse package and adhere to the tidyverse style guide

Other Topics:[397]

The syllabus reading discusses three additional topics:

- Why you probably should not model coverage options with GLMs.
- Why territories are not a good fit for the GLM framework.
- Ensembling.

Coverage Options:[398]

Insureds can choose coverage options such as deductible amount or limit of liability.[399]
There are corresponding deductible credits or increased limits factors.[400]
You probably should not model the rating factors for coverage options with GLMs.

For example, a GLM might indicate that one should charge more for a higher deductible. There may be something systematic about insureds with higher deductibles that may make them a worse risk relative to others in their class.[401] In which case, the coefficients estimated by the GLM are reflecting some of this increased risk due to antiselection effects.

To the extent that the factor indicated by the GLM differs from the pure effect on loss potential, it will affect the way insureds choose coverage options in the future. Thus, the selection dynamic will change and the past results would not be expected to be replicated for new policies.

Thus factors for coverage options should be estimated outside the GLM, using traditional actuarial techniques.[402] The resulting factors should then be included in the GLM as an offset, as has been discussed previously.

Examples of GLM Output:

On my webpage I have posted a file in which I discuss some examples of GLM output taken from Chapter 10 and Appendix F of Basic Ratemaking, on the syllabus of the Basic Ratemaking exam.[403] While these examples should not be tested directly on your exam, you may find it helpful to briefly review them.

[397] See Chapter 9 of Generalized Linear Models for Insurance Rating.
[398] See Section 8.1 of Generalized Linear Models for Insurance Rating.
[399] These can be distinguished from characteristics of the insured.
[400] In general, the insured should pay less for less coverage and more for more coverage.
[401] "The choice of high deductible may be the result of a high risk appetite on the part of an insured, which would manifest in other areas as well. Alternately, the underwriter, recognizing an insured as a higher risk, may have required the policy to be written at a higher deductible."
[402] This is the recommendation of Goldburd, Khare, and Tevet.
Even if the final factors for coverage options are not estimated within the GLM, I think the results of including coverage options in a GLM may reveal something interesting and potentially important to the actuary.
[403] Also included are some related problems from past Basic Ratemaking exams.

Territory Modeling:[404]

Territories are not a good fit for the GLM framework.

There may have hundreds of territories, which requires many levels in the GLM. Therefore, the authors recommend the use of other techniques, such as spatial smoothing, to model territories.[405]

One should include the territory relativities produced by the separate model as an offset in the GLM used to determine classification relativities. Similarly, one should include classification relativities produced by the GLM as an offset in the separate model used to determine territory relativities.[406]

Ideally this should be an iterative process.[407]

Ensembling:[408]

Two (or more) teams model the same item; they build separate models working independently. The models are evaluated and found to be approximately equal in quality.

Combining the answers from both models is likely to perform better than either individually.[409] **A model that combines information from two or more models is called an ensemble model**.

A simple means of ensembling is to average the separate model predictions.[410] “Predictive models each have their strengths and weaknesses. Averaged together, they can balance each other out, and the gain in performance can be significant.”

[404] See Section 9.2 of Generalized Linear Models for Insurance Rating.
I believe the authors are discussing determining territory relativities rather than constructing territories from smaller geographical units such as zipcode. However, they may be discussing doing both together.

[405] The authors do not discuss any details of spatial smoothing or other techniques.

[406] In determining territories one should adjust the pure premiums for a zipcode by its average class rating factor. Chapter 11 of Basic Ratemaking by Werner and Modlin have a discussion of determining territories.

[407] If they being updated at the same time, both models should be run, using the other as an offset, until they reach an acceptable level of convergence.

[408] See Section 9.3 of Generalized Linear Models for Insurance Rating.

[409] Of course it is costly to have two teams build two separate models.
“Done properly, though, ensembles can be quite powerful; if resources permit, it may be worth it.”

[410] The authors do not discuss more complicated methods of ensembling.

A More Realistic and Complex Example:

Consider the following data on claim severity for personal auto insurance:[411]

Observation	Age Group	Vehicle-Use	Severity	Claim Count
1	17–20	Pleasure	250.48	21
2	17–20	Drive to Work < 10 miles	274.78	40
3	17–20	Drive to Work > 10 miles	244.52	23
4	17–20	Business	797.80	5
5	21–24	Pleasure	213.71	63
6	21–24	Drive to Work < 10 miles	298.60	171
7	21–24	Drive to Work > 10 miles	298.13	92
8	21–24	Business	362.23	44
9	25–29	Pleasure	250.57	140
10	25–29	Drive to Work < 10 miles	248.56	343
11	25–29	Drive to Work > 10 miles	297.90	318
12	25–29	Business	342.31	129
13	30–34	Pleasure	229.09	123
14	30–34	Drive to Work < 10 miles	228.48	448
15	30–34	Drive to Work > 10 miles	293.87	361
16	30–34	Business	367.46	169
17	35–39	Pleasure	153.62	151
18	35–39	Drive to Work < 10 miles	201.67	479
19	35–39	Drive to Work > 10 miles	238.21	381
20	35–39	Business	256.21	166
21	40–49	Pleasure	208.59	245
22	40–49	Drive to Work < 10 miles	202.80	970
23	40–49	Drive to Work > 10 miles	236.06	719
24	40–49	Business	352.49	304
25	50–59	Pleasure	207.57	266
26	50–59	Drive to Work < 10 miles	202.67	859
27	50–59	Drive to Work > 10 miles	253.63	504
28	50–59	Business	340.56	162
29	60+	Pleasure	192.00	260
30	60+	Drive to Work < 10 miles	196.33	578
31	60+	Drive to Work > 10 miles	259.79	312
32	60+	Business	342.58	96

[411] Data taken from Exhibit 1 of “A Systematic Relationship Between Minimum Bias and Generalized Linear Models,” by Stephen J. Mildenhall, PCAS 1999, not on the syllabus.

There are 8 age categories and 4 vehicle use types.
Thus there are a large number of ways to set up a GLM.
I will make age 40-49 and drive to work less than 10 miles as the base levels.

I will use the following definitions of variables:
X_0 corresponds to the base levels.
X_1 is one if 17-20 years old and zero otherwise.
X_2 is one if 21-24 years old and zero otherwise.
X_3 is one if 25-29 years old and zero otherwise.
X_4 is one if 30-34 years old and zero otherwise.
X_5 is one if 35-39 years old and zero otherwise.
X_6 is one if 50-59 years old and zero otherwise.
X_7 is one if 60+ years old and zero otherwise.
X_8 is one if Pleasure Use and zero otherwise.
X_9 is one if Drive to Work > 10 and zero otherwise.
X_{10} is one if Business Use and zero otherwise.

A Gamma Distribution with an identity link function was fit to these data:[412]

Parameter	Fitted Value	Standard Error	p-Value
β_0	203.522	6.54517	0
β_1	62.9056	37.0291	8.9%
β_2	66.1851	19.4111	0
β_3	46.1676	12.5584	0
β_4	33.2979	11.3777	0.3%
β_5	-15.289	9.57527	11.0%
β_6	3.57547	8.79087	68.4%
β_7	-1.84956	9.5907	84.7%
β_8	-8.63574	8.22596	29.4%
β_9	45.1086	7.43089	0
β_{10}	122.802	13.4003	0

[412] The fitted severities are: 257.79, 266.43, 311.54, 389.23, 261.07, 269.70, 314.82, 392.51, 241.05, 249.69, 294.80, 372.49, 228.19, 236.82, 281.93, 359.62, 179.60, 188.23, 233.34, 311.04, 194.89, 203.52, 248.63, 326.32, 198.46, 207.10, 252.21, 329.90, 193.04, 201.67, 246.78, 324.47.

Based on their large p-values, β_5, β_6, β_7, and β_8 are not significantly different than zero.

Let us test a model in which we eliminate the corresponding variables.
The reduced model will have:
Age 35-39 combined with 40-49.
Age 50-60 combined with 60+.
Pleasure use combined with Drive to Work < 10 miles.
Another GLM with Gamma Distribution with an identity link function was fit to these data.[413] [414]

The unscaled deviance for the original model with more variables is 31.2438 [415]
The unscaled deviance for the new model with less variables is 37.0310.

We have two nested models. GLM 1 is a special case of GLM 2.
Then the test statistic (asymptotically) follows an F-Distribution with numbers of degrees of freedom equal to: ν_1 = the difference in number of parameters = 3,
and ν_2 = number of degrees of freedom for the more complex model
= (number of observations) - (number of parameters) = 32 - 7 - 3 = 22.

$\hat{\phi}_B$ = estimated dispersion parameter for the bigger (more complex) model
= D_B / ν_B = 31.2438/22 = 1.420.[416]

The test statistic is: $\dfrac{D_S - D_B}{\text{(number of added parameters) } \hat{\phi}_B} = \dfrac{(37.0310 - 31.2438) / 3}{1.420} = 1.358.$

Using a computer, the p-value is 45.8%.
Thus we do <u>not</u> reject the null hypothesis of using the simpler model with fewer parameters.[417]

[413] The fitted parameters are: 196.36, 67.41, 71.78, 50.88, 38.42, 6.13, 47.01, 125.74.

[414] The fitted severities are: 263.77, 310.77, 389.51, 268.14, 315.15, 393.88, 247.24, 294.248, 372.98, 234.78, 281.79, 360.52, 196.36, 243.37, 322.10, 202.49, 202.49, 249.49, 328.23.

[415] A computer was used to fit both models and to calculate the unscaled deviances.

[416] The syllabus reading does not discuss how to estimate ϕ; this is one way.

[417] One could now compare additional models with different subsets of the original variables.
One could also fit models using different distributional forms and/or link functions.

Example of Homeowners Rating Factors Used in the United Kingdom:[418]

Personal lines rates in the United Kingdom have long been based on GLMs.
One important aspect to using GLMs is to find relevant variables.
Here is a list of some rating variables that are used for Homeowners Insurance.

Postal code (so geodemographic and geophysical factors can be derived)[419]
Amount of insurance
Number of rooms / bedrooms
Wall type
Roof type
State of repair
Extensions
Ownership status (rent/own)
Occupancy in day
Neighborhood watch scheme
Approved locks, alarms, smoke detectors
Deductibles
Endorsements purchased (e.g. riders for jewelry, oriental rugs)
How long held insurance / when last claimed

Policyholder details:

- Age
- Sex
- Marital status
- Number of children
- Occupation
- Residency
- Criminal convictions
- Claims in past 2 or past 5 years

Smokers present in house
Non-family members sharing house
Length of time living at property
Use (principal residence / secondary residence / business / rented)
Coverage selected (buildings/contents/both)
Source of business (e.g. agent, internet, etc.)

[418] "Homeowners Modeling" by Claudine Modlin, presentation at the 2006 CAS Seminar on Predictive Modeling.
[419] Geodemographics are the average characteristics in an area. Examples are: population density, length of homeownership, average age of residents, and average family income. Geophysical factors can include soil type, and weather data such as the maximum wind speed, the average rainfall, and the average snowfall.

Homeowners Perils:

There can be advantages to modeling the different homeowners perils separately.[420]
One can either model pure premium or separately model frequency and severity.

Some variables may have different effects on different perils. For example, increased population density may be related to an increased frequency for theft claims while being related to a decreased frequency of fire claims.

Some variables may have a significant effect on one peril but not another. For example, more children in the house may be related to an increased frequency of liability while being unrelated to the frequency for wind.

Here is an example of data by peril for the United States.

Peril	Frequency (in percent)	Median Claim Amount
Fire	0.310	4,152
Lightning	0.527	899
Wind	1.226	1,315
Hail	0.491	4,484
Water-Weather Related	0.491	1,481
Water-NonWeather[377]	1.332	2,167
Liability	0.187	1,000
Other	0.464	875
Theft-Vandalism	0.812	1,119
Total	5.889	1,661

The percent of losses expected by peril varies considerably by geographical location. For example, the expected percent from wind (from hurricanes and other storms) is higher than average on the coast of Florida. For example, the expected percent from theft is higher than average in the center of a large city.

Recently, homeowners insurers have begun to implement rating plans that have separate base rates for each major peril covered and the individual rating variable relativities are applied to the applicable base rate (e.g., burglar alarm discount applies to the theft base rate only).

[420] See for example, “Predictive Modeling of Multi-Peril Homeowners Insurance,” by Edward W. Frees, Glenn Meyers, and A. David Cummins, in Variance Volume 6 / Issue 1. They show that the perils are not independent.

Problems:

3.1. (1.5 points)
Five Generalized Linear Models have been fit to the same set of 50 observations.

Model	Number of Fitted Parameters	Scaled Deviance
A	6	335.8
B	8	331.9
C	10	325.2
D	12	321.4
E	14	317.0

Which model has the best AIC (Akaike Information Criterion)?

3.2. (0.5 points) Briefly discuss how to pick the base level of a categorical variable.

3.3. (1 point) When a log link is used, it is usually appropriate to take the natural logs of continuous predictors before including them in the model, rather than placing them in the model in their original forms. Discuss why.

3.4. (1.5 points) Fully discuss the use of weights in GLMs.

3.5. (0.5 points) Briefly discuss a primary strength of GLMs versus univariate analyses.

3.6. (0.5 points) A continuous predictor x_1 has a coefficient of $\beta_1 = 0.4$ in a logistic model. For a unit increase in x_1, what is the estimated change in the odds?

3.7. (1 point) Compare and contrast the Poisson and the Negative Binomial Distributions.

3.8. (0.5 points) With respect to GLMs, briefly discuss aliasing.

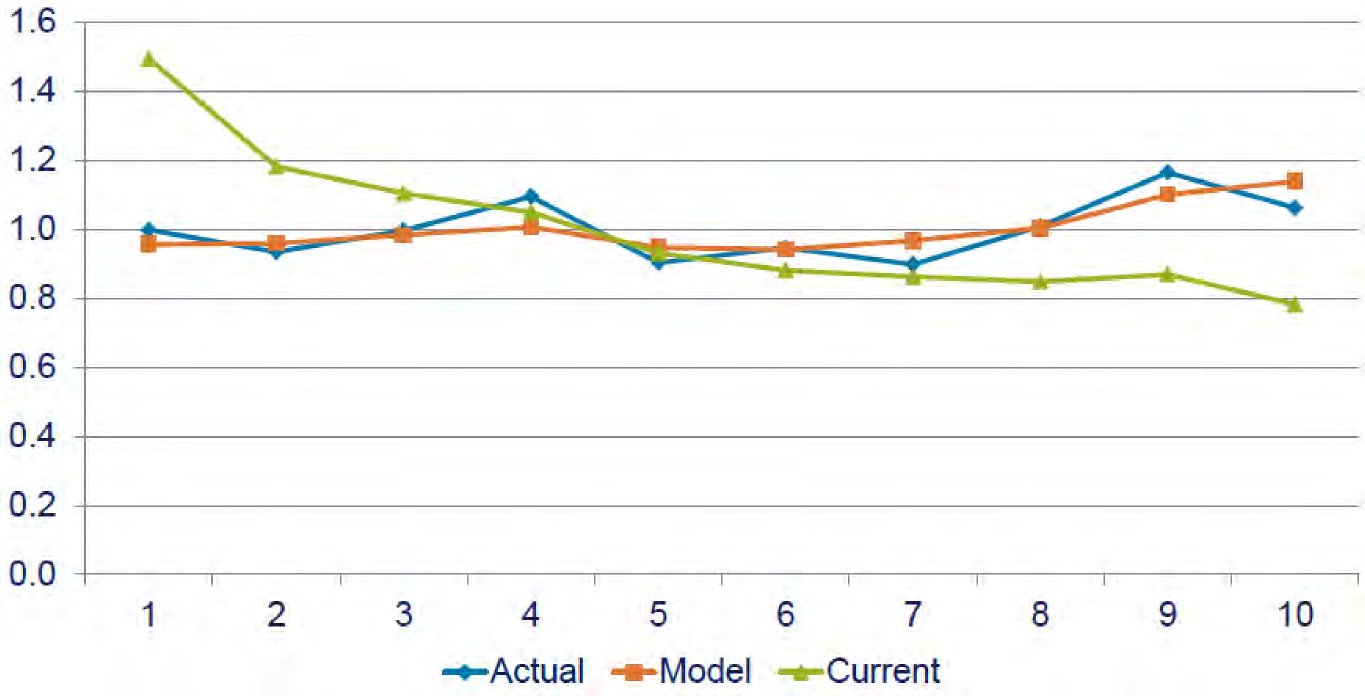
3.9. (0.5 points) List two limitations of GLMs.

3.10. (1 point) One possible fix for nonlinearity in a continuous variable is not to model it as continuous at all; rather, a new categorical variable is created where levels are defined as intervals over the range of the original variable. Briefly discuss two drawbacks to this approach.

3.11. (1.5. points) A GLM has been fit using a Poisson Distribution with $\hat{\beta}_1 = 0.02085$ with standard error 0.00120.
Using instead an overdispersed Poisson the estimate of ϕ is 7.9435.
For this second model, determine a 95% confidence interval for β_1.

3.12. (1 point) Discuss the Tweedie Distribution.

3.13. (1 point) You are given a double lift chart, sorted by ratio of the model prediction over the current plan prediction. Discuss the lift of the proposed model compared to the current plan.



3.14. (1 point) The flexibility afforded by the ability to use a link function is a good thing because it gives us more options in specifying a model, thereby providing greater opportunity to construct a model that best reflects reality. However, when using GLMs to produce insurance rating plans, an added benefit is obtained when the link function is specified to be the natural log function. Briefly discuss this added benefit.

3.15. (1 point) A logistic regression has been fit to some data. For a certain threshold:

		Predicted Claims		
		No	Yes	Total
Actual Claim	No	6000	2000	8000
	Yes	300	700	1000
	Total	6300	2700	9000

What point would be plotted in the ROC curve?

3.16. (2 points) List and briefly discuss four components of a predictive modeling project.

3.17. (1.5 points)
(a) (0.5 points) Define the partial residuals.
(b) (1 point) Discuss partial residual plots.

3.18. (0.5 points) Briefly contrast the following two GLMs:
$\mu = \exp[\beta_0 + \beta_1 X_1 + \beta_2 X_2]$.
$\mu = \exp[\beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_1 X_2]$.

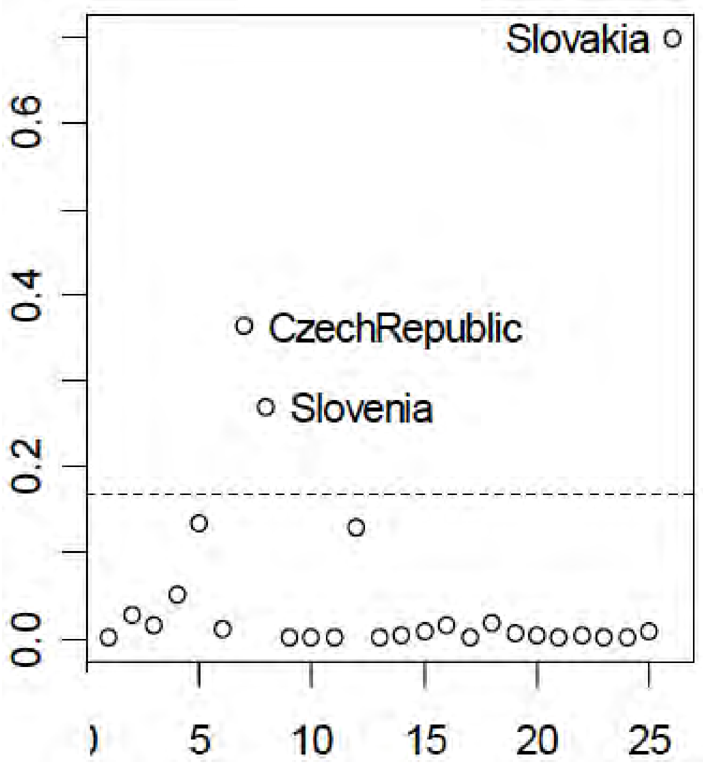
3.19. (1 point) Any data set of sufficient size is likely to have errors.
Briefly discuss two of the steps that should always be taken to attempt to catch and remedy some of the more common errors that can occur.

3.20. (0.5 points) List two types of Exploratory Data Analysis (EDA) plots and their purposes.

3.21. (1 point) Discuss some reasons to use frequency and severity models rather than a pure premium model.

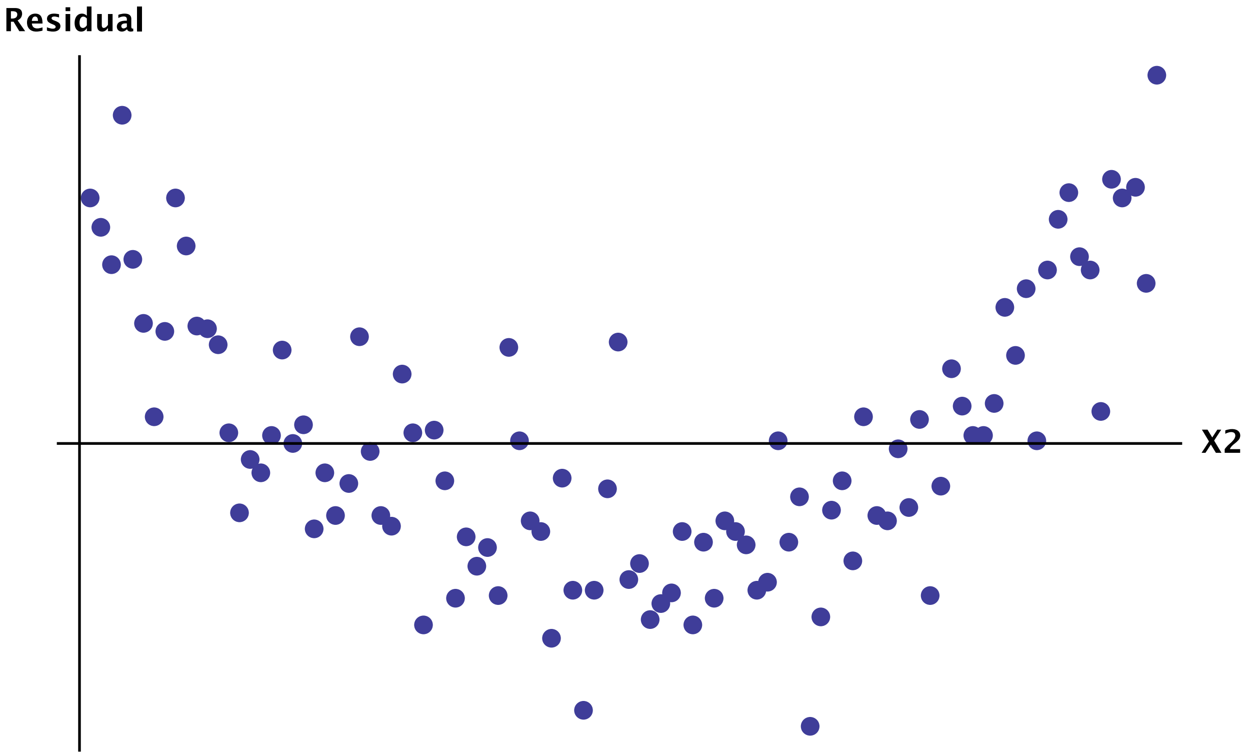
3.22. (1.5 points) Fully discuss the use of an offset term in GLMs.

3.23. (0.5 points) Discuss the following graph of Cook's Distance for 26 observations:



3.24. (1 point) Define the saturated and the null models, and discuss them with respect to scaled deviance.

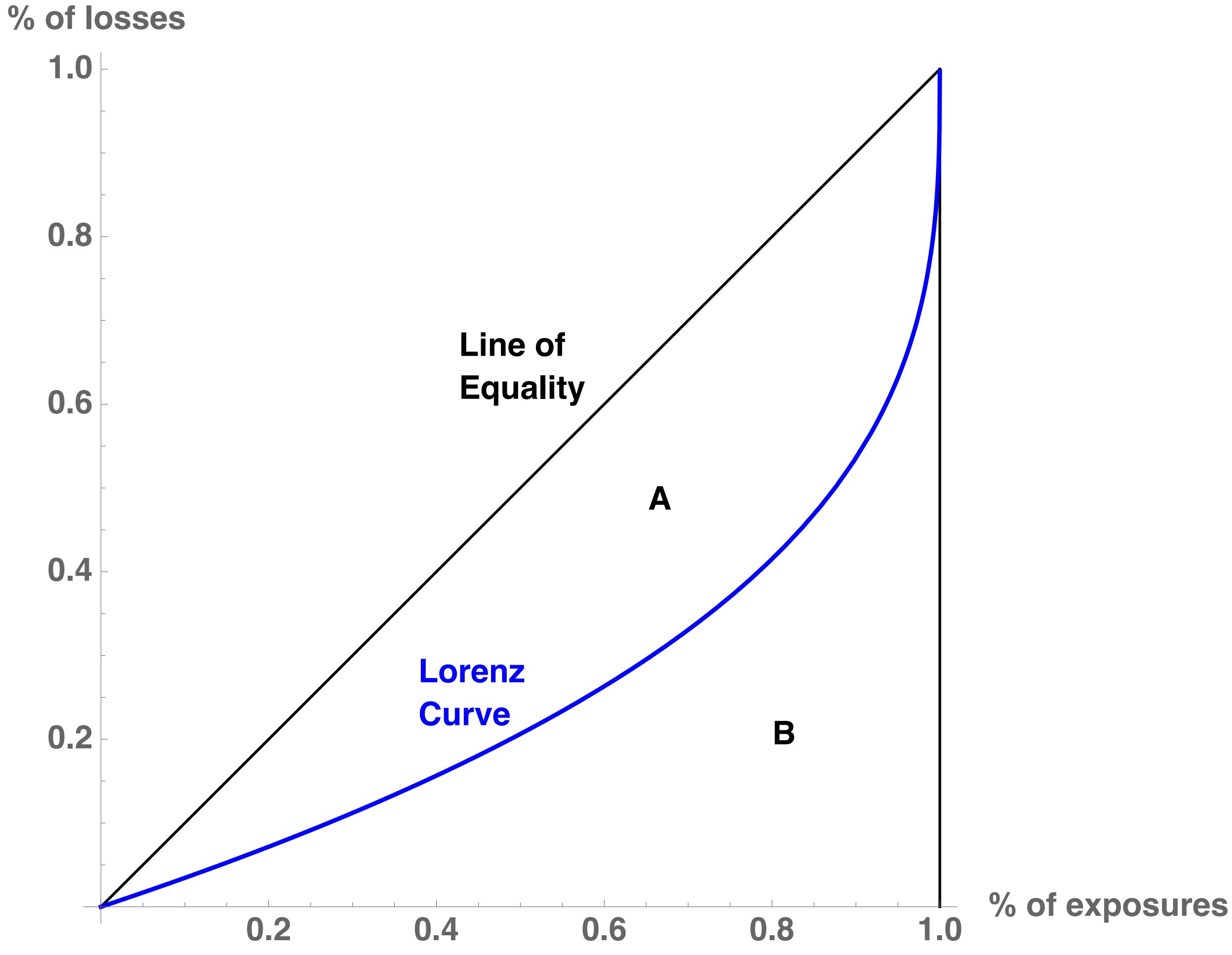
3.25. (1 point) Briefly comment on the following plot of deviance residuals of a model as a function of a predictor variable X_2:



3.26. (2 points) A GLM using a Tweedie Distribution and a log link function is being used to model pure premiums of private passenger automobile property damage liability insurance.
There are 100,000 observations.
10 parameters including an intercept were fit.
The unscaled deviance is 233,183.65.
Credit score as a categorical variable is added to the model, with a total of 6 categories.
The unscaled deviance for this more complex model is 233,134.37, and the estimated dispersion parameter is 2.371.
Discuss how you would use an F-Test to determine whether credit score should be added to this model.

3.27. (2 points) The following 5 returns on a stock price are observed:
-0.154, 0.239, -0.064, -0.328, 0.195.
Construct the corresponding Normal Q-Q Plot.

3.28. (0.5 points) Areas have been labeled in the following graph of a Lorenz Curve. Determine the Gini index.



3.29. (0.5 points) With respect to GLMs, briefly discuss pricing coverage options such as deductibles or increased limits.

3.30. (0.5 points) Give an example of a hinge function.

3.31. (0.5 points) Five logistic regressions have been fit to the same data. ROC curves have been drawn for each model.

Model	Number of Parameters	AUROC
A	1	0.58
B	2	0.66
C	3	0.73
D	4	0.79
E	5	0.75

Which model is preferred?

3.32. (1 point) For a GLM, the estimated mean for an individual is 35, with variance 5.
Determine a 95% confidence interval for the estimated mean.

3.33. (1.5 points)
Five different Generalized Linear Models, have been fit to the same set of 400 observations.

Model	Number of Fitted Parameters	LogLikelihood
A	3	-730.18
B	4	-726.24
C	5	-723.56
D	6	-721.02
E	7	-717.50

Which model has the best BIC (Bayesian Information Criterion)?

Use the following information for the following four questions:

- There is data on commercial building insurance claims frequency.
- A Poisson GLM was fit using the log link function.
- A categorical predictor used is building occupancy class, coded 1 through 4, with 1 being the base class.
- A binary predictor used is sprinklered status, with 1 being yes and 0 being no.
- A continuous predictor used is: ln[amount of insurance / 200,000] = ln[AOI / 200,000].
- The fitted intercept is β_0 = -3.8.
- The fitted parameters for building occupancy classes 2, 3, and 4 are: β_1 = 0.3, β_2 = 0.5, β_3 = 0.1.
- The fitted parameter for sprinklers is: β_4 = -0.5.
- The fitted parameter for ln[AOI / 200,000] is: β_5 = 0.4.
- An interaction term between sprinkler status and ln[AOI / 200,000] is included in the model; the fitted parameter is: β_6 = -0.1.

3.34. (1 point) Determine the fitted frequency for a $100,000 building in occupancy class 1 without sprinklers.

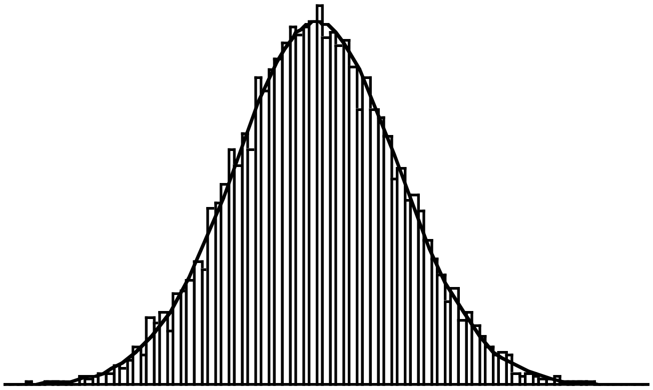
3.35. (1 point) Determine the fitted frequency for a $250,000 building in occupancy class 2 with sprinklers.

3.36. (1 point) Determine the fitted frequency for a $300,000 building in occupancy class 3 without sprinklers.

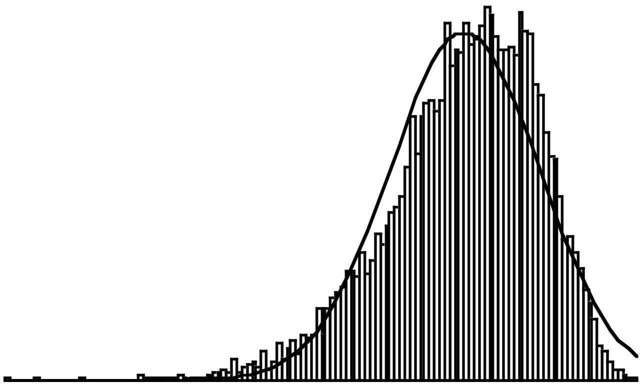
3.37. (1 point) Determine the fitted frequency for a $600,000 building in occupancy class 4 with sprinklers.

3.38. (1 point) The following are histograms of deviance residuals for GLMs.
Which of the following histograms represents the best model?

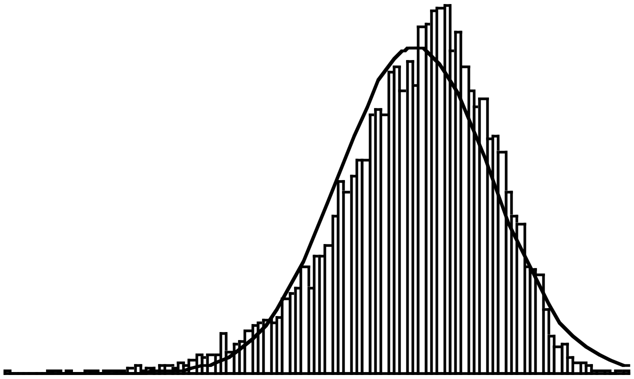
A.



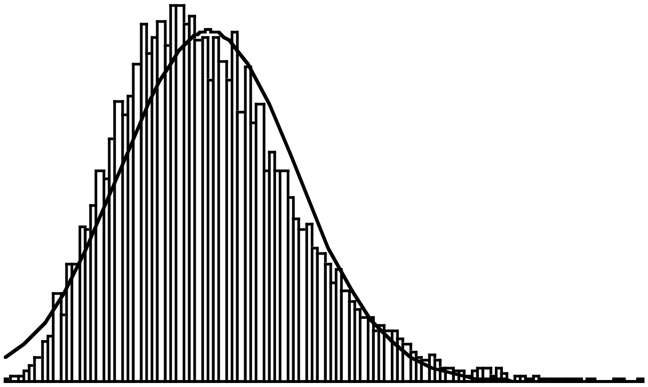
B.



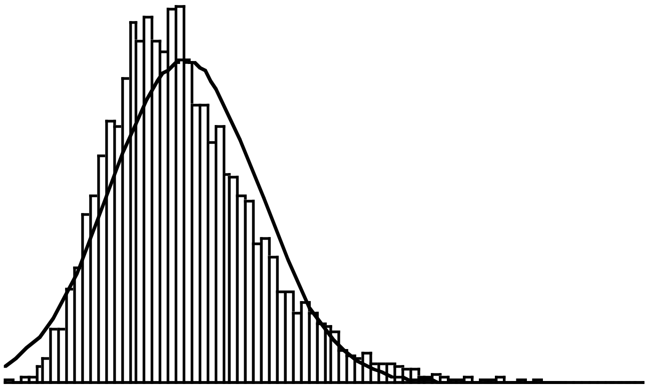
C.



D.

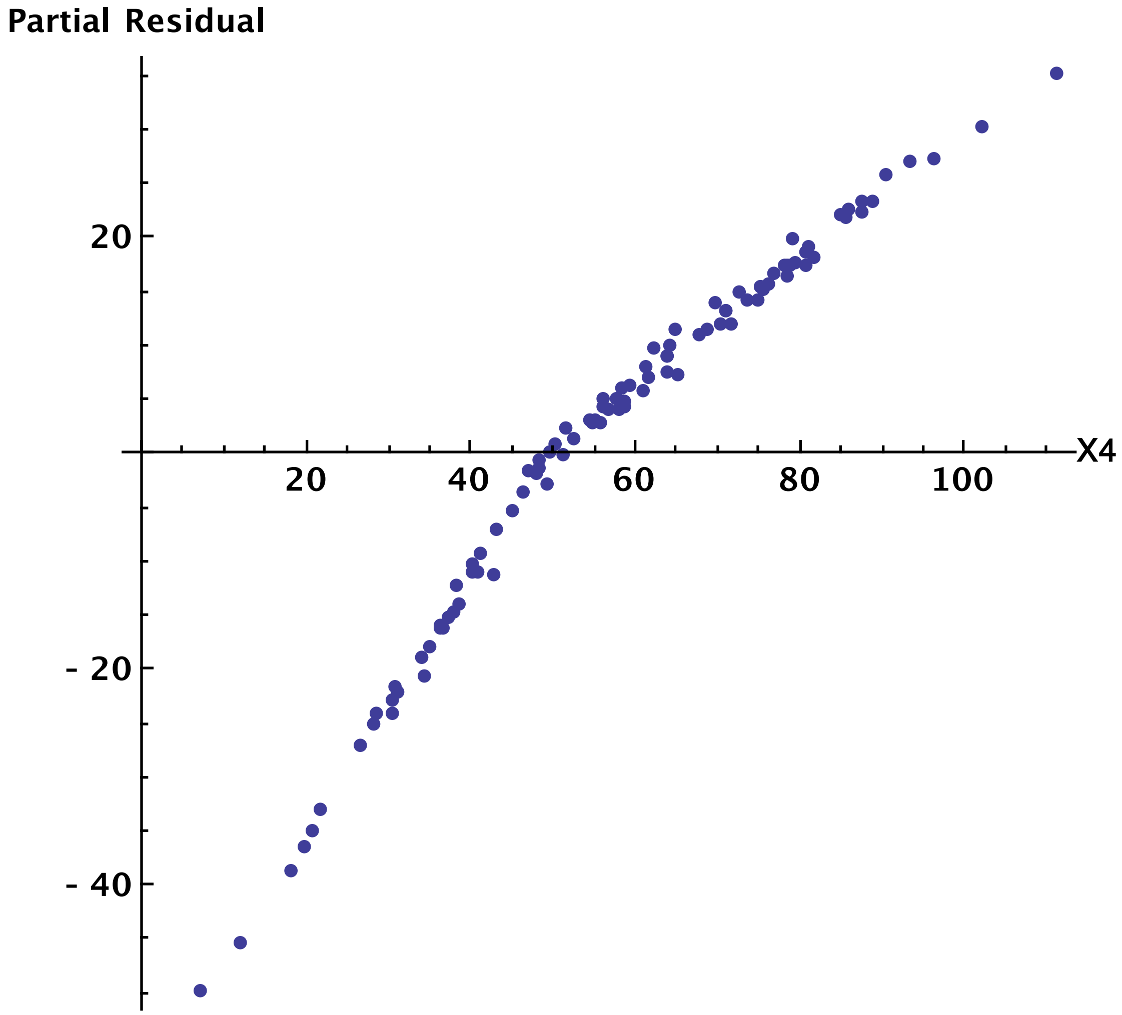


E.



3.39. (2 points) You are constructing a Generalized Linear Model.
(a) (0.5 point) If the model is additive, what link function would you use?
(b) (0.5 point) If the model is multiplicative, what link function would you use?
(c) (0.5 point) If the variance is proportional to the mean, what distribution would you use?
(d) (0.5 point) If the standard deviation is proportional to the mean, what distribution would you use?

3.40. (1 point) For a GLM, here is a partial residual plot for the predictor variable X_4:



Briefly discuss the meaning of this plot.
If necessary, what is a possible solution?

3.41. (1.5 points) With respect to GLMs, discuss the training, validation, and test sets.

3.42. (2 points) Exponential families have a relationship between their mean and variance:
$V(Y_i) = \phi\, V(\mu_i) / \omega_i$, where $V(\mu)$ is the variance function.
List different exponential families and their variance functions.

3.43. (6 points) You are given the following 20 breaking strengths of wires:
500, 750, 940, 960, 1100, 1130, 1150, 1170, 1190, 1240, 1260, 1350, 1400, 1450, 1490, 1520, 1550, 1580, 1850, 2000.
With the aid of a computer, construct a Normal Q-Q Plot.

3.44. (5 points) You have the following data on reported occurrences of a communicable disease in two areas of the country at 2 month intervals:

Months	Area A	Area B
2	8	14
4	8	19
6	10	16
8	11	21
10	14	23
12	17	27
14	13	28
16	15	29
18	17	33
20	15	31

Let X_1 = ln(months). Let X_2 = 0 for Area A and 1 for Area B.

Assume the number of occurrences Y_i are Poisson variables with means μ_i, and

$\ln(\mu_i) = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i}$.

Set up the equations to be solved in order to fit this model via maximum likelihood.

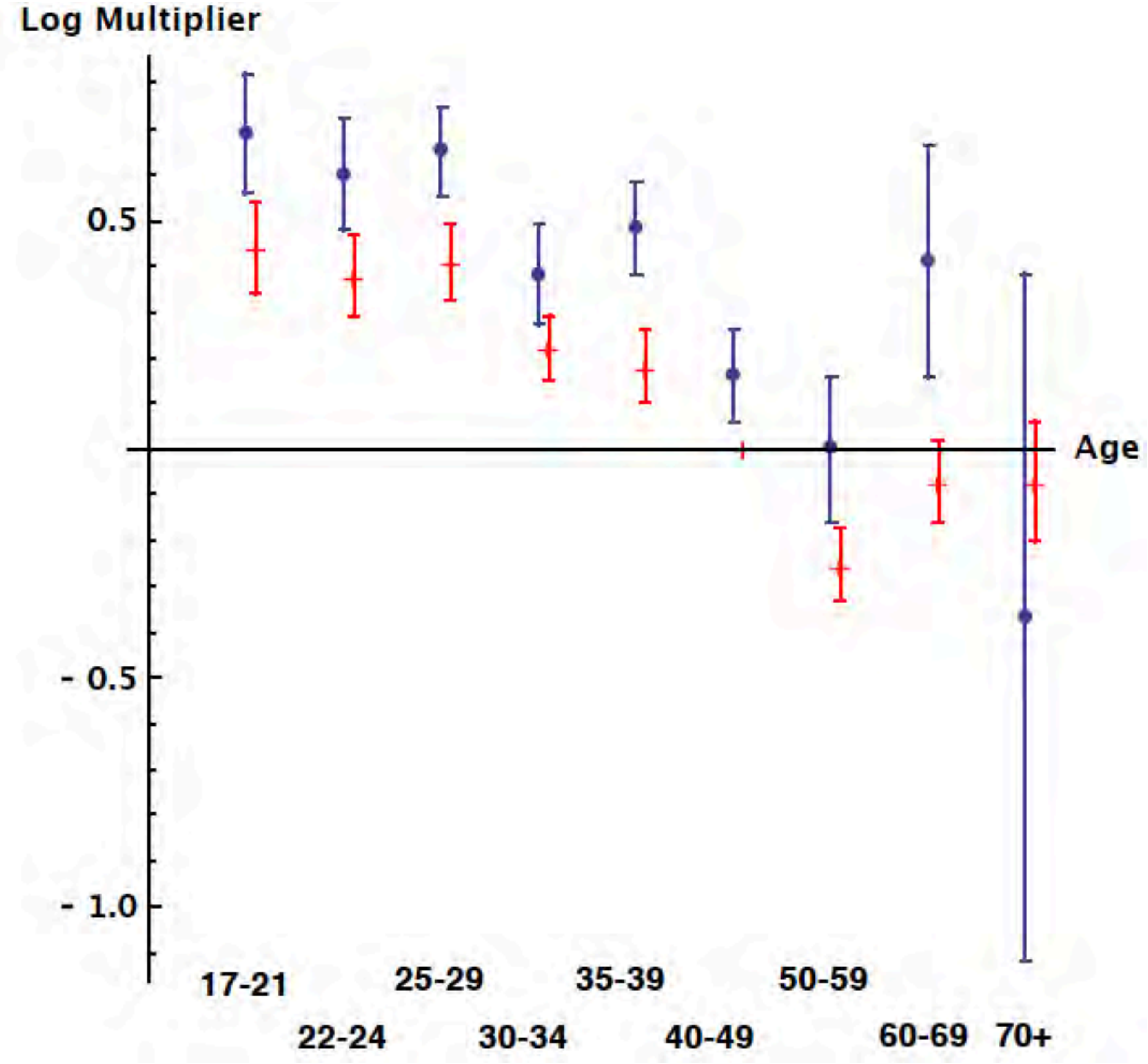
3.45. (1 point) Which of the following statements are true with respect to Generalized Linear Models?

1. Errors are assumed to be Normally Distributed.
2. The link function defines the relationship between the expected response variable and the linear combination of the predictor variables.
3. The use of a log link function assumes the rating variables relate multiplicatively to one another.

3.46. (1.5 points) Generalized Linear Models with a overdispersed Poisson error structure and a log link function have been fit in order to model claim frequency for Homeowners Insurance.
The models use many variables. The homes have been split into four age categories.
A model that uses age has an unscaled deviance of 3306.9,
and an estimated dispersion parameter of 1.83.
An otherwise similar model that does not use age has an unscaled deviance of 3320.2.
The null hypothesis is to use the model that does not include age.
The alternative hypothesis is to use the model that does include age.
Calculate the F-test statistic.
Discuss how you would perform the test.

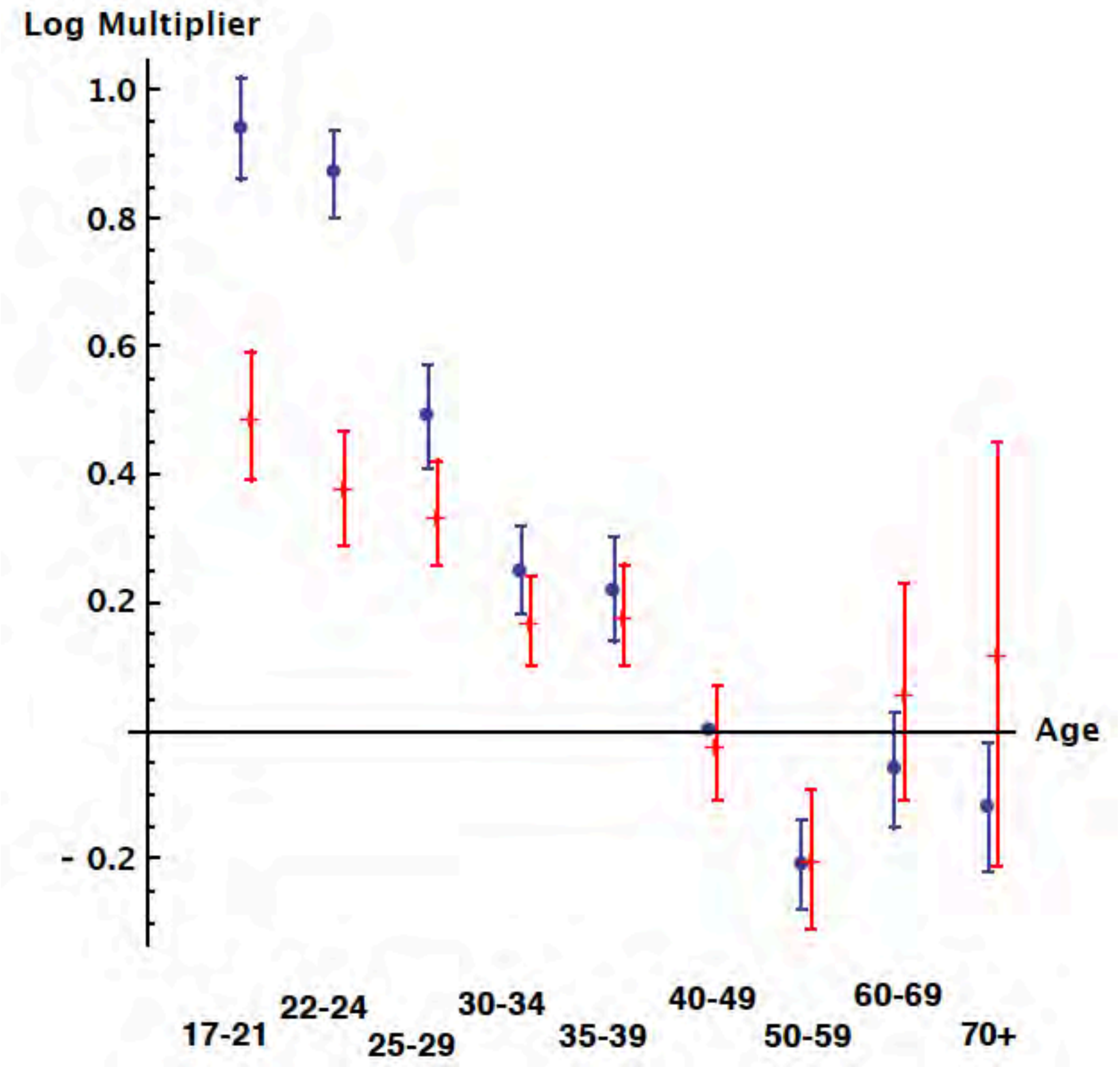
3.47. (1 point) Discuss model lift.

<u>3.48</u>. (1.5 points) The following graph displays the modeled log of the frequency relativity by age for two different frequency of premium payment: yearly in red pluses, and four times a year in blue dots. Also approximate 95% confidence intervals are shown for each case.



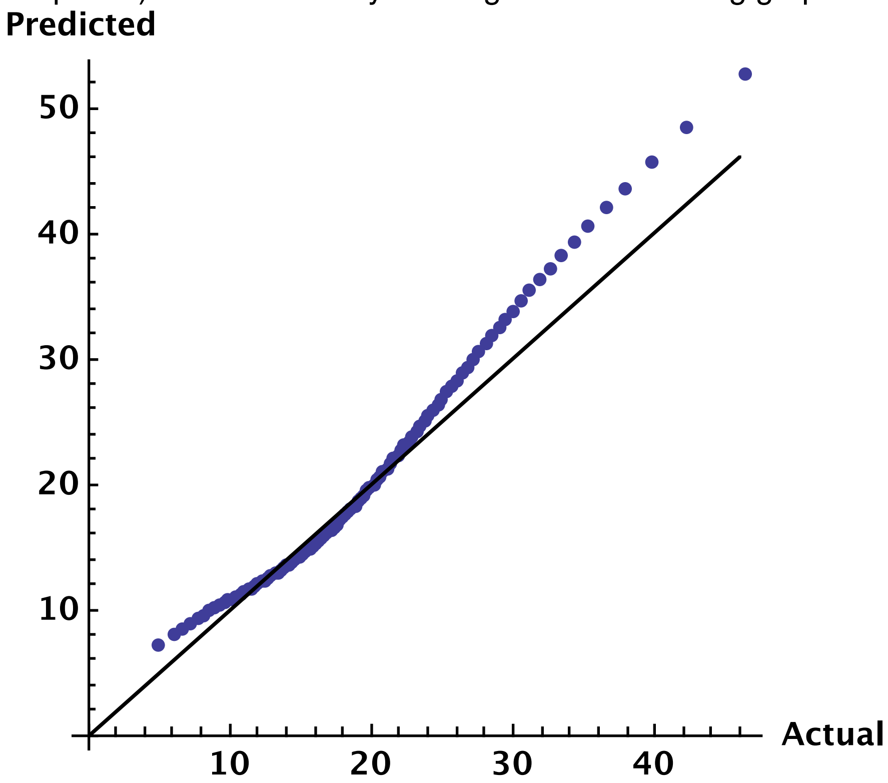
Question continued on the next page.

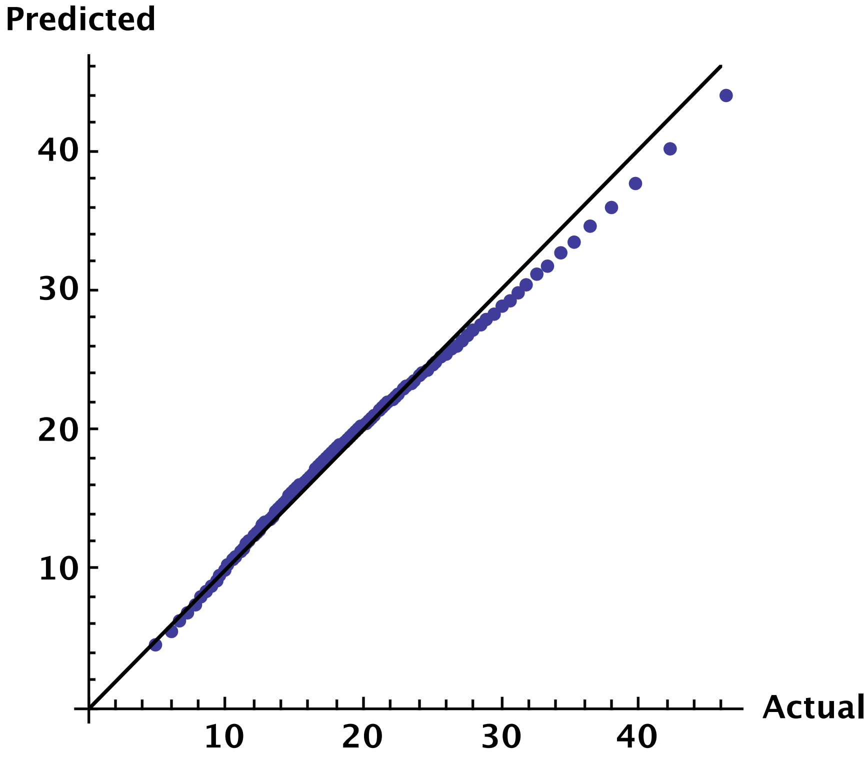
The following similar graph displays the modeled log of the frequency relativity by age for males in blue dots and females in red pluses.
Also approximate 95% confidence intervals are shown for each case.



Briefly compare and contrast the interaction of age of driver and payment frequency with the interaction of age of driver and gender.

3.49. (0.5 points) For two GLMs you are given the following graphs based on holdout data:



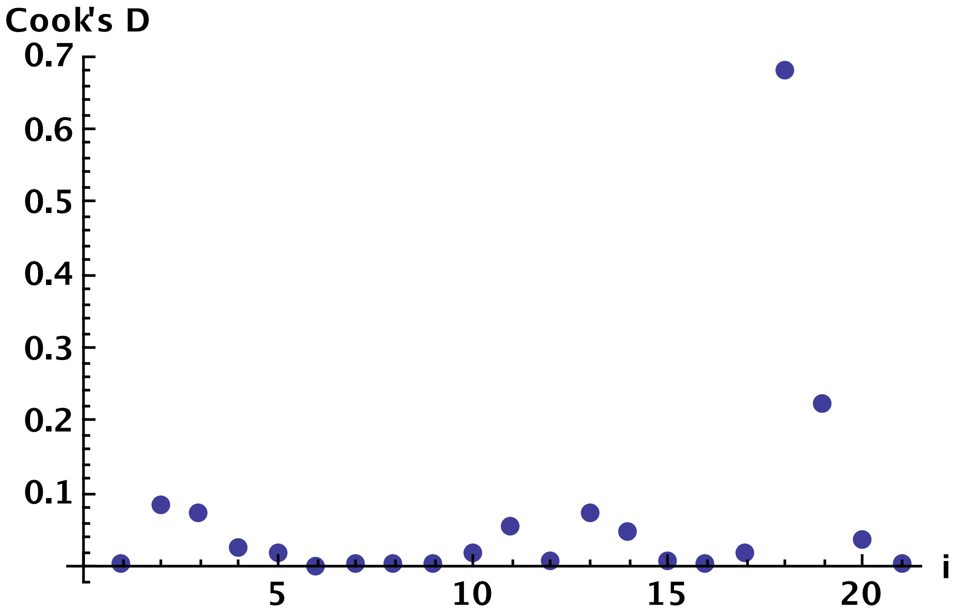


Which model do you prefer and why?

3.50. (2 points) There are three age groups of cars: A, B, C.
There are also three size categories of cars: small, medium, large.
Specify the following structural components of a generalized linear model.
i. Design matrix
ii. Vector of model parameters

3.51. (2 points) Briefly discuss, compare and contrast under-fitting and over-fitting a model.

3.52. (0.5 points) Discuss the following graph of Cook's Distance for 21 observations:



3.53. (2 points)
Use the following information on two Generalized Linear Models fit to the same 100 data points:

Number of Fitted Parameters	Loglikelihood
6	-321.06
7	-319.83

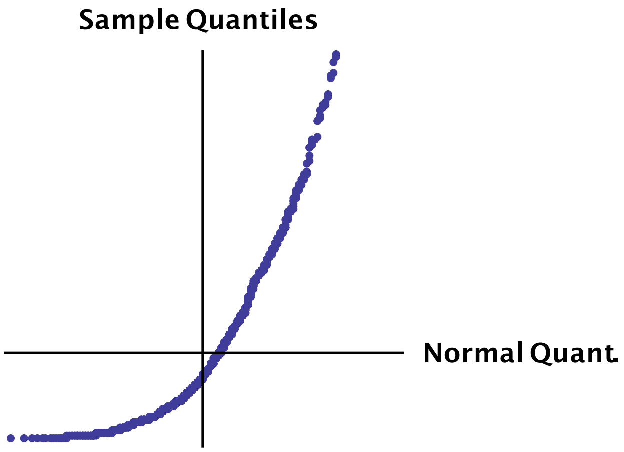
(a) Based on AIC (Akaike Information Criterion), which model is preferred?
(b) Based on BIC (Bayesian Information Criterion), which model is preferred?

3.54.(2 points) A GLM uses a Poisson Distribution.
One of the observations of the response variable is 11.
The corresponding fitted value is 9.5.
Determine the corresponding Deviance residual.

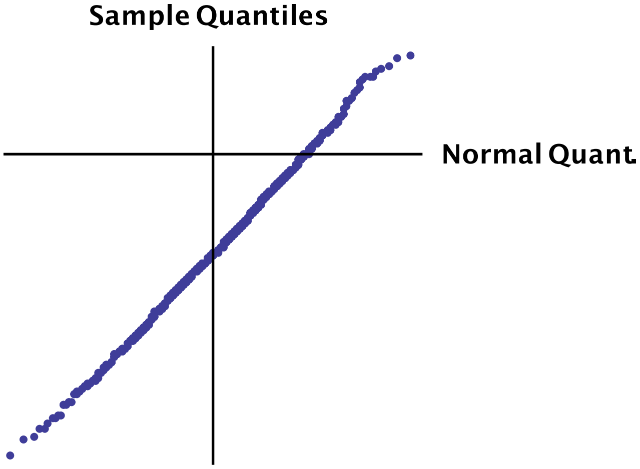
Hint: $D = 2 \sum_{i=1}^{n} \{y_i \ln[y_i / \hat{\mu}_i] - (y_i - \hat{\mu}_i)\}$.

3.55. (1 point) Which of the following Normal Q-Q Plots is most likely to be of data drawn from a Normal Distribution?

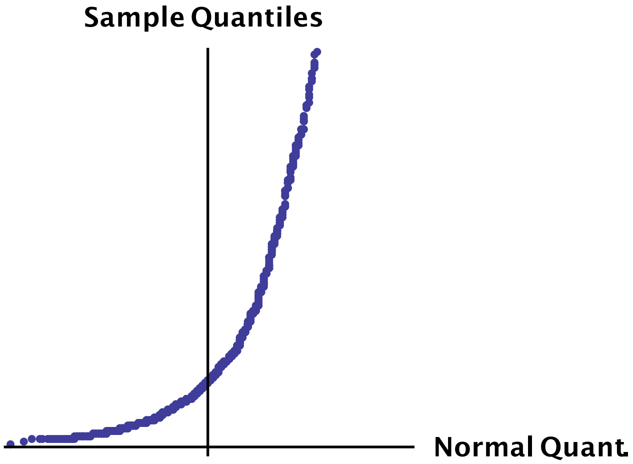
A.



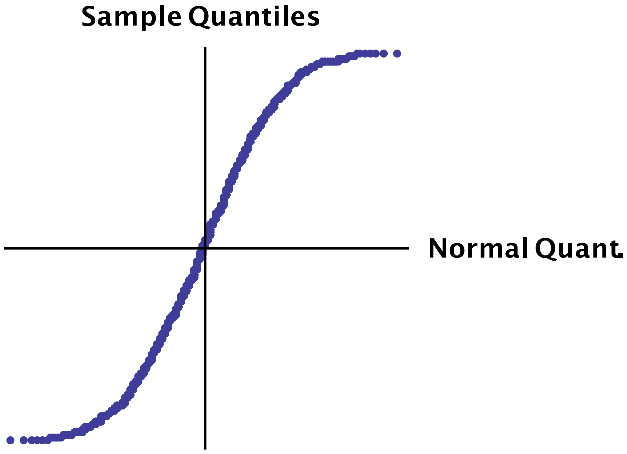
B.



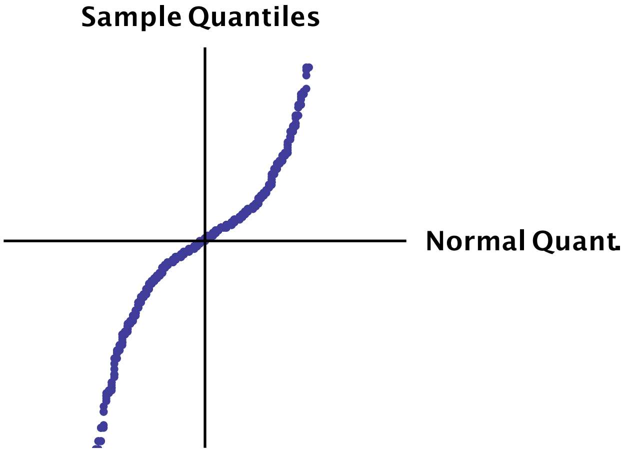
C.



D.



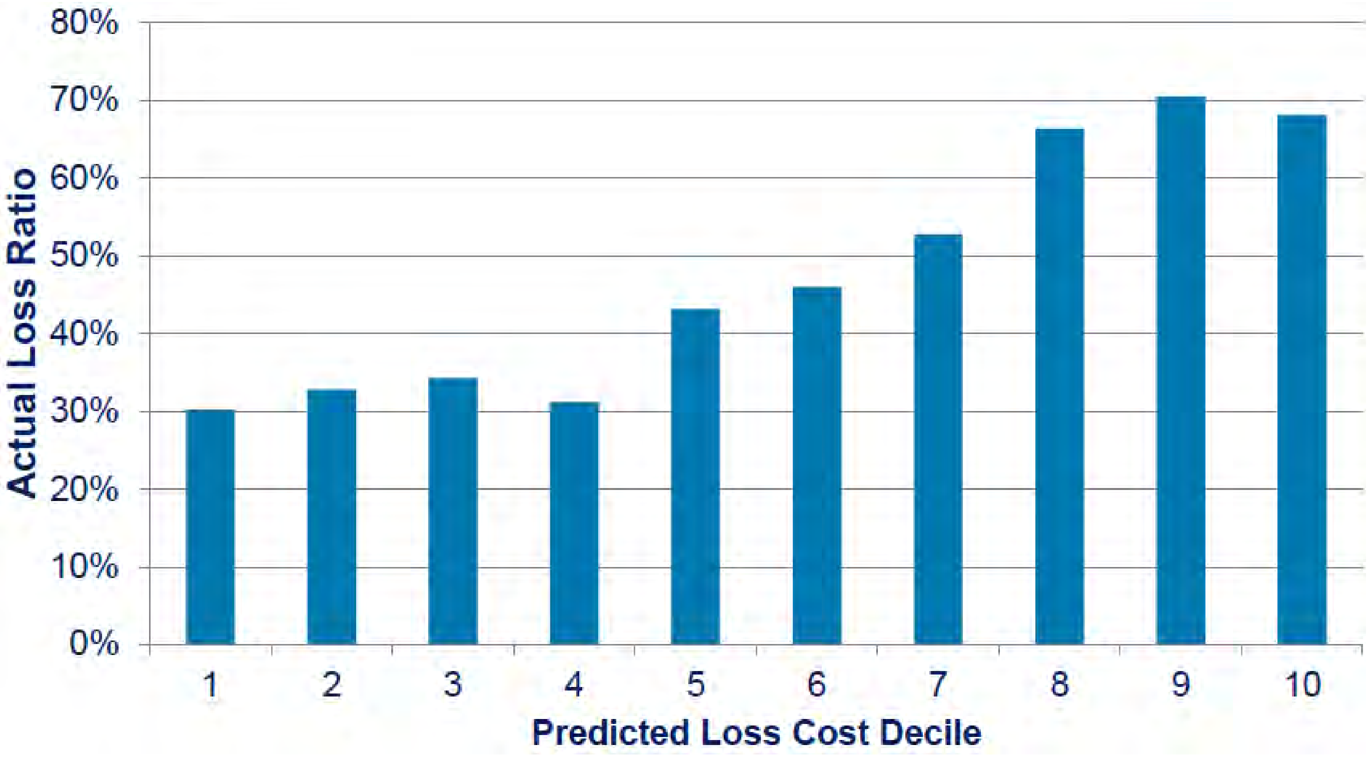
E.



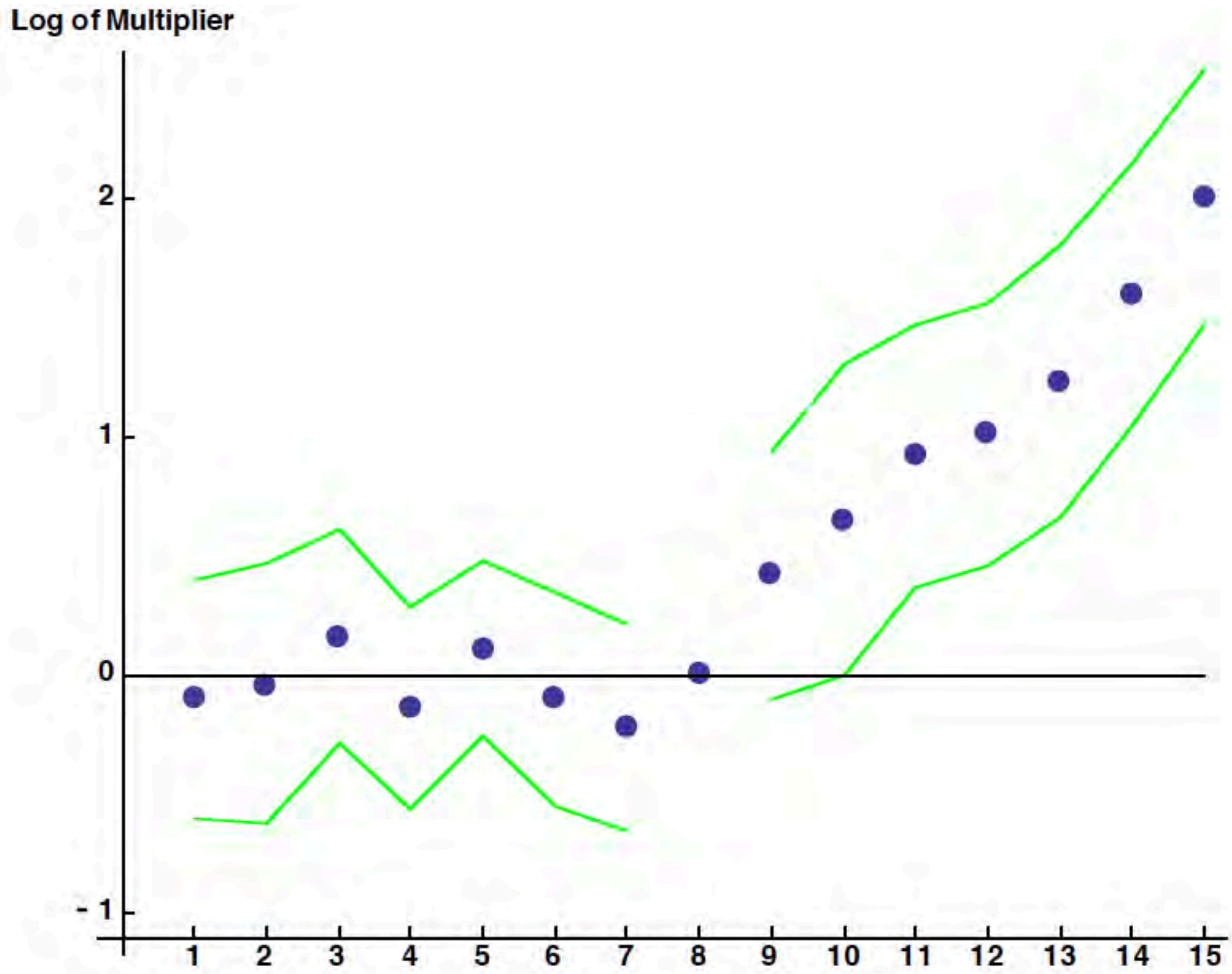
3.56. (2.5 points) For each of the following situations, give the typical generalized linear model form. State the distributional form of the error and link function typically used.
(a) Claim Frequencies.
(b) Claim Counts.
(c) Average Claim Sizes
(d) Probability of Policy Renewal
(e) Pure Premiums

3.57. (0.5 points) You are comparing two rating plans.
The first has a Gini Index of 0.48, while the second has a Gini Index of 0.55.
Which rating plan is preferred?

3.58. (1 point) You are given the following loss ratio chart for a proposed rating plan.
Discuss the lift of the proposed plan compared to the current plan.



3.59. (1 point) Below is a graph of a GLM fit to data, showing the natural log of the fitted multiplicative factors for levels of a variable. (8 is the base level.)
Also shown are approximate 95% confidence intervals.
Briefly discuss what this graph tells the actuary about the fitted model.



3.60. (5 points)
The observed claim frequencies for urban vs rural and male vs female drivers are:

Claim frequency	Urban	Rural
Male	0.200	0.100
Female	0.125	0.050

There are equal exposures in each of the four cells.
We will fit a GLM using a Poisson Distribution.
(a) (2.5 points) For an additive model, determine the maximum likelihood equations to be solved.
(b) (2.5 points) For an multiplicative model, determine the maximum likelihood equations to be solved.

3.61. (1 point) A logistic regression has been fit to some data. For a certain threshold:

		Predicted Claims		
		No	Yes	Total
Actual Claim	No	40,000	10,000	50,000
	Yes	1200	1800	3000
	Total	41,200	11,800	53,000

What point would be plotted in the ROC curve?

Use the following information for the next two questions:

X:	1	5	10	25
Y:	5	15	50	100

Y_1, Y_2, Y_3, Y_4 are independently Normally distributed with means $\mu_i = \beta X_i$, i = 1, 2, 3, 4, and common variance σ^2.

3.62. (2 points) Determine $\hat{\beta}$ via maximum likelihood.

3.63. (3 points) Estimate the standard deviation of $\hat{\beta}$.

3.64. (1.5 points) A GLM is used to model claim size.
You are given the following information about the model:

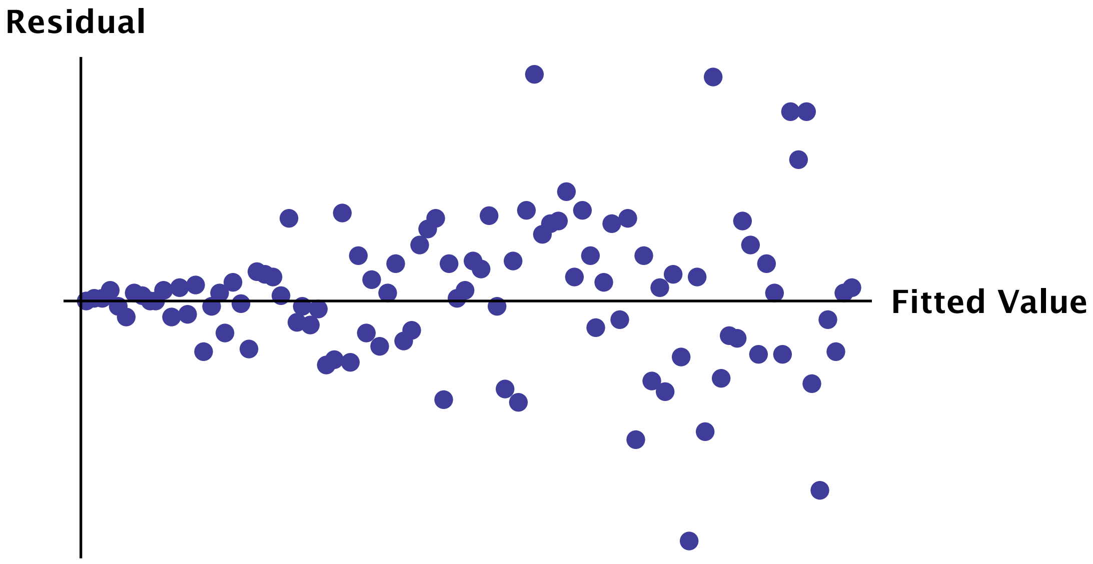
- Claim size follows an Inverse Gaussian distribution.
- Log is the selected link function.
- The dispersion parameter is estimated to be 0.00510.
- Territory and gender are used in the model.
- Selected Model Output:

Variable	$\hat{\beta}$
Intercept	8.03
Territory D	0.18
Gender - Male	0.22

Calculate the standard deviation of the predicted claim size for a male in Territory D.

3.65. (2 points) List four ways that an actuary can analyze the appropriateness of a Generalized Linear Model.

3.66. (1 point) Briefly comment on the following plot of deviance residuals of a model as a function of the fitted values:



3.67. (1 points) You have fit a Generalized Linear Model using an exponential family. What is the scaled deviance?

3.68. (1 point) A GLM has been fit with a log link function.
Age is used, grouped into categories.
Gender is used.
There are categories of Use of Vehicle.
Territories are used.
The expected pure premium for the base is $207.
For the age group 24-26 the coefficient is 0.43.
For Male the coefficient is 0.22.
For Pleasure Use (No Driving to Work) the coefficient is -0.32.
For Territory H the coefficient is 0.36.
Determine the expected pure premium for a male, 24-26 years old, Pleasure Use, in Territory H.

3.69. (1 point) Define and briefly discuss ensemble models.

3.70. (2 points) A GLM uses an Inverse Gaussian Distribution.
One of the observations of the response variable is 288.
The corresponding fitted value is 361.
The estimated θ is 1/121.
Determine the corresponding Deviance residual.

Hint: For the Inverse Gaussian, $D = \theta \sum_{i=1}^{n} \frac{(y_i - \hat{y}_i)^2}{\hat{y}_i^2 \ y_i}$.

3.71. (2 points) A GLM using a Gamma Distribution and a log link function is being used to model severity of personal injury claims. There are 25,000 observations.
3 parameters were fit: an intercept, time until settlement, and whether there is legal representation.
The unscaled deviance is 24,359. A variable is added to the model, equal to the product of the time until settlement and the legal representation variable. (This is an interaction variable.)
The unscaled deviance is now 24,352. The estimated dispersion parameter is 1.22.
Determine whether this additional variable should be added to this model.
You may use the following:
If X follows an F-Distribution with 1 and n degrees of freedom,
then $\sqrt{X}$ follows a t-distribution with n degrees of freedom.
For n large, a t-distribution is approximately a Standard Normal Distribution.
Selected percentiles of the Standard Normal Distribution:

	Values of z for selected values of Pr(Z < z)						
z	0.842	1.036	1.282	1.645	1.960	2.326	2.576
Pr(Z < z)	0.800	0.850	0.900	0.950	0.975	0.990	0.995

3.72. (1.5 points) Fully discuss model stability and some ways to assess it.

3.73. (8 points) You are given 19 data points:
258, 636, 652, 814, 833, 860, 895, 937, 950, 1009,
1020, 1059, 1103, 1113, 1127, 1139, 1246, 1335, 1770.
You wish to compare this data to a Normal Distribution with μ = 1000 and σ = 300.
With the aid of a computer, draw a Q-Q plot.

3.74. (2 points) A GLM uses a Gamma Distribution.
The estimated shape parameter α is 5.
One of the observations of the response variable is 113.
The corresponding fitted value is 102.4.
Determine the corresponding Deviance residual.

Hint: $D = 2\,\alpha \sum_{i=1}^{n} \{-\ln[y_i / \hat{y}_i] + (y_i - \hat{y}_i)/\hat{y}_i\}$.

3.75. (4 points) For private passenger automobile liability claim frequency, you use three factors: gender, age of driver, and territory.
There are 4 levels for driver age, and 3 territories.
A GLM with a log link function is fit.
An intercept term is used.
Let β_1 correspond to the intercept term, β_2 correspond to male,
and assign the other parameters as follows:

Age of driver			Territory	
Factor level	Parameter		Factor level	Parameter
17-21	β_3		A	β_6
22-29	β_4		B	
30-59			C	β_7
60+	β_5			

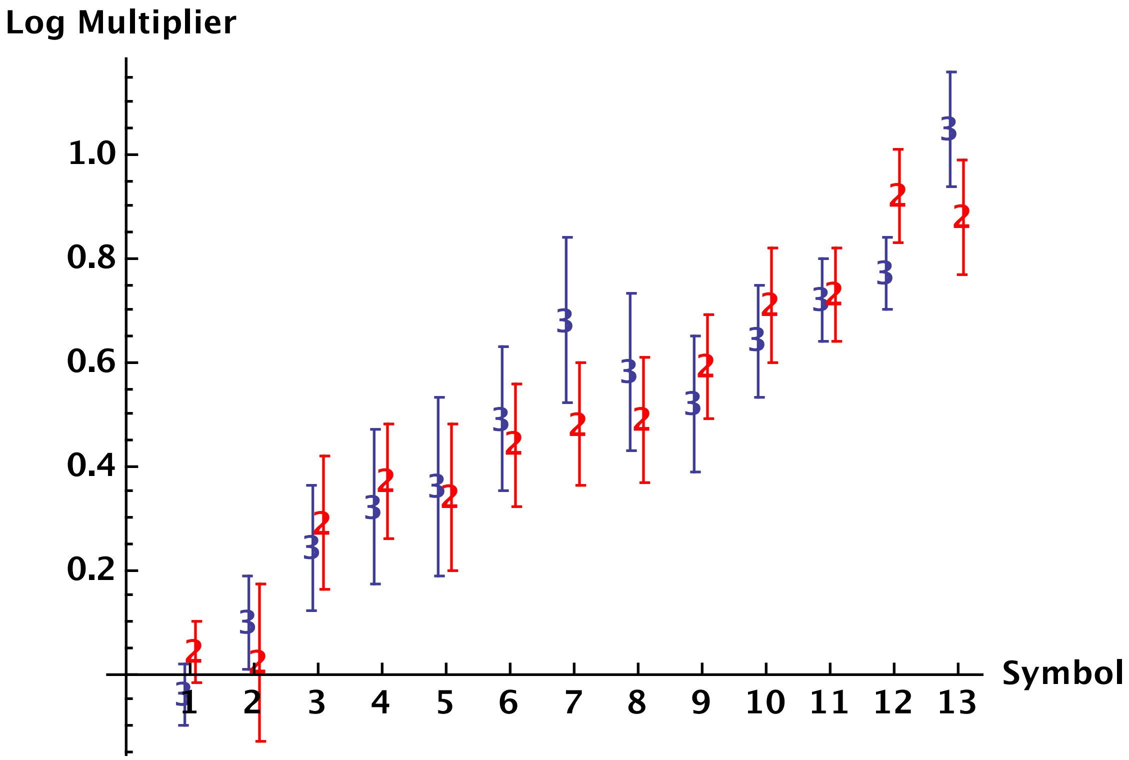
(a) (3 points) What is the design matrix?
(b) (0.5 point) In terms of the fitted parameters, what is the estimated frequency for a 30-59 year old female driver in Territory B?
(c) (0.5 point) In terms of the fitted parameters, what is the estimated frequency for a 22-29 year old male driver in Territory C?

3.76. (1.5 points) Five Generalized Linear Models have been fit to the same set of 200 observations.

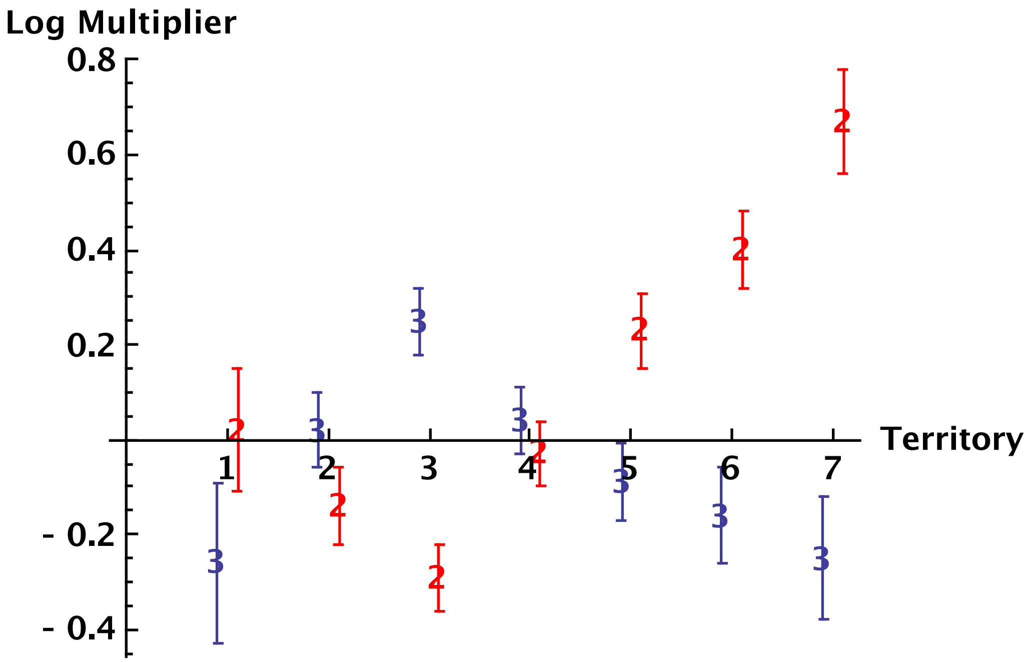
Model	Number of Fitted Parameters	LogLikelihood
A	3	-359.17
B	4	-357.84
C	5	-356.42
D	6	-354.63
E	7	-353.85

Which model has the best AIC (Akaike Information Criterion)?

3.77. (1.5 points) The following graph displays the modeled log of the relativity by vehicle symbol, for a base level of the other predictor variables in a GLM, for two separate years of data. Approximate 95% confidence intervals are shown.



Here is a second similar graph for a different model, by Territory:



Briefly compare and contrast what the two graphs tell the actuary about each model.

3.78. (1.5 points) Before embarking on a GLM modeling project, it is important to understand the correlation structure among the predictors.
Discuss why this is important and what actions may be indicated.

3.79. (1 point) Multiplicative models are the most common type of rating structure used for pricing insurance, due to a number of advantages they have over other structures.
Briefly discuss two advantages of a multiplicative rating structure.

3.80. (1.5 points) A GLM using a Gamma Distribution has been fit for modeling severity of medical malpractice claims. There are 1000 observations.
50 parameters were fit, including an intercept.
It uses gender and 6 categories of age of claimant.
The unscaled deviance is 1120.3 and the estimated dispersion parameter of 0.395.
An otherwise similar GLM excluding gender and age of claimant has an unscaled deviance of 1128.1.
Discuss how you would use an F-Test to determine whether age and gender should be used in this model.

3.81. (1.5 point) Briefly discuss limitations on the use of the loglikelihood and deviance to compare the fit of two GLMs.

3.82. (1 point) An insurer sells “Disgrace Insurance” which covers a business against the possibility that their celebrity spokesperson may engage in disgraceful behavior or expressions.
You are putting together Generalized Linear Models (GLMs) to try to develop a rating algorithm.
Assuming you have plenty of good data, list some variables you would include in your testing of possible GLMs.

3.83. (1 point) Compare and contrast the Gamma and the Inverse Gaussian Distributions.

3.84. (1 point) A GLM uses a Normal Distribution.
One of the observations of the response variable is 71.
The corresponding fitted value is 74.8.
The estimated σ is 23.
Determine the corresponding Deviance residual.

Hint: $D = \frac{1}{\sigma^2} \sum_{i=1}^{n} (y_i - \hat{\mu}_i)^2$.

3.85. (1.5 points) An actuary has historical information relating to personal loan default rates.
A logistic model (GLM with a logit link function) was used to estimate the probability of default for a given customer.
The two variables determined to be significant were the size of loan in thousands of dollars and the credit score of the customer.
β_0 corresponds to the intercept term, β_1 corresponds to size of loan, and

and β_2 corresponds to credit score

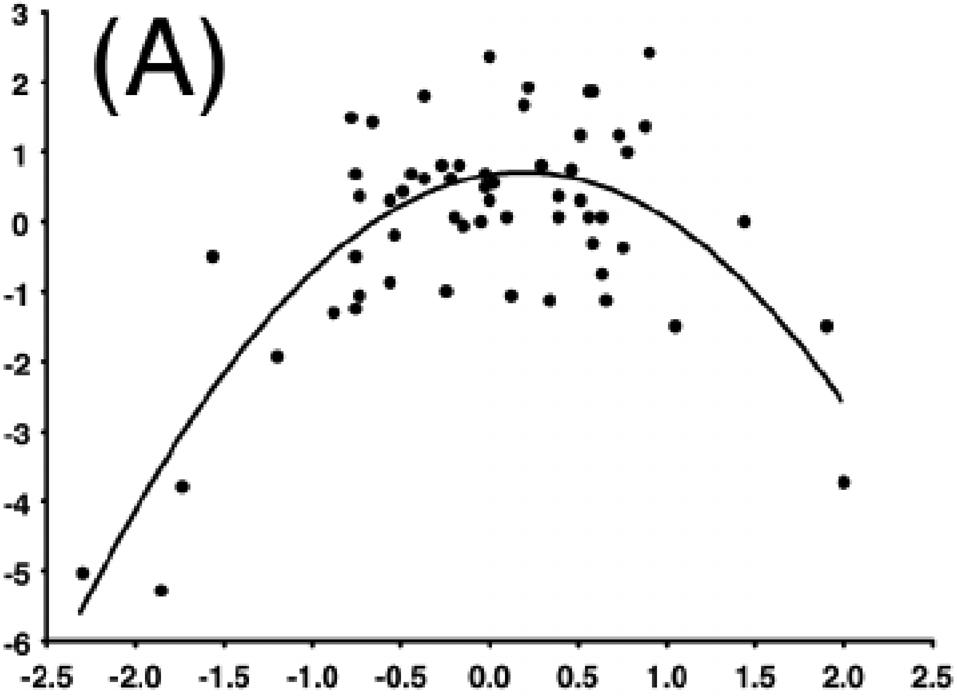
The parameter estimates were determined to be as follows:

β_0	9.5
β_1	0.01
β_2	-0.02

a. (0.75 point) Calculate the estimated default rate for a customer who has credit score of 670 and took out a loan for $180,000.

b. (0.75 point) Calculate the estimated default rate for a customer who has credit score of 760 and took out a loan for $100,000.

3.86. (1 point) For a GLM, here is a partial residual plot for the predictor variable X_1:



Briefly discuss the conclusion from this plot.
If necessary, what is a possible solution?

3.87. (6 points) We model average claim severity by type and horsepower of the car:

- Type: Sedan or SUV
- Horsepower: Low, Medium, or High

We observe an equal number of vehicles of each of the six possible types,
and the observed average claim severities are:

	Sedan	SUV
Low Horsepower	800	1,500
Medium Horsepower	900	1,700
High Horsepower	1,100	2,000

We will fit a GLM using a Gamma Distribution.
(a) (3 points) For an additive model, determine the maximum likelihood equations to be solved.
(b) (3 points) For an multiplicative model, determine the maximum likelihood equations to be solved.

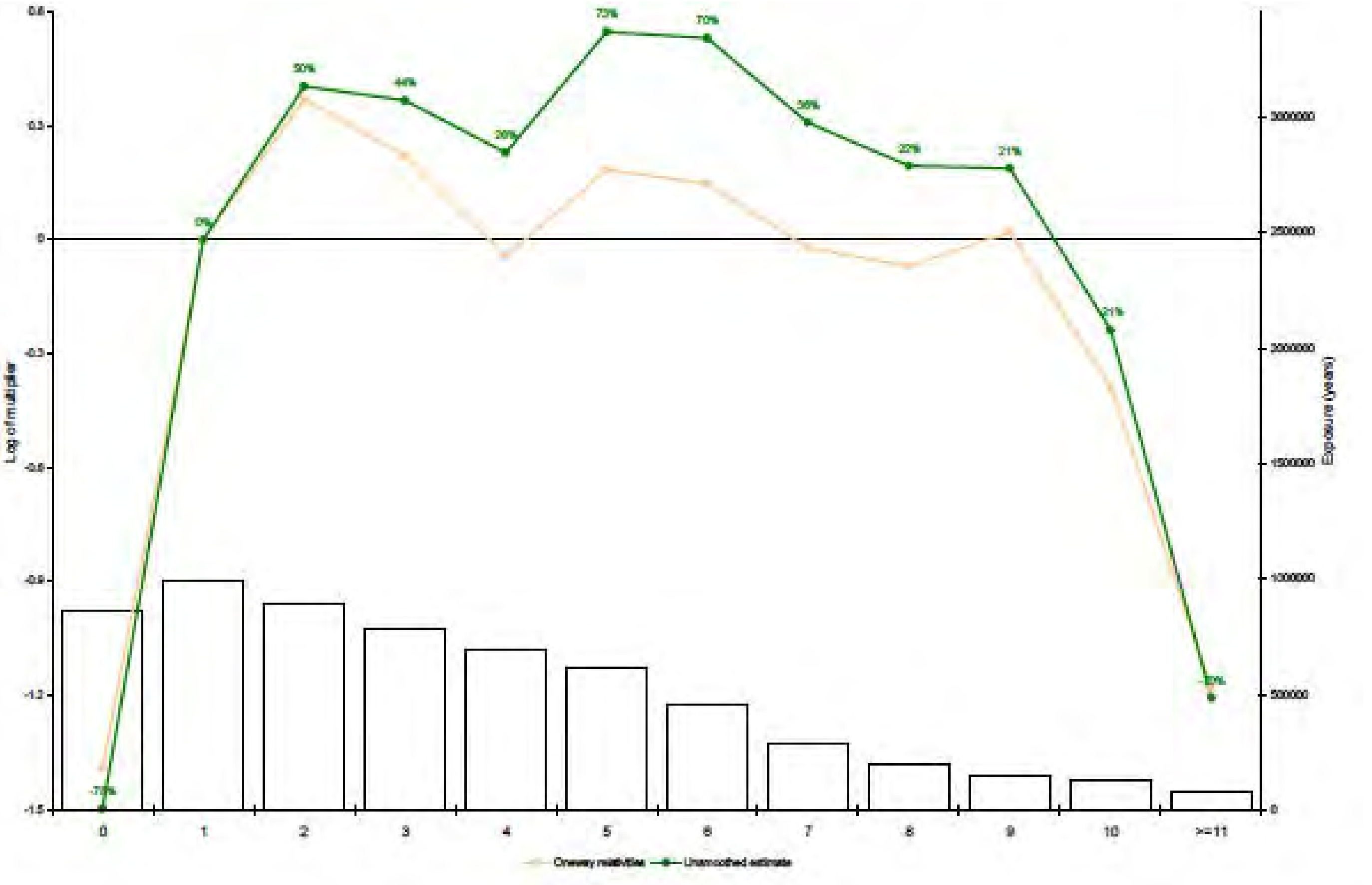
3.88. (2 points) A GLM using an Inverse Gaussian Distribution and an inverse link function is being used to model severity of private passenger automobile property damage liability claims.
There are 2000 observations.
14 parameters including an intercept were fit.
The unscaled deviance is 1848.5.
A categorical variable is added to the model based on vehicle type, with a total of 10 categories.
The unscaled deviance for this more complex model is 1833.0, and the estimated dispersion parameter is 0.93.
Discuss how you would use an F-Test to determine whether vehicle type should be added to this model at the 5% significance level.

3.89. (1.5 points) Five Generalized Linear Models have been fit to the same set of 250 observations.

Model	Number of Fitted Parameters	Scaled Deviance
A	6	1679.1
B	8	1666.4
C	10	1655.9
D	12	1646.2
E	14	1634.5

Which model has the best BIC (Bayesian Information Criterion)?

3.90. (1 point)
A Generalized Linear Model was fit to data on lapse rates for life insurance policies.
Three predictor variables were included in the GLM:
calendar year of exposure, policy duration, and product class.
The graph below displays logs of the relativities by policy duration.
For each band, the black bars at bottom show exposure, quantified on the righthand axis.
The GLM results are in green, and are relative to the base level for policy duration.
The yellow line (lighter line) is what would have been generated by a 'one-way' analysis: i.e., considering just policy duration, without any other factors.



Briefly discuss a likely reason why the green and yellow lines differ.

3.91. (0.5 points) A continuous predictor x_2 has a coefficient of $\beta_2 = -0.3$ in a logistic model.
For a unit increase in x_2, what is the estimated change in the odds?

3.92. (1 point) We are fitting a GLM to private passenger automobile liability pure premiums.
Female drivers age 31 to 59 in a rural territory may have observed pure premiums higher or lower than their fitted values.
Unmarried male drivers age 17 to 21 in an urban territory may have observed pure premiums higher or lower than their fitted values.
Contrast the effect on fitting the GLM of the modeling errors from these two groups.

3.93. (1 point) A logistic regression has been fit to some data. For a certain threshold:

		Predicted Fraud		
		No	Yes	Total
Actual Fraud	No	70,000	10,000	80,000
	Yes	3000	2000	5000
	Total	73,000	12,000	85,000

What point would be plotted in the ROC curve?

3.94. (1 point) How would the standard error help to analyze the results of fitting a Generalized Linear Model (GLM)?

3.95. (1 point) For a rating plan, briefly discuss how to construct a Lorenz Curve and how to compute the Gini Index.

3.96. (4 points) Assume a set of three observations:
For $z = 1$, we observe 4. For $z = 2$, we observe 7. For $z = 3$, we observe 8.
Fit to these observations a Generalized Linear Model with a Poisson Distribution and a log link function. In other words, assume that each observation is a Poisson random variable, with mean λ and $\ln(\lambda) = \beta_0 + \beta_1 z$.

3.97. (1 point) In addition to statistical significance, give other considerations for variable selection.

3.98. (3.5 points) A personal auto class system has three class dimensions:
- Sex: Male vs female
- Age: Youthful vs adult vs retired
- Territory: Urban vs suburban vs rural

An actuary sets rate relativities from the experience of 20,000 cars.
- Urban is the base level in the territory dimension.
- Adult is the base level in the age dimension.
- Male is the base level in the sex dimension.

a. (0.5 point) How many elements does the vector of covariates have in a multiplicative model?
b. (0.5 point) How many elements does the vector of covariates have in an additive model?
c. (1 point) Specify each element of the vector of parameters, with $\beta_0 \Leftrightarrow$ the base class.
d. (0.5 point) How many columns does the design matrix have?
e. (0.5 point) How many rows does the design matrix have if each record is analyzed separately?
f. (0.5 point) For grouped data, how many rows does the design matrix have?

3.99. (2 points) Answer the following with respect to deviance residuals of a GLM.
(a) (0.5 points) Define the deviance residual.
(b) (0.5 points) Give an intuitive interpretation of deviance residuals.
(c) (1 point) Discuss how deviance residuals can be used to check the fit of a model.

3.100. (4 points) You have the following data on the renewal of homeowners insurance policies with the ABC Insurance Company:

Number of Years Insured	Number of Policies	Number of Policies Renewed
1	1000	900
2	900	820
3	800	740
4	700	660
5	600	580

Let X = number of years insured with ABC Insurance Company.
A Generalized Linear Model using a Binomial Distribution with a logit link function will be fit to this data, including an intercept term.
Determine the equations to be solved in order to fit this model via maximum likelihood.

3.101. (0.5 points) The variance of a distribution from the exponential family can be expressed using the following formula: $\mathrm{Var}(y_i) = \frac{\phi\ V(\mu_i)}{\omega_i}$.

Define the parameters ϕ and ω_i in the formula above.

Use the following information for the next five questions:

X		2	5	8	9
Y		10	6	11	13

Y_1, Y_2, Y_3, Y_4 are independently Normally distributed with means $\mu_i = \beta_0 + \beta_1 X_i$, i = 1, 2, 3, 4, and common variance σ^2.

3.102. (2 points) Determine $\hat{\beta}_1$ via maximum likelihood.

3.103. (2 points) Determine $\hat{\beta}_0$ via maximum likelihood.

3.104. (2 points) Determine $\hat{\sigma}$ via maximum likelihood.

3.105. (3 points) Estimate the standard deviation of $\hat{\beta}_1$.

3.106. (3 points) Estimate the standard deviation of $\hat{\beta}_0$.

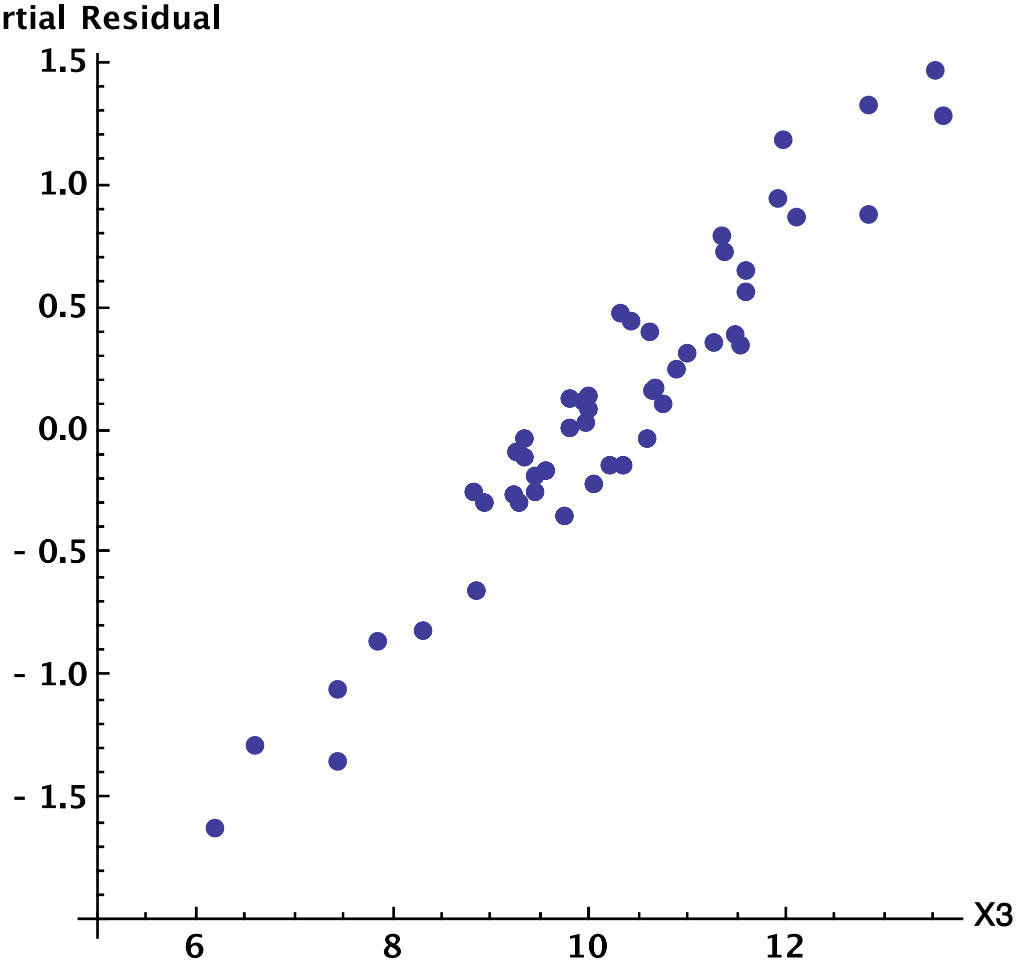
3.107. (1 point) Five Generalized Linear Models have been fit to the same set of observations.
Each model uses the same number of parameters.
Which of these models is preferred?

Model	Scaled Deviance
A	3609.5
B	3611.0
C	3606.3
D	3602.1
E	3605.8

3.108. (1 point) Discuss the overdispersed Poisson Distribution.

3.109. (1 point) A common statistical rule of thumb is to reject the null hypothesis where the p-value is 0.05 or lower. Is this appropriate for a typical insurance modeling project?
Why or why not?

3.110. (1 point) For a GLM, here is a partial residual plot for the predictor variable X_3:



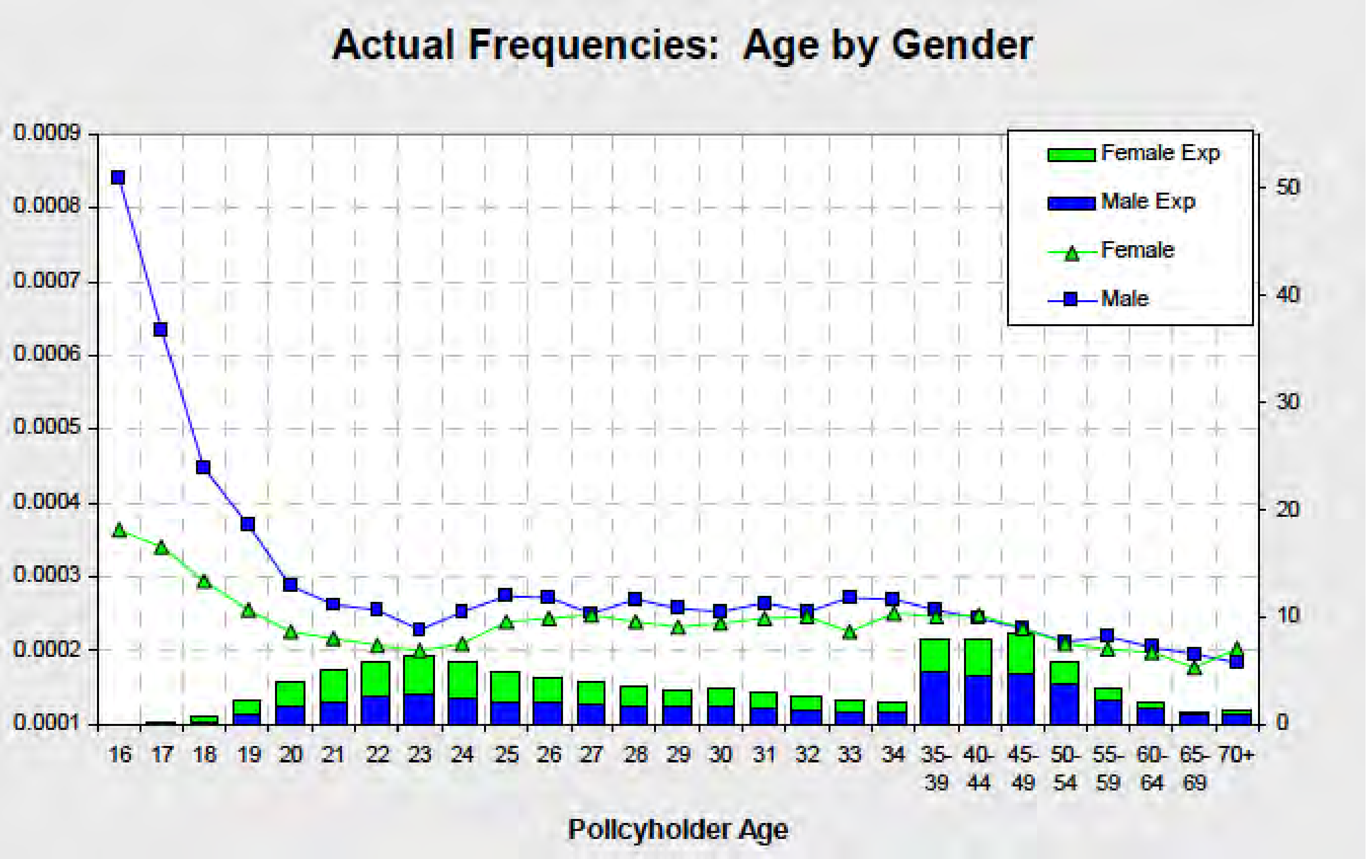
Briefly discuss the meaning of this plot.
If necessary, what is a possible solution?

3.111. (0.5 points) With respect to GLMs, briefly discuss multicollinearity.

3.112. (1 point) An actuary is determining the rates by class and territory.
With respect to GLMs, briefly discuss determining territory relativities.

3.113. (1 point)
Define a holdout sample of data, and briefly discuss how it can be used in GLM validation.

3.114. (1 point) The following graph shows claim frequency for private passenger automobile insurance by gender and age. (The rectangles represent the number of exposures.)



Briefly discuss the implications for modeling frequency via a Generalized Linear Model.

3.115. (2 points) Using Generalized Linear Models, an actuary Edward Conners has developed a policy renewal model for private passenger automobile insurance written by the Some States Insurance Company. There are two predictor variables:

z_1 = the number of years the insured has been with Some States.

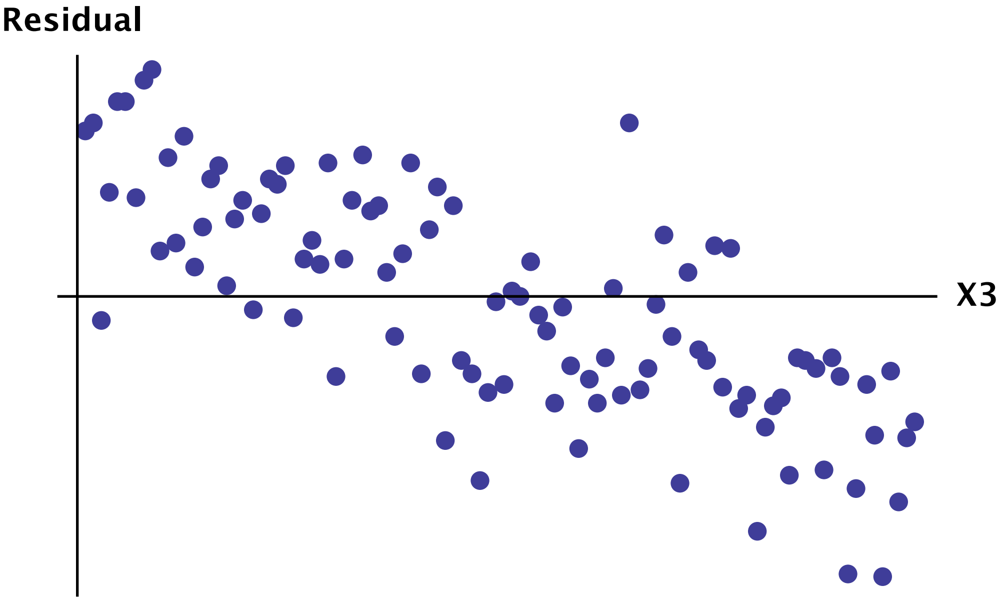
z_2 = the age of the principal operator of the vehicle.

The predicted probability of policy renewal is: $\dfrac{\text{Exp}[0.6 + 0.05\ z_1 + 0.02\ z_2]}{1 + \text{Exp}[0.6 + 0.05\ z_1 + 0.02\ z_2]}$.

(a) For a principal operator who is 30 years old, what is the multiplicative relativity of 1 year with Some States compared to 10 years with Some States?

(b) For a principal operator who is 50 years old, what is the multiplicative relativity of 1 year with Some States compared to 10 years with Some States?

3.116. (1 point) Briefly comment on the following plot of deviance residuals of a model as a function of a predictor variable X_3:



3.117. (6 points) You are given the following information on the labor force participation of 10 married women between the ages of 25 and 35:

Child of Age 6 or Less	Years of Education	Participating in the Labor Force
No	12	Yes
No	14	No
No	15	Yes
No	16	No
No	17	Yes
Yes	10	No
Yes	11	No
Yes	13	Yes
Yes	15	No
Yes	16	Yes

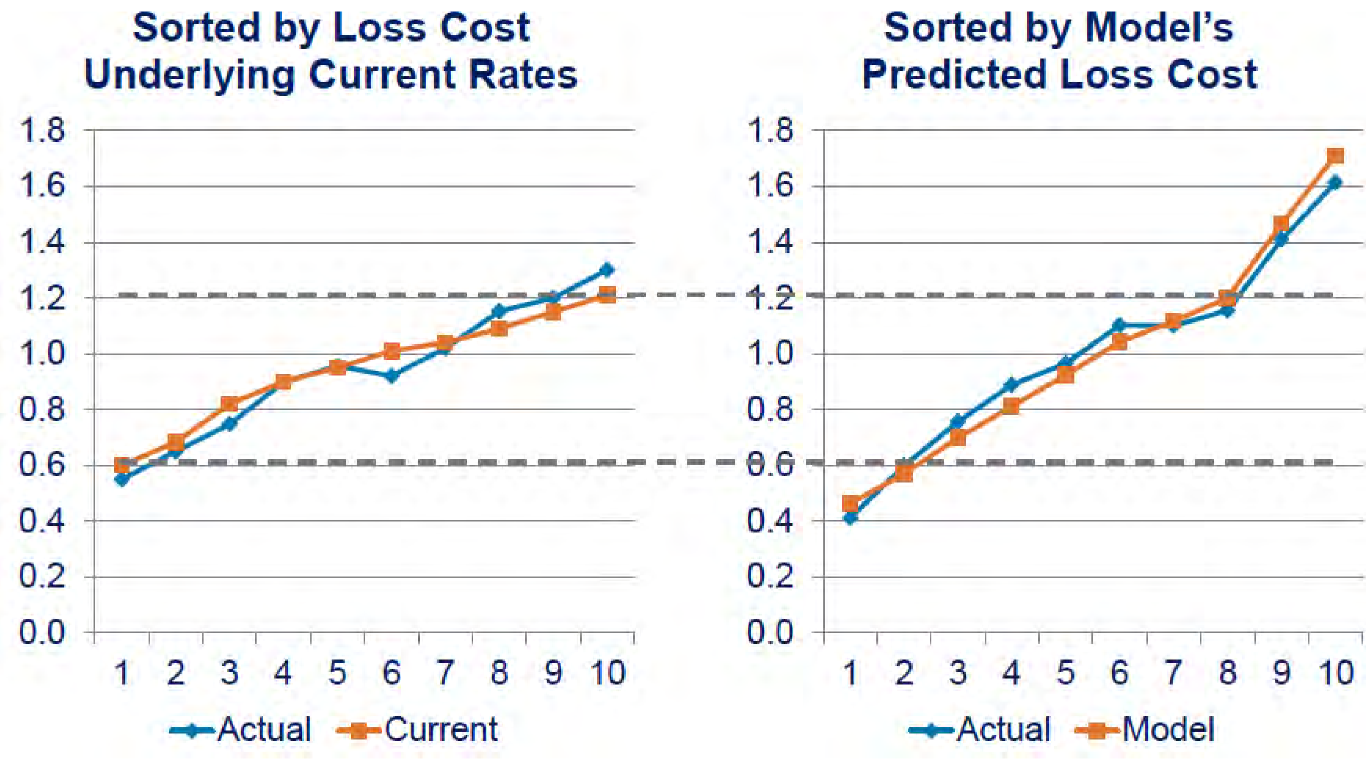
A Generalized Linear Model using a Binomial Distribution with a logit link function will be fit to this data, including an intercept term.

a. (1 point) What are the design matrix and the response vector?

b. (5 points) Determine the equations to be solved in order to fit this model via maximum likelihood.

3.118. (1 point) Les N. DeRisk is an actuary. Les has scrubbed and adjusted the data he will be using for classification ratemaking for a certain line of insurance.
Les will run a Generalized Linear Model. List 3 things Les has to specify.

3.119. (1.5 points) You are given two simple quantile plots, one sorted by the current plan and one sorted by a proposed plan.
Discuss the lift of the proposed plan compared to the current plan.



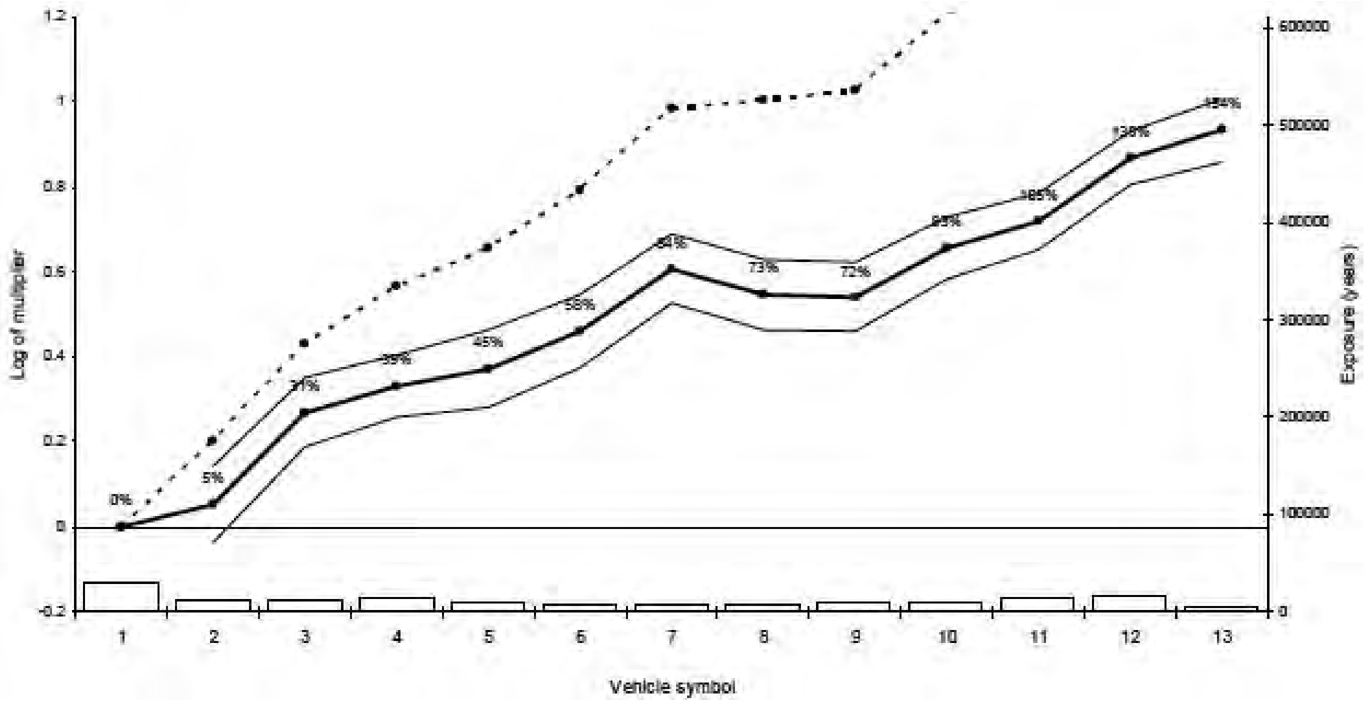
3.120. (0.5 point) Give an example of a situation where a GLM with a Binomial distribution and logit link function would be used.

Use the following information for the next two questions:
- A GLM using a Gamma Distribution and a log link function has been fit for modeling severity of auto claims.
- The explanatory variables are: x_1 driver age, and x_2 marital status where 1 = married.
- The fitted coefficients are: $\beta_0 = 8.80$, $\beta_1 = -0.03$, $\beta_2 = -0.15$.
- The estimated $\phi = 0.3$.

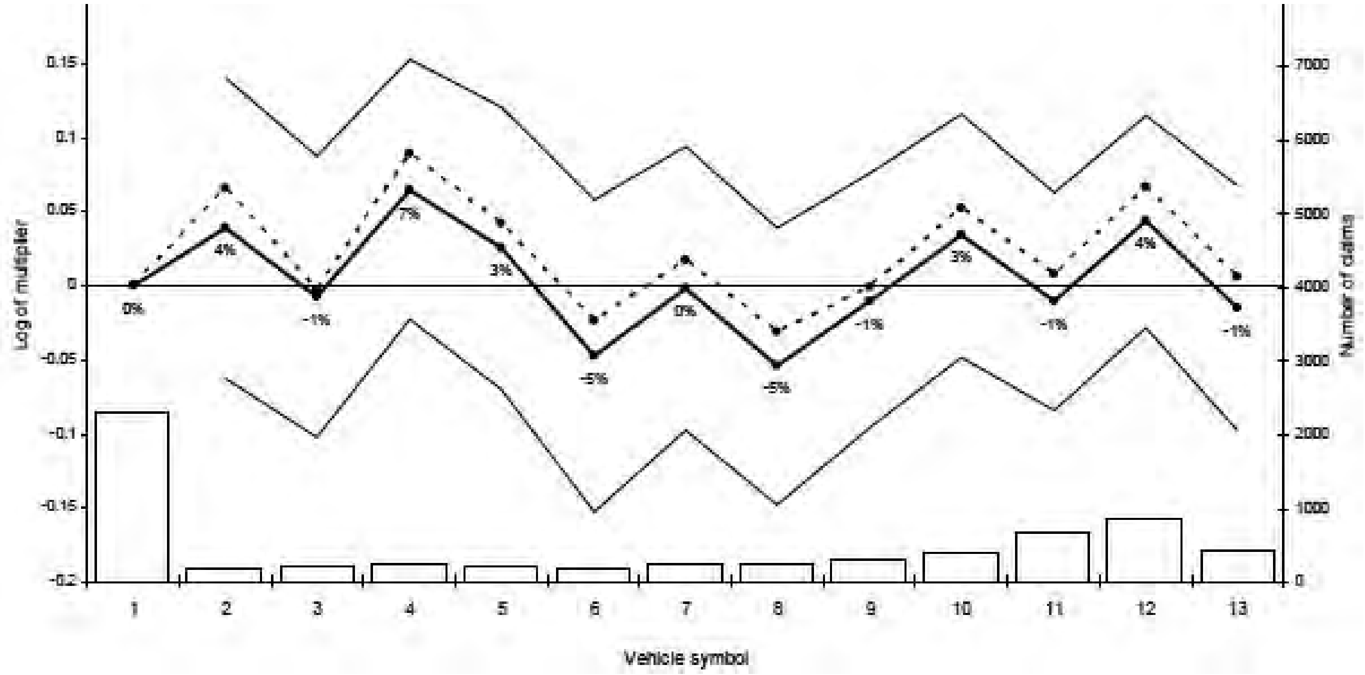
3.121. (1 point) Determine the estimated mean severity for a 30 year old married driver.

3.122. (1 point) Determine the estimated variance of severity for a 40 year old unmarried driver.

3.123. (2 points) The following graph displays the modeled log of the relativity by vehicle symbol, for a base level of the other predictor variables in a GLM.
The bold line shows the fitted parameter estimates.
Lines indicates two standard errors on either side of the parameter estimate.
The dotted line show the relativities implied by a simple one-way analysis. The distribution of exposure for all business considered is also shown as a bar chart at the bottom.



Here is a second similar graph for a different model.



Briefly compare and contrast what the two graphs tell the actuary about each model.

3.124. (1 point) For a line of insurance, an actuary fits separate GLMs to different perils. Discuss one way to combine separate models by peril in order to get a model for all perils.

3.125. (2 points) Claim counts for private passenger automobile insurance are Poisson.
The mean frequency, m, depends on age and gender.
Briefly discuss and contrast the following two models, where x is age.
(a) (1 point) $\log(\mu) = \alpha_i + \beta x$, where α_1 and α_2 depend on gender.
(b) (1 point) $\log(\mu) = \alpha_i + \beta_i x$, where α_1, α_2, β_1, and β_2 depend on gender.

3.126. (2 points) A GLM uses a Binomial Distribution.
For m = 8, an observation of the response variable is 3.
The corresponding fitted value q is 0.2.
Determine the corresponding deviance residual.

Hint: $D = 2 \sum_{i=1}^{n} \{y_i \ln[\frac{y_i}{\hat{y}_i}] + (m_i - y_i) \ln[\frac{m_i - y_i}{m_i - \hat{y}_i}]\}$.

3.127. (1 point) List and briefly discuss two potential drawbacks of using using piecewise linear functions (hinge functions) in GLMs.

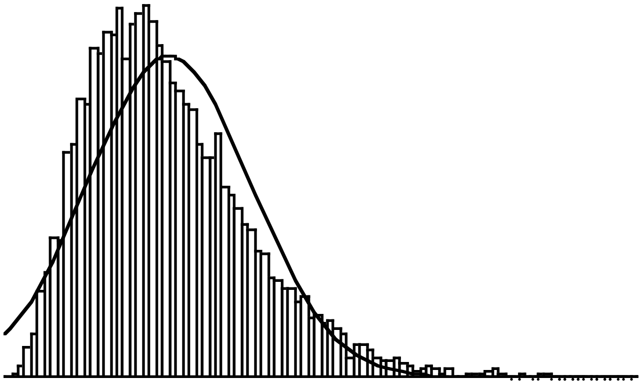
3.128. (1.5 points) An insurer uses a GLM for classification ratemaking.
The insurer is considering using a different GLM instead.
You are given the following data on five insureds.

Insured	Actual Loss Cost	Loss Cost Predicted by Proposed Model	Earned Premium at Present Rates
1	28,000	26,000	43,000
2	25,000	32,000	49,000
3	42,000	37,000	57,000
4	36,000	43,000	61,000
5	48,000	41,000	66,000

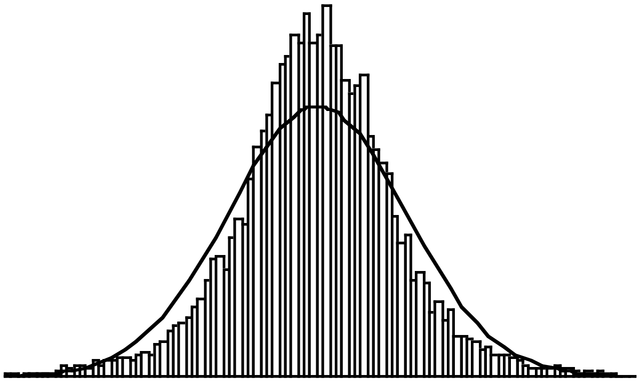
Construct a Loss Ratio Chart that management can use to assess lift.
Identify the basis of sorting the data.

3.129. (1 point) The following a histograms of deviance residuals for GLMs.
Which of the following histograms represents the best model?

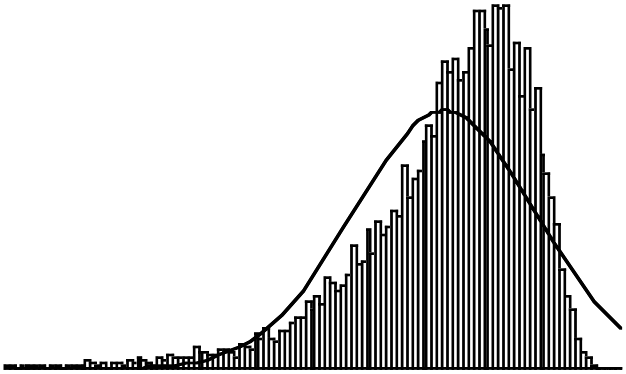
A.



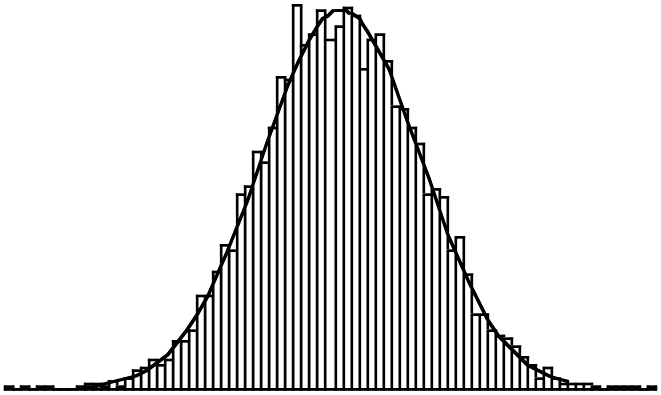
B.



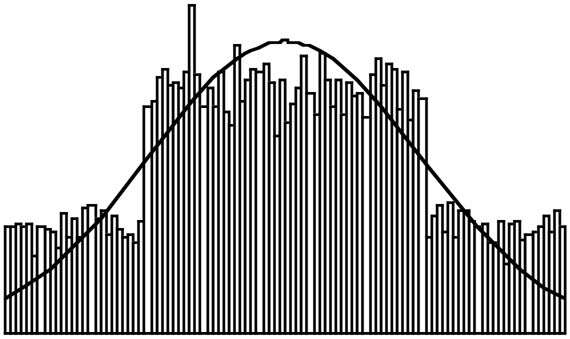
C.



D.



E.



3.130. (1 point) Geoff Linus Modlin is an actuary using Generalized Linear Models (GLMs) to determine classification rates for private passenger automobile insurance.
Geoff notices that the relativity for drivers aged 19 is different between two GLMs based on the same data. Briefly discuss why that can be the case.

3.131. (1 point) You observe 36 monthly returns on a stock.
The 9th value from smallest to largest is 0.004.
What is the corresponding point in the Normal Q-Q Plot?

3.132. (1.5 points) With respect to GLMs, fully discuss variance inflation factors (VIF).

3.133. (1.5 points) Dollar Bill Bradley, an actuary at the Knickerbocker Insurance Company, has fit a Generalized Linear Model with a overdispersed Poisson error structure and a log link function in order to model claim frequency for automobile liability insurance.
His model has a unscaled deviance of 2196.1 and estimated dispersion parameter of 2.22.
Bill now introduces into the model an additional categorical variable with five categories:
1. Insured has homeowners insurance with Knickerbocker.
2. Insured has homeowners insurance with another insurer.
3. Insured has renters insurance with Knickerbocker.
4. Insured has renters insurance with another insurer.
5. Other
With this additional variable, the model has a unscaled deviance of 2179.3 and estimated dispersion parameter of 2.09.
The null hypothesis is to use the simpler model.
The alternative hypothesis is to use the more complicated model.
Determine the F-test statistic and discuss how you would perform the statistical test.

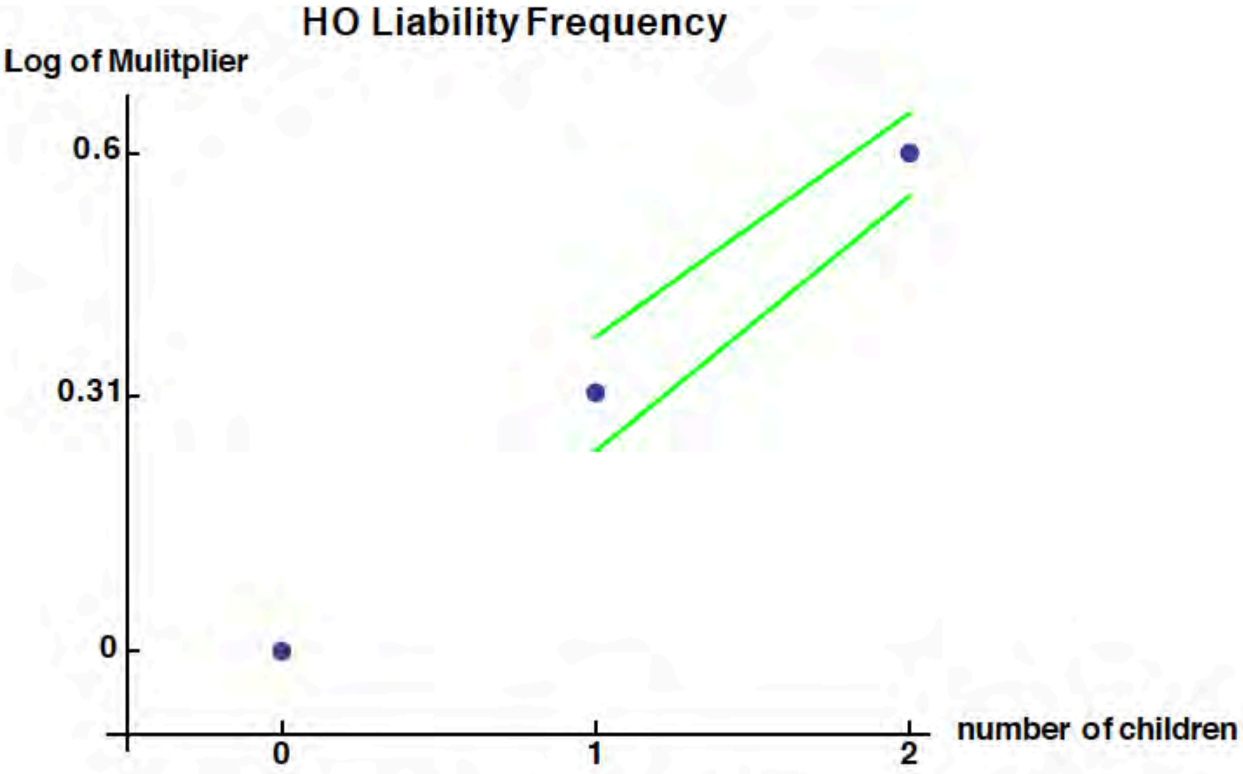
3.134. (1.5 points) Fully discuss cross validation as used with GLMs, including any limitations

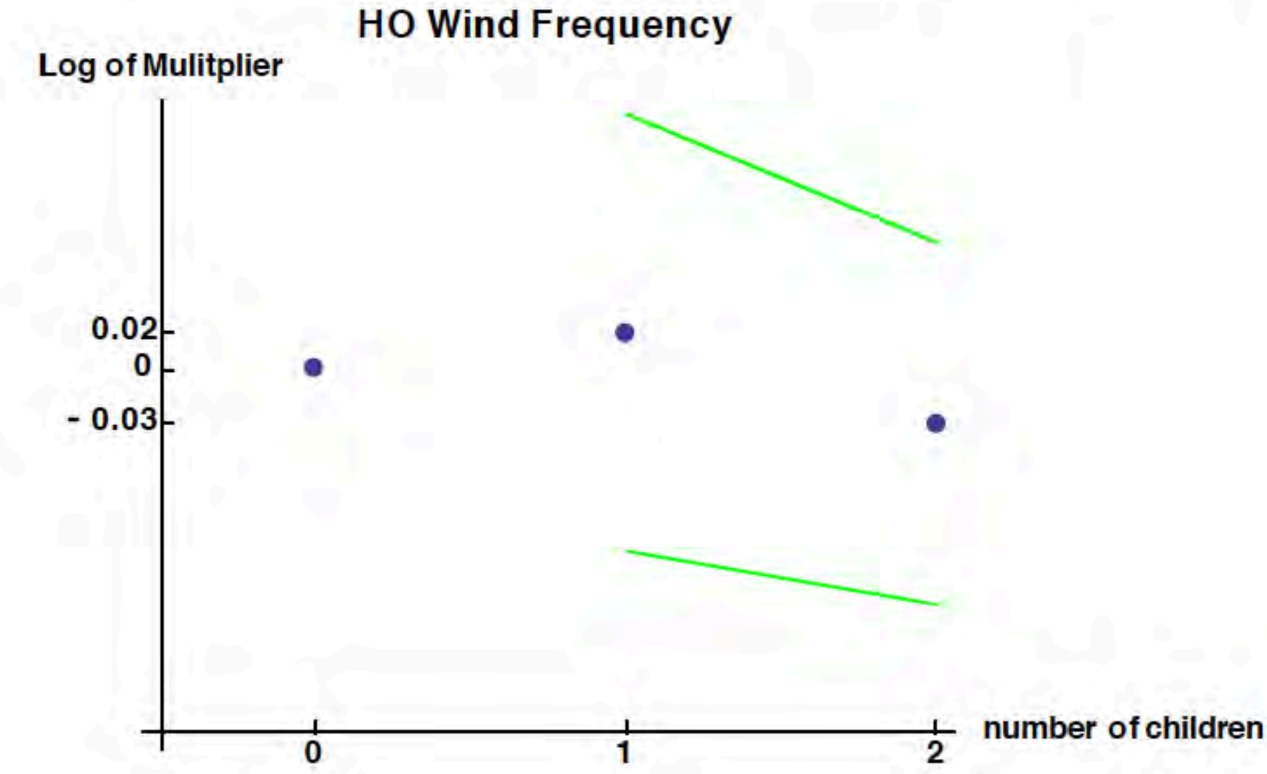
3.135. (1 point) A GLM has been fit in order to predict blood pressure of individuals.

Variable	Coefficient	VIF
Constant	-12.87	
Age	0.7033	1.76
Weight	0.9699	10.42
Body Surface Area	3.780	6.33
Duration of Hypertension	0.0684	1.24
Basal Pulse	-0.0845	4.41
Stress Index	0.00341	1.83

Briefly discuss this output.

3.136. (2 points) Below are graphs of GLMs fit to Homeowners frequency data, showing the natural log of the fitted multiplicative factors for one or two children in the house relative to none. Also shown are approximate 95% confidence intervals.
Briefly compare and contrast what the two graphs tell the actuary about each model.





3.137. (2.5 points) An actuary at a private passenger auto insurance company wishes to use a generalized linear model to create an auto severity model using the data below.

	Dollars of Loss	
Gender	Territory A	Territory B
Male	700,000	500,000
Female	400,000	300,000

	Number of Claims	
Gender	Territory A	Territory B
Male	800	700
Female	600	500

The model will include three parameters: β_1, β_2, β_3, where β_1 is the average severity for males, β_2 is the average severity for Territory A, and β_3 is an intercept.

Assuming β_3 = 570.356, solve a generalized linear model with a normal error structure and identity link function for β_1.

3.138. (1.5 points)

Five Generalized Linear Models have been fit to the same set of 60 observations.

Model	Number of Fitted Parameters	LogLikelihood
A	2	-220.18
B	3	-217.40
C	4	-214.92
D	5	-213.25
E	6	-211.03

Which model has the best BIC (Bayesian Information Criterion)?

3.139. (1 point) You fit a GLM using year as one of the predictor variables.
The values of year in your data are: 2010, 2011, 2012, 2013, and 2014.
You pick 2012 as the base level.
Applying statistical tests you determine that the coefficients for 2011 and 2014 are not significant.
Discuss what would you do.

3.140. (3 points) You are given the following wage distribution table:

Ratio to SAWW	Cumulative Portion of Workers	Cumulative Portion of Wages
0.10	0.18%	0.01%
0.20	0.93%	0.13%
0.30	3.53%	0.79%
0.40	6.85%	1.96%
0.50	11.33%	4.00%
0.60	18.49%	7.98%
0.70	28.57%	14.56%
0.80	40.05%	23.13%
0.90	48.99%	30.75%
1.00	57.47%	38.80%
1.10	64.98%	46.69%
1.20	71.14%	53.76%
1.30	76.34%	60.25%
1.40	80.99%	66.51%
1.50	85.33%	72.80%
1.75	92.86%	84.92%
2.00	96.91%	92.48%
2.25	98.73%	93.41%
2.50	99.28%	94.41%
3.00	99.66%	95.79%
4.00	99.87%	97.28%
5.00	99.93%	98.05%
6.00	99.96%	98.52%
7.00	99.97%	98.84%

With the aid of a computer, draw the corresponding Lorenz curve.

3.141. (2 points) Assume there are two models, Model A and Model B, both of which produce an estimate of the expected loss cost (pure premium) for each policyholder.
Discuss using Simple Quantile Plots to compare the two models A and B.
How are Simple Quantile Plots created?
How would one determine the winning model?

3.142. (1.5 points)
A logistic model was built to predict the probability of a claim being fraudulent.
(a) Briefly define the discrimination threshold.
(b) Briefly discuss the selection of what discrimination threshold to use.

3.143. (4 points) An actuary is considering using a generalized linear model to estimate the expected frequency of a recently introduced insurance product.
Given the following assumptions:

- The expected frequency for a risk is assumed to vary by territory and gender.
- A log link function is used.
- A Poisson error structure is used.
- β_0 is the intercept.
- β_1 is the effect of gender = Female.
- β_2 is the effect of Territory = B.

	Number of Claims	
Gender	Territory A	Territory B
Male	1200	1100
Female	800	900

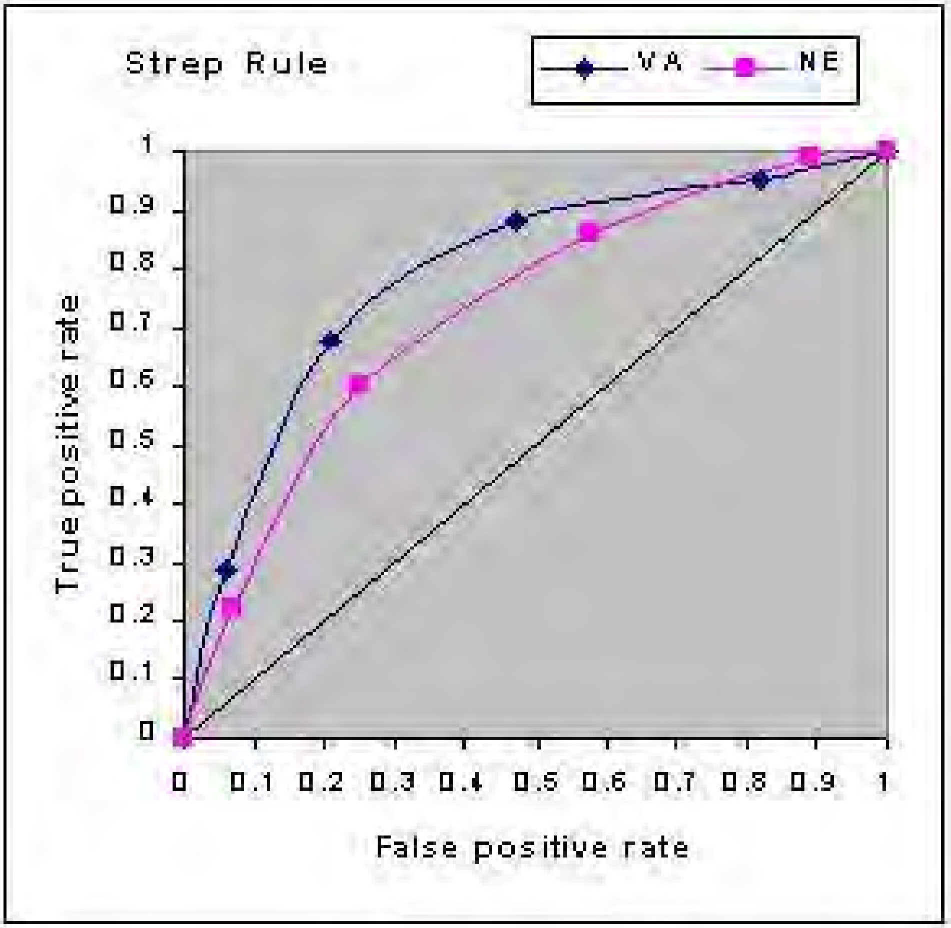
	Number of Exposures	
Gender	Territory A	Territory B
Male	24,000	15,000
Female	20,000	13,000

Given that β_0 = -3.0300, determine the expected frequency of a female risk in Territory B.

3.144. (1 point) Briefly discuss interaction in GLMs and give an example of an interaction term.

3.145. (2 points) A GLM has been used to develop an insurance rating plan.
There are only two classes A and B, with equal numbers of exposures.
The predicted pure premium for Class A is less than that for class B
(a) Determine the Gini Index if the actual losses for the two classes are equal.
(b) Determine the Gini Index if the actual losses for Class A are 0 and for Class B are positive.

3.146. (0.5 points) The following ROC curves are for two medical tests for strep throat:



Which test do you prefer and why?

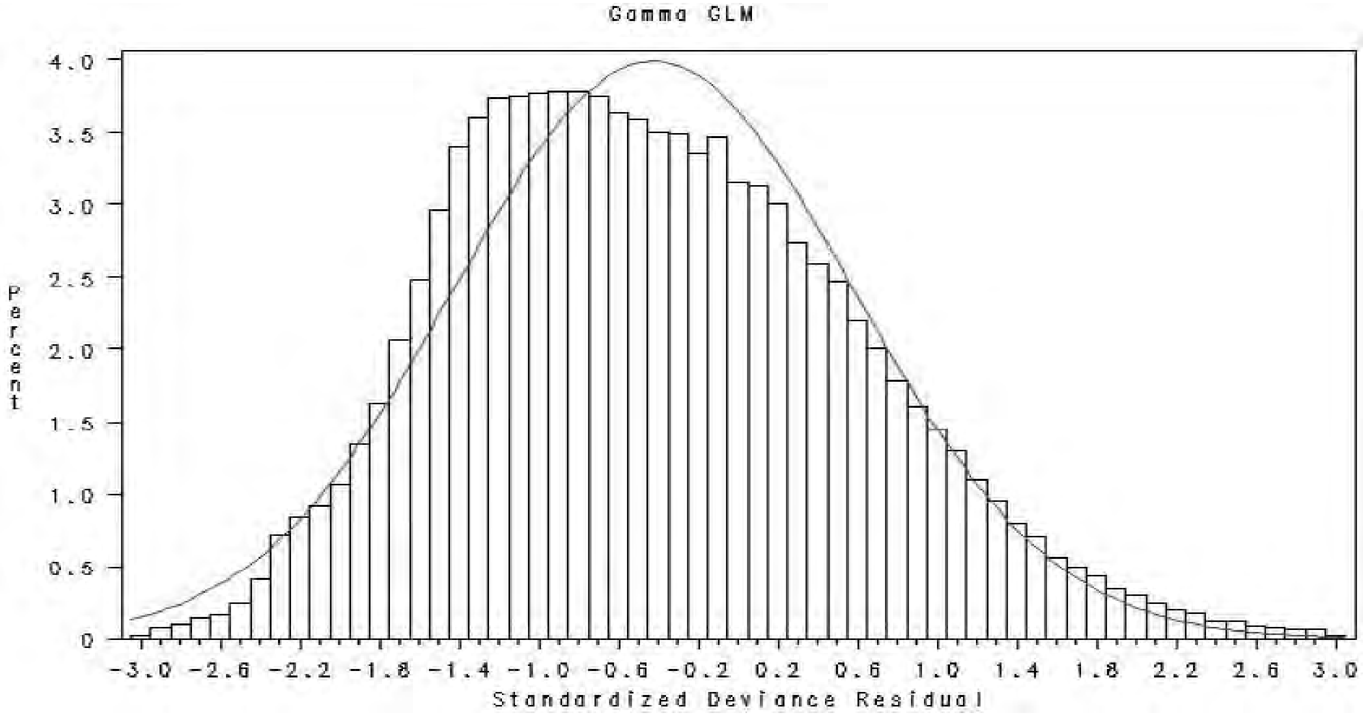
3.147. (1.5 points)

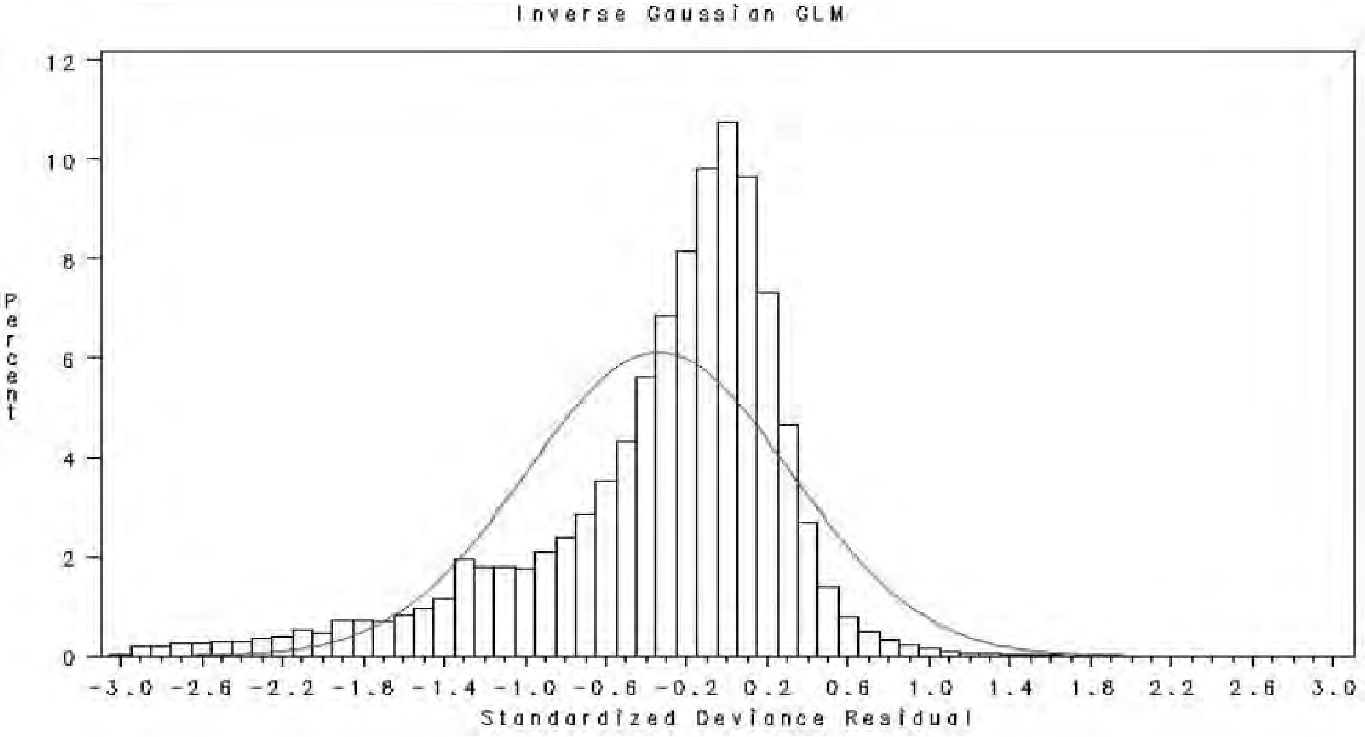
A GLM has been fit using a Poisson Distribution with $\hat{\beta}_1 = 5.624$ with standard error 0.1978.

Using instead an overdispersed Poisson the estimate of ϕ is 3.071.
For this second model, determine a 95% confidence interval for β_1.

3.148. (1.75 points) An analyst has fit several different variations of a GLM to a large dataset in order to predict pure premiums.
For each model variation listed below, draw a simple quintile plot based on the training data.
Label the axes and identify each data series.
i. A saturated model
ii. A null model
iii. A model that could be used in practice

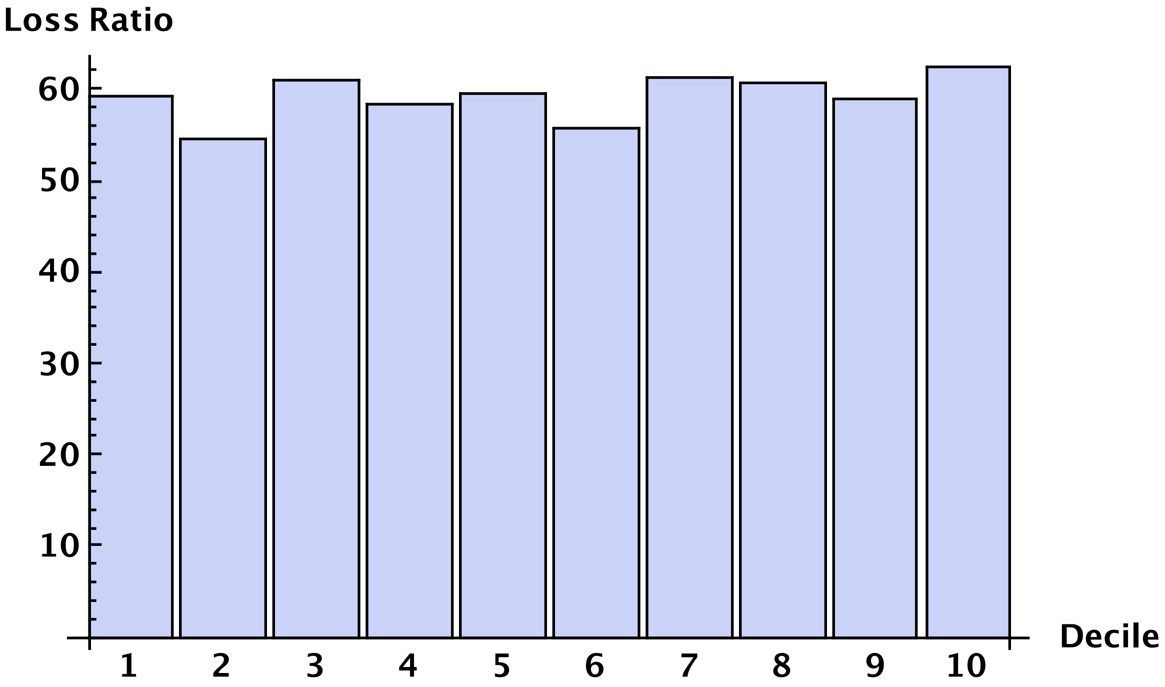
3.149. (1 point) Otherwise similar GLMs have been fit, one using a Gamma Distribution and the other using an Inverse Gaussian Distribution. Based on the following histograms of standardized deviance residuals which model do you prefer and why.





3.150. (1 point) The following loss ratio chart for a proposed rating plan was created by:
1. Sorting the dataset based on the model prediction, in other words modeled loss ratios.
2. Bucketing the data into deciles, such that each decile has approximately the same volume of exposures.
3. Within each bucket, calculate the actual loss ratio (under the current plan) for risks within that bucket.

Discuss the lift of the proposed plan compared to the current plan.



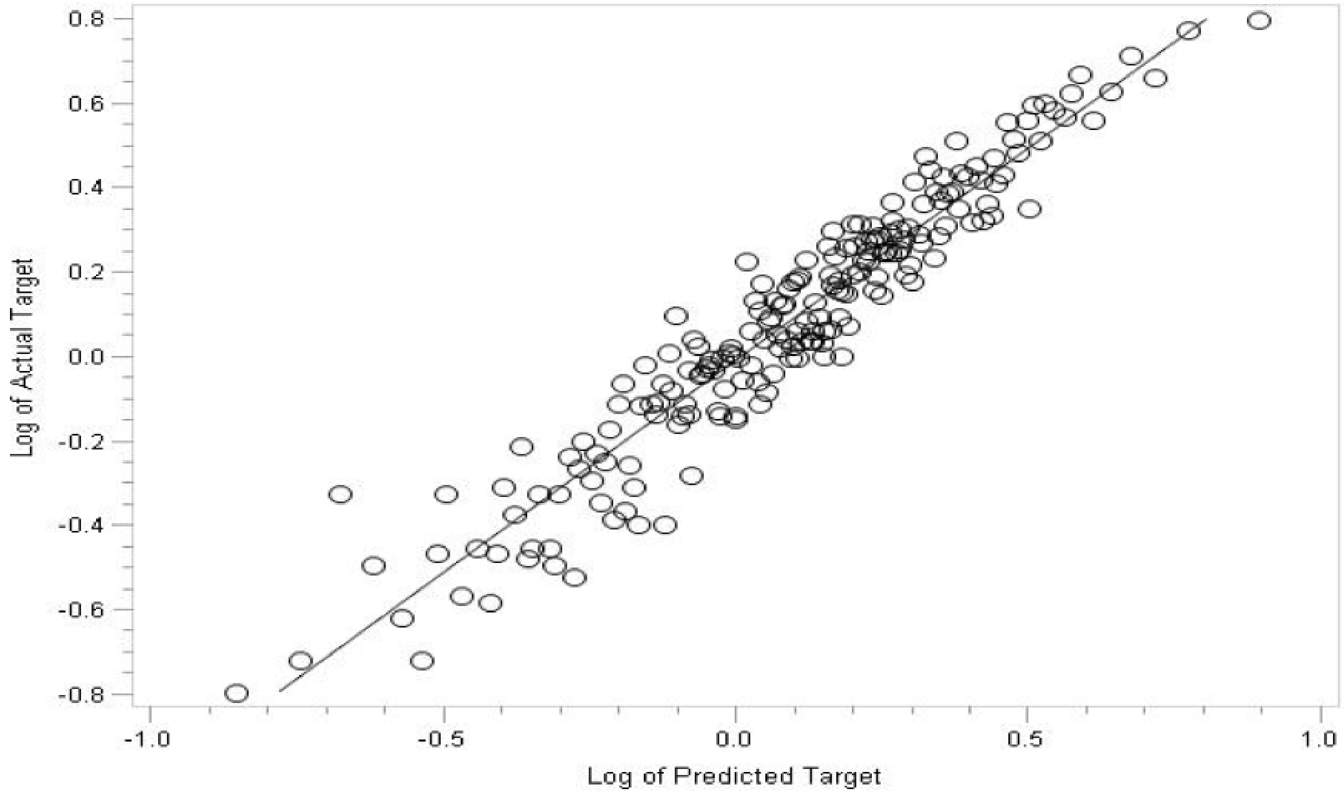
Use the following information for the next two questions:
Three Generalized Linear Models have been fit to the same set of 5000 observations.

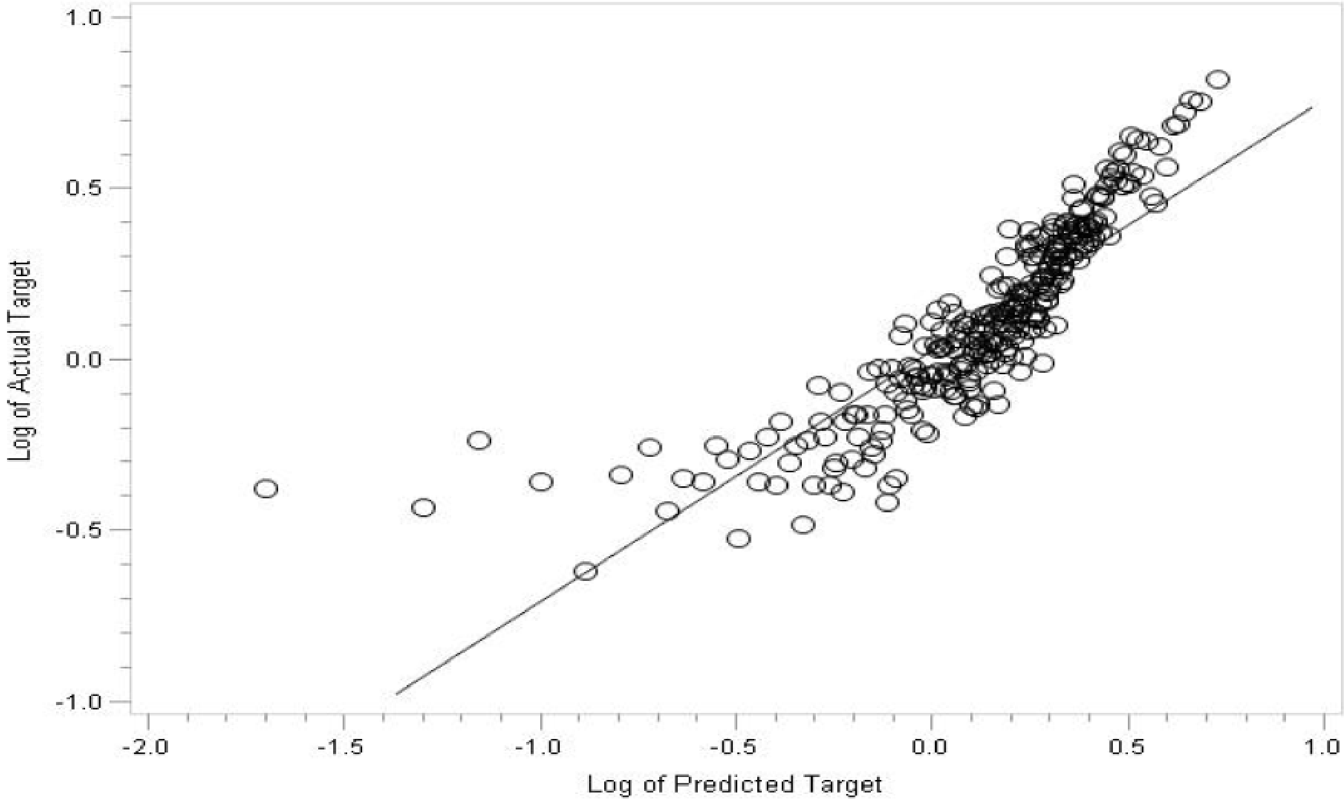
Model	Number of Fitted Parameters	LogLikelihood
A	5	-9844.16
B	10	-9822.48
C	15	-9815.70

3.151. (1 point) Which model has the best AIC (Akaike Information Criterion)?

3.152. (1 point) Which model has the best BIC (Bayesian Information Criterion)?

3.153. (1 point) Below are plots of Actual vs. Predicted for two different GLMs.





Which model do you prefer and why.

3.154. (4 points) A GLM has been used to develop an insurance rating plan.
The results are given below:

Risk	Exposures	Model Predicted Pure Premium	Actual Pure Premium
1	3	7000	6000
2	7	1000	4000
3	8	4000	2000
4	11	5000	8000
5	12	3000	1000
6	16	6000	8000
7	19	8000	6000
8	24	2000	4000

Plot the Lorenz curve for this rating plan.
Label each axis and the coordinates of each point on the curve.

3.155. (2 points) You are given a GLM of collision claim size with the following potential explanatory variables only:

- ● Vehicle price, which is a continuous variable modeled with a second order polynomial
- ● Vehicle Age which is a categorical variable with 8 levels
- ● Average driver age, which is a continuous variable modeled with a first order polynomial
- ● Number of drivers, which is a categorical variable with three levels
- ● Gender, which is a categorical variable with two levels
- ● There is only one interaction in the model, which is between gender and average driver age.

Determine the number of parameters in this model.

3.156. (1.5 points) Discuss how to construct a double lift chart.

3.157. (1.5 points)
Generalized Linear Models have been fit both with and without a certain predictor variable.

Model	With	Without
Unscaled Deviance	8,901.4414	8,905.6226
Degrees of Freedom	18,169	18,175
Scale Parameter	0.4523	0.4327

The null hypothesis is to use the simpler model.
The alternative hypothesis is to use the more complicated model.
Calculate the F-test statistic.
Discuss how you would perform the test.

3.158. (2 points) You are given the following GLM output:

Response variable	Pure Premium
Response distribution	Gamma
Link	log
Estimated alpha	2.2

Parameter		df	$\hat{\beta}$
Intercept		1	5.07
Risk Group		2	
	Group 1	0	0.00
	Group 2	1	0.21
	Group 3	1	0.48
Vehicle Symbol	1		
	Symbol 1	1	-0.36
	Symbol 2	0	0.00

Calculate the variance of the pure premium for an insured in Risk Group 3
with Vehicle Symbol 1.

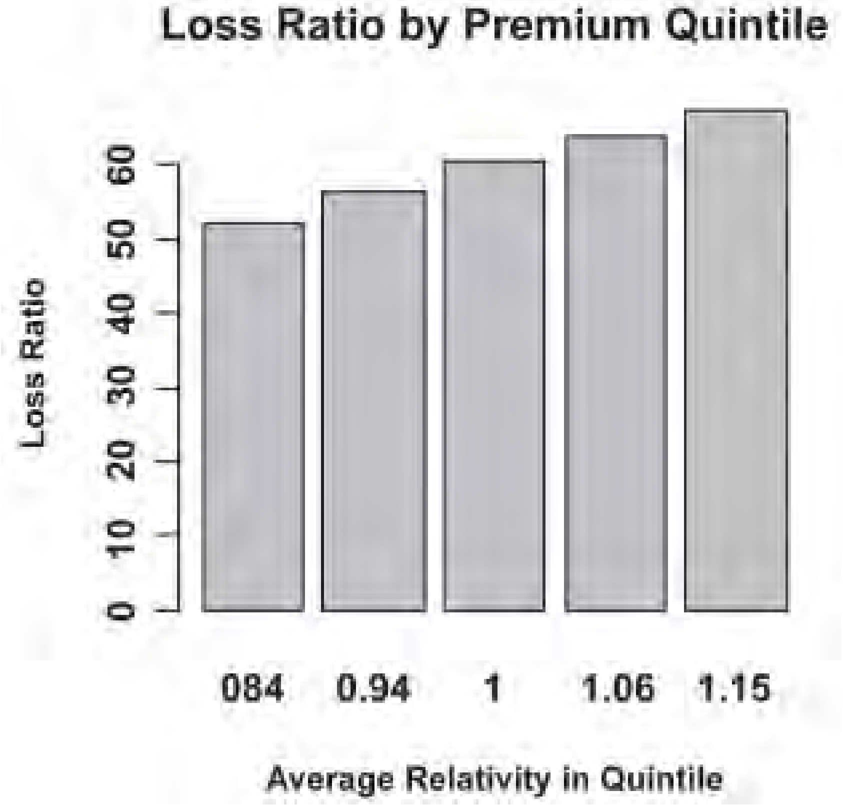
3.159. (1 point) Two GLMs with somewhat different sets of variables have been fit to the same data. Model 1 has a Gini index of 0.16, while Model 2 has a Gini index of 0.12.
Briefly discuss which rating plan has better lift.

3.160. (1 point) Discuss how to construct a loss ratio chart.

3.161. (1 point) An actuary fits a GLM to a large amount of data on pure premiums for private passenger automobile insurance. The model includes driver age.
The actuary wants to test adding a new variable, number of years claims-free:
0, 1, 2, 3, 4 or more.
The new variable will only be used for drivers at least 25 years old.
The actuary fits an otherwise similar model that includes number of years claims-free to the same data. The effect of driver age in the second model is significantly less than in the first model.
Briefly discuss why this may have occurred.

3.162. (1 point) The following loss ratio chart for a proposed rating plan was created by:
1. Sorting the dataset based on the model prediction.
2. Bucketing the data into quintiles, such that each quintile has approximately the same volume of exposures.
3. Within each bucket, calculate the actual loss ratio (under the current plan) for risks within that bucket.

Discuss the lift of the proposed plan compared to the current plan.



3.163. (1 point) An actuary is modeling pure premiums, using a GLM with a log ink function. Deductible relativities have been determined separately, and their effect will be included in the GLM via an offset.

The fitted GLM uses two predictors x_1 and x_2; $\hat{\beta}_0 = 6$, $\hat{\beta}_1 = 0.1$, and $\hat{\beta}_2 = -0.2$.

Calculate the fitted pure premium for a policy with a deductible relativity of 0.8, $x_1 = 13$ and $x_2 = 3$.

3.164. (1 point) Laurel and Hardy are each fitting GLMs to the same 100 observations.
Laurel proposes using a GLM with 100 parameters.
Hardy proposes using a GLM with one parameter, the overall mean.
Discuss which if either of their proposals makes sense.

3.165. (1 point) An actuary, Simon Leroy is modeling private passenger automobile liability insurance via a GLM. All of the current rating variables are included in the model; the current territory relativities are included via an offset.
Simon adds to the model the number of years a driver has been claims free.
The indicated relativities are:

	Relativity	Percent of Premiums
0 years claims free	1.50	9%
1 year claims free	1.35	4%
2 years claims free	1.18	5%
3 or more years claims free	1.00	82%

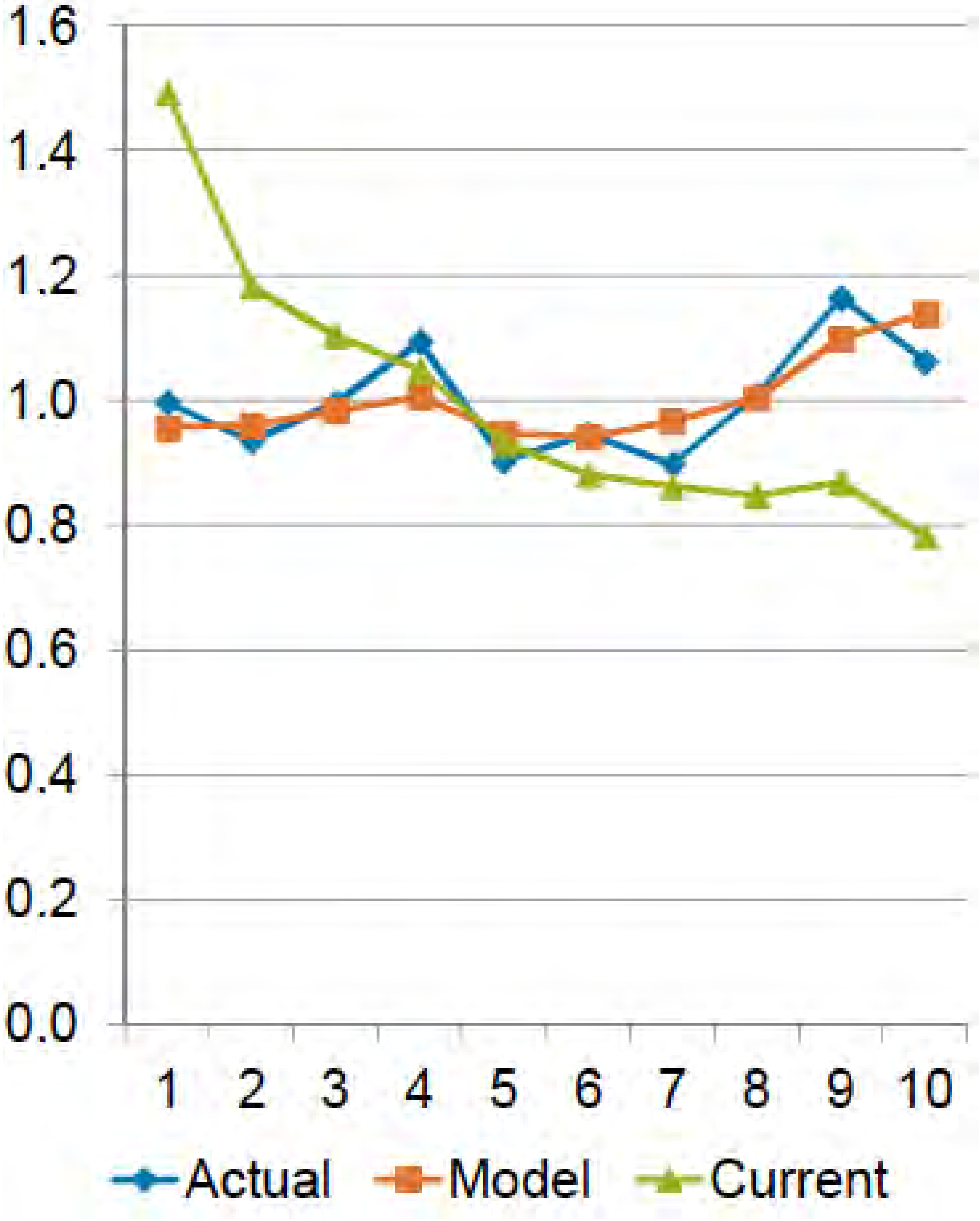
What can you infer about the credibility of a single private passenger driver?

3.166. (1 point) An actuary is modeling private passenger automobile insurance.
Both the number of operators listed on the policy and the age of the youngest operator listed on the policy will be included in the model. Discuss a potential difficulty with this.

3.167. (1.5 points) Compare and contrast AIC and BIC.
What do the authors of Generalized Linear Models for Insurance Rating conclude with respect to the building of GLMs for actuarial work?

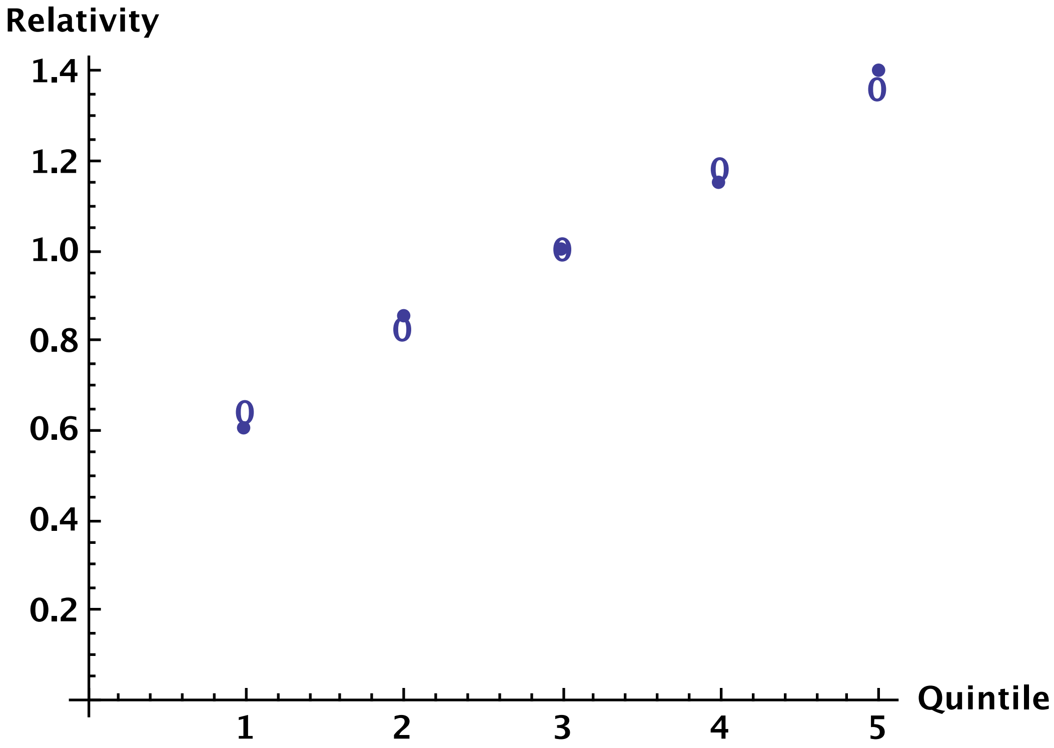
3.168. (3 points) Answer the following questions about working residuals and GLMs.
(a) (0.5 points) Define working residuals.
(b) (1 point) Fully discuss the main advantage of working residuals.
(c) (0.5 points) Define working weights.
(d) (0.5 points) Briefly discuss the purpose of the working weights.
(e) (0.5 points) List three useful types of plots of working residuals.

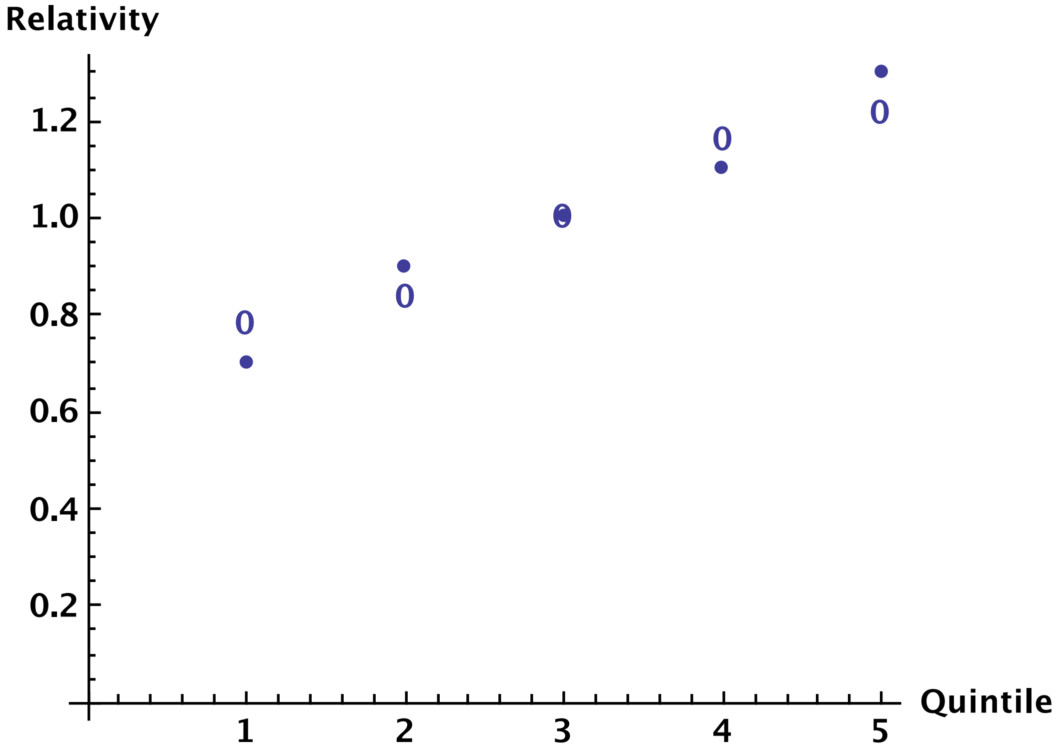
<u>3.169</u>. (1 point) You are given the following double lift chart:



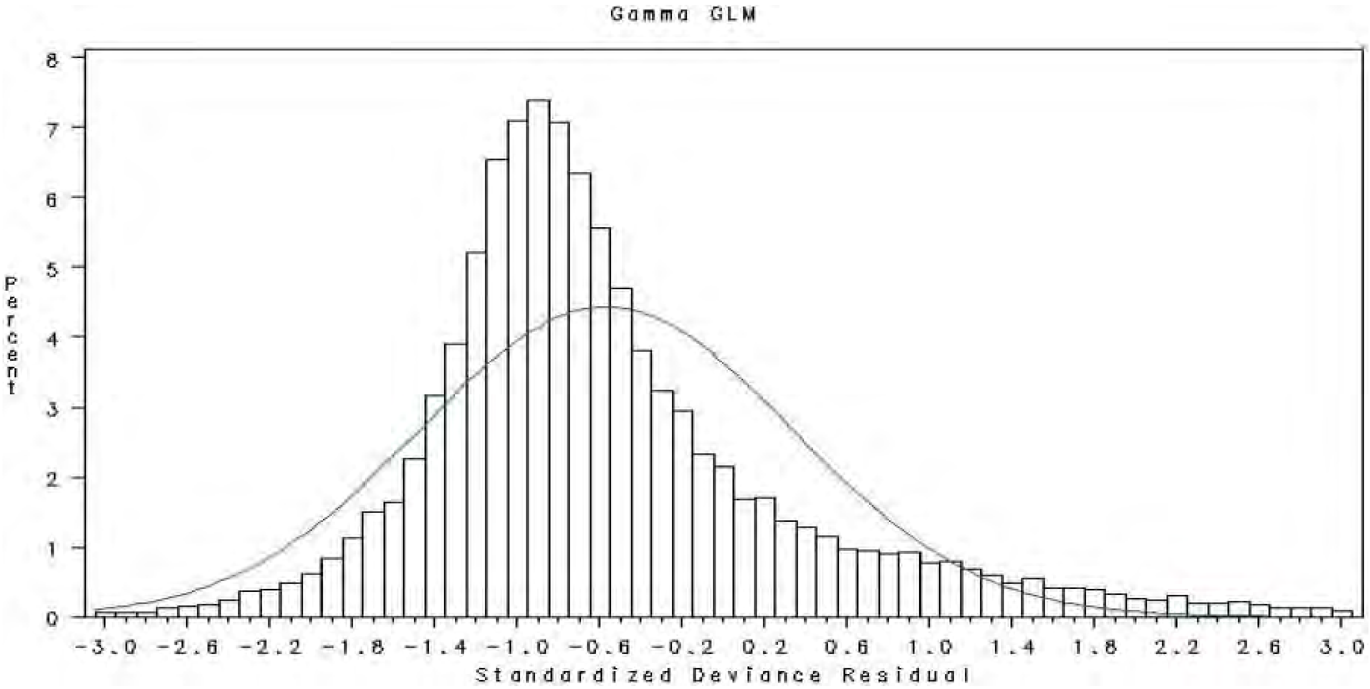
Briefly discuss what conclusion you draw and and why.

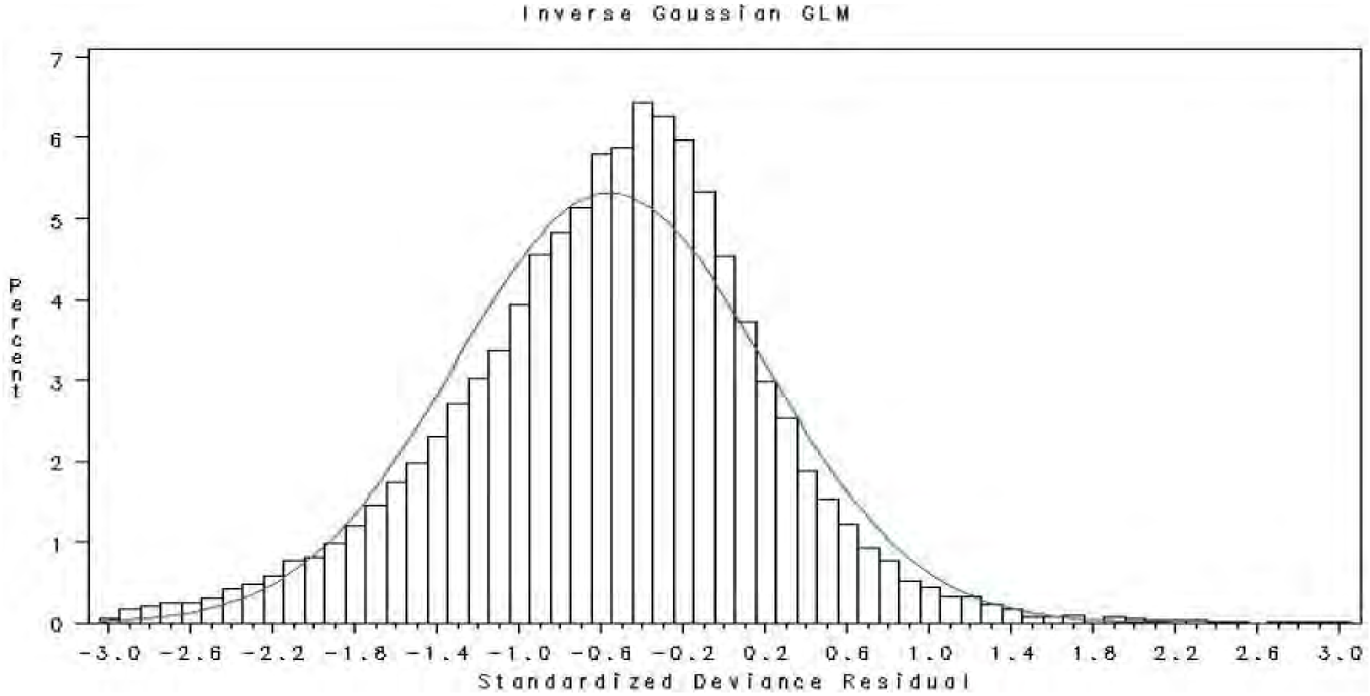
3.170. (1 point) Below are shown two simple quantile plots, the first for Plan A and the second for Plan B. In each case, the model plan predictions are shown by dots and the actual by o. Which plan is preferable and why?





3.171. (1 point) Otherwise similar GLMs have been fit, one using a Gamma Distribution and the other using an Inverse Gaussian Distribution. Based on the following histograms of standardized deviance residuals which model do you prefer and why.





3.172. (2 points) You are given the following information about an insurance policy:

- The probability of a policy renewal, p(X), follows a logistic model with an intercept and one explanatory variable.
- $\beta_0 = 1$
- $\beta_1 = 0.31$

Calculate the odds of renewal at x = 8.

3.173. (3 points) A logistic model was built to predict the probability of a claim being fraudulent. Consider the predicted probabilities for the 15 claims below to be a representative sample of the total model.

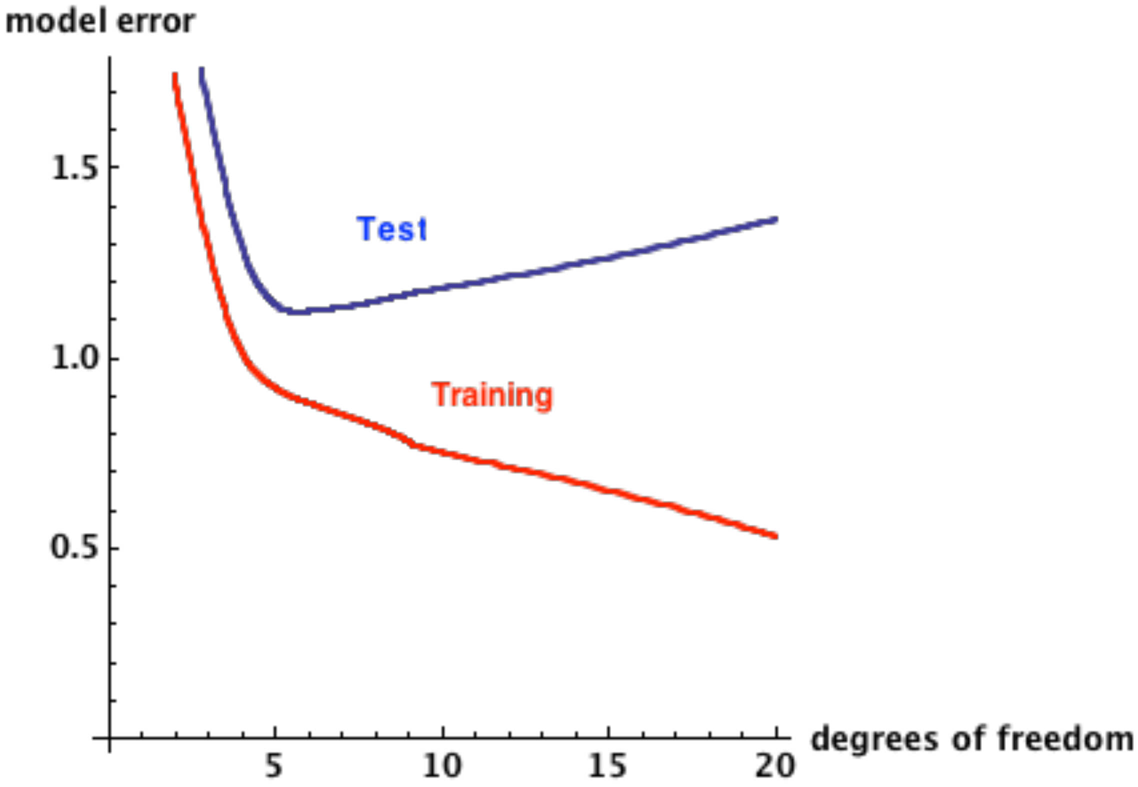
Claim Number	Actual Fraud Indicator	Predicted Probability of Fraud
1	N	37%
2	N	46%
3	N	23%
4	N	13%
5	Y	89%
6	N	5%
7	Y	21%
8	N	74%
9	Y	75%
10	Y	69%
11	N	57%
12	Y	54%
13	N	53%
14	N	83%
15	N	49%

a. (1.5 point) Construct confusion matrices for discrimination thresholds of 0.30 and 0.60.

b. (1.5 points) Plot the Receiver Operating Characteristic (ROC) curve with the discrimination thresholds of 0.30 and 0.60.
Label each axis and the coordinates and discrimination threshold of each point on the curve.

3.174. (1 point) Olaf is an actuary with the Arendelle Insurance Company.
Olaf is revising the classification relativities using a GLM with a log link function.
Recently another actuary Anna had revised the territory relativities.
Olaf with take these territory definitions and relativities as given.
Discuss how Olaf should make use of offsets in his modeling of classification relativities.

<u>3.175</u>. (0.75 points) An actuary has split data into training and test groups for a model. The chart below shows the relationship between model performance and model complexity. Model performance is represented by model error and model complexity is represented by degrees of freedom.



Briefly discuss the optimal balance of complexity and performance.

3.176. (2 points) An actuary is analyzing a partial residual plot of the driver age variable. The plot appears to be non-linear.

a. (1 point) Briefly describe two approaches that can be used to improve the fit of the driver age variable.

b. (1 point) Briefly describe a downside to each of the two approaches discussed in part a. above.

<u>3.177</u>. (2 points) You are given the following information for a GLM of customer retention:

Response variable	Retention
Response distribution	Binomial
Link	Logit

Parameter	df	$\hat{\beta}$
Intercept	1	2.182
Years with Insurer	1	
1	0	0.000
>1	1	1.137
Last Rate Change	2	
<0%	0	0.000
0%-10%	1	-0.422
>10%	1	-0.901

Calculate the probability of retention for a policy with the insurer for 4 years and with a prior rate change of 7%.

3.178. (2 points) An actuary is comparing the output of two generalized linear models to develop a new rating plan for personal auto. Model statistics are shown below:

	Saturated Model	Model A	Model B
Log-Likelihood	-100	-130	-123
Estimated Dispersion Parameter	0.65	0.61	0.63

- Each model is fit to the same set of 50 data points.
- Model A uses 5 parameters (including an intercept).
- Model A is a nested model of Model B,
  where Model B has an additional variable for driver age.
- Driver age is fit using 5 bins.
- The critical value to be used from the F-distribution is 2.600.

Using two statistical tests, recommend whether or not driver age should be included in the rating plan.

3.179. (2 points) Your coworker, Clifford Clavin, has fit a GLM using the following model form:

$g(Y) = \beta_0 + \beta_1 X_1 + \beta_2 X_2$

The fitted parameters were:

$\hat{\beta}_0$	1
$\hat{\beta}_1$	2
$\hat{\beta}_2$	-5

You know that Cliff used a canonical link function, but do not know which of the following three error distribution Cliff used:

I: Gamma

II: Poisson

III: Normal

Cliff is currently on vacation and can not be reached; Cliff's limited notes do not help.

Determine the correct ordering of the three possible models' predicted values at the observed point $(X_1, X_2) = (0.50, 0.29)$.

3.180. (1.25 points) An actuary is planning to add a credit-based insurance score from FIDO to a model that estimates the probability of a policy having a claim. The actuary has decided to offset all of the current model variables before fitting the new variable.

Given the following:

- The current model (without the FIDO insurance score variable) is a logit-link binomial GLM (logistic regression).
- The logit link function is defined as $g(\mu) = \ln(\frac{\mu}{1 - \mu})$.
- The FIDO insurance score is a continuous variable having a value between 1 and 100.
- The current fitted values and FIDO insurance score for three policies as well as regression results from the fit of the FIDO insurance score variable are given below:

Policy Number	Fitted Probability Without Insurance Score	FIDO Insurance Score
1	2.3%	34
2	11.2%	66
3	4.5%	88

Variable	Parameter Estimate
Intercept	1.581
FIDO Insurance Score	-0.032

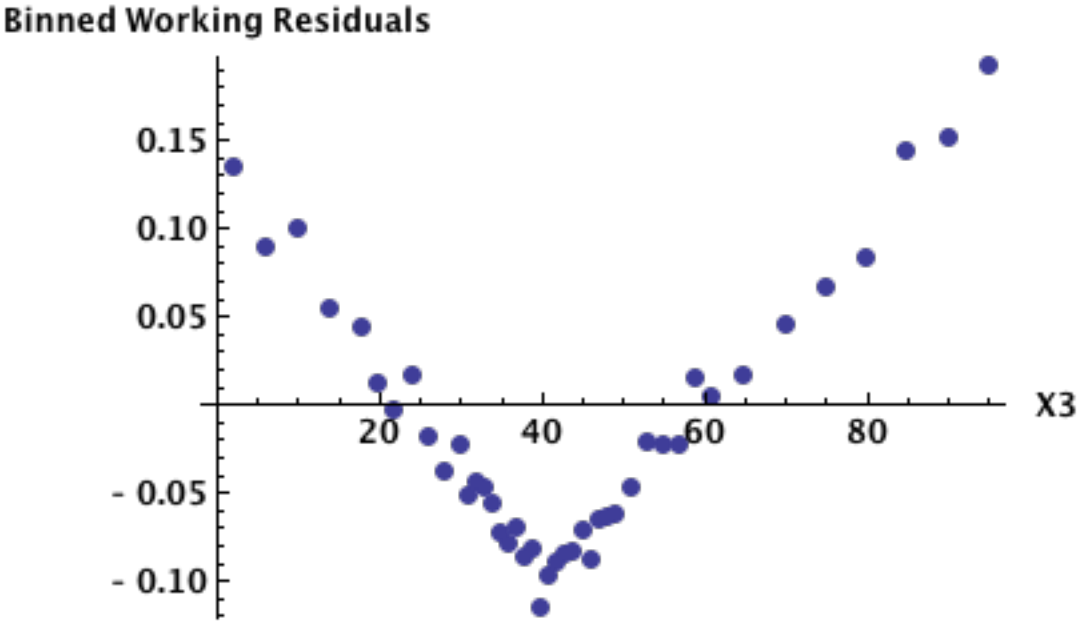
a. (0.5 point)

Calculate the offset term to be used in the regression for each of the three policies above.

b. (0.75 point)

Calculate the revised fitted probability of having a claim for each of the three policies above.

3.181. (1 point) A GLM has been fit to some data.
The following is a plot of binned working residuals versus one of the predictor variables X_3:



Briefly discuss what this plot tells one about the appropriateness of the fitted model.

3.182. (3 points) You are given the following data on the percent of likes on a dating app:

Group	Men	Women
Top 1%	16%	11%
Top 5%	41%	31%
Top 10%	58%	46%
Bottom 50%	4%	8%

For example, the 5% of men with the most likes get 41% of all likes for men.
Draw separate Lorenz curves for men and women.
Do men or women have the larger Gini index?

3.183. (2.5 points) An actuary creates a generalized linear model (GLM) to estimate commercial property claim frequency by occupancy class and amount of insurance (AOI) for sprinklered and non-sprinklered risks. Given the following:

- Occupancy class is a categorical variable with four levels: class 1, 2, 3 and 4.
- Sprinklered status is a categorical variable with two levels: sprinklered and non-sprinklered.
- The natural log of AOI, ln(AOI), is a continuous variable.
- The log link function is selected.
- An interaction variable is included as ln(AOI) for sprinklered and zero otherwise.
- The model results are as follows:

Parameter	Coefficient
Intercept	-8.200
Occupancy class 2	0.200
Occupancy class 3	0.300
Occupancy class 4	0.500
Sprinklered	0.700
ln(AOI)	0.400
Sprinklered: Yes, ln(AOI)	-0.100

a. (0.75 point) Calculate the ratio of the estimated model frequency of a sprinklered property to that of a non-sprinklered property for AOI = 150,000 and occupancy class 3.

b. (0.5 point) Calculate the intercept term if AOI is centered at the base level of 300,000.

c. (0.5 point) Calculate the coefficient of sprinklered if AOI is centered at the base level of 300,000.

d. (0.75 point)

Briefly describe two advantages of centering variables of a GLM at their base levels.

3.184. (1 point) Model documentation is important; list three purposes.

3.185. (1.5 points) A model has been developed to distinguish legitimate emails from spam. The following is the confusion matrix resulting from applying this model to some test data:

	Predicted Class	
True Class	Legitimate Email	Spam
Legitimate Email	58.2%	2.5%
Spam	3.0%	36.3%

Determine the specificity and sensitivity of this test.

3.186. (5 points) An actuary has built two generalized linear models to predict loss costs.

The data has been sorted based on the ratio: $\frac{\text{prediction for Model 1}}{\text{prediction for Model 2}}$,

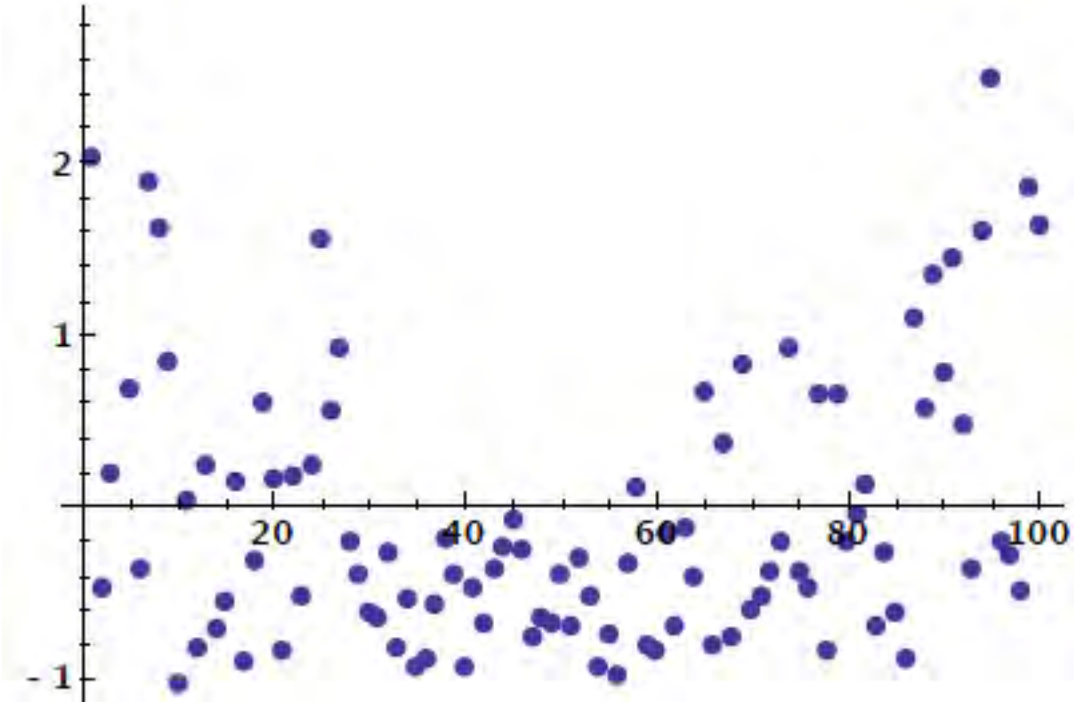
and then has been grouped into deciles with approximately the same number of exposures.
The results are shown below:

Decile	Actual Pure Premium	Model 1 Pure Premium	Model 2 Pure Premium
1	$118.88	$109.62	$115.08
2	$141.58	$121.73	$125.95
3	$129.37	$115.13	$117.95
4	$107.00	$117.76	$119.68
5	$117.91	$115.58	$116.57
6	$113.02	$118.84	$119.08
7	$130.21	$121.57	$121.11
8	$123.52	$126.99	$125.70
9	$121.75	$124.94	$121.36
10	$135.65	$134.13	$123.65
Total	$123.88	$120.62	$120.61

Construct a double lift chart.

3.187. (0.5 point) A claim fraud GLM has been developed.
Briefly describe how the severity of claims will impact the selection of an appropriate discrimination threshold to use together with the model.

3.188. (1 point) A GLM has been fit to some data.
The following is a plot of binned working residuals versus the linear predictor in the model:



Briefly discuss what this plot tells one about the appropriateness of the fitted model.

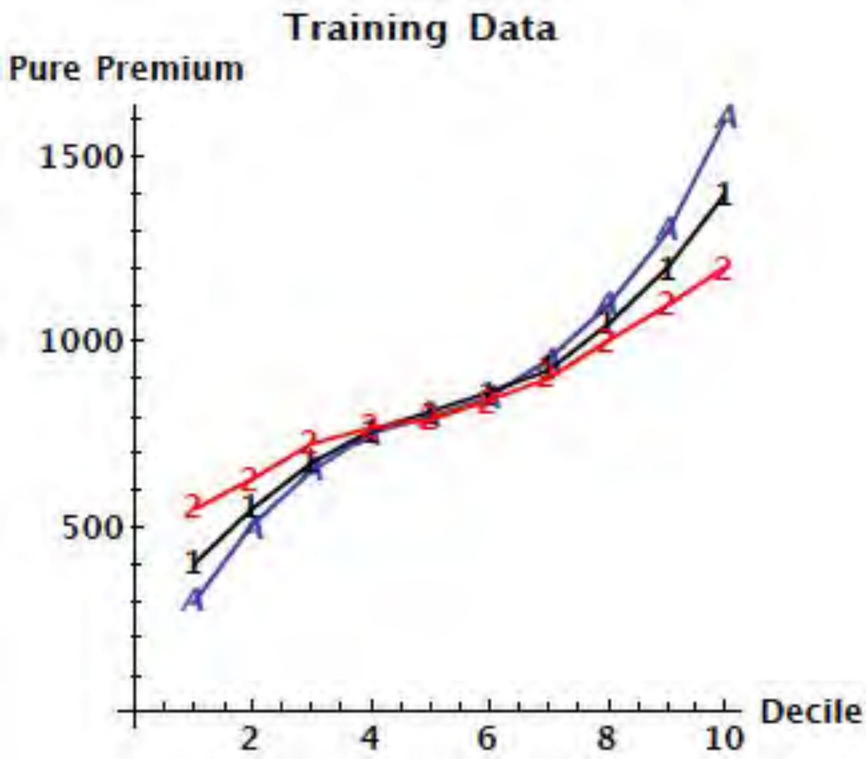
3.189. (1 point) List and briefly discuss three characteristics of Natural Cubic Splines.

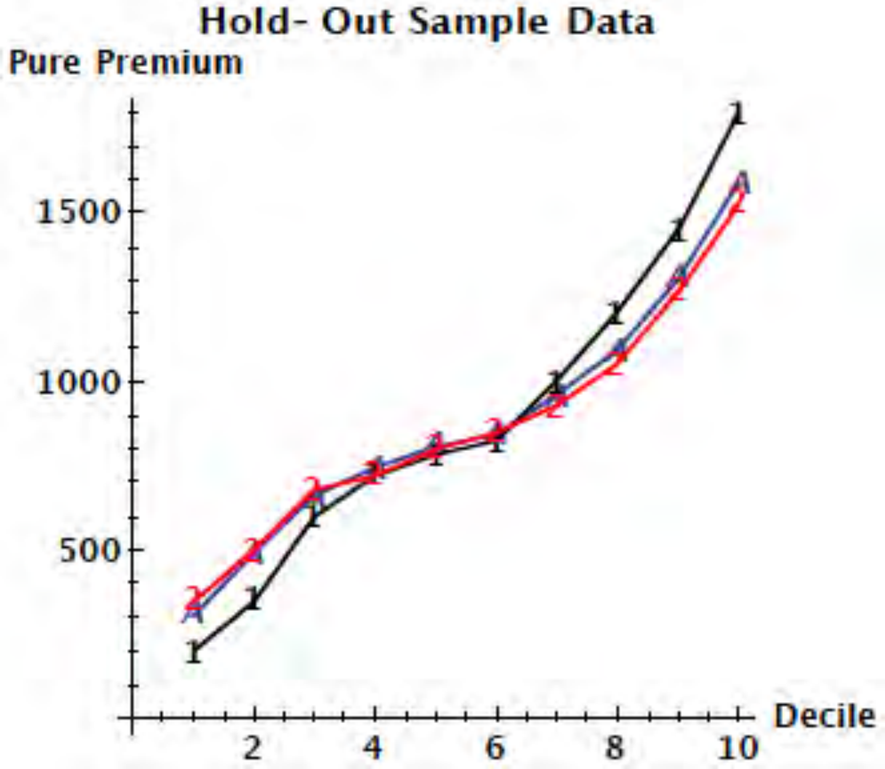
3.190. (4 points) A GLM has been used to develop an insurance rating plan.
The results are given below:

Risk	Exposures	Model Predicted Pure Premium (000)	Actual Pure Premium (000)
1	15	56	60
2	19	30	36
3	21	49	42
4	24	38	49
5	27	43	28
6	29	64	63
7	31	77	79
8	34	52	39

Plot the Lorenz curve for this rating plan.
Label each axis and the coordinates of each point on the curve.

3.191. (1.5 points) The following graphs show two competing generalized linear models (GLMs) predictions versus the data used in modeling ("training") and a hold-out sample.
Assess each of the models. A = Actual Data. 1 = Model 1. 2 = Model 2.
Data in each graph has been sorted into equal volume deciles,
ranked from low to high actual loss.


Training Data


Hold- Out Sample Data

3.192. (2.5 points) An actuary has fit a GLM using a Poison Distribution with log link function.
Exposures were used as the weights.
The actuary is creating a plot of working residuals in order to assess the model fit.
The following eight observations will be binned together.
Compute the binned working residual for this bin.

Observed	Predicted	Exposures
4	3.3	11
3	3.7	9
6	5.5	15
2	4.1	7
5	5.2	12
4	3.4	8
2	2.6	14
4	3.0	10

3.193. (1.5 points) An insurer uses a GLM for classification ratemaking.
You are given the following data on five insureds.

Insured	Actual Loss Cost	Loss Cost Predicted by the Model	Exposures
1	$38,000	$36,000	100
2	$36,000	$42,000	120
3	$52,000	$57,000	130
4	$46,000	$49,000	150
5	$58,000	$51,000	180

Construct a Simple Quantile Plot; sort the data based on predicted pure premium.

3.194. (1 point) List and briefly discuss two potential drawbacks of using polynomials in GLMs.

3.195. (3 points) A GLM has been fit.
The ninth response is 0.4, and the corresponding prediction is 0.5.
Determine the ninth working residual for each of the following link functions.
(a) Identity Link Function
(b) Log Link Function,
(c) Logit Link Function.

3.196. (2 points) Hari Seldon is an actuary.
Hari was given the task of fitting a GLM in order to model pure premium using a log link function.
Hari was to take as a given the current deductible relativities and the current territory base rates.

Deductible	Credit	
500	0	Base
1000	6%	
2500	11%	

Territory	Base Rate	
A	400	Base
B	600	
C	900	

Calculate the appropriate offset that Hari should have used for each combination of deductible and territory.

3.197. (1 point) One can use a Tweedie Distribution in a GLM.
Discuss two ways to determine the Tweedie p parameter.

3.198. (2.5 points) The following confusion matrix shows the result from a claim fraud model with a discrimination threshold of 40%:

	Predicted	
Actual	Yes	No
Yes	172	90
No	88	302

a. (0.5 point) Calculate the sensitivity and specificity from the above data.
b. (1.5 points) Plot the receiver operating characteristic (ROC) curve with the discrimination threshold of 40%. Label each axis, the coordinates, and the discrimination thresholds of 100%, 40%, and 0% on the curve.
c. (0.5 point) What is the ROC curve for each of the following two models:
i. A model with no predictive power
ii. A hypothetical "perfect" model

3.199. (6 points) A GLM has been fit. The seventh observation was given a weight of 60.
The seventh response is 0.8, and the corresponding prediction is 0.7.
Determine the seventh working weight for each of the following cases.
(a) Poison with log link function
(b) Gamma with log link function
(c) Tweedie with p = 1.6 and log link function
(d) Normal with identity link function
(e) Bernoulli with logit link function
(f) Inverse Gaussian with inverse link function

3.200. (3 points) An actuary has built two generalized linear models to predict loss costs.
Output for each model are shown below:

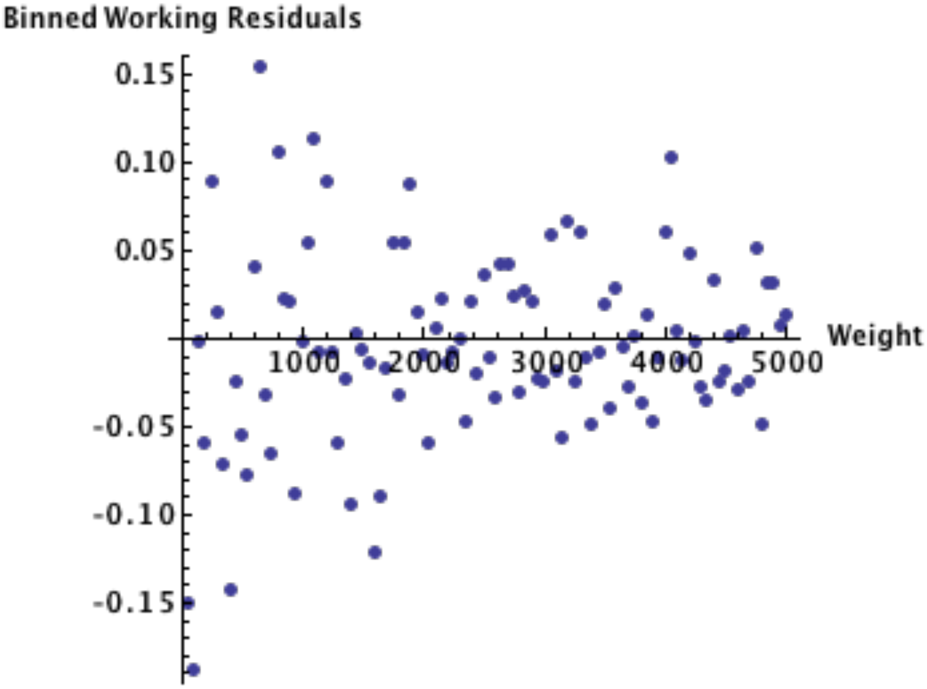
Observation	Actual Loss Cost	Model A Loss Cost	Model B Loss Cost	Exposures
1	$15,000	$16,000	$18,000	50
2	$20,000	$25,000	$22,000	70
3	$42,000	$31,000	$37,000	80
4	$44,000	$48,000	$39,000	100
5	$39,000	$38,000	$41,000	140

Construct a double lift chart.

3.201. (2 points) You have completed a modeling project.
(a) Many people may have questions about this model. List two types of such people external to your organization and two types of such people internal to your organization.
(b) List four features your documentation of your model should have.

3.202. (2 points) A test to detect antibodies to a particular virus has a 90% specificity and 95% sensitivity. This test is applied a population of 1000 people, of whom 20% actually have antibodies to this virus (and have thus been exposed to this virus.)
What are the expected results of applying this test?
Show the resulting confusion matrix.

3.203. (1 point) A GLM has been fit to some data.
The following is a plot of binned working residuals versus the weight variable used in the model:



Briefly discuss what this plot tells one about the appropriateness of the fitted model.

3.204. (2 points) 2% of people of a certain age have a particular type of cancer.
A test for this type of cancer has a sensitivity of 95% and a specificity of 90%.
(a) If a person of this age tests <u>positive</u> for this type of cancer, what is the probability that they have this type of cancer?
(b) If a person of this age tests <u>negative</u> for this type of cancer, what is the probability that they do <u>not</u> have this type of cancer?

3.205. (3 points) For a population, the bottom 60% earn 30% of the income.
Using only this information, determine the Gini Index.

3.206. (9, 11/03, Q.25) (2 points)
a. (1 point) Explain why one-way analysis of risk classification relativities can produce indicated relativities that are inaccurate and inconsistent with the data.
b. (1 point) Describe an approach to calculating risk classification relativities that would reduce the error produced by a one-way analysis.

3.207. (9, 11/06, Q.5) (4 points)
a. (3 points) Compare the random component, the systematic component, and the link functions of a linear model to those of a generalized linear model.
b. (1 point) Describe two reasons why the assumptions underlying linear models are difficult to guarantee in application.

3.208. (9, 11/07, Q.4a) (1 point) There are a variety of methods available to a ratemaking actuary when determining classification rates.
Compare the Generalized Linear Model to the Classical Linear Model with respect to the following:

i. The distribution of the response variable.
ii. The relationship between the mean and variance of the response variable.

3.209. (9, 11/08, Q.3) (2 points) When using a Generalized Linear Model one of the concerns of which the practitioner must be aware is the presence of aliasing within the model.
a. (1 point) Discuss the two types of aliasing and provide an example of how each can arise in a model.
b. (1 point) An actuary is using a Generalized Linear Model to determine possible interactions between pure premiums. While reviewing the model, the actuary observes the following pure premiums for liability coverage:

Liability Pure Premium			
	Vehicle Size		
Territory	Small	Medium	Large
North	100	150	250
South	80	110	290
East	90	170	200
West	180	260	540

Assuming equal exposure distribution across all combinations of territory and vehicle size, demonstrate how aliasing can be used to exclude a level from either the territory or the vehicle size variable.

3.210. (2 points) Use the information in the previous question, 9, 11/08, Q.3.
Take North and Medium as the base levels.
Specify the following structural components of a generalized linear model:
Definition of Variables, Design matrix, Vector of responses, Vector of model parameters.

3.211. (9, 11/09, Q.3) *(3 points) Consider a simple private passenger auto classification system that has two rating variables: territory (urban or rural) and gender (male or female).*
The observed average claim severities are:

Gender	*Urban*	*Rural*
Male	*$400*	*$250*
Female	*$200*	*$100*

Y, the response variable, is the average claim severity. Male (x1), Female (x2), Urban (x3) and Rural (x4) are the 4 covariates. A uniquely defined model is:

$$Y = \beta_1 X1 + \beta_2 X2 + \beta_3 X3 + e.$$

a. (2 points) Using the classical linear model, derive the equations to solve for the parameters $\beta_{1,}$ β_2 and β_3 using the sum of squared errors. (Do NOT solve the equations.)

b. (1 point) Briefly describe two underlying assumptions of the classical linear model. Explain why the model may not be able to guarantee these assumptions.

3.212. *(9 points) Use the information in the previous question, 9, 11/09, Q.3.*
As per the exam question, use the following variables: Male (X_1), Female (X_2), Urban (X_3).

a. (2 points) Specify the following structural components of a generalized linear model: Design matrix, Vector of responses, Vector of model parameters.

b. (2 points) Determine the equations that would need to be solved in order to fit the model.
Assume a Gamma Distribution and the identity link function.
Assume equal exposures for each cell.

c. (2 points) Determine the equations that would need to be solved in order to fit the model.
Assume a Gamma Distribution and the inverse link function.
Assume equal exposures for each cell.

d. (3 points) Determine the equations that would need to be solved in order to fit the model.
Assume a Inverse Gaussian Distribution and the squared inverse link function.
Assume equal exposures for each cell.

For the Inverse Gaussian : $f(x) = \sqrt{\frac{\theta}{2\pi}}\ \frac{\exp\left[-\frac{\theta\left(\frac{x}{\mu} - 1\right)^2}{2x}\right]}{x^{1.5}}$*, mean =* μ*, variance =* μ^3 / θ.

3.213. (9, 11/10, Q.3) (3.5 points)
The following chart represents claim frequencies for a commercial auto book of business:

	Claim Frequencies (1,000 Vehicle-Years)		
	Private Passenger	Light Truck	Medium Truck
Territory A	10	12	20
Territory B	5	10	18

a. (2 points) Complete the first step in solving a generalized linear model by specifying the design matrix and vector of beta parameters.

b. (0.5 point) For each of the Poisson and gamma error structures, describe the relationship between the variance and the expected value and how these relationships differ.

c. (1 point) Once the link function and error structure have been selected, describe the process to determine the final beta parameters.

3.214. (8, 11/11, Q.3) *(1.5 points) An actuary is considering performing a one-way analysis to provide pricing guidance for an insurance company's personal auto book of business.*
a. (0.5 point) Briefly describe two shortcomings associated with one-way analyses.
b. (1 point) Provide an example of how each shortcoming in part a. above may arise.

3.215. (8, 11/12, Q.2) (2.25 points) A private passenger auto insurance company orders a report whenever it writes a policy, showing what other insurance the policyholder has purchased. The following table shows claim frequencies (per 100 earned car-years) for bodily injury liability coverage, split by whether the policyholder has a homeowners policy and whether the policyholder had a prior auto policy:

	Homeowners Policy	
Prior Auto Policy	Yes	No
Yes	3	5
No	8	12

The table does not include the experience of policyholders with missing data.
a. (1.25 points) Specify the following structural components of a generalized linear model that estimates frequencies for this book of business.
i. Error distribution
ii. Link function
iii. Vector of responses
iv. Vector of model parameters
v. Design matrix
b. (1 point) Describe how the missing data may cause problems for the company in developing the model, and suggest a solution.

3.216. (8, 11/12, Q.4) (1.75 points) An actuary has historical information relating to customer retention. A logistic model was used to estimate the probability of renewal for a given customer. The two variables determined to be significant were the size of rate change and number of phone calls the insured made to the company. The parameter estimates were determined to be as follows:

Rate Change	Parameter Estimate
Decrease to 3.9% increase	0.3323
4.0% to 6.9% increase	0
Increase of 7.0% or more	-0.4172

Number of Phone Calls in Past Year	Parameter Estimate
0	0
1	-0.2128
2+	-0.4239
Intercept Term	1.793

a. (0.75 point) Calculate the renewal probability for a customer who has a 7% rate increase and called the company twice in the past year.

b. (1 point) The company needs policyholder retention to be above 78% to maintain growth and expense ratio goals. A possible strategy is to add the number of phone calls to the classification plan and use the model results to determine the rate increase.

Construct an argument either in favor of or against the strategy above, describing two reasons for that position.

3.217. (8, 11/13, Q.2) (3.5 points)

An actuary at a private passenger auto insurance company wishes to use a generalized linear model to create an auto frequency model using the data below.

	Number of Claims	
Gender	Territory A	Territory B
Male	700	600
Female	400	420

	Number of Exposures	
Gender	Territory A	Territory B
Male	1,400	1,000
Female	1,000	1,200

The model will include three parameters: β_1, β_2, β_3, where β_1 is the average frequency for males, β_2 is the average frequency for Territory A, and β_3 is an intercept.

a. (0.5 point) Define the design matrix [X].

b. (0.25 point) Define the vector of responses [Y].

c. (2.25 points) Assuming $\beta_3 = 0.35$, solve a generalized linear model with a normal error structure and identity link function for β_1.

d. (0.5 point) The actuary determines that the analysis results would be improved by assuming a Poisson error structure with a log link function.
Identify two reasons this structure may better suit this data.

3.218. (8, 11/14, Q.3) (2 points) The random component of a generalized linear model must come from the exponential family of distributions. The variance of a distribution from the exponential family can be expressed using the following formula: $\mathrm{Var}(Y_i) = \dfrac{\phi\ V(\mu_i)}{\omega_i}$

a. (0.5 point) Define the parameters ϕ and ω_i in the formula above.

b. (1 point) For each of the data sets below, identify the error distribution that should be used to model the data. Briefly explain why that error distribution is appropriate.

i. Severity

ii, Policy Renewal Retention

c. (0.5 point) For each of the error distributions in part b. above, provide an example of how w_i should be assigned for the type of data being modeled.

3.219. (CAS S Sample Exam 2015, Q.4) (2 points)

An actuary wants to estimate the probability of a home insurance policy having a claim by using a logistic regression model. He has the following pieces of information from 1,000 historical policies:

- Cost of the home, in $000s
- Age of the home, in years
- Whether or not there was a claim on the policy

The actuary is considering a number of different model specifications. Below are the models he is considering along with the calculated scaled deviance of each model:

Model #	Included Variables	Scaled Deviance
1	Intercept + Cost	1085.0
2	Intercept + Cost + Age	1084.8
3	Intercept + Cost + (Cost * Age)	1083.0
4	Intercept + Cost + Cost^2 + Cost^3	1081.9
5	Intercept + Cost + Cost^2 + Cost^3 + Cost^4	1081.6

Determine the optimal model using the Bayesian Information Criterion.

Note: I have rewritten this past exam question, in order to match the syllabus of your exam.

3.220. (2 points) In the previous question, determine the optimal model using instead the Akaike Information Criterion.

3.221. (CAS S, 11/15, Q.32) (2 points)

A GLM is used to model claim size. You are given the following information about the model:

- Claim size follows a Gamma distribution.
- Log is the selected link function.
- The dispersion parameter f is estimated to be 2.
- Model Output:

Variable	$\hat{\beta}$
(Intercept)	2.32
Location - Urban	0.00
Location - Rural	-0.64
Gender - Female	0.00
Gender - Male	0.76

Calculate the variance of the predicted claim size for a rural male.

3.222. (CAS S, 11/15, Q.33) (2 points)

You are given the following output from a GLM to estimate the probability of a claim:

- Distribution selected is Binomial.
- Link selected is Logit.

Parameter	β
Intercept	-1.485
Vehicle Body	
Coupe	-0.881
Roadster	-1.047
Sedan	-1.175
Station wagon	-1.083
Truck	-1.118
Utility	-1.330
Driver's Gender	
Male	-0.025
Area	
B	0.094
C	0.037
D	-0.101

Calculate the estimated probability of a claim for:

- Driver Gender: Female
- Vehicle Body: Sedan
- Area: D

3.223. (CAS S, 11/15, Q.34) (1 point)
You are given the following information for a model of vehicle claim counts by policy:

- The response distribution is Poisson and the model has a log link function.
- The model uses two categorical explanatory variables: Number of Youthful Drivers and Number of Adult Drivers.
- The parameters of the model are given:

Parameter		Degrees of Freedom	$\hat{\beta}$
Intercept		1	-2.663
Number of Youthful Drivers			
	0		
	1	1	0.132
Number of Adult Drivers			
	1		
	2	1	-0.031

Calculate the predicted claim count for a policy with one adult driver and one youthful driver.

3.224. (CAS S, 11/15, Q.35) (2 points)
You are given a GLM of liability claim size with the following potential explanatory variables only:

- Vehicle price, which is a continuous variable modeled with a third order polynomial
- Average driver age, which is a continuous variable modeled with a first order polynomial
- Number of drivers, which is a categorical variable with four levels
- Gender, which is a categorical variable with two levels
- There is only one interaction in the model, which is between gender and average driver age.

Determine the maximum number of parameters in this model.

3.225. (CAS S, 11/15, Q.36) (2 points) You are given the following information for two potential logistic models used to predict the occurrence of a claim:

- Model 1: (AIC = 262.68)

Parameter	$\hat{\beta}$
(Intercept)	-3.264
Vehicle Value ($000s)	0.212
Gender-Female	0.000
Gender-Male	0.727

- Model 2: (AIC = 263.39)

Parameter	$\hat{\beta}$
(Intercept)	-2.894
Gender-Female	0.000
Gender-Male	0.727

- AIC is used to select the most appropriate model.

Calculate the probability of a claim for a male policyholder with a vehicle valued $12,000 by using the selected model.

3.226. (CAS S, 11/15, Q.38) (2 points)
You are testing the addition of a new categorical variable into an existing GLM.
You are given the following information:

- The change in model scaled deviance after adding the new variable is -53.
- The change in AIC after adding the new variable is -47.
- The change in BIC after adding the new variable is -32.
- Prior to adding the new variable, the model had 15 parameters.

Calculate the number of observations in the model.
Note: I have rewritten this past exam question, in order to match the syllabus of your exam.

3.227. (8, 11/15, Q.3) (2.5 points) An actuary is considering using a generalized linear model to estimate the expected frequency of a recently introduced insurance product.
Given the following assumptions:

- The expected frequency for a risk is assumed to vary by state and gender.
- A log link function is used.
- A Poisson error structure is used.
- The likelihood function of a Poisson is

$$l(y;\, \mu) = \sum \ln f(y_i;\, \mu_i) = \sum \{-\mu_i + y_i \ln[\mu_i] - \ln[y_i!]\}$$

- β_1 is the effect of gender = Male.
- β_2 is the effect of gender = Female.
- β_3 is the effect of State = State A.

	Claim Frequency	
	State A	State B
Male	0.0920	0.0267
Female	0.1500	0.0500

Given that $\beta_3 = 1.149$, determine the expected frequency of a male risk in State A.

3.228. (CAS S, 5/16, Q.29) (2 points) You are given the following information for a fitted GLM:

Response variable	Occurrence of Accidents
Response distribution	Binomial
Link	Logit

Parameter		df	$\hat{\beta}$
Intercept		1	x
Driver's Age		2	
	1	1	0.288
	2	1	0.064
	3	0	0
Area		2	
	A	1	-0.036
	B	1	0.053
	C	0	0
Vehicle Body		2	
	Bus	1	1.136
	Other	1	-0.371
	Sedan	0	0

The probability of a driver in age group 2, from area C and with vehicle body type Other, having an accident is 0.22.
Calculate the odds ratio of the driver in age group 3, from area C and with vehicle body type Sedan having an accident.

3.229. (CAS S, 5/16, Q.30) (2 points) You are given the following information for a fitted GLM:

Response variable	Occurrence of Accidents
Response distribution	Binomial
Link	Logit

Parameter	df	$\hat{\beta}$	se
Intercept	1	-2.358	0.048
Area	2		
Suburban	0	0.000	
Urban	1	0.905	0.062
Rural	1	-1.129	0.151

Calculate the modeled probability of an Urban driver having an accident.

3.230. (CAS S, 5/16, Q.31) (2 points) You are given the following information for a fitted GLM:

Response variable	Claim size
Response distribution	Gamma
Link	Log
Estimated alpha	1

Parameter		df	$\hat{\beta}$
Intercept		1	2.100
Zone		4	
	1	1	7.678
	2	1	4.227
	3	1	1.336
	4	0	0.000
	5	1	1.734
Vehicle Class		6	
	Convertible	1	1.200
	Coupe	1	1.300
	Sedan	0	0.000
	Truck	1	1.406
	Minivan	1	1.875
	Station wagon	1	2.000
	Utility	1	2.500
Driver Age		2	
	Youth	1	2.000
	Middle age	0	0.000
	Old	1	1.800

Calculate the predicted claim size for an observation from Zone 3,
with Vehicle Class Truck and Driver Age Old.

3.231. (CAS S, 5/16, Q.32) (2 points) You are given the following information for a fitted GLM:

Response variable	Claim size
Response distribution	Gamma
Link	Log
Estimated ϕ	1

Parameter		df	$\hat{\beta}$
Intercept		1	2.100
Zone		4	
	1	1	7.678
	2	1	4.227
	3	1	1.336
	4	0	0.000
	5	1	1.734
Vehicle Class		6	
	Convertible	1	1.200
	Coupe	1	1.300
	Sedan	0	0.000
	Truck	1	1.406
	Minivan	1	1.875
	Station wagon	1	2.000
	Utility	1	2.500
Driver Age		2	
	Youth	1	2.000
	Middle age	0	0.000
	Old	1	1.800

Calculate the variance of a claim size for an observation from Zone 4,
with Vehicle Class Sedan and Driver Age Middle age

3.232. (CAS S, 5/16, Q.33) (2 points)
You are given the following information for a GLM of customer retention:

Response variable	Retention
Response distribution	Binomial
Link	Logit

Parameter		df	$\hat{\beta}$
Intercept		1	1.530
Number of Drivers		1	
	1	0	0.000
	>1	1	0.735
Last Rate Change		2	
	<0%	0	0.000
	0%-10%	1	-0.031
	>10%	1	-0.372

Calculate the probability of retention for a policy with 3 drivers and a prior rate change of 5%.

3.233. (CAS S, 5/16, Q.35) (2 points) You are given the following information about three candidates for a Poisson frequency GLM on a group of condominium policies:

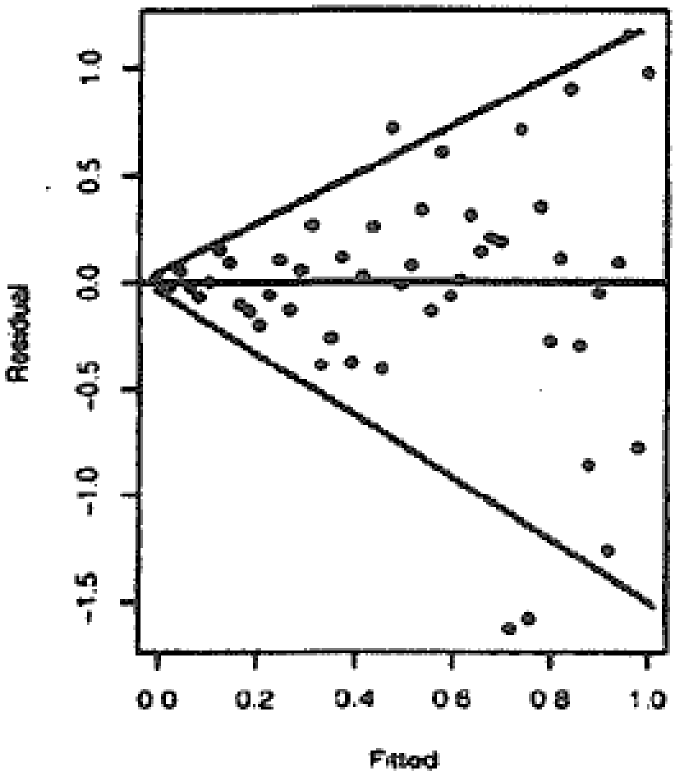
Model	Variables in the Model	DF	Log Likelihood	AIC	BIC
1	Risk Class	5	-47,704	95,418	95,473.61182
2	Risk Class + Region		-47,495		
3	Risk Class + Region + Claim Indicator	10	-47,365	94,750	

- Insureds are from one of five Risk Class: A, B, C, D, E
- Condominium policies are located in several regions
- Claim Indicator is either Yes or No
- All models are built on the same data

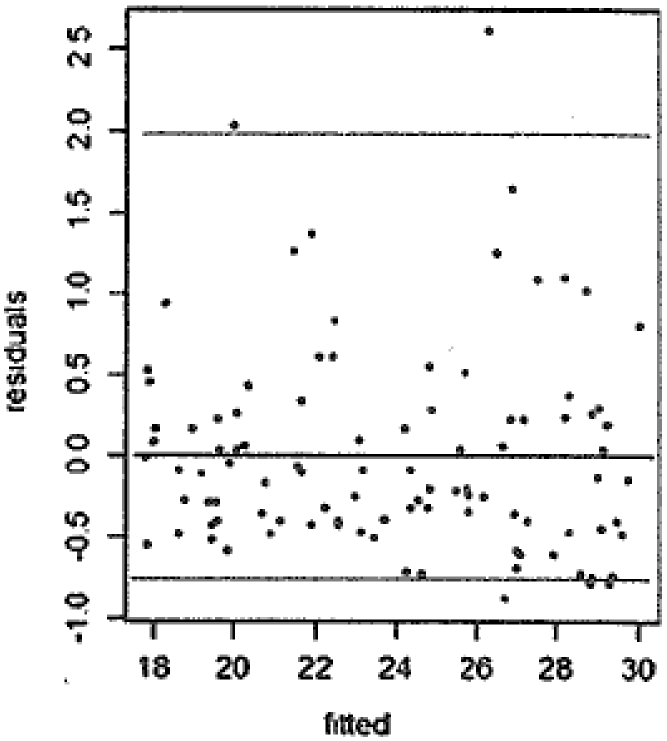
Calculate the absolute difference between the AIC and the BIC for Model 2.

3.234. (CAS S, 5/16, Q.36) (2 points) You are given the following two graphs comparing the fitted values to the residuals of two different linear models:

Graph 1



Graph 2



Determine which of the following statements are true.

I. Graph 1 indicates the data is homoscedastic

II. . Graph 1 indicates the data is heteroskedastic (a lack of homoscedasticity)

III. Graph 2 indicates the data is non-normal

3.235. (CAS S, 5/16, Q.37) (2 points)

Determine which of the following GLM selection considerations is true.

A. The model with the largest AIC is always the best model in model selection process.

B. The model with the largest BIC is always the best model in model selection process.

C. The model with the largest scaled deviance is always the best model in model selection process.

D. Other things equal, when the number of observations > 1000, AIC penalizes more for the number of parameters used in the model than BIC.

E. Other things equal, when number of observations > 1000, BIC penalizes more for the number of parameters used in the model than AIC.

Note: I have rewritten this past exam question, in order to match the syllabus of your exam.

3.236. (CAS S, 5/16, Q.38) (2 points) You are testing the addition of a new categorical variable into an existing GLM, and are given the following information:

- A is the change in AIC and B is the change in BIC after adding the new variable.
- $B > A + 25$
- There are 1500 observations in the model.

Calculate the minimum possible number of levels in the new categorical variable.

3.237. (CAS S, 5/16, Q.41) (1 point) A Poisson regression model with log link is used to estimate the number of diabetes deaths. The parameter estimates for the model are:

Response variable	Number of Diabetes Deaths
Response distribution	Poisson
Link	Log

Parameter	df	$\hat{\beta}$	p-value
Intercept	1	-15.000	< 0.0001
Gender: Female	1	-1.200	< 0.0001
Gender: Male	1	0.000	
Age	1	0.150	< 0.0001
Age^2	1	0.004	< 0.0001
Age ×Gender: Female	1	0.012	< 0.0001
Age ×Gender: Male	0	0.000	

Calculate the expected number of deaths for a population of 100,000 females age 25.

3.238. (8, 11/16, Q.4) (3 points) An actuary is conducting a generalized linear model (GLM) analysis on historical personal automobile data in order to develop a rating plan.

a. (1.5 points)
Argue against the following factors being included as predictors in the actuary's GLM analysis:
i. Limit of liability.
ii. Number of coverage changes during the current policy period.
iii. ZIP code of the garaging location of the automobile.

b. (1 point) The actuary is modeling pure premium with a log-link function and a Tweedie error distribution ($1 < p < 2$). Provide two arguments against the inclusion of deductible as a predictor in the actuary's GLM analysis.

c. (0.5 point) Other than including deductible as a predictor in the GLM, describe how to determine deductible relativities and how such relativities can be incorporated in a GLM.

3.239. (8, 11/16, Q.5) (2.25 points)
A GLM has been used to develop an insurance rating plan.
The results are given below:

Risk	Model Predicted Loss	Actual Loss
1	2,000	2,050
2	500	220
3	1,500	1,480
4	800	850
5	200	400

a. (1.75 points) Plot the Lorenz curve for this rating plan.
Label each axis and the coordinates of each point on the curve.

b. (0.5 point) Briefly describe how the Gini index is calculated and what the Gini index measures when applied to an insurance rating program. Do not calculate the Gini index.

3.240. (8, 11/16, Q.6) (2.5 points) An actuary has constructed a three-variable Tweedie GLM with a log-link function to estimate loss ratios for commercial property new business.
The actuary wants to create a second model for renewal business that will include all of the variables from the new business model, plus a variable for the prior year claim count.
The actuary requires that the coefficients of the variables: Average Building Age, log(Manual Premium), and Location Count, are consistent between the new and renewal models.
The fitted new business model parameters are as follows:

Variable	Name	Estimate
	intercept	0.910
Average Building Age (Years)	age	0.013
log(Manual Premium)	logprem	-0.187
Location Count	loccnt	0.062

a. (0.75 point) Calculate the modeled loss ratio for a new business policy with a manual premium of $25,000, an average building age of four years, and having eight locations.

b. (0.75 point) Briefly describe how to produce the renewal business model, and specify the resulting equation for the renewal business modeled loss ratio.

c. (1 point) Identify and briefly describe two techniques that the actuary can use to assess the stability of the new variable in the renewal business model.

3.241. (8, 11/16, Q.7) (1.5 points) A company is considering modifying its rating plan to include factors by age group. Below are statistics for the base model and for the new model.

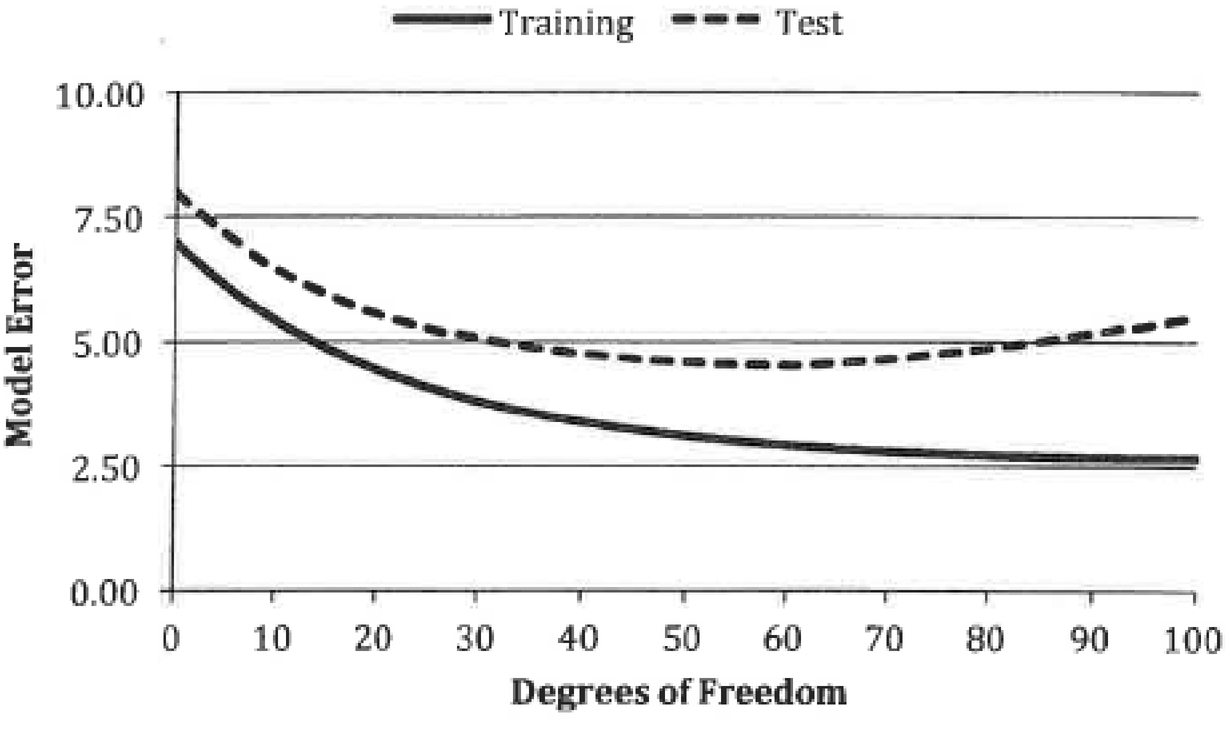
Statistic	Base Model	New Model
Loglikelihood	-750	-737.5
Scaled Deviance	500	475
Parameters	10	15
Data points	1,000,000	1,000,000

a. (1 point) Calculate the Akaike Information Criterion (AIC) and the Bayesian Information Criterion (BIC) for both models.

b. (0.25 point) Explain whether AIC or BIC is a more reliable test statistic as an indicator of whether to adopt the new model.

c. (0.25 point) Recommend and briefly justify whether to adopt the new model.

3.242. (8, 11/17, Q.4) (1.75 points) An actuary has split data into training and test groups for a model. The chart below shows the relationship between model performance and model complexity. Model performance is represented by model error and model complexity is represented by degrees of freedom.



a. (0.5 point) Briefly describe two reasons for splitting modeling data into training and test groups.

b. (0.75 point) Briefly describe whether each of the following model iterations has an optimal balance of complexity and performance.

i. Model iteration 1: 10 degrees of freedom
ii. Model iteration 2: 60 degrees of freedom
iii. Model iteration 3: 100 degrees of freedom

c. (0.5 points) Identify and briefly describe one situation where it is an advantage to split the data by time rather than by random assignment.

3.243. (8, 11/17, Q.5) (1.75 points) An analyst has fit several different variations of a logistic GLM to a dataset containing 1,000 records of fraudulent claims and 9,000 records of legitimate claims.
For each model variation listed below, draw a quintile plot based on the training data.
Label the axes and identify each data series.
i. A saturated model
ii. A null model
iii. A model that could be used in practice

3.244. (8, 11/17, Q.6) (3.5 points) A logistic model was built to predict the probability of a claim being fraudulent. Consider the predicted probabilities for the 10 claims below to be a representative sample of the total model.

Claim Number	Actual Fraud Indicator	Predicted Probability of Fraud
1	Y	11%
2	N	23%
3	N	15%
4	N	70%
5	Y	91%
6	Y	30%
7	N	11%
8	Y	75%
9	N	58%
10	N	27%

a. (1 point) Construct confusion matrices for discrimination thresholds of 0.50 and 0.25.
b. (1.5 points) Plot the Receiver Operating Characteristic (ROC) curve with the discrimination thresholds of 0.50 and 0.25.
Label each axis and the coordinates and discrimination threshold of each point on the curve.
c. (0.5 point) Describe an advantage and a disadvantage of selecting a discrimination threshold of 0.25 instead of 0.50.
d. (0.5 point) Describe whether a discrimination threshold of 0.25 or 0.50 is more appropriate for a line of business with low frequency and high severity.

3.245. (MAS-1, 5/18, Q.25) (2.2 points)
Three separate GLMs are fit using the following model form: $g(Y) = \beta_0 + \beta_1 X_1 + \beta_2 X_2$
The following error distributions were used for the three GLMs.
Each model also used their canonical link functions:

Model I: gamma
Model II: Poisson
Model III: binomial

When fit to the data, all three models resulted in the same parameter estimates:

$\hat{\beta}_0$	2.0
$\hat{\beta}_1$	1.0
$\hat{\beta}_2$	-1.0

Determine the correct ordering of the models' predicted values at observed point $(X_1, X_2) = (2, 1)$.

3.246. (MAS-1, 5/18, Q.39) (2.2 points) A GLM was used to estimate the expected losses per customer across gender and territory. The following information is provided:

- The link function selected is log
- Q is the base level for Territory
- Male is the base level for Gender
- Interaction terms are included in the model

The GLM produced the following predicted values for expected loss per customer:

	Territory	
Gender	Q	R
Male	148	545
Female	446	4,024

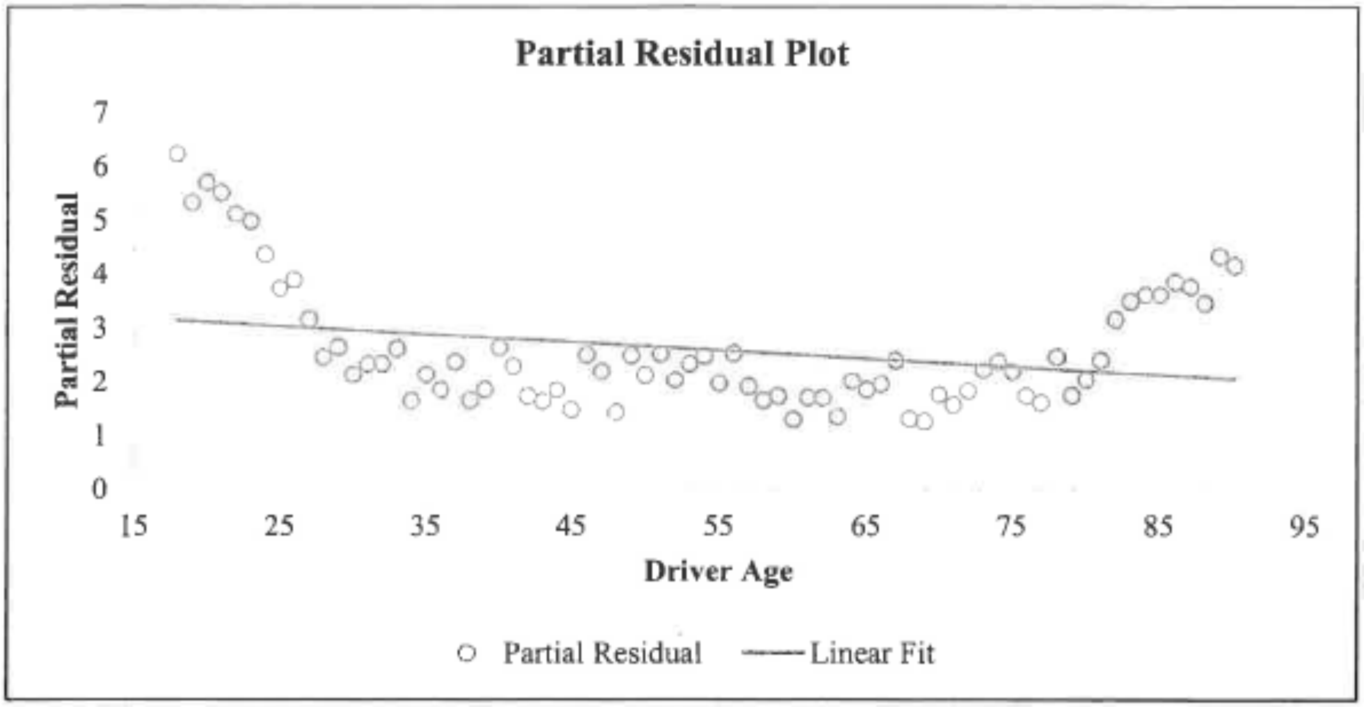
Calculate the estimated beta for the interaction of Territory R and Female.

3.247. (MAS-1, 5/18, Q.27) (2.2 points)
You are given the following information about an insurance policy:

- The probability of a policy renewal, p(X), follows a logistic model with an intercept and one explanatory variable.
- $\beta_0 = 5$
- $\beta_1 = -0.65$

Calculate the odds of renewal at x = 5.

3.248. (8, 11/18, Q.5) (1.5 points)
An actuary is analyzing a partial residual plot of the driver age variable, which is shown below:



a. (1 point)
Adding polynomial terms is one approach to address the non-linearity in the driver age variable. Briefly describe two other alternative approaches and how they can be used to improve the fit of the driver age variable shown above.
b. (0.5 point) Briefly describe a downside to each of the two alternative approaches recommended in part a. above.

3.249. (8, 11/18, Q.6) (2.5 points) An actuary is comparing the output of two generalized linear models to develop a new rating plan for personal auto. Model statistics are shown below:

	Saturated Model	Model A	Model B
Log-Likelihood	-1,000	-1,500	-1,465
Estimated Dispersion Parameter	1.75	1.75	1.75

- Model A is a nested model of Model B,
  where Model B has an additional variable for driver age.
- Driver age is fit using a second-order polynomial.
- The critical value to be used from the F-distribution is 3.183.

a. (2 points) Using two statistical tests, recommend whether or not driver age should be included in the rating plan.
b. (0.5 point) Describe why the deviance statistic alone should not be used to assess model fit.
Note: I have slightly rewritten this past exam question.

3.250. (8, 11/18, Q.7) (2.5 points) An actuary is planning to add a credit-based insurance score to a model that estimates the probability of a policy having a claim. The actuary has decided to offset all of the current model variables before fitting the new variable.

Given the following:

- The current model (without the insurance score variable) is a logit-link binomial GLM (logistic regression).
- The logit link function is defined as $g(\mu) = \ln(\frac{\mu}{1 - \mu})$
- The insurance score is a continuous variable having a value between 1 and 100.
- The current fitted values and insurance score for three policies as well as regression results from the fit of the insurance score variable are given below:

Policy Number	Fitted Probability Without Insurance Score	Insurance Score
1	1.3%	78
2	20.3%	92
3	2.5%	35

Variable	Parameter Estimate
Intercept	1.250
Insurance Score	-0.020

a. (0.5 point)
Calculate the offset term to be used in the regression for each of the three policies above.

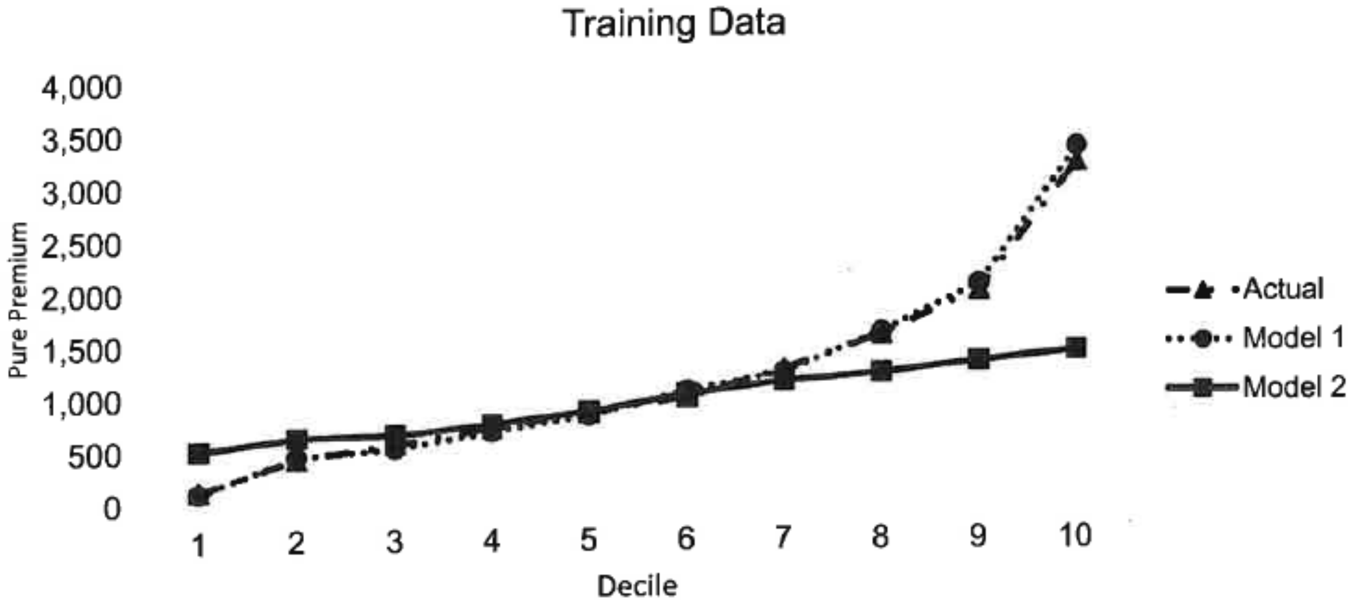
b. (0.75 point)
Calculate the revised fitted probability of having a claim for each of the three policies above.

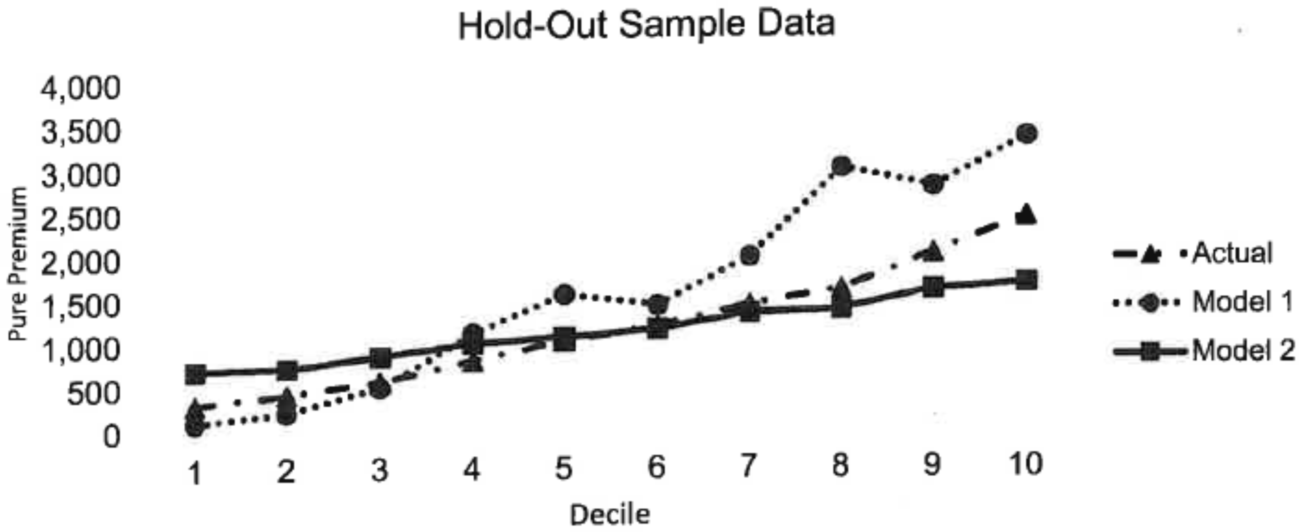
c. (0.5 point) Identify the range of:
i. the logit function
ii. the logistic function

d. (0.25 point) Briefly explain why logistic regression is often used to model probabilities.

e. (0.5 point) Identify and briefly describe one situation in which the use of an offset is preferable to (re)fitting all variables.

3.251. (5, 5/19, Q.9) (1 point)
The following graphs show two competing generalized linear models' (GLMs) predictions versus the data used in modeling ("training") and a hold-out sample. Data in each graph has been sorted into equal volume deciles, ranked from low to high actual loss.

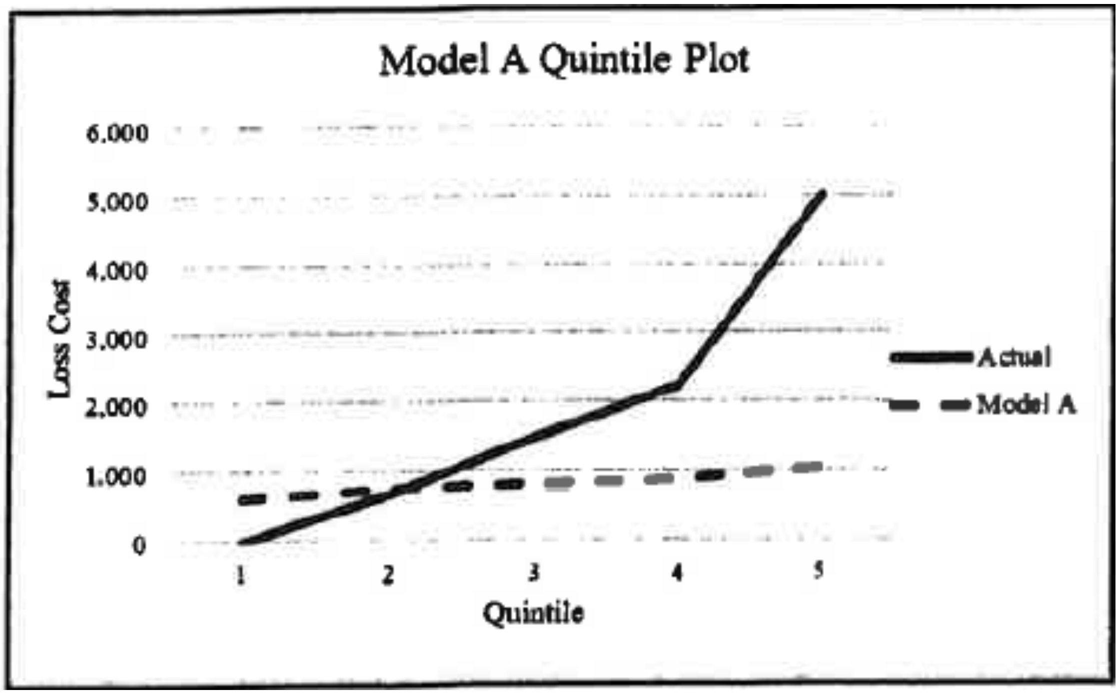




Assess each of the models.

3.252. (8, 11/19, Q.2) (2.75 points) An actuary has built two generalized linear models to predict loss costs. Management has requested a series of model validation plots to demonstrate the appropriateness of each of the new models.
Output for each model, simple quintile plots, and a double lift plot are shown below:

Observation	Actual Loss Cost	Model A Loss Cost	Model B Loss Cost	Earned Premium
1	1,500	825	900	1,800
2	675	765	800	1,450
3	0	615	350	2,375
4	2,250	900	3,000	2,625
5	5,000	1,050	3,700	4,875



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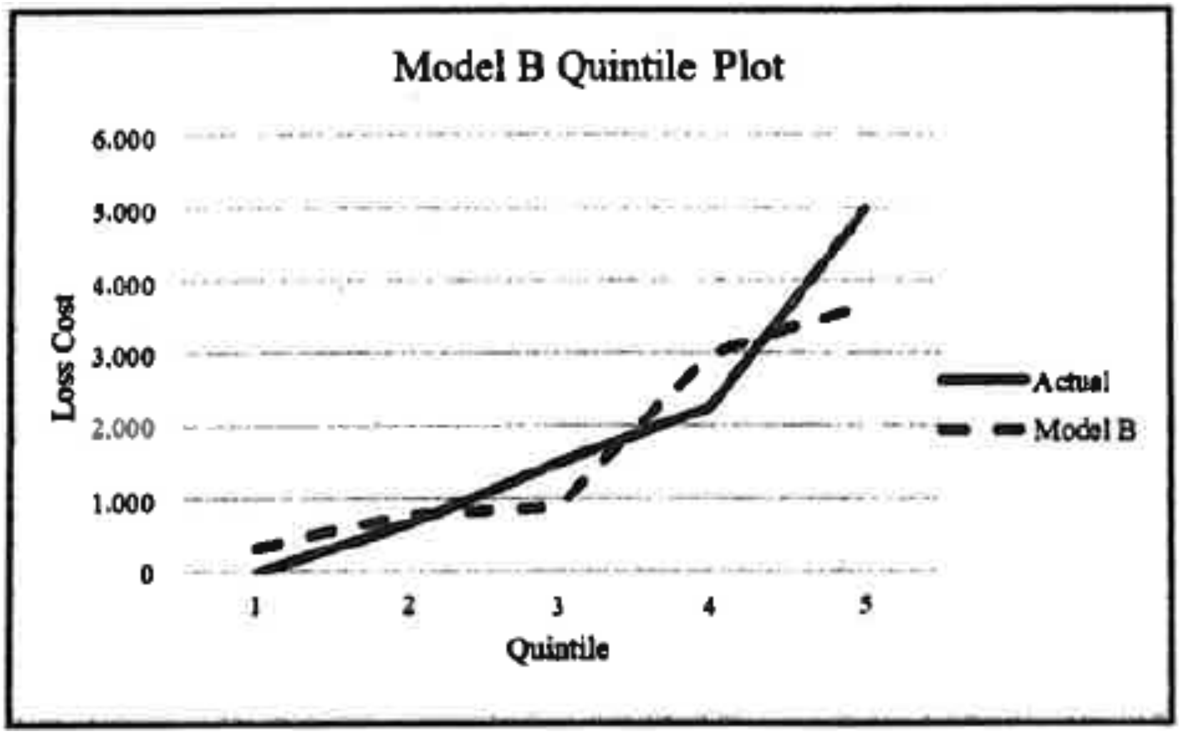

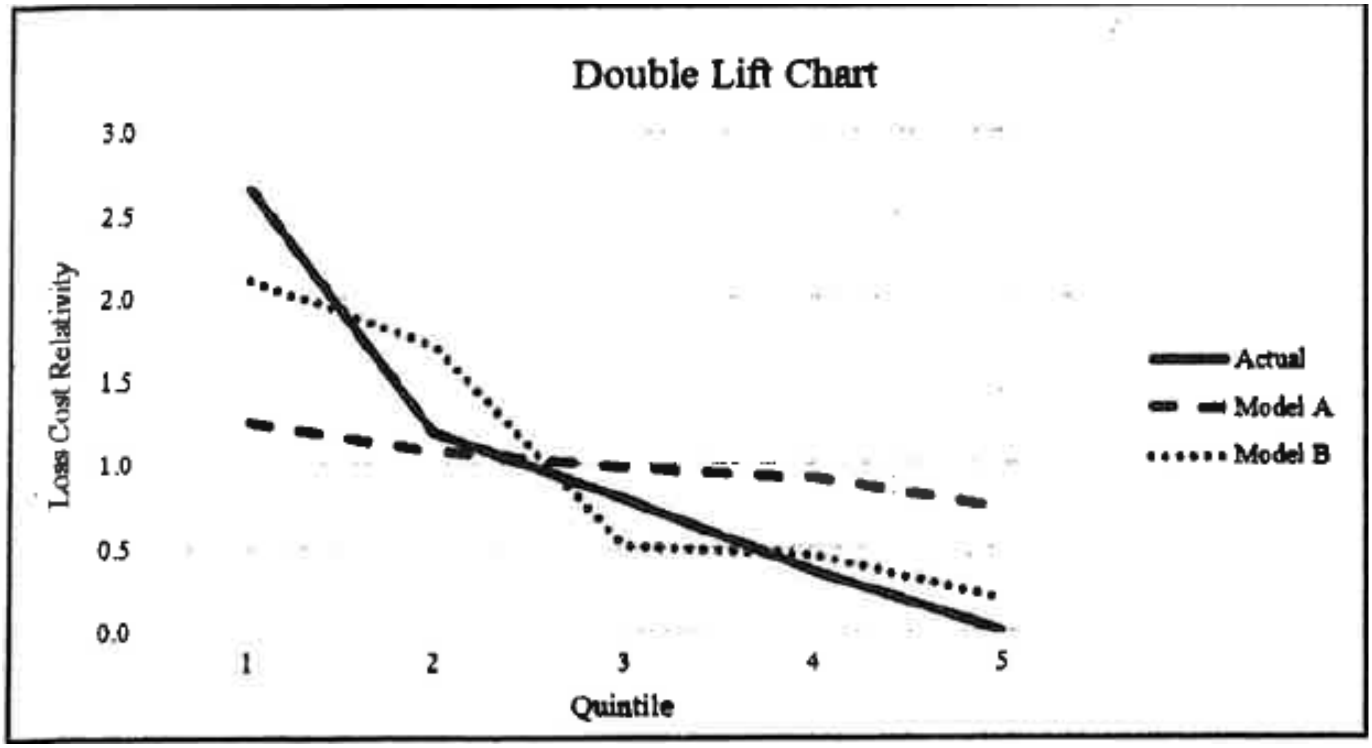

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Given the following:

- The actuary has already provided management with the simple quintile plots and the double lift chart shown above.
- The company has implemented several segmented rate changes over the last three years.

a. (1 point) For each model, provide a loss ratio plot that management can use to assess lift. Identify the basis of sorting the data.

b. (0.75 point) Briefly describe one drawback of each type of model validation plot that the actuary has provided to management, including the plot produced in part a. above.

c. (1 point) Using all three types of model validation plots provided to management, recommend which model should be implemented. Do not perform any calculations.

3.253. (8, 11/19, Q.5) (2.75 points) The following confusion matrix shows the result from a claim fraud model with a discrimination threshold of 25%:

	Predicted	
Actual	Yes	No
Yes	72	162
No	63	1203

a. (0.5 point) Identify a link function that can be used for a generalized linear model that has a binary target variable and briefly explain why this link function is appropriate.

b. (0.5 point) Calculate the sensitivity and specificity from the above data.

c. (1.5 points) Plot the receiver operating characteristic (ROC) curve with the discrimination threshold of 25%. Label each axis, the coordinates, and the discrimination thresholds of 100%, 25%, and 0% on the curve.
In addition, plot the ROC curve for each of the following two models:
i. A model with no predictive power
ii. A hypothetical "perfect" model

d. (0.25 point)
Briefly describe how the severity of claims will impact the selection of the model threshold.

3.254. (8, 11/19, Q.6) (2 points) An actuary creates a generalized linear model (GLM) to estimate commercial property claim frequency by occupancy class and amount of insurance (AOI) for sprinklered and non-sprinklered risks. Given the following:

- Occupancy class is a categorical variable with four levels: class 1, 2, 3 and 4.
- Sprinklered status is a categorical variable with two levels: sprinklered and non-sprinklered.
- The natural log of AOI, ln(AOI), is a continuous variable.
- The log link function is selected.
- An interaction variable is included as ln(AOI) for sprinklered and zero otherwise.
- The model results are as follows:

Parameter	Coefficient
Intercept	-8.4607
Occupancy class 2	0.2714
Occupancy class 3	0.3620
Occupancy class 4	0.0395
Sprinklered	0.7228
ln(AOI)	0.4311
Sprinklered: Yes, ln(AOI)	-0.0960

a. (0.75 point) Calculate the ratio of the estimated model frequency of a sprinklered property to that of a non-sprinklered property for AOI = 200,000 and occupancy class 2.

b. (0.75 point) Calculate the intercept term if AOI is centered at the base level of 200,000.

c. (0.5 point)
Briefly describe two advantages of centering variables of a GLM at their base levels.

Solutions:

3.1. Ignoring the loglikelihood of the saturated model, which is a constant,
AIC = Scaled Deviance + (number of parameters)(2).
For example, AIC = 335.6 + (6)(2) = 347.6.

Model	Number of Parameters	Scaled Deviance	AIC
A	6	335.60	347.60
B	8	331.90	347.90
C	10	325.20	345.20
D	12	321.40	345.40
E	14	317.00	**345.00**

Since AIC is smallest for model E, model E is preferred.

3.2. When using categorical variables, it is important to set the base level to be one with populous data, so that our measures of significance will be most accurate.
By choosing the base level to be one with lots of data, the estimates of the coefficients for the non-base levels are more stable.

3.3. This allows the scale of the predictors to match the scale of the entity they are linearly predicting, which in the case of a log link is the log of the mean of the outcome.
When a logged continuous predictor is placed in a log link model, the resulting coefficient becomes a power transform of the original variable. The coefficient b_1 becomes an exponent applied to the original variable x_1.
Including continuous predictors in their logged form allows a log link GLM flexibility in fitting the appropriate response curve. On the other hand, if the variable x is not logged, the response curve for any positive coefficient will always have the same basic shape: exponential growth, that is, increasing at an increasing rate.

3.4. Frequently, the dataset going into a GLM will include records that represent the averages of the outcomes of groups of similar risks rather than the outcomes of individual risks.
In such instances, it is intuitive that records that represent a greater number of risks should carry more weight in the estimation of the model coefficients, as their outcome values are based on more data. GLMs accommodate that by allowing the user to include a weight variable, which specifies the weight given to each record in the estimation process.
The weight is the number of exposures for frequency or pure premium models.
For severity models, the weight is the number of claims.
The weight variable, usually denoted ω, formally works its way into the math of GLMs as a modification to the assumed variance: $\mathrm{Var}[y_i] = \frac{\phi\ V(\mu_i)}{\omega_i}$.

3.5. Determining accurate estimates of relativities in the presence of moderately correlated rating variables is a primary strength of GLMs versus univariate analyses. Unlike univariate methods, the GLM will be able to sort out each variable's unique effect on the outcome, as distinct from the effect of any other variable that may correlate with it, thereby ensuring that no information is double-counted.

3.6. exp[0.4] - 1 = 49.2%.
<u>Comment</u>: For a logistic model: Odds = μ / (1 - μ).

3.7. Both are discrete distributions used to model frequency.
Both have support from zero to infinity. Both have $\phi = 1$.
The Negative Binomial Distribution has an additional parameter $k > 0$, called the overdispersion parameter.
The Poisson Distribution has variance function $V(\mu) = \mu$, while the Negative Binomial Distribution has variance function $V(\mu) = \mu(1 + \kappa\mu)$. Thus the Negative Binomial Distribution has a variance greater than its mean, while the Poisson has a variance equal to its mean.
The Negative Binomial Distribution has a heavier righthand tail than the Poisson Distribution.
<u>Comment</u>: One way a Negative Binomial Distribution can arise is as a Gamma mixture of Poissons.

3.8. Where two predictors are perfectly correlated, they are said to be aliased, and the GLM will not have a unique solution.

3.9. 1. GLMs assign <u>full</u> credibility to the data.
2. GLMs assume that the randomness of outcomes are uncorrelated.

3.10. "**Continuity in the Estimates is Not Guaranteed.** Allowing each interval to move freely may not always be a good thing. The ordinal property of the levels of the binned variable have no meaning in the GLM; there is no way to force the GLM to have the estimates behave in any continuous fashion, and each estimate is derived independently of the others. Therefore, there is a risk that some estimates will be inconsistent with others due to random noise."
Variation within Intervals is Ignored. Since each bin is assigned a single estimate, the GLM ignores any variation that may exist within the bins.
<u>Comment</u>: See Section 5.4.2 of <u>Generalized Linear Models for Insurance Rating</u>.

3.11. The fitted parameter(s) are the same, while the standard errors are multiplied by $\sqrt{7.9435}$.

The standard error of $\hat{\beta}_1$ is: $0.00120\sqrt{7.9435} = 0.00338$.

95% confidence interval for β_1: 0.02085 ± (1.96) (0.00338) = **0.02085 ± 0.00662**.

<u>Comment</u>: One could instead use: 0.02085 ± (2) (0.00338) = 0.02085 ± 0.00676.

3.12. The Tweedie Distribution is an (linear) exponential family,
used for modeling pure premiums.
Besides the usual parameters μ and ϕ, the Tweedie Distribution has a power parameter p.
The variance function for Tweedie is $V(\mu) = \mu^p$. For use in GLMs we usually take $1 < p < 2$.
The Tweedie Distribution can be represented as a compound Poisson with a Gamma severity.
One rather interesting characteristic of the Tweedie distribution is that several of the other exponential family distributions are in fact special cases of Tweedie, dependent on the value of p.

3.13. It is clear that the proposed model more accurately predicts actual pure premium by decile than does the current rating plan. Specifically, consider the first decile. It contains the risks that the model thinks are best relative to the current plan. As it turns out, the model is correct. Similarly, in the 10th decile, the model more accurately predicts pure premium than does the current plan.
<u>Comment</u>: Graph taken from “Introduction to Predictive Modeling Using GLMs A Practitioner’s Viewpoint,” a presentation by Dan Tevet and Anand Khare.

3.14. The use of a log link results in the linear predictor, which begins as a series of additive terms, transforming into a series of multiplicative factors when deriving the model prediction. Multiplicative models are the most common type of rating structure used for pricing insurance, due to a number of advantages they have over other structures.

3.15. The sensitivity is: $\dfrac{\text{true positives}}{\text{total times there is an event}}$ = 700 / 1000 = 0.70.

The specificity is: $\dfrac{\text{true negatives}}{\text{total times there is not an event}}$ = 6000 / 8000 = 0.75.

For this threshold, we graph the point: (1 - specificity , sensitivity) = **(0.25, 0.70)**.

3.16. 1. Setting of objectives and goals.
Determine the goals. Determine appropriate data to collect. Determine the time frame.
What are key risks and how can they be mitigated?
Who will work on the project; do they have the necessary knowledge and expertise?

2. Communicating with key stakeholders.
Legal and regulatory compliance. Information. Technology (IT) Department.
Underwriters. Agents.

3. Collecting and processing the necessary data for the analysis.
Time-consuming. Data is messy. Often an iterative process. The data should also be split into at least two subsets, so that the model can be tested on data that was not used to build it.
Formulate a strategy for validating the model.

4. Conducting exploratory data analysis (EDA).
Spend some time to better understand the nature of the data and the relationships between the target and explanatory variables. Helpful EDA plots include:
Plotting each response variable versus the target variable to see what (if any) relationship exists.
Plotting continuous response variables versus each other, to see the correlation between them.

5. Specifying the form of the predictive model.
What type of predictive model works best?
What is the target variable, and which response variables should be included?
Should transformations be applied to the target variable or to any of the response variables?
Which link function should be used?

6. Evaluating the model output.
Assessing the overall fit of the model.
Identifying areas in which the model fit can be improved.
Analyzing the significance of each predictor variable, and removing or transforming variables accordingly.
Comparing the lift of a newly constructed model over the existing model or rating structure.

7. Validating the model.
Assessing fit with plots of actual vs. predicted on holdout data. Measuring lift.
For Logistic Regression, use Receiver Operating Characteristic (ROC) Curves.

8. Translating the model results into a product.
For GLMs, often the desired result is a rating plan.
The product should be clear and understandable.
Are there other rating factors included in the rating plan that were not part of the GLM?

9. Maintaining the model.
Models should be periodically rebuilt in order to maximize their predictive accuracy, but in the interim it may be beneficial to merely refresh the existing model using newer data.

10. Rebuilding the model.
More frequently one would update the classification relativities without updating the rating algorithm or classification definitions. Less frequently, one would do a more complete update, investigating changing the classification definitions, the predictor variables used, and/or the rating algorithm.

Comment: See Chapter 3 of Generalized Linear Models for Insurance Rating.

3.17. (a) Concentrate on one of the explanatory variables X_j.

The partial residuals are: (ordinary residual) $g'(\hat{\mu}_i) + x_{ij}\,\hat{\beta}_j$.

(b) In a Partial Residual Plot, we plot the partial residuals versus the variable of interest.
If there seems to be curvature rather than linearity in the plot, that would indicate a departure from linearity between the explanatory variable of interest and $g(\mu)$, adjusting for the effects of the other independent variables.

3.18. The second model includes an interaction term.
In the second model, the effect of X_1 depends on the level of X_2 and vice-versa.
In contrast, for the first model, the effects of X_1 and X_2 are independent.

3.19. "**Check for duplicate records.** If there are any records that are exactly identical, this likely represents an error of some sort. This check should be done prior to aggregation and combination of policy and claim data."
"**Cross-check categorical fields against available documentation.** If data base documentation indicates that a roof can be of type A, B, or C, but there are records where the roof type is coded as D, this must be investigated. Are these transcription errors, or is the documentation out of date?"
"**Check numerical fields for unreasonable values.** For every numerical field, there are ranges of values that can safely be dismissed as unreasonable, and ranges that might require further investigation. A record detailing an auto policy covering a truck with an original cost (new) of $30 can safely be called an error. But if that original cost is $5,000, investigation may be needed."
Comment: Quoted from Section 4.2 of Generalized Linear Models for Insurance Rating t.
"**Decide how to handle each error or missing value that is discovered.** The solution to duplicate records is easy, delete the duplicates. But fields with unreasonable or impossible values that cannot be corrected may be more difficult to handle."

3.20.
1. Plot each response variable versus the target variable, to see what if any relationship exists.
2. Plot continuous response variables versus each other, to see the correlation between them.

3.21. Advantages of the frequency/severity approach over pure premium modeling:
● Provides the actuary with more insight.
● Each of frequency and severity is more stable than pure premium.
Disadvantages of pure premium modeling versus the frequency/severity approach:
● Some interesting effects may go unnoticed.
● Pure premium modeling can lead to underfitting or overfitting.
● The Tweedie distribution used to model pure premium contains the implicit assumption that an increase in pure premiums is made up of an increase in both frequency and severity.

3.22. In general, an offset factor is a vector of known amounts which adjusts for known effects not otherwise included in the GLM. For example, one could take the current territories and territory relativities as givens, and include an offset term in a GLM for the current territory relativity. In general: $g(\mu_i) = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + ...+ \beta_p x_{ip}$ + offset.

An offset is used with a Poisson Distribution and a log link function, and there are exposures associated with each observation. In that case, the offset term is ln(exposure) = $\ln(n_i)$.

Then the model is: $\ln(Y_i) = \ln(n_i) + \eta_{i.} \Leftrightarrow Y_i = n_i \exp[\eta_i]$.

3.23. The observation for Slovakia has by far the biggest Cook's Distance, and is thus the most influential. The observations for the Czech Republic and Slovenia are less influential than Slovakia, but more influential than the others.

3.24. The saturated model has an equal number of predictors as there are records in the dataset. Since the saturated model predicts each record perfectly it is the theoretical best a model can possibly do.

The null model has only an intercept and no predictors. The null model produces the same prediction for every record: the grand mean.

The scaled deviance for the saturated model is zero, while the scaled deviance of the null model can be thought of as the total deviance inherent in the data. The scaled deviance for your model will lie between those two extremes.

3.25. The deviance residuals seem to be on average positive for small and large values of X_2, while being on average negative for middle values of X_2. Such a pattern is not good. This indicates that one should investigate other possible forms of the model, for example, a model including a term involving X_2^2.

3.26. Adding credit score adds 6 - 1 = 5 parameters to the model.

Test statistic is:

$$F = \frac{D_S - D_B}{(\text{number of added parameters})\ \hat{\phi}_B} = \frac{(233{,}183.65 - 233{,}134.37) / 5}{2.371} = 4.157.$$

The number of degrees of freedom in the numerator is 5.

The number of degrees of freedom in the denominator is:

number of observations minus the number of parameters in the bigger model

= 100,000 - 16 = 99,984.

We compare the test statistic to an F-distribution with 5 and 99,984 degrees of freedom.

The null hypothesis is to use the simpler model.

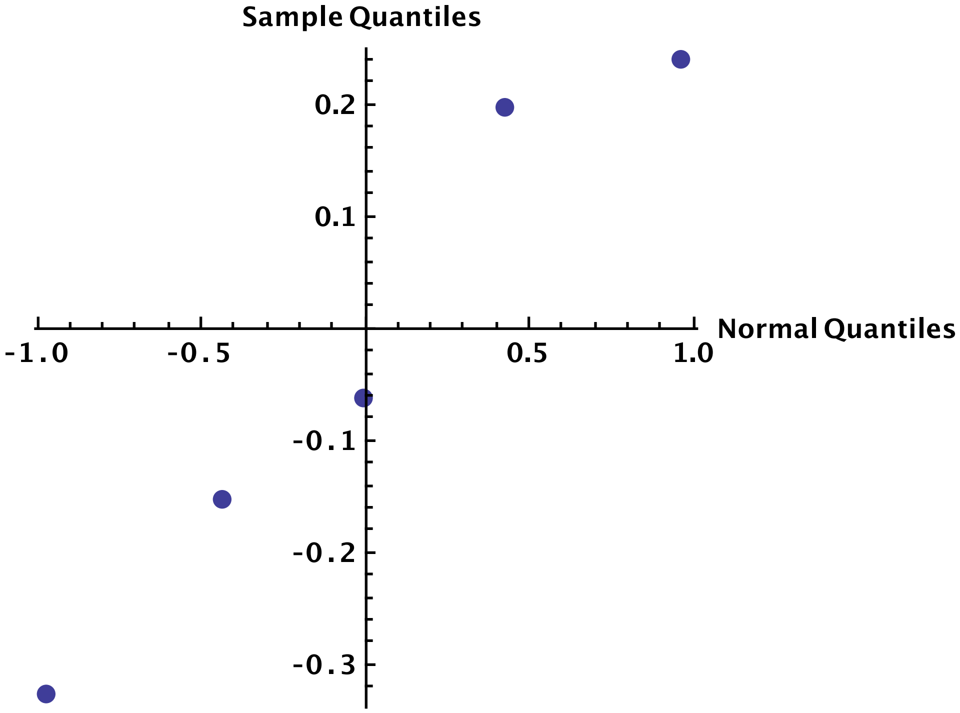
The alternate hypothesis is to use the more complex model including credit score.

We reject the null hypothesis when the F-Statistic is big.

Comment: Using a computer, the p-value of this test is 0.09%.

Thus one would use the more complex model including credit score rather than the simpler model.

3.27. Arranged from smallest to largest: -0.328, -0.154, -0.064, 0.195, 0.239.
Plot ($Q_{i/6}$, $x_{(i)}$).
$Q_{1/6}$ = -0.967, since Φ[-0.967] = 1/6. $Q_{2/6}$ = -0.431. $Q_{3/6}$ = 0. $Q_{4/6}$ = 0.431. $Q_{5/6}$ = 0.967.
Thus the five plotted points are:
(-0.967, -0.328), (-0.431, -0.154), (0, -0.064), (0.431, 0.195), (0.967, 0.239).
Here are the 5 points plotted:



There is too little data to decide whether or not these stock price returns are Normally distributed.

3.28. Gini index = **2A**.

<u>Comment</u>: Gini index = $\frac{\text{Area A}}{\text{Area A + Area B}}$.

However, Area A + Area B add up to a triangle with area 1/2.

Therefore, Gini index = $\frac{\text{Area A}}{\text{Area A + Area B}}$ = 2A

= twice the area between the Lorenz Curve and the line of equality = 1 - 2B.

<u>3.29</u>. Factors for coverage options should be estimated outside the GLM, using traditional actuarial techniques. The resulting factors should then be included in the GLM as an offset.

3.30. Min[X - c, 0], where c is some constant and X is a variable.
For example, Min[X_2 - 13, 0] is a hinge function.

3.31. Model D is preferred. Bigger Area Under ROC Curve (AUROC) is better.

3.32. $35 \pm 1.96\sqrt{5}$ = **(30.62, 39.38)**.

3.33. BIC = (-2) (maximum loglikelihood) + (number of parameters)ln[400].
For example, BIC = (-2)(-730.18) + 3 ln[400] = 1478.33.

Model	Number of Parameters	Loglikelihood	BIC
A	3	-730.18	1478.33
B	4	-726.24	**1476.45**
C	5	-723.56	1477.08
D	6	-721.02	1477.99
E	7	-717.50	1476.94

Since BIC is smallest for model B, model B is preferred.

3.34. exp[-3.8 + (0.4)ln[1/2]] = **1.7%**.
Comment: Loosely based on Table 12 in Generalized Linear Models for Insurance Rating, by Goldburd, Khare and Tevet.

3.35. exp[-3.8 + 0.3 - 0.5 + (0.4) ln[2.5/2] - (0.1) ln[2.5/2]] = **2.0%**.

3.36. exp[-3.8 + 0.5 + (0.4)ln[3/2]] = **4.3%**.

3.37. exp[-3.8 + 0.1 - 0.5 + (0.4)ln[6/2] - (0.1) ln[6/2]] = **2.1%**.

3.38. Histogram A most closely matches the Normal Distribution.

3.39. a) Identity link function.
b) Log link function.
c) Poisson Distribution.
d) For the variance proportional to the square of the mean, use the Gamma Distribution.

3.40. The partial residual plot is not linear; thus, we should do something to improve the model. Since the slope seems to change somewhere around 50 or 60, we could use a hinge function: Min[0, X_4 - 50] or Min[0, X_4 - 60].
Comment: In general, we could instead group the variable, or add polynomial terms to the model.

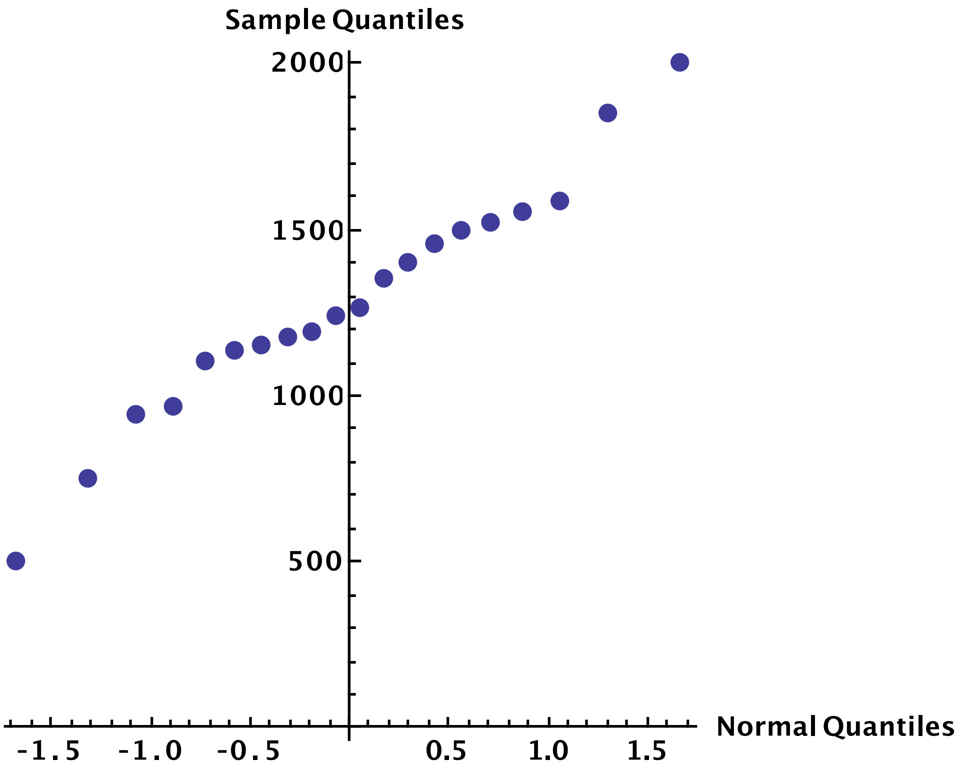
3.41. We can divide the original data into three sets.
We fit GLMs to the training data, until we have one or more good candidate models.
Then we see how these models perform on the validation set.
Based on what we find out, we can go back and fit some other GLMs to the training data.
The validation set is used to refine the models during the building process.
The test set (holdout data) is held out until the end.
We compare the performance of models on the test set to pick a final model to use.

3.42.

Distribution	V(μ)
Normal	$\mu^0 = 1$
Poisson	$\mu^1 = \mu$
Gamma	μ^2
Binomial (one trial)	μ (1-μ)
Inverse Gaussian	μ^3
Tweedie	μ^p, p < 0, 1 < p < 2, or p > 2.

Alternately, for the Binomial Distribution, V(μ) = μ (1 - μ/m).

3.43. $Q_{1/21}$ = -1.668, since Φ[-1.668] = 1/21.
Thus the first plotted point is: (-1.668 , 500).
The Q-Q Plot:



3.44. For a Poisson, $f(n) = e^{-\lambda}\lambda^n/n!$.

$\ln f(n) = -\lambda + n\ln\lambda - \ln(n!) = -\exp[\beta_0 + \beta_1X_{1i} + \beta_2X_{2i}] + n_i(\beta_0 + \beta_1X_{1i} + \beta_2X_{2i}) - \ln(n_i!)$.

loglikelihood = $-\sum\exp[\beta_0 + \beta_1X_{1i} + \beta_2X_{2i}] + \sum Y_i(\beta_0 + \beta_1X_{1i} + \beta_2X_{2i})$ + constants.

Setting the partial derivatives of the loglikelihood with respect to β_0, β_1, and β_2 equal to zero:

$0 = -\sum\exp[\beta_0 + \beta_1X_{1i} + \beta_2X_{2i}] + \sum Y_i$.

$0 = -\sum X_{1i}\exp[\beta_0 + \beta_1X_{1i} + \beta_2X_{2i}] + \sum Y_iX_{1i}$.

$0 = -\sum X_{2i}\exp[\beta_0 + \beta_1X_{1i} + \beta_2X_{2i}] + \sum Y_iX_{2i}$.

$\sum Y_i = 8 + 8 + 10 + ... + 33 + 31 = 369$.

$\sum Y_iX_{1i} = 8\ln(2) + 8\ln(4) + 10\ln(6) + ... + 33\ln(18) + 31\ln(20) = 872.856$.

$\sum Y_iX_{2i} = 14 + 19 + ... + 33 + 31 = 241$.

$\exp[\beta_0 + \beta_1X_{1i} + \beta_2X_{2i}] = \exp[\beta_0]\exp[\beta_1X_{1i}]\exp[\beta_2X_{2i}] = \exp[\beta_0]\exp[X_{1i}]^{\beta_1}\exp[\beta_2X_{2i}]$.

The first equation becomes:

$\exp[\beta_0]\,\{2^{\beta_1} + 4^{\beta_1} + ... + 20^{\beta_1} + 2^{\beta_1}\exp[\beta_2] + 4^{\beta_1}\exp[\beta_2] + 20^{\beta_1}\exp[\beta_2]\} = 369. \Rightarrow$

$\exp[\beta_0]\ (1 + \exp[\beta_2])\,\{2^{\beta_1} + 4^{\beta_1} + 6^{\beta_1} + ... + 20^{\beta_1}\} = 369$.

The second equation becomes:

$\exp[\beta_0]\ (1 + \exp[\beta_2])\,\{\ln(2)2^{\beta_1} + \ln(4)4^{\beta_1} + \ln(6)6^{\beta_1} + ... + \ln(20)20^{\beta_1}\} = 872.856$.

The third equation becomes:

$\exp[\beta_0]\ \exp[\beta_2]\,\{2^{\beta_1} + 4^{\beta_1} + 6^{\beta_1} + ... + 20^{\beta_1}\} = 241$.

Comment: Well beyond what you should be asked on your exam!
A Poisson variable with a logarithmic link function.
Dividing the 1st and 3rd equations:

$(1 + \exp[\beta_2])/\exp[\beta_2] = 369/241. \Rightarrow \beta_2 = \ln(241/148) = 0.6328$.

Using a computer, the fitted parameters are: $\beta_0 = 1.684$, $\beta_1 = 0.3784$, $\beta_2 = 0.6328$.

One can verify that these values satisfy the three equations.
Example taken from Applied Regression Analysis by Draper and Smith.

3.45. While one may assume that the errors are Normally Distributed, in a GLM one could assume a different distribution of errors, such as Gamma or Poisson.
Thus Statement #1 is not true.
Statements #2 and #3 are true.

3.46. With four age categories, we add 4 -1 = 3 parameters.

Test statistic is: $F = \dfrac{D_S - D_B}{(\text{number of added parameters})\ \hat{\phi}_B} = \dfrac{3320.2 - 3306.9}{(3)\ (1.83)} = 2.42.$

The number of degrees of freedom in the numerator is 3.
The number of degrees of freedom in the denominator is:
number of observations minus the number of parameters in the bigger model.
We compare the test statistic to the appropriate F-distribution.
We reject the null hypothesis if the test statistic is sufficiently big.

3.47. “Broadly speaking, model lift is the economic value of a model. The phrase “economic value” doesn’t necessarily mean the profit that an insurer can expect to earn as a result of implementing a model, but rather it refers to a model’s ability to prevent adverse selection. The lift measures ... attempt to visually demonstrate or quantify a model’s ability to charge each insured an actuarially fair rate, thereby minimizing the potential for adverse selection.
Model lift is a relative concept, so it requires two or more competing models. That is, it doesn’t generally make sense to talk about the lift of a specific model, but rather the lift of one model over another.
In order to prevent overfitting, model lift should always be measured on holdout data.”
Comment: Quoted from Section 7.2 of Generalized Linear Models for Insurance Rating.

3.48. The effects of age and gender interact strongly. For example, the relationship between male and female relativities is very different for young drivers than it is for middle-aged drivers.
In contrast, the effects of frequency of payment and age do not appear to interact significantly; there seems to be approximately the same relationship for each age group.
Comment: The graphs are adapted from “A Practitioner's Guide to Generalized Linear Models,“ by Duncan Anderson, Sholom Feldblum, Claudine Modlin, Doris Schirmacher, Ernesto Schirmacher, and Neeza Thandi.

3.49. The second model is preferred since the predictions are closer to the actual than in Model 1.
Comment: See Figure 21 in Generalized Linear Models for Insurance Rating.

3.50. Let X_1 = 1 if age group A, and 0 otherwise.
X_2 = 1 if age group B, and 0 otherwise.
X_3 = 1 if small, and 0 otherwise.
X_4 = 1 if medium, and 0 otherwise.
X_5 = 1 if large, and 0 otherwise.
Then the design matrix is:

$$\begin{pmatrix} \text{A/small} \\ \text{A/medium} \\ \text{A/large} \\ \text{B/small} \\ \text{B/medium} \\ \text{B/large} \\ \text{C/small} \\ \text{C/medium} \\ \text{C/large} \end{pmatrix} \Leftrightarrow \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

For example, the first row corresponds to age group A and small:
$X_1 = 1$, $X_2 = 0$, $X_3 = 1$, $X_4 = 0$, and $X_5 = 0$.
The last row corresponds to age group C and large: $X_1 = 0$, $X_2 = 0$, $X_3 = 0$, $X_4 = 0$, and $X_5 = 1$.

The vector of parameters is: $\begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \end{pmatrix}$.

Alternately, define medium and age group C as the base level.
Then the constant, b_0, would apply to all observations.
Let X_1 = 1 if age group A, and 0 otherwise.
X_2 = 1 if age group B, and 0 otherwise.
X_3 = 1 if small, and 0 otherwise.
X_4 = 1 if large, and 0 otherwise.
Then the design matrix is:

$$\begin{pmatrix} \text{A/small} \\ \text{A/medium} \\ \text{A/large} \\ \text{B/small} \\ \text{B/medium} \\ \text{B/large} \\ \text{C/small} \\ \text{C/medium} \\ \text{C/large} \end{pmatrix} \Leftrightarrow \begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The first column of ones corresponds to the constant term which applies to all observations. For example, the first row corresponds to age group A and small:
$X_0 = 1$, $X_1 = 1$, $X_2 = 0$, $X_3 = 1$, $X_4 = 0$.
The last row corresponds to age group C and large: $X_0 = 1$, $X_1 = 0$, $X_2 = 0$, $X_3 = 0$, $X_4 = 1$.

The vector of parameters is: $\begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix}$.

Comment: There is no unique answer. I have given two out of the many possible answers. There are 3 age categories and 3 size categories, so we need to have either 3 + 3 - 1 = 5 covariates, or 4 covariates and a constant term.
The data would be arranged in a grid such as:

	Small	Medium	Large
Age A	???	???	???
Age B	???	???	???
Age C	???	???	???

The response vector would have 9 rows and one column, containing the observations in the same order as the rows of the design matrix.

3.51. "If the modeler retains variables in the model that reflect a non-systematic effect on the response variable (i.e., noise) or over-specifies the model with high order polynomials, the result is over-fitting. Such a model will replicate the historical data very well (including the noise) but is not going to predict future outcomes reliably (as the future experience will most likely not have the same noise).
Conversely, if the model is missing important statistical effects (the extreme being a model that contains no explanatory variables and fits to the overall mean), the result is under-fitting. This model will predict future outcomes (e.g., in the extreme case mentioned above, the future mean) reliably but hardly help the modeler explain what is driving the result."
"Considerable disparity between actual and expected results on the hold-out sample may indicate that the model is over or under-fitting."

Underfit. ⇔ Too few Parameters. ⇔ Does not use enough of the useful information.

Overfit. ⇔ Too many Parameters. ⇔ Reflects too much of the noise.
In general, the actuary wants to avoid both underfitting and overfitting models.
Comment: See page 182 of Basic Ratemaking, on Exam 5.

3.52. The 18th observation has by far the biggest Cook's Distance, and is thus the most influential.

3.53. (a) AIC = (-2) (maximum loglikelihood) + (number of parameters)(2).
For the first model, AIC = (-2)(-321.06) + (6)(2) = 654.12.
For the second model, AIC = (-2)(-319.83) + (7)(2) = 653.66.
Since AIC is smaller for the second model, the second model is preferred.
(b) BIC = (-2) (maximum loglikelihood)) + (number of parameters) ln(number of data points).
For the first model, BIC = (-2)(-321.06) + (6) ln[100] = 669.75.
For the second model, BIC = (-2)(-319.83) + (7) ln[100] = 671.90.
Since BIC is smaller for the first model, the first model is preferred.
Comment: An example where using AIC and BIC lead to different conclusions.

3.54. $d_i^2 = 2\ \{y_i \ln[y_i / \hat{\mu}_i\ ]\text{-}\ (y_i - \hat{\mu}_i)\ \}$ = 2{11 ln[11/9.5] - (11 - 9.5)} = 0.2253.

Since 11 - 9.5 > 0, we take d_i as positive. $d_i = \sqrt{0.2253}$ = **0.475**.

3.55. Graph B is closest to a straight line.
Comment: If the data was drawn from a Normal Distribution with m ≠ 0, then we would expect the plotted points to be close to a straight line, but not a straight line through the origin.

3.56. (a) Poisson with log link function.
(b) Poisson or Negative Binomial with log link function.
(c) Gamma with log link function.
(d) Binomial with logit link function.
(e) Tweedie with log link function.
Comment: Claim frequency is claim count per exposure. If each insured has the same number of exposures, then a model of claim counts and claim frequency are mathematically equivalent.

3.57. Larger Gini index is better, all else being equal. The second rating plan is preferred
Comment: The higher the Gini index, the better the model is at identifying risk differences.

3.58. If the current rating plan were perfect, then all risks should have the same loss ratio. The fact that the proposed model is able to segment the data into lower and higher loss ratio buckets is a strong indicator that it is outperforming the current rating plan.
Comment: Graph taken from "Introduction to Predictive Modeling Using GLMs A Practitioner's Viewpoint," a presentation by Dan Tevet and Anand Khare.

3.59. For levels 1 to 7 of the variable, the log of the multiplier is not significantly different than zero; in other words the relativity is not significantly different from one. Also for levels 1 to 7, there is no consistent pattern. Thus perhaps, levels 1 to 8 of this variable should be grouped into one level for purposes of the model; this would be treated as the new base.
In contrast, for levels 9 to 15 there is pattern of increasing relativities. For levels 11 to 15 the relativities are significantly different from one. Given the pattern, one could also use the indicated relativities for levels 9 and 10.
Comment: As always, more testing may lead to a different conclusion. For example, it would be interesting to compare the results for different years of data to see if they are consistent.
For example, the levels of the variable could be groups of annual income.

3.60. There are many ways to define the variables.
Let us define X_1 = 1 if male and zero otherwise.
X_2 = 1 if female and zero otherwise.
X_3 = 1 if urban and zero otherwise.
For the Poisson, $f(x) = \lambda^x e^{-\lambda} / x!$. $\ln f(x) = x \ln(\lambda) - \lambda - \ln(x!) = x \ln(\mu) - \mu$ - constants.
(a) We use an identity link function. The estimated means are:

	Urban	Rural
Male	$\beta_1 + \beta_3$	β_1
Female	$\beta_2 + \beta_3$	β_2

Ignoring constants, the loglikelihood is:
$0.2 \ln(\beta_1 + \beta_3) - (\beta_1 + \beta_3) + 0.1 \ln(\beta_1) - (\beta_1) + 0.125 \ln(\beta_2 + \beta_3) - (\beta_2 + \beta_3) + 0.05 \ln(\beta_2) - (\beta_2)$.
Setting the partial derivative with respect to β_1 equal to zero: $0.2/(\beta_1 + \beta_3) + 0.1/\beta_1 = 2$.
Setting the partial derivative with respect to β_2 equal to zero: $0.125/(\beta_2 + \beta_3) + 0.05/\beta_2 = 2$.
Setting the partial derivative with respect to β_3 equal to zero: $0.2/(\beta_1 + \beta_3) + 0.125/(\beta_2 + \beta_3) = 2$.
(b) We use an log link function. The estimated means are:

	Urban	Rural
Male	$\exp[\beta_1 + \beta_3]$	$\exp[\beta_1]$
Female	$\exp[\beta_2 + \beta_3]$	$\exp[\beta_2]$

Ignoring constants, the loglikelihood is:
$0.2(\beta_1 + \beta_3) - \exp[\beta_1 + \beta_3] + 0.1\beta_1 - \exp[\beta_1] + 0.125(\beta_2 + \beta_3) - \exp[\beta_2 + \beta_3] + 0.05\ \beta_2 - \exp[\beta_2]$.
Setting the partial derivative with respect to β_1 equal to zero: $\exp[\beta_1 + \beta_3] + \exp[\beta_1] = 0.3$.
Setting the partial derivative with respect to β_2 equal to zero: $\exp[\beta_2 + \beta_3] + \exp[\beta_2] = 0.175$.
Setting the partial derivative with respect to β_3 equal to zero: $\exp[\beta_1 + \beta_3] + \exp[\beta_2 + \beta_3] = 0.325$.
Comment: Using a computer, the fitted parameters in part (a) are:
$\beta_1 = 0.105556$, $\beta_2 = 0.047500$, $\beta_3 = 0.084444$.
The fitted frequencies are: 0.1900, 0.1056, 0.1319, 0.0475.
Using a computer, the fitted parameters in part (b) are:
$\beta_1 = -2.35665$, $\beta_2 = -2.89565$, $\beta_3 = 0.77319$.
The fitted frequencies are: 0.2053, 0.0947, 0.1197, 0.0553.

3.61. The sensitivity is: $\dfrac{\text{true positives}}{\text{total times there is an event}}$ = 1800/3000 = 0.6.

The specificity is: $\dfrac{\text{true negatives}}{\text{total times there is not an event}}$ = 40,000/50,000 = 0.8.

For this threshold, we graph the point: (1 - specificity , sensitivity) = **(0.2, 0.6)**.

3.62. $f(y) = \exp[-(y - \mu)^2/(2\sigma^2)]/\{\sigma\sqrt{2\pi}\}$. $\ln f(Y_i) = -(Y_i - \beta X_i)^2/(2\sigma^2) - \ln(\sigma) - \ln(2\pi)/2$.

Loglikelihood is: $-\sum(Y_i - \beta X_i)^2/(2\sigma^2) - n \ln(\sigma) - n \ln(2\pi)/2$.

Set the partial derivative of the loglikelihood with respect to b equal to zero:

$0 = \sum X_i(Y_i - \beta X_i)/\sigma^2. \Rightarrow \sum X_i Y_i = \beta \sum X_i^2. \Rightarrow \hat{\beta} = \sum X_i Y_i / \sum X_i^2 = 3080/751 =$ **4.10**.

<u>Comment</u>: Matches the linear regression model with no intercept, $\hat{\beta} = \sum X_i Y_i / \sum X_i^2$.

3.63. Set the partial derivative of the loglikelihood with respect to s equal to zero:

$0 = \sum(Y_i - \beta X_i)^2/\sigma^3 - n/\sigma. \Rightarrow \sigma^2 = \sum(Y_i - \beta X_i)^2/n =$

$$\frac{\{5 - (1)(4.1)\}^2 + \{15 - (5)(4.1)\}^2 + \{50 - (10)(4.1)\}^2 + \{100 - (25)(4.1)\}^2}{4} = 29.58.$$

$\hat{\beta} = \sum X_i Y_i / \sum X_i^2$. $\text{Var}[\hat{\beta}] = \text{Var}[\sum X_i Y_i / \sum X_i^2] = \sum \text{Var}[X_i Y_i / \sum X_i^2] = \sum X_i^2 \text{Var}[Y_i] / (\sum X_i^2)^2 =$

$\sum X_i^2 \sigma^2 / (\sum X_i^2)^2 = \sigma^2 / \sum X_i^2 = 29.58/751 = 0.0394$.

$\text{StdDev}[\hat{\beta}] = \sqrt{0.0394} =$ **0.198.**

<u>Comment</u>: In the linear regression version of this same example, one would estimate the variance of the regression as: $\sigma^2 = \sum \hat{\varepsilon}_i^2 \ / \ (N - 1) =$

$$\frac{\{5 - (1)(4.1)\}^2 + \{15 - (5)(4.1)\}^2 + \{50 - (10)(4.1)\}^2 + \{100 - (25)(4.1)\}^2}{4 - 1} = 39.4.$$ This is an unbiased estimate of σ^2, which is <u>not</u> equal to that from maximum likelihood which is biased.

3.64. Estimated mean severity for a male in Territory D is: exp[8.03 + 0.18 + 0.22] = 4583.
For the Inverse Gaussian Distribution, $\text{Var}[Y] = \phi\mu^3 = (0.00510)(4583^3) = 490{,}930{,}199$.
$\text{StdDev}[Y] = \sqrt{490{,}930{,}199} =$ **22,157**.

3.65. 1. Actuarial judgement. Does the model make sense; is the model reasonable.
2. Statistical Tests such as the F-Test.
3. Graph the modeled relativities plus or minus two standard errors.
We would like the range between plus and minus two standard errors to be relatively narrow.
4. Check the consistency of the model run on different years of data.
5. Check the predictive accuracy of the model on a hold-out data set.
<u>Comment</u>: There are other possible answers.

3.66. The variance of the residuals appears to increasing with the fitted values, indicating heteroscedasticity (a lack of homoscedasticity.) This is not good, and one should try to refine the current model.

3.67. The scaled deviance is equal to twice the difference between the maximum achievable loglikelihood (i.e., the loglikelihood where the fitted value is equal to the observation for every record) and the loglikelihood of the model.
Alternately, the scaled deviance is equal to twice the difference between the loglikelihood of the saturated model and the loglikelihood of the fitted model.

3.68. $\ln[\mu/207] = 0.43 + 0.22 - 0.32 + 0.36 = 0.69$.
$\mu = 207 \exp[0.69] =$ **$413**.
Comment: This is a multiplicative model with four categorical variables.

3.69. A model that combines information from two or more models is called an ensemble model.
Two (or more) teams model the same item; they build separate models working independently.
Combining the answers from both models is likely to perform better than either individually.
A simple means of ensembling is to average the separate model predictions.

3.70. $d_i^2 = \theta \dfrac{(y_i - \hat{y}_i)^2}{\hat{y}_i^2 \ y_i} = \dfrac{1}{121} \dfrac{(288 - 361)^2}{361^2 \ 288} = 0.000001173.$

Since $288 - 361 < 0$, we take $d_i < 0$. $d_i =$ **-0.00108**.

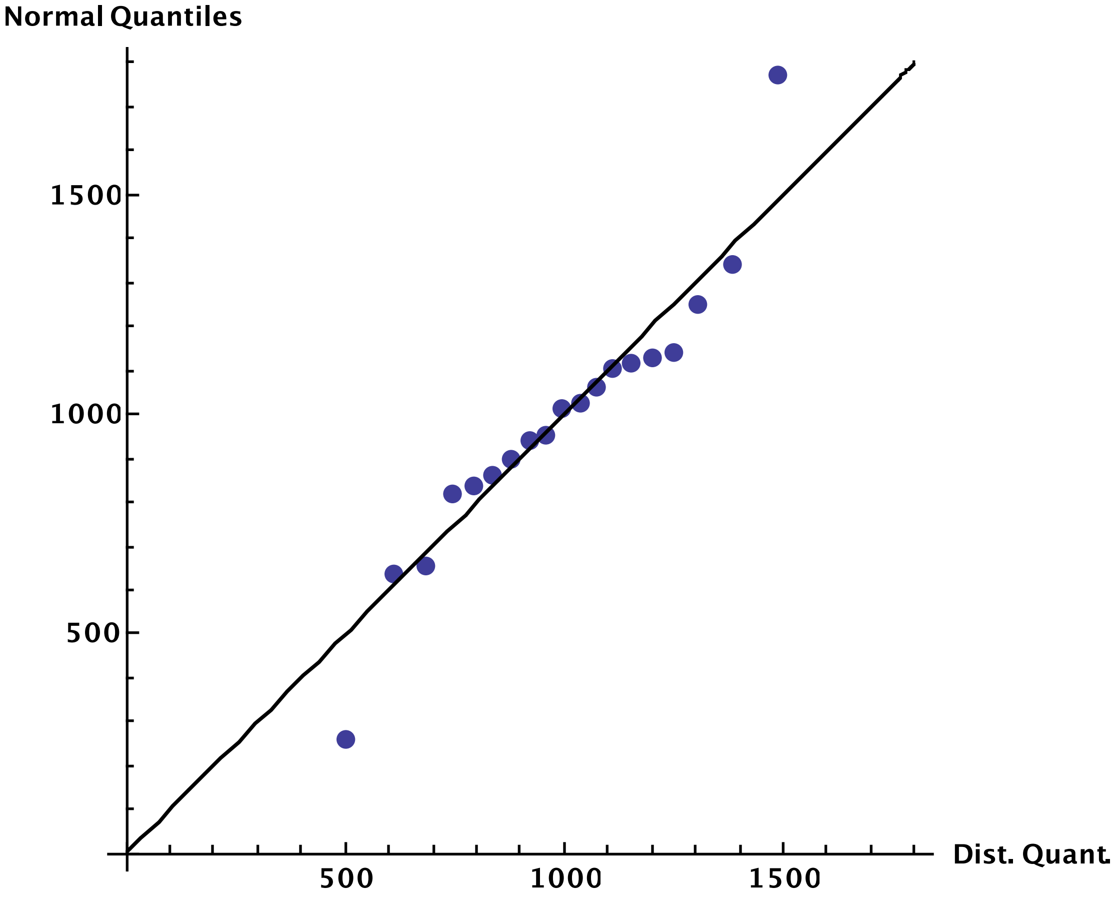
3.71. Test statistic is: $F = \dfrac{D_S - D_B}{(\text{number of added parameters})\ \hat{\phi}_B} = \dfrac{(24{,}359 - 24{,}352) / 1}{1.22} = 5.738.$

The number of degrees of freedom in the numerator is 1.
The number of degrees of freedom in the denominator is:
number of observations minus the number of parameters in the bigger model
$= 20{,}000 - 4 - 1 = 19{,}995$.
This is equivalent to a two sided t-test at $\sqrt{5.738} = 2.395$, with 19,995 degrees of freedom.

Using the Normal approximation, the p-value is: $(2)\ (1 - \Phi[2.395])$.
Since $2.326 < 2.395 < 2.576$, the two-sided p-value is between 2% and 1%.
Thus at a 2% significance level we should use the more complex model with the added variable, but at a 1% significance level we should use the simpler model without the additional variable.
Comment: Using a computer, the p-value of this test is 1.66%.
The null hypothesis is to use the simpler model. The alternate hypothesis is to use the more complex model. We reject the null hypothesis if the test statistic is sufficiently big.

3.72. The actuary would like the GLM to be stable; in other words, the predictions of the model should not be overly sensitive to small changes in the data.
An observation is influential if it has a large effect on the fitted model.
The larger the value of Cook's distance, the more influential the observation.
The actuary should rerun the model excluding the most influential points to see their impact on the results. If this causes large changes in some of the parameter estimates, the actuary should consider for example whether to give these influential observations less weight.
Cross-validation can also be used to assess the stability of a GLM. A single model can be run on the set of folds. The results of the models fit to these different subsets of the data ideally should be similar. The amount by which these results vary is a measure of the stability of the model.
Bootstrapping via simulation can also be used to assess the stability of a GLM. The original data is randomly sampled with replacement to create a new set of data of the same size. One then fits the GLM to this new set of data. By repeating this procedure many times one can estimate the distribution of the parameter estimates of the GLM; we can estimate the mean, variance, confidence intervals, etc.

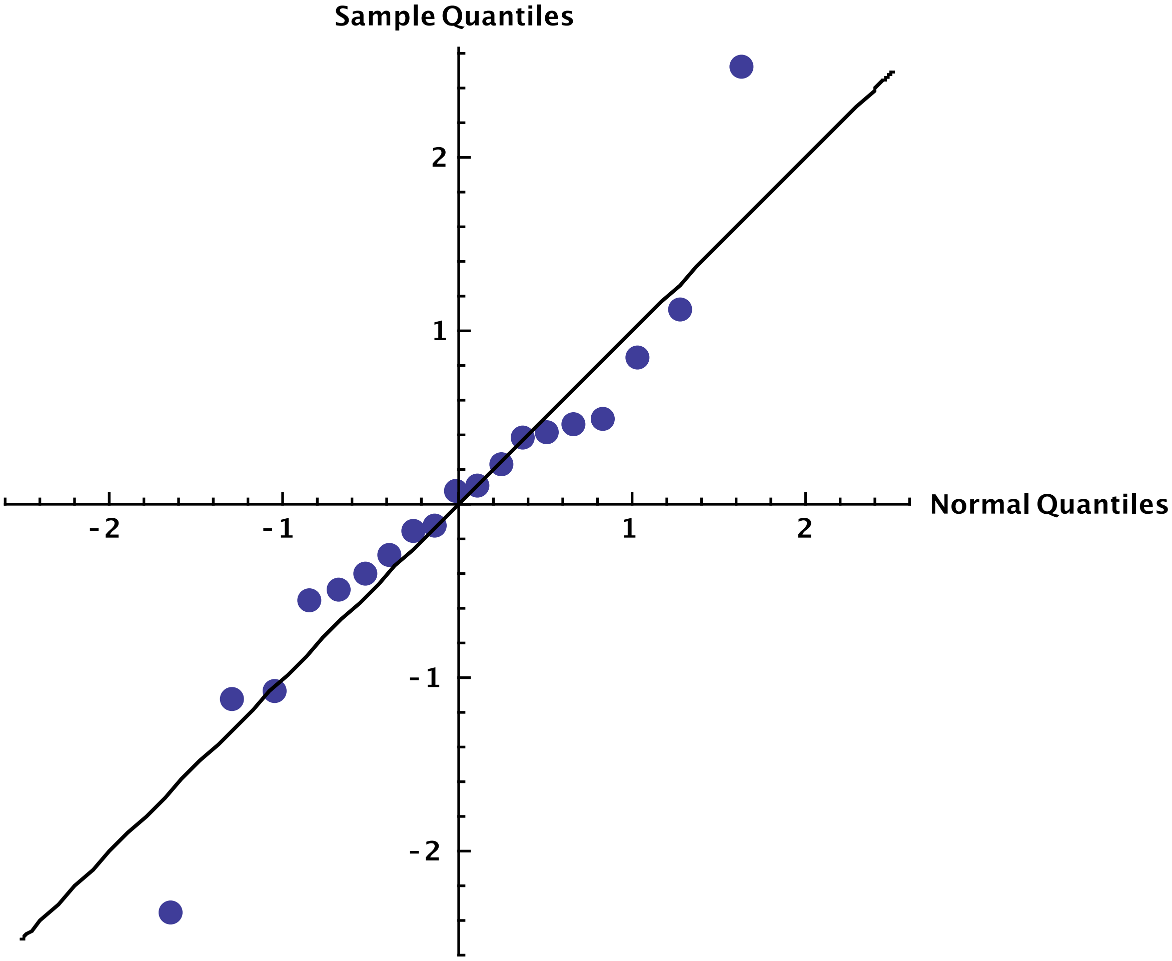
3.73. $\Phi[-1.645] = 1/20$. Thus for the given Normal, $Q_{0.05} = 1000 - (1.645)(300) = 506.5$.
The 19 plotted points are: (506.5, 258), (615.5, 636), (689.1, 652), (747.5, 814),
(797.7, 833), (842.7, 860), (884.4, 895), (924.0, 937), (962.3, 950), (1000.0, 1009),
(1037.7, 1020), (1076.0, 1059), (1115.6, 1103), (1157.3, 1113), (1202.3, 1127),
(1252.5, 1139), (1310.9, 1246), (1384.5, 1335), (1493.5, 1770).
The resulting Q-Q plot:



Alternately, one could standardize the data, by subtracting the sample mean of 987.158 and dividing by the square root of the sample variance of 96,057.8.

For example, $(258 - 987.158) / \sqrt{96{,}057.8} = -2.353$.

Then one compares the standardized data to the quantiles of the Standard Normal Distribution.
(-1.645, -2.353), (-1.282, -1.133), (-1.036, -1.081), (-0.842, -0.559), (-0.674, -0.497), (-0.524, -0.410), (-0.385, -0.297), (-0.253, -0.162), (-0.126, -0.120), (0, 0.070), (0.126, 0.106), (0.253, 0.232), (0.385, 0.374), (0.524, 0.406), (0.674, 0.451), (0.842, 0.490), (1.036, 0.835), (1.282, 1.122), (1.645, 2.526).
The resulting Q-Q plot:



Comment: With the exception of the first and last plotted points, the points stay close to the 45 degree comparison line, indicating that this data may be normally distributed.

3.74. $d_i^2 = (2)(5)\{-\ln[113/102.4] + (113 - 102.4)/102.4\} = 0.050145$.

Since 113 - 102.4 > 0, we d_i as positive. $d_i = \sqrt{0.050145} = \mathbf{0.224}$.

3.75. a) The total number of cells is: (2)(4)(3) = 24. So the design matrix would have 24 rows.
Each row has a one in the first column; the intercept term applies to all insureds.
For example, the first row has one in columns 3 and 6 corresponding to age 17-21 and Territory A.

$$\left(\begin{array}{ccccccc}
1 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 1 & 0 & 1
\end{array}\right)
\quad
\begin{array}{c}
\text{F 17-21 A} \\
\text{F 22-29 A} \\
\text{F 30-59 A} \\
\text{F 60+ A} \\
\text{M 17-21 A} \\
\text{M 22-29 A} \\
\text{M 30-59 A} \\
\text{M 60+ A} \\
\text{F 17-21 B} \\
\text{F 22-29 B} \\
\text{F 30-59 B} \\
\text{F 60+ B} \\
\text{M 17-21 B} \\
\text{M 22-29 B} \\
\text{M 30-59 B} \\
\text{M 60+ B} \\
\text{F 17-21 C} \\
\text{F 22-29 C} \\
\text{F 30-59 C} \\
\text{F 60+ C} \\
\text{M 17-21 C} \\
\text{M 22-29 C} \\
\text{M 30-59 C} \\
\text{M 60+ C}
\end{array}$$

b) 30-59 year old female driver in Territory B is the base. Estimated frequency is $\exp[\hat{\beta}_1]$.

c) For 22-29 year old male driver in Territory C, the estimated frequency is:
$\exp[\hat{\beta}_1 + \hat{\beta}_2 + \hat{\beta}_4 + \hat{\beta}_7]$.

Comment: One can arrange the rows of the design matrix differently, as long as everything is consistent. Since there is an intercept term, and since each of the factors is a categorical variable, each has one less parameter than its number of levels.
We have chosen 30-59 year old female driver in Territory B as the base; some other choice could have been made.

3.76. AIC = (-2) (maximum loglikelihood) + (number of parameters)(2).
For example, AIC = (-2)(-359.17) + (3)(2) = 724.34.

Model	Number of Parameters	Loglikelihood	AIC
A	3	-359.17	724.34
B	4	-357.84	723.68
C	5	-356.42	722.84
D	6	-354.63	**721.26**
E	7	-353.85	721.70

Since AIC is smallest for model D, model D is preferred.

3.77. In the first graph, the relativities indicated by the separate years are similar to each other. Also the relativities for each year display a similar pattern of increase with vehicle symbol, which makes sense. Vehicle symbol appears to be a significant factor for the first model; it is likely to be a good predictor of future experience.
In the second graph, the relativities indicated by separate years are <u>not</u> consistent. Territory does <u>not</u> appear to be a significant factor for the second model.
<u>Comment</u>: The graphs are adapted from ones showing more information in Sections 2.40-2.41 of "A Practitioner's Guide to Generalized Linear Models,"by Duncan Anderson, Sholom Feldblum, Claudine Modlin, Doris Schirmacher, Ernesto Schirmacher, and Neeza Thandi, <u>not</u> on the syllabus.

3.78. If two predictors are highly correlated (have a correlation coefficient close to plus or minus one) coefficients may behave erratically. Furthermore, the standard errors associated with those coefficients will be large, and small perturbations in the data may swing the coefficient estimates wildly. Such instability in a model should be avoided. As such it is important to look out for instances of high correlation prior to modeling, by examining two-way correlation tables.
Where high correlation is detected, means of dealing with this include the following:
• For any group of correlated predictors, remove all but one from the model.
• Preprocess the data using dimensionality reduction techniques such as principal component analysis.
<u>Multicollinearity</u>: A more subtle potential problem may exist where two or more predictors in a model may be strongly predictive of a third, a situation known as multicollinearity. The same instability problems as above may result. A useful statistic for detecting multicollinearity is the variance inflation factor (VIF), which can be output by most statistical packages. A common statistical rule of thumb is that a VIF greater than 10 is considered high.
<u>Aliasing</u>: Where two predictors are perfectly correlated, they are said to be aliased, and the GLM will not have a unique solution. Where they are nearly perfectly correlated, the model will be highly unstable; the fitting procedure may fail to converge, and even if the model run is successful the
estimated coefficients will be nonsensical. Such problems can be avoided by looking out for and properly handling correlations among predictors, as discussed above.
<u>Comment</u>: See Section 2.9 of <u>Generalized Linear Models for Insurance Rating</u>.
Not necessary to say all of the above rather than some of the above.

3.79. 1. They are simple and practical to implement.
2. Having additive terms in a model can result in negative premiums, which doesn't make sense. With a multiplicative plan you guarantee positive premium without having to implement clunky patches like minimum premium rules.
3. A multiplicative model has more intuitive appeal. It doesn't make much sense to say that having a violation should increase your auto premium by $500, regardless of whether your base premium is $1,000 or $10,000.
Rather it makes more sense to say that the surcharge for having a violation is 10%.
Comment: For these and other reasons, log link models, which produce multiplicative structures, are usually the most natural model for insurance risk.
"As for the link function, it is usually the case that the desirability of a multiplicative rating plan trumps all other considerations, so the log link is almost always used. One notable exception is where the target variable is binary (i.e., occurrence or non-occurrence of an event), for which a special link function (logistic) must be used."

3.80. In order to incorporate age, avoiding aliasing, we need 6 - 1 = 5 variables.
In order to incorporate gender, we would need one more variable for a total of 6.
So getting rid of age and gender would produce a model with 6 fewer parameters.

Test statistic is: $F = \dfrac{D_S - D_B}{(\text{number of added parameters})\ \hat{\phi}_B} = \dfrac{(1128.1 - 1120.3) / 6}{0.395} = 3.291.$

The number of degrees of freedom in the numerator is 6.
The number of degrees of freedom in the denominator is:
number of observations minus the number of parameters in the bigger model
= 1000 - 50 = 950.
We compare the test statistic to an F-distribution with 6 and 950 degrees of freedom.
The null hypothesis is to use the simpler model, the one without age and gender.
The alternate hypothesis is to use the more complex model.
We reject the null hypothesis if the test statistic is sufficiently big.
Comment: Using a computer, the p-value of this test is 3.3%.

3.81. "Firstly, when comparing two models using log-likelihood or deviance, the comparison is valid only if the data sets used to fit the two models are exactly identical. If a new variable has missing values for some records, the default behavior of most model fitting software is to toss out those records when fitting the model. In that case, the resulting measures of fit are no longer comparable, since the second model was fit with fewer records than the first.
For any comparisons of models that use deviance it is also necessary that the assumed distribution must be identical as well."
Comment: See Section 6.1.3 of Generalized Linear Models for Insurance Rating.

3.82. Age of spokesperson, gender of spokesperson, marital status of the spokesperson, time he has been a spokesperson, type of celebrity (actor, singer, athlete, etc.), criminal record of the spokesperson, past drug/alcohol abuse of the spokesperson, etc.
Comment: There are other reasonable answers.
This is often sold as death, disability, and disgrace insurance.

3.83. Both are continuous distributions used to model severity. Both are right-skewed, with a sharp peak and a long tail to the right, and a lower bound at zero.
The Gamma Distribution has variance function $V(\mu) = \mu^2$, while the Inverse Gaussian Distribution has variance function $V(\mu) = \mu^3$.
The Inverse Gaussian Distribution has a sharper peak and a wider tail than the Gamma Distribution.
Therefore, the Inverse Gaussian Distribution is appropriate for situations where the skewness of the severity curve is more extreme.
Comment: The skewness for the Gamma distribution is always twice times the coefficient of variation, while the skewness for the Inverse Gaussian distribution is always three times the coefficient of variation.

3.84. For the Normal $d_i^2 = \frac{1}{\sigma^2}\ (y_i - \hat{\mu}_i)^2 = \{(71 - 74.8)/23\}^2$.

Since $71 - 74.8 < 0$, we take $d_i < 0$. $d_i = (71 - 74.8)/23 =$ **-0.165**.

3.85. a) $9.5 + (0.01)(180) + (-0.02)(670) = -2.1$.
Using the inverse of the logit link function, the probability of default is:
$\frac{\exp(-2.1)}{1 + \exp(-2.1)} =$ **10.9%**.
b) $9.5 + (0.01)(100) + (-0.02)(760) = -4.7$.
Probability of default is: $\frac{\exp(-4.7)}{1 + \exp(-4.7)} =$ **0.9%**.
Comment: Similar to 8, 11/12, Q.4a. Not intended as a realistic model.

3.86. The partial residual plot is not linear; thus, we should do something to improve the model.
We could group the variable X_1, converting it into a categorical variable.
We could add polynomial terms such X_1^2 to the model.
We could use piecewise linear functions such as: Min[0, X_1 + 1] and Min[0, X_1 - 1].

3.87. There are many ways to define the variables.
Let us define X_1 = 1 if low horsepower and zero otherwise.
X_2 = 1 if medium horsepower and zero otherwise.
X_3 = 1 if high horsepower and zero otherwise.
X_4 = 1 if sedan and zero otherwise.

For the Gamma Distribution, $f(y) = \theta^{-\alpha} y^{\alpha-1} e^{-y/\theta} / \Gamma(\alpha)$.
$\ln f(y) = (\alpha-1)\ln(y) - y/\theta - \alpha\ln(\theta) - \ln[\Gamma(\alpha)] = (\alpha-1)\ln(y) - y/(\mu/\alpha) - \alpha\ln(\mu/\alpha) - \ln[\Gamma(\alpha)]$
$= (\alpha-1)\ln(y) - \alpha y/\mu - \alpha\ln(\mu) + \alpha\ln(\alpha) - \ln[\Gamma(\alpha)]$.
a) With the identity link function: $\mu = \beta_1X_1 + \beta_2X_2 + \beta_3X_3 + \beta_4X_4$.
Ignoring terms that do not involve the betas, the loglikelihood is:
$-\alpha 800/(\beta_1 + \beta_4) - \alpha \ln(\beta_1 + \beta_4) - \alpha 900/(\beta_2 + \beta_4) - \alpha \ln(\beta_2 + \beta_4) - \alpha 1100/(\beta_3 + \beta_4) - \alpha \ln(\beta_3 + \beta_4)$
$- \alpha 1500/(\beta_1) - \alpha \ln(\beta_1) - \alpha 1700/(\beta_2) - \alpha \ln(\beta_2) - \alpha 2000/(\beta_3) - \alpha \ln(\beta_3)$.
Setting the partial derivative with respect to β_1 equal to zero:
$800/(\beta_1 + \beta_4)^2 + 1500/\beta_1^2 = 1/(\beta_1 + \beta_4) + 1/\beta_1$.
Setting the partial derivative with respect to β_2 equal to zero:
$900/(\beta_2 + \beta_4)^2 + 1700/\beta_2^2 = 1/(\beta_2 + \beta_4) + 1/\beta_2$.
Setting the partial derivative with respect to β_3 equal to zero:
$1100/(\beta_3 + \beta_4)^2 + 2000/\beta_3^2 = 1/(\beta_3 + \beta_4) + 1/\beta_3$.
Setting the partial derivative with respect to β_4 equal to zero:
$800/(\beta_1 + \beta_4)^2 + 900/(\beta_2 + \beta_4)^2 + 1100/(\beta_3 + \beta_4)^2 = 1/(\beta_1 + \beta_4) + 1/(\beta_2 + \beta_4) + 1/(\beta_3 + \beta_4)$.
(b) We use a log link function. $\mu = \exp[\beta_1X_1 + \beta_2X_2 + \beta_3X_3 + \beta_4X_4]$.
Ignoring terms that do not involve the betas, the loglikelihood is:
$-\alpha 800\exp[-\beta_1 - \beta_4] - \alpha(\beta_1+\beta_4) - \alpha 900\exp[-\beta_2 - \beta_4] - \alpha(\beta_2+\beta_4) - \alpha 1100\exp[-\beta_3 + \beta_4] - \alpha(\beta_3+\beta_4)$
$- \alpha 1500\exp[-\beta_1] - \alpha(\beta_1) - \alpha 1700\exp[-\beta_2] - \alpha(\beta_2) - \alpha 2000\exp[-\beta_3] - \alpha(\beta_3)$.
Setting the partial derivative with respect to β_1 equal to zero:
$800\exp[-\beta_1 - \beta_4] + 1500\exp[-\beta_1] = 2$.
Setting the partial derivative with respect to β_2 equal to zero:
$900\exp[-\beta_2 - \beta_4] + 1700\exp[-\beta_2] = 2$.
Setting the partial derivative with respect to β_3 equal to zero:
$1100\exp[-\beta_3 - \beta_4] + 2000\exp[-\beta_3] = 2$.
Setting the partial derivative with respect to β_4 equal to zero:
$800\exp[-\beta_1 - \beta_4] + 900\exp[-\beta_2 - \beta_4] + 1100\exp[-\beta_3 - \beta_4] = 3$.
Comment: Using a computer, the fitted parameters in part (a) are:
$\beta_1 = 1567.71$, $\beta_2 = 1688.03$, $\beta_3 = 1914.41$, $\beta_4 = -784.60$.
The fitted severities are: 783.11, 903.43, 1129.81, 1567.71, 1688.03, 1914.41.
Using a computer, the fitted parameters in part (b) are:
$\beta_1 = 7.30933$, $\beta_2 = 7.43082$, $\beta_3 = 7.61246$, $\beta_4 = -0.620811$.
The fitted severities are: 803.13, 906.88, 1087.51, 1494.17, 1687.20, 2023.24.

3.88. Adding vehicle type adds 10 - 1 = 9 parameters to the model.

Test statistic is: $F = \frac{D_S - D_B}{(\text{number of added parameters})\ \hat{\phi}_B} = \frac{(1848.5 - 1833.0) / 9}{0.93} = 1.852$.

The number of degrees of freedom in the numerator is 9.
The number of degrees of freedom in the denominator is:
number of observations minus the number of parameters in the bigger model
= 2000 - 23 = 1977.
We compare the test statistic to an F-distribution with 9 and 1977 degrees of freedom.
The null hypothesis is to use the simpler model, the one without vehicle type.
The alternate hypothesis is to use the more complex model.
We reject the null hypothesis at 5% if the test statistic is bigger than the 5% critical value, which is where the F-distribution is 95%.
Comment: Using a computer, the p-value of this test is 5.5%.
Thus we would not reject the null hypothesis at 5%.
If we reduced the number of vehicle type categories by combining some of the 10 categories we used, it might turn out that now we should use vehicle type at the 5% significance level.

3.89. Ignoring the loglikelihood of the saturated model, which is a constant,
BIC = Scaled Deviance + (number of parameters) ln[250].
For example, BIC = 1679.1 + 6 ln[250] = 1712.23.

Model	Number of Parameters	Scaled Deviance	BIC
A	6	1679.10	1712.23
B	8	1666.40	**1710.57**
C	10	1655.90	1711.11
D	12	1646.20	1712.46
E	14	1634.50	1711.80

Since BIC is smallest for model B, model B is preferred.

3.90. The difference between the yellow univariate line and the green GLM line, which better represents the underlying reality, arises from correlation between policy duration shown in the graph and the two other factors in the model.
Comment: One does not have to understand the life insurance details in order to answer the question asked.

3.91. exp[-0.3] - 1 = -25.9%.
Comment: For a logistic model: Odds = μ / (1 - μ).

3.92. Female drivers age 31 to 59 in a rural territory have lower (process) variances than unmarried male drivers age 17 to 21 in an urban territory.
Therefore, the fitted model shifts to agree more closely with the observed values for the first group compared to the second group.
A GLM is more concerned with differences between observed and fitted where the (process) variances in observations are smaller. A GLM is less concerned with differences between observed and fitted where the variances in observations are larger.

3.93. The sensitivity is: 2000/5000 = 0.40.
The specificity is: 70,000 / 80,000 = 0.875.
For this threshold, we graph the point: (1 - specificity , sensitivity) = **(0.125, 0.40)**.

3.94. "Two standard errors from the parameter estimates are akin to a 95% confidence interval. This means the GLM parameter estimate is a point estimate, and the standard errors show the range in which the modeler can be 95% confident the true answer lies within."
Comment: See page 179 of Basic Ratemaking, on Exam 5.

3.95. The Lorenz curve for the rating plan is determined as follows:
1. Sort the dataset based on the model predicted loss cost.
2. On the x-axis, plot the cumulative percentage of exposures.
3. On the y-axis, plot the cumulative percentage of losses.
Draw a 45-degree line connecting (0, 0) and (1, 1), called the line of equality.
The Gini index is twice the area between the Lorenz curve and the line of equality.

3.96. $\ln(\lambda) = \beta_0 + \beta_1 z. \Rightarrow \lambda = \exp[\beta_0 + \beta_1 z]$.

For the Poisson Distribution: $f(y) = e^{-\lambda} \lambda^y / y!$.
$\ln f(y) = -\lambda + y\ln(\lambda) - \ln(y!) = -\exp[\beta_0 + \beta_1 z] + y(\beta_0 + \beta_1 z) - \ln(y!)$.
The loglikelihood is the sum of the contributions from the three observations:
$-\exp[\beta_0 + \beta_1] - \exp[\beta_0 + 2\beta_1] - \exp[\beta_0 + 3\beta_1] + 4(\beta_0 + \beta_1) + 7(\beta_0 + 2\beta_1) + 8(\beta_0 + 3\beta_1)$
$- \ln(4!) - \ln(7!) - \ln(8!)$.
To maximize the loglikelihood, we set its partial derivatives equal to zero.
Setting the partial derivative with respect to β_0 equal to zero:
$0 = -\exp[\beta_0 + \beta_1] - \exp[\beta_0 + 2\beta_1] - \exp[\beta_0 + 3\beta_1] + 19$.
Setting the partial derivative with respect to b_1 equal to zero:
$0 = -\exp[\beta_0 + \beta_1] - 2\exp[\beta_0 + 2\beta_1] - 3\exp[\beta_0 + 3\beta_1] + 42$.
Thus we have two equations in two unknowns:
$\exp[\beta_0 + \beta_1]\{1 + \exp[\beta_1] + \exp[2\beta_1]\} = 19$.
$\exp[\beta_0 + \beta_1]\{1 + 2\exp[\beta_1] + 3\exp[2\beta_1]\} = 42$.
Dividing the second equation by the first equation:
$\{1 + 2\exp[\beta_1] + 3\exp[2\beta_1]\}/\{1 + \exp[\beta_1] + \exp[2\beta_1]\} = 42/19$.
$\Rightarrow 19 + 38\exp[\beta_1] + 57\exp[2\beta_1] = 42 + 42\exp[\beta_1] + 42\exp[2\beta_1]. \Rightarrow 15\exp[2\beta_1] - 4\exp[\beta_1] - 23 = 0$.
Letting $v = \exp[\beta_1]$, this equation is: $15v^2 - 4v - 23 = 0$, with positive solution:
$v = (4 + \sqrt{1396})/30 = 1.3788$.
$\exp[\beta_1] = 1.3788. \Rightarrow \beta_1 = 0.3212$.
$\Rightarrow \exp[\beta_0] = 19/\{\exp[\beta_1] + \exp[2\beta_1] + \exp[3\beta_1]\} = 19/\{1.3788 + 1.3788^2 + 1.3788^3\} = 3.2197$.
$\Rightarrow \beta_0 = 1.1693$.
$\lambda = \exp[\beta_0 + \beta_1 z] = \exp[\beta_0] \exp[\beta_1]^z = (3.2197)(1.3788^z)$.
For $z = 1$, $\lambda = 4.439$. For $z = 2$, $\lambda = 6.121$. For $z = 3$, $\lambda = 8.440$.
Comment: An ordinary linear regression fit to these same observations turns out to be:
$y = 2.333 + 2x$, with fitted values: 4.333, 6.333, and 8.333.

3.97. Examples include:

- Will it be cost-effective to collect the value of this variable when writing new and renewal business?
- Does inclusion of this variable in a rating plan conform to actuarial standards of practice and regulatory requirements?
- Can the electronic quotation system be easily modified to handle the inclusion of this variable in the rating formula?

3.98. a. We would have one parameter for gender, two parameters for age, and two parameters for territory. In addition we would have a parameter related to the base level.
A total of **6** parameters.
(2-1) + (3-1) + (3-1) + 1 = 6.
Sex Age Terr. Base
b. A total of **6** parameters. The link function does not affect the number of parameters.
c. β_0 is the intercept term that applies to all insureds.
β_1 corresponds to Female.
β_2 corresponds to Youthful.
β_3 corresponds to Retired.
β_4 corresponds to Suburban.
β_5 corresponds to Rural.
(There are many other possible orders for the parameters.)
d. With 6 parameters, the design matrix has **6** columns.
e. With 20,000 cars, the design matrix has **20,000** rows.
f. The number combinations are: (2)(3)(3) = 18. Thus the design matrix has **18** rows.
(I have assumed that none of these cells is empty.
I have assumed that there are no records with missing classification information.)

3.99. (a) The squared deviance residual for any given record is defined as that record's contribution to the unscaled deviance, adjusted for the sign of actual minus predicted; the deviance residual is taken to be negative where actual is less expected, and positive where actual is more than expected.
(b) Intuitively, we can think of the deviance residual as the residual adjusted for the shape of the assumed GLM distribution, such that its distribution will be approximately normal if the assumed GLM distribution is correct.
(c) In a well-fit model, we expect deviance residuals to follow no predictable pattern, and be normally distributed, with constant variance.
One could plot the deviance residuals versus the fitted values or versus an important predictor variable, in order to see whether there is a pattern.
We can check for the normality of the deviance residuals via either a histogram or q-q plot.

3.100. $p/(1-p) = \exp[\beta_0 + \beta_1 X]$. $\Rightarrow 1/p - 1 = \exp[-\beta_0 - \beta_1 X]$. $\Rightarrow p = 1/(1 + \exp[-\beta_0 - \beta_1 X])$.

$\Rightarrow 1 - p = \exp[-\beta_0 - \beta_1 X] / (1 + \exp[-\beta_0 - \beta_1 X]) = 1 / (1 + \exp[\beta_0 + \beta_1 X])$.

For a Binomial with parameters m and p, $f(n) = p^n (1-p)^{m-n}\, m! / \{(n!)(m-n)!\}$.
$\ln f(n) = n \ln p + (m-n)\ln(1-p) + \ln(m!) - \ln(n!) - \ln[(m-n)!] = n \ln[p/(1-p)] + m \ln(1-p) +$ constants $=$
$n(\beta_0 + \beta_1 X) - m \ln[(1 + \exp[\beta_0 + \beta_1 X])] +$ constants.

loglikelihood $= \sum n_i(\beta_0 + \beta_1 X_i) - \sum m_i \ln[(1 + \exp[\beta_0 + \beta_1 X_i])] +$ constants.
Setting the partial derivatives of the loglikelihood with respect to β_0 and β_1 equal to zero:

$0 = \sum n_i - \sum m_i \exp[\beta_0 + \beta_1 X_i]/(1 + \exp[\beta_0 + \beta_1 X_i])$.

$0 = \sum n_i X_i - \sum m_i X_i \exp[\beta_0 + \beta_1 X_i]/(1 + \exp[\beta_0 + \beta_1 X_i])$.

$\sum n_i = 900 + 820 + 740 + 660 + 580 = 3700$.

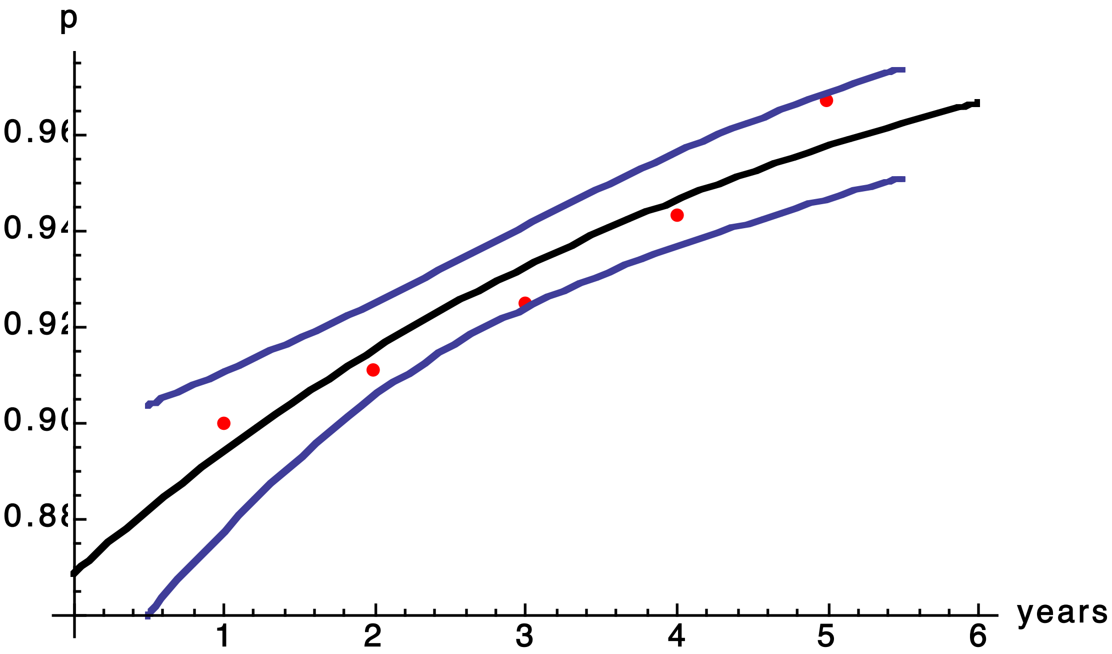
$\sum n_i X_i = (1)(900) + (2)(820) + (3)(740) + (4)(660) + (5)(580) = 10,300$.
The first equation becomes:
$3700 = 1000/(1 + \exp[-\beta_0 - \beta_1]) + 900/(1 + \exp[-\beta_0 - 2\beta_1]) + 800/(1 + \exp[-\beta_0 - 3\beta_1])$
$+ 700/(1 + \exp[-\beta_0 - 4\beta_1]) + 600/(1 + \exp[-\beta_0 - 5\beta_1])$.
The second equation becomes:
$10300 = 1000/(1 + \exp[-\beta_0 - \beta_1]) + 1800/(1 + \exp[-\beta_0 - 2\beta_1]) + 2400/(1 + \exp[-\beta_0 - 3\beta_1])$
$+ 2800/(1 + \exp[-\beta_0 - 4\beta_1]) + 3000/(1 + \exp[-\beta_0 - 5\beta_1])$.
<u>Comment</u>: An example of a Logistic Regression.
Using a computer, the maximum likelihood fit is: $\beta_0 = 1.88543$ and $\beta_1 = 0.245509$.

The covariance matrix of the fitted parameters is: $\begin{array}{c} \beta_0 \\ \beta_1 \end{array} \begin{pmatrix} 0.0154836 & -0.00501396 \\ -0.00501396 & 0.00212092 \end{pmatrix}$.

Thus the standard error of β_0 is: $\sqrt{0.0154836} = 0.1244$,

and the standard error of β_1 is: $\sqrt{0.00212092} = 0.04605$.

Here is a graph of the data, the fitted curve, and 95% confidence intervals:



3.101. ϕ is the dispersion parameter, which scales the variance.
ω_i is a (prior) weight, representing the amount of data we have for observation i; the variance is inversely proportional to the volume of data.

3.102. & 3.103. $f(y) = \exp[-(y - \mu)^2/(2\sigma^2)] / \{\sigma\sqrt{2\pi}\}$.

$\ln f(Y_i) = -(Y_i - \beta_0 - \beta_1 X_i)^2/(2\sigma^2) - \ln(\sigma) - \ln(2\pi)/2$.

Loglikelihood is: $-\sum(Y_i - \beta_0 - \beta_1 X_i)^2/(2\sigma^2) - n \ln(\sigma) - n \ln(2\pi)/2$.

Set the partial derivative of the loglikelihood with respect to β_0 equal to zero:

$0 = \sum(Y_i - \beta_0 - \beta_1 X_i)/\sigma^2. \Rightarrow \sum Y_i = n\beta_0 + \beta_1 \sum X_i. \Rightarrow \beta_0 = \bar{Y} - \beta_1 \bar{X}$.

Set the partial derivative of the loglikelihood with respect to β_1 equal to zero:

$0 = \sum X_i(Y_i - \beta_0 - \beta_1 X_i)/\sigma^2. \Rightarrow \sum X_i Y_i = \beta_0 \sum X_i + \beta_1 \sum X_i^2. \Rightarrow \sum X_i Y_i = (\bar{Y} - \beta_1 \bar{X})\sum X_i + \beta_1 \sum X_i^2$.

$\Rightarrow \hat{\beta}_1 = \{\sum X_i Y_i - \bar{Y} \sum X_i\} / \{\sum X_i^2 - \bar{X} \sum X_i\} = \{255 - (10)(24)\} / \{174 - (6)(24)\} = 15/30 = \mathbf{0.5}$.

$\Rightarrow \hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X} = 10 - (0.5)(6) = \mathbf{7}$.

Comment: Matches the linear regression model with an intercept.
For example, in deviations form:
$\bar{X} = 24/4 = 6$. $x = X - \bar{X} = -4, -1, 2, 3$. $\bar{Y} = 40/4 = 10$. $y = Y - \bar{Y} = 0, -4, 1, 3$.

$\hat{\beta} = \sum x_i y_i / \sum xi^2 = 15/30 = 0.5$. $\hat{\alpha} = \bar{Y} - \hat{\beta}\bar{X} = 10 - (0.5)(6) = 7$.

3.104. Set the partial derivative of the loglikelihood with respect to σ equal to zero:
$0 = \sum(Y_i - \beta_0 - \beta_1 X_i)^2/\sigma^3 - n/\sigma$. $\Rightarrow \sigma^2 = \sum(Y_i - \beta_0 - \beta_1 X_i)^2/n = \sum(Y_i - 7 - (0.5)X_i)^2/4 =$

$$\frac{\{10 - 7 - (0.5)(2)\}^2 + \{6 - 7 - (0.5)(5)\}^2 + \{11 - 7 - (0.5)(8)\}^2 + \{13 - 7 - (0.5)(9)\}^2}{4} = 18.5/4 = 4.625.$$

$\Rightarrow \hat{\sigma} = \sqrt{4.625} =$ **2.15**.

3.105. & 3.106. Let $x = X - \bar{X} = -4, -1, 2, 3$, and $y = Y - \bar{Y} = 0, -4, 1, 3$.
Then, $\sum X_i Y_i - \bar{Y}\sum X_i = \sum X_j Y_j - \sum Y_j \sum X_i/n = \sum Y_j(X_j - \bar{X}) = \sum Y_j x_j$.
Also, $\sum X_i^2 - \bar{X}\sum X_i = \sum X_i(X_i - \bar{X}) = \sum X_i x_i = \sum(X_i - \bar{X})x_i + \sum \bar{X} x_i = \sum x_i^2 + \bar{X}\sum x_i = \sum x_i^2 + \bar{X}(0)$
$= \sum x_i^2$.
$\hat{\beta}_1 = \{\sum X_i Y_i - \bar{Y}\sum X_i\} / \{\sum X_i^2 - \bar{X}\sum X_i\} = \sum Y_i x_i / \sum x_i^2$.
$\text{Var}[\hat{\beta}_1] = \text{Var}[\sum Y_i x_i / \sum x_i^2] = \sum \text{Var}[Y_i x_i]/\{\sum x_i^2\}^2 = \sum x_i^2 \text{Var}[Y_i] / \{\sum x_i^2\}^2 = \sigma^2 \sum x_i^2/\{\sum x_i^2\}^2 = \sigma^2/\sum x_i^2$
$= 4.625/30 = 0.1542$. $\text{StdDev}[\hat{\beta}_1] = \sqrt{0.1542} =$ **0.393**.

$\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X} = (Y_1 + Y_2 + Y_3 + Y_4)/4 - (\sum Y_i x_i / \sum x_i^2)(6) =$
$(Y_1 + Y_2 + Y_3 + Y_4)/4 - (-4Y_1 - Y_2 + 2Y_3 + 3Y_4)(6/30) = 1.05Y_1 + 0.45Y_2 - 0.15Y_3 - 0.35Y_4$.
Recalling that the Y_i are independent and each have variance σ^2:
$\text{Var}[\hat{\beta}_0] = \sigma^2(1.05^2 + 0.45^2 + 0.15^2 + 0.35^2) = 1.45\sigma^2 = (1.45)(4.625) = 6.706$.
$\text{StdDev}[\hat{\beta}_0] = \sqrt{6.706} =$ **2.59**.
Comment: Beyond what you should be asked on your exam.

One can show in general that $\text{Var}[\hat{\beta}] = \sigma^2 / \sum x_i^2$ and $\text{Var}[\hat{\alpha}] = \sigma^2 \sum X_i^2 / (N \sum x_i^2)$.
While the maximum likelihood results are similar, they do not match linear regression:

$\hat{Y} = \hat{\alpha} + \hat{\beta} X = 8, 9.5, 11, 11.5$. $\hat{\varepsilon} = Y - \hat{Y} = 2, -3.5, 0, 1.5$. $\text{ESS} = \sum \hat{\varepsilon}_i^2 = 18.5$.

$s^2 = \text{ESS} / (N - 2) = 18.5 / (4 - 2) = 9.25$.
$\text{Var}[\hat{\beta}] = s^2 / \sum x_i^2 = 9.25/30 = 0.3083$. $s_{\hat{\beta}} = \sqrt{0.3083} = 0.555$.

$\text{Var}[\hat{\alpha}] = s^2 \sum X_i^2 / (N \sum x_i^2) = (9.25)(174) / ((4)(30)) = 13.41$. $s_{\hat{\alpha}} = \sqrt{13.41} = 3.66$.

3.107. Since they have all been fit to the same data and have the same number of parameters, the model with the smallest Scaled Deviance is best. This is **Model D.**
Comment: If we were to apply either AIC or BIC, in this case the ranks of the models would be the same as that of their scaled deviances.

3.108. When the variance is greater than the mean we can use an overdispersed Poisson with $\phi > 1$.
$\text{Var}[Y_i] = \phi\ E[Y_i]$. For $\phi > 1$, variance is greater than the mean. While this does not correspond to the likelihood of any exponential family, otherwise the GLM mathematics works.
Using an overdispersed Poisson (ODP), we get the same estimated betas as for the usual Poisson regression. However, the standard errors of all of the estimated parameters are multiplied by $\sqrt{\phi}$.
Comment: When the variance is greater than the mean, one could use a Negative Binomial Distribution, which has a variance greater than its mean.
Often the results of using an overdispersed Poisson and a Negative Binomial will be similar.

3.109. While a 5% probability value may seem small, it allows for a 1-in-20 chance of a variable being accepted as significant when it is not. Since in a typical insurance modeling project we are testing many variables, this threshold may be too high to protect against the possibility of spurious effects making it into the model.
For example, if we are testing the potential usefulness of 40 possible predictor variables, then if we use a p-value of 5%, even if none of the variables actually predict the outcome, on average two of these 40 variables will be selected as significant.
Comment: See Section 2.3.2 of Generalized Linear Models for Insurance Rating.
"Spurious correlations exist when the historical correlation between two variables is random or coincidental. In these cases, one variable cannot reliably be used to inform a projection of the other variable going forward. For example, over the past year the number of California Department of Insurance rate regulation actuaries has increased, as has California average rainfall. Unfortunately, however, we cannot expect to influence future California rainfall by hiring additional actuaries."
Quoted from "Predictive Analytics: Regulatory Review" by Rachel Hemphill
in the AAA Casualty Quarterly, Summer 2017.

3.110. The partial residual plot seems linear; thus, no action is indicated.

3.111. A potential problem may exist where two or more predictors in a model may be strongly predictive of a third, a situation known as multicollinearity. Instability problems may result, since the information contained in the third variable is also present in the model in the form of the combination of the other two variables. However, the variable may not be highly correlated with either of the other two predictors individually, and so this effect will not show up in a correlation matrix, making it more difficult to detect.

3.112. Territories are not a good fit for the GLM framework.
One should include the territory relativities produced by the separate model as an offset in the GLM used to determine classification relativities. Similarly, one should include classification relativities produced by the GLM as an offset in the separate model used to determine territory relativities.

3.113. A hold-out sample is data that was not used in the development of the model so that it could be used to test the effectiveness of the model. (This could either be a random sample of the original data, or an additional year of data.) One compares the expected outcome of the model with results on the hold-out sample. The extent to which the model results track closely to results on the hold-out sample for a large part of the portfolio is an indication of how well the model validates.

3.114. You can use age groups, but probably want to group fewer ages together for the younger ages. (Unfortunately, the volume of data is smaller for the very youngest ages, so there is a trade-off between homogeneity and credibility.) For ages above about 25, the affect of gender is relatively small and similar by age. In contrast, for younger ages the affect of gender is large and differs by age. Thus a simple multiplicative model with a single relativity for male compared to female will not work. One would need to have a gender relativity that varied by age. (This may be possible to accomplish this by having an interaction term in the GLM.)

3.115. (a) For $z_1 = 1$ and $z_2 = 30$, renewal probability is:

$$\frac{\text{Exp}[0.6 + (0.05)(1) + (0.02)(30)]}{1 + \text{Exp}[0.6 + (0.05)(1) + (0.02)(30)]} = 0.7773.$$

For $z_1 = 10$ and $z_2 = 30$, renewal probability is:

$$\frac{\text{Exp}[0.6 + (0.05)(10) + (0.02)(30)]}{1 + \text{Exp}[0.6 + (0.05)(10) + (0.02)(30)]} = 0.8455.$$

0.7773 / 0.8455 = **0.919**.

(b) For $z_1 = 1$ and $z_2 = 50$, renewal probability is:

$$\frac{\text{Exp}[0.6 + (0.05)(1) + (0.02)(50)]}{1 + \text{Exp}[0.6 + (0.05)(1) + (0.02)(50)]} = 0.8389.$$

For $z_1 = 10$ and $z_2 = 50$, renewal probability is:

$$\frac{\text{Exp}[0.6 + (0.05)(10) + (0.02)(50)]}{1 + \text{Exp}[0.6 + (0.05)(10) + (0.02)(50)]} = 0.8909.$$

0.8389 / 0.8909 = **0.942**.

Comment: Not intended as a realistic model of policy renewal.
In general for a particular GLM, the relativities for one predictor variable can depend on the level(s) of the other predictor variable(s).
This model was based on the logit link function. If instead the log link function had been used, the model would have been multiplicative, and the indicated multiplicative relativities would not have depended on the other predictor variable. If instead the identity link function had been used, the model would have been additive, and the indicated additive relativities would not have depended on the other predictor variable.

3.116. The deviance residuals seem to decrease on average with X_3.
The lack of independence of the deviance residuals and X_3 is not good.
One should investigate refining the model.

3.117. Let X_0 correspond to the constant term.
Let X_1 be 1 if there is child. Let X_2 be the years of education.

a. $$X = \begin{pmatrix} 1 & 0 & 12 \\ 1 & 0 & 14 \\ 1 & 0 & 15 \\ 1 & 0 & 16 \\ 1 & 0 & 17 \\ 1 & 1 & 10 \\ 1 & 1 & 11 \\ 1 & 1 & 13 \\ 1 & 1 & 15 \\ 1 & 1 & 16 \end{pmatrix} \qquad Y = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

b. $p/(1-p) = \exp[\beta_0 + \beta_1 X_1 + \beta_2 X_2]$. $\Rightarrow 1/p - 1 = \exp[-(\beta_0 + \beta_1 X_1 + \beta_2 X_2)]$.

$\Rightarrow p = 1/ (1 + \exp[-(\beta_0 + \beta_1 X_1 + \beta_2 X_2)])$.

$\Rightarrow 1 - p = \exp[-(\beta_0 + \beta_1 X_1 + \beta_2 X_2)] / (1 + \exp[-(\beta_0 + \beta_1 X_1 + \beta_2 X_2)])$

$= 1 / (1 + \exp[\beta_0 + \beta_1 X_1 + \beta_2 X_2])$.

For a Bernoulli (yes/no) with parameter p, $f(y) = p^y(1-p)^{1-y}$.
$\ln f(y) = y \ln p + (1-y)\ln(1-p) = y \ln[p/(1-p)] + \ln(1-p) =$
$y(\beta_0 + \beta_1 X_1 + \beta_2 X_2) - \ln[1 + \exp[\beta_0 + \beta_1 X_1 + \beta_2 X_2]]$.

loglikelihood = $\sum y_i(\beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i}) - \sum \ln[1 + \exp[\beta_0 + \beta_1 X_{1i} + \beta_2 X_{i2}]]$.

Setting the partial derivatives of the loglikelihood with respect to β_0, β_1, and β_2 equal to zero:

$0 = \sum y_i - \sum \exp[\beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i}]/(1 + \exp[\beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i}])$.

$0 = \sum y_i X_{1i} - \sum X_{1i} \exp[\beta_0 + \beta_1 X_i]/(1 + \exp[\beta_0 + \beta_1 X_{1i} + \beta b_2 X_{2i}])$.

$0 = \sum y_i X_{2i} - \sum X_{2i} \exp[\beta_0 + \beta_1 X_i]/(1 + \exp[\beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i}])$.

$\sum y_i = 1 + 0 + 1 + 0 + 1 + 0 + 0 + 1 + 0 + 1 = 5$.

$\sum y_i X_{1i} = 2$.

$\sum y_i X_{2i} = 12 + 15 + 17 + 13 + 16 = 73$.

The first equation becomes:

$5 = 1/(1 + \exp[-\beta_0 - 12\beta_2]) + 1/(1 + \exp[-\beta_0 - 14\beta_2]) + 1/(1 + \exp[-\beta_0 - 15\beta_2])$
$+ 1/(1 + \exp[-\beta_0 - 16\beta_2]) + 1/(1 + \exp[-\beta_0 - 17\beta_2]) + 1/(1 + \exp[-\beta_0 - \beta_1 - 10\beta_2])$
$+ 1/(1 + \exp[-\beta_0 - \beta_1 - 11\beta_2]) + 1/(1 + \exp[-\beta_0 - \beta_1 - 13\beta_2]) + 1/(1 + \exp[-\beta_0 - \beta_1 - 15\beta_2])$
$+ 1/(1 + \exp[-\beta_0 - \beta_1 - 16\beta_2])$.

The second equation becomes:

$2 = 1/(1 + \exp[-\beta_0 - \beta_1 - 10\beta_2]) + 1/(1 + \exp[-\beta_0 - \beta_1 - 11\beta_2]) + 1/(1 + \exp[-\beta_0 - \beta_1 - 13\beta_2])$
$+ 1/(1 + \exp[-\beta_0 - \beta_1 - 15\beta_2]) + 1/(1 + \exp[-\beta_0 - \beta_1 - 16\beta_2])$.

The third equation becomes:

$73 = 12/(1 + \exp[-\beta_0 - 12\beta_2]) + 14/(1 + \exp[-\beta_0 - 14\beta_2]) + 15/(1 + \exp[-\beta_0 - 15\beta_2])$
$+ 16/(1 + \exp[-\beta_0 - 16\beta_2]) + 17/(1 + \exp[-\beta_0 - 17\beta_2]) + 10/(1 + \exp[-\beta_0 - \beta_1 - 10\beta_2])$
$+ 11/(1 + \exp[-\beta_0 - \beta_1 - 11\beta_2]) + 13/(1 + \exp[-\beta_0 - \beta_1 - 13\beta_2])$
$+ 15/(1 + \exp[-\beta_0 - \beta_1 - 15\beta_2]) + 16/(1 + \exp[-\beta_0 - \beta_1 - 16\beta_2])$.

Comment: In a practical application, one would have at least several hundred data points.
Using a computer, the fitted parameters are:
$\beta_0 = -3.65238$, $\beta_1 = -0.373673$, $\beta_2 = 0.275467$.
The fitted probabilities of workplace participation are:
0.4142, 0.5509, 0.6177, 0.6803, 0.7370, 0.2190, 0.2697, 0.3906, 0.5265, 0.5942.
For example, with a child and 10 years of education, the estimated probability of participating in the workforce is:

$$\frac{\exp[-3.65238 - (1)(0.373673) + (10)(0.275467)]}{1 + \exp[-3.65238 - (1)(0.373673) + (10)(0.275467)]} = \frac{\exp[-1.271383]}{1 + \exp[-1.271383]} = 21.90\%.$$

3.118. 1. The variables to be considered.
2. The distributional form of the errors.
3. The link function.
4. Whether he is modeling frequency, severity, or pure premium.
5. Whether he will be modeling all of the perils together or he will be modeling one of the major perils separately.

Comment: There are probably other good answers.

3.119. 1. Predictive accuracy: for the right panel graph, the plotted loss costs correspond more closely between the two lines than for the left panel graph, indicating that the proposed model seems to predict actual loss costs better than the current rating plan does.
2. Monotonicity: the current plan has a reversal in the 6th decile, whereas the model has no significant reversals.
3. Vertical distance between the first and last quantiles: the spread of actual loss costs for the current plan is about 0.6 to 1.2, which is not very much. The spread of the proposed model is larger.
Thus, by all three metrics, the new plan outperforms the current one.
Comment: Graphs taken from “Introduction to Predictive Modeling Using GLMs A Practitioner’s Viewpoint,” a presentation by Dan Tevet and Anand Khare.

3.120. Modeling personal auto probability of policy renewal.
Modeling fraud on claims.
Comment: Many other possible answers.

3.121. exp[8.8 + (-0.03)(30) - 0.15] = **2322**.

3.122. mean = exp[8.8 + (-0.03)(40)] = 1998. Variance = ϕ mean^2 = (0.3)(1998^2) = **1,197,601**.

3.123. Approximately 95% of the time the actual relativity should be within the bands two standard errors on either side of the parameter estimate.
In the first graph the bands are relatively narrow. Also the relativities display an increase with vehicle symbol, which makes sense. Vehicle symbol appears to be a significant factor for the first model.
In the second graph, the bands are wide. Also the relativities display no consistent pattern with vehicle symbol. Vehicle symbol does not appear to be a significant factor for the second model. There are no parameter estimates more than two standard errors from zero.
In other words, the results are consistent with a multiplicative relativity of one for all symbols.
Comment: The graphs are taken from “A Practitioner's Guide to Generalized Linear Models,“ by Duncan Anderson, Sholom Feldblum, Claudine Modlin, Doris Schirmacher, Ernesto Schirmacher, and Neeza Thandi. We note that in the first graph the one-way (univariate analysis) comes up with different relativities than the GLM, presumably because vehicle symbol is correlated with other significant predictor variables in the GLM. The bottom righthand of the original of the first graph shows a p-value of 0%, indicating that vehicle symbol is significant. The original of the second graph shows a p-value of 52.5% indicating that vehicle symbol is not significant in this second model.

3.124. One way to combine separate models by peril in order to get a model for all perils:
1. Use the separate models by peril to generate predictions of expected loss due to each peril for some set of exposure data.
2. Add the peril predictions together to form a combined loss cost for each record.
3. Run a model on that data, using the combined loss cost calculated in Step 2 as the target, and the union of all the individual model predictors as the predictor variables.

3.125. (a) $\mu = \text{Exp}[\alpha_i + \beta x] = \text{Exp}[\alpha_i]\ \text{Exp}[\beta x]$.

This is a multiplicative model, with relativities for gender and relativities for age.
The age relativities are the same for males and females.
If $\beta < 0$, then the relative frequencies decline exponentially with age.
(b) $\mu = \text{Exp}[\alpha_i + \beta_i x] = \text{Exp}[\alpha_i]\ \text{Exp}[\beta_i x]$.
Similar to the previous model, except now the age relativities differ by gender.
For example, the relativity for age 20 relative to age 30 is:
$\text{Exp}[20\beta_i] / \text{Exp}[30\beta_i] = \text{Exp}[-10\beta_i]$, which differs by gender.
(If $\beta_1 = \beta_2$, then this reduces to the previous model.)

Comment: Even for $\beta_i < 0$, this is not a realistic model of expected claim frequencies by driver age. Instead one would group the ages into for example, 17-20, 21-24, etc., and treat the age groups as categorical variables.

3.126. $d_i^2 = 2\ \{y_i \ln[\frac{y_i}{\hat{y}_i}] + (m_i - y_i) \ln[\frac{m_i - y_i}{m_i - \hat{y}_i}]\} = 2\ \{(3) \ln[3/1.6] + (8 - 3) \ln[(8-3)/(8-1.6)]\}$

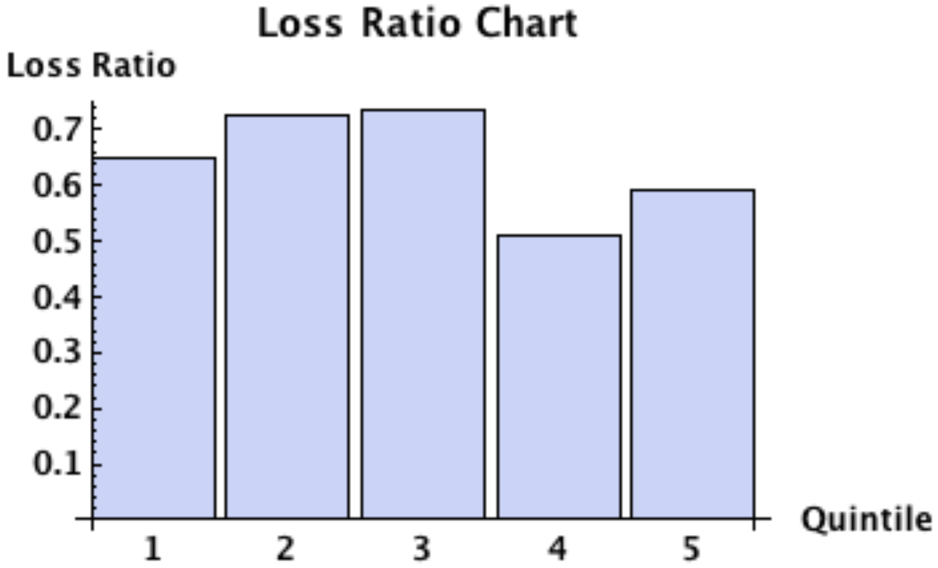
$= 1.3031$. Since $3 - (8)(0.2) > 0$, we take $d_i > 0$. $d_i = \sqrt{1.3031} =$ **1.142**.

3.127. "One major drawback of this approach is that the break points must be selected by the user. Generally, break points are initially guesstimated by visual inspection of the partial residual plot, and they may be further refined by adjusting them to improve some measure of model fit such as deviance. However, the GLM provides no mechanism for estimating them automatically."
"Another potential downside is that while the fitted response curve is continuous, its first derivative is not—in other words, the fit line does not exhibit the smooth quality we would expect, but rather abruptly changes direction at our selected breakpoints."
Comment: Quoted from Section 5.4.4 of Generalized Linear Models for Insurance Rating.

3.128. Sort the data based on the loss ratio predicted by the proposed model.

Insured	Actual Loss Cost	Actual Loss Ratio	Model Loss Cost	Model Loss Ratio	Earned Premium at Present Rates
1	28,000	65.1%	26,000	60.5%	43,000
2	25,000	51.0%	32,000	65.3%	49,000
3	42,000	73.7%	37,000	64.9%	57,000
4	36,000	59.0%	43,000	70.5%	61,000
5	48,000	72.7%	41,000	62.1%	66,000

For the proposed model, the order of predicted loss ratios is: 1, 5, 3, 2, 4.
The corresponding actual loss ratios are: 65.1%, 72.7%, 73.7%, 51.0%, 59.0%.



Comment: Similar to 8, 11/19, Q.2a.
One would work with many more than 5 observations; I would not draw any conclusions based on such a small amount of data.

3.129. D. Histogram D most closely matches the Normal Distribution.

3.130. The results of a GLM depend on the choice of link functions. So perhaps the two models have different link functions. The results of a GLM depend on the choice of predictor variables. So perhaps the two models have different sets of predictor variables other than driver age.
The results of a GLM depend on the choice of the assumed distributional form of the errors. So perhaps the two models have different distributional forms of their errors.
Comment: Commonly the actuary analyzes the relativities for driver age assuming all of the other predictor variables in the GLM are at the base level. If one varies the levels of the other predictor variables in the GLM, then relativities between driver ages will also usually vary.

3.131. Plot (Φ^{-1}[i/37)], $x_{(i)}$).
$Q_{9/37} = Q_{0.243}$ = -0.696, since Φ[-0.696] = 0.243.
Thus the plotted point is: **(-0.696, 0.004)**.

3.132. A useful statistic for detecting multicollinearity is the variance inflation factor (VIF). The VIF for any predictor is a measure of how much the squared standard error for the predictor is increased due to the presence of collinearity with other predictors. It is determined by running a linear model for each of the predictors using all the other predictors as inputs, and measuring the predictive power of those models.
A common statistical rule of thumb is that a VIF greater than 10 is considered high. However, where large VIFs are indicated, it is important to look deeper into the collinearity structure in order to make an informed decision about how best to handle it in the model.

3.133. The new categorical variable has five categories, so adds 4 degrees of freedom.

$$\text{Test statistic is: } F = \frac{D_S - D_B}{(\text{number of added parameters})\ \hat{\phi}_B} = \frac{(2196.1 - 2179.3) / 4}{2.09} = 2.010.$$

The number of degrees of freedom in the numerator is 4.
The number of degrees of freedom in the denominator is:
number of observations minus the number of parameters in the bigger model.
We compare the test statistic to an F-distribution.
The null hypothesis is to use the simpler model.
We reject the null hypothesis if the test statistic is big.

3.134. Cross Validation is another technique for data splitting.
Split the data into for example 10 groups. Each group is called a fold. For each fold:
• Train the model using the other folds.
• Test the model using the given fold.
Several models can be compared by running the procedure for each of them on the same set of folds and comparing their relative performances for each fold.
However, cross validation is often of limited usefulness for most insurance modeling applications.Using cross validation in place of a holdout set is only appropriate where a purely automated variable selection process is used. The actuary usually applies a great deal of care and judgment in selecting the variables to be included in the model. If using cross validation, this actuarial judgement should be applied separately to each of the data sets created by leaving out one fold. This is not really practical.
For most actuarial modeling, the use of a holdout set is preferred to the use of cross validation.
Comment: See Section 4.3.4 of Generalized Linear Models for Insurance Rating.
Purely automated variable selection processes should be used with appropriate caution.

3.135. A common statistical rule of thumb is that a VIF greater than 10 is considered high.
Thus, there is probably multicollinearity related to Weight; two or more predictors in the model are probably strongly predictive of Weight. This may cause instability problems with the model. This situation should be investigated further.
It may help to either remove Weight from the model or to preprocess the data using dimensionality reduction techniques such as principal components analysis.
Comment: The VIF of 6.33 for Body Surface Area may also warrant some investigation.

3.136. In the first graph for liability losses, the number of children seems to have a significant impact on frequency. The 95% confidence intervals do not include a log of the multiplier of 0; in other words the multiplier is significantly different from one. Also while one child increases the frequency compared to none, two children also increase the frequency compared to one. It seems as if the number of children in the household is a useful variable for modeling liability frequency for Homeowners.

In the second graph for wind losses, the number of children seems to have a insignificant impact on frequency. The 95% confidence intervals do include a log of the multiplier of 0; in other words the multiplier is not significantly different from one. Also while one child increases the frequency compared to none, two children decreases the frequency compared to one. The number of children in the household is not a useful variable for modeling wind frequency for Homeowners.

Comment: There is no logical relationship between the number of children and wind losses.

A child (or any relative) who lives in the house is covered for any liability claim he or she causes. Also having children in the house may lead to more neighborhood children coming on your property with the potential for liability claims if they are injured on your property. Thus there is some logical relationship between the number of children in the household and the frequency of liability claims for Homeowners.

Presumably, the liability relativity for three children would be higher than for two children. (Three children was not shown in the graph in order to keep things simple.)

One would want to apply statistical tests to see if the number of children in the household is a useful variable for modeling liability frequency. Also one would want to check the consistency over time of the indicated relativities.

3.137. With a Normal error function and an identity link function, this is the same as a multiple regression. The squared error is:

$$800\,(\beta_1 + \beta_2 + \beta_3 - 700{,}000/800)^2 + 600\,(\beta_2 + \beta_3 - 400{,}000/600)^2$$
$$+\ 700\,(\beta_1 + \beta_3 - 500{,}000/700)^2 + 500\,(\beta_3 - 300{,}000/500)^2.$$

We are given that $\beta_3 = 570.356$, thus the squared error is:

$800\,(\beta_1 + \beta_2 - 304.644)^2 + 600\,(\beta_2 - 96.311)^2 + 700\,(\beta_1 - 143.930)^2 + 500\,(-29.644)^2.$

Setting the partial derivative with respect to β_1 equal to zero:

$0 = 1600\,(\beta_1 + \beta_2 - 304.644) + 1400\,(\beta_1 - 143.930). \Rightarrow 3000\,\beta_1 + 1600\,\beta_2 = 688{,}932.$

Setting the partial derivative with respect to β_2 equal to zero:

$0 = 1600\,(\beta_1 + \beta_2 - 304.644) + 1200\,(\beta_2 - 96.311). \Rightarrow 1600\,\beta_1 + 2800\,\beta_2 = 603{,}004.$

$\Rightarrow \beta_2 = (603{,}004 - 1600\,\beta_1) / 2800.$

Plugging back into the first equation: $3000\,\beta_1 + 1600\,(603{,}004 - 1600\,\beta_1) / 2800 = 688{,}932.$

$$\Rightarrow \beta_1 = \frac{(2800)(688{,}932) - (1600)(603{,}004)}{(3000)(2800) - (1600)(1600)} = 964{,}203{,}200 / 5{,}840{,}00 = \mathbf{165.103}.$$

$\Rightarrow \beta_2 = 121.014.$

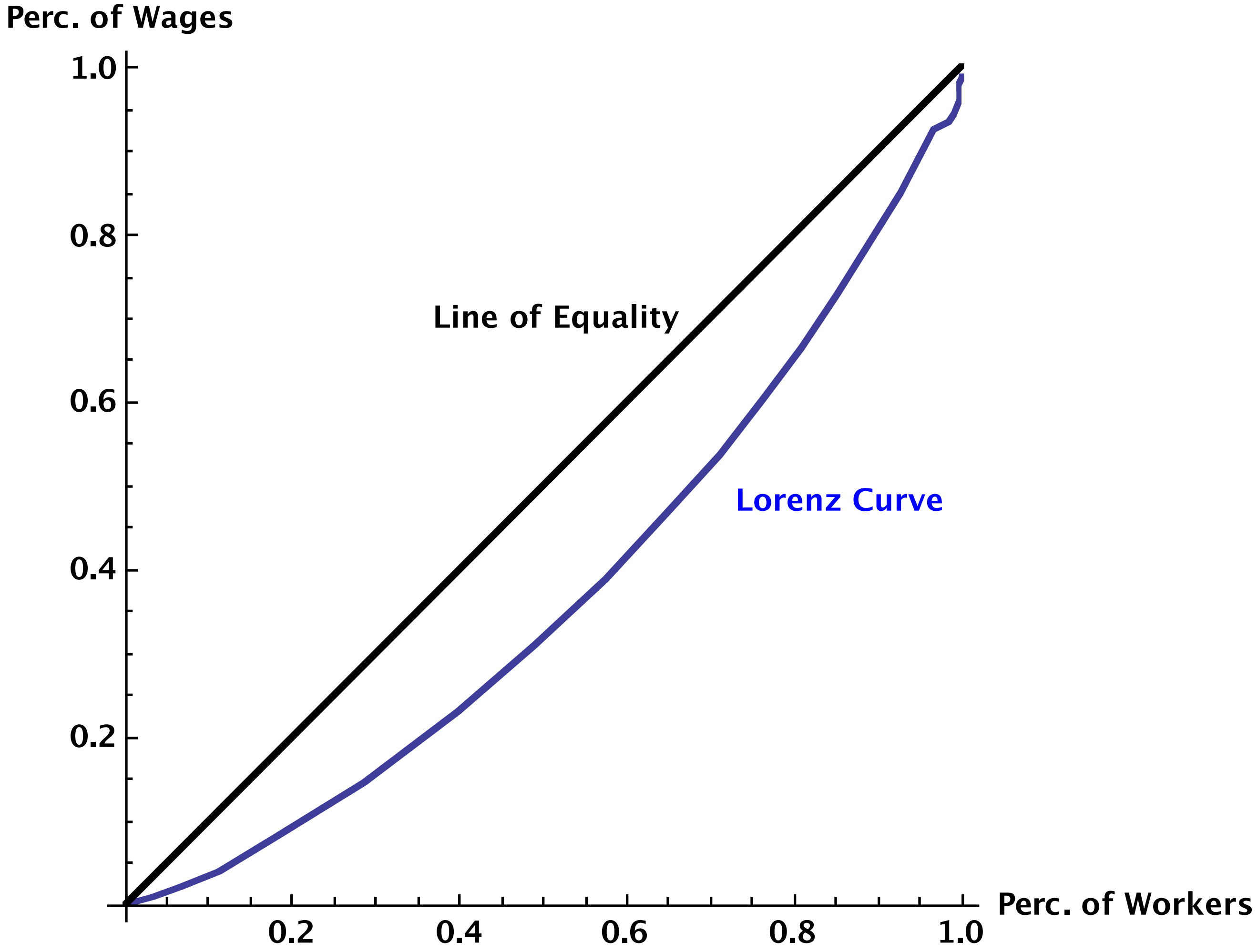
3.138. BIC = (-2) (maximum loglikelihood) + (number of parameters)ln[60].
For example, BIC = (-2)(-220.18) + 2 ln[60] = 448.55.

Model	Number of Parameters	Loglikelihood	BIC
A	2	-220.18	448.55
B	3	-217.40	447.08
C	4	-214.92	**446.22**
D	5	-213.25	446.97
E	6	-211.03	454.81

Since BIC is smallest for model C, model C is preferred.

3.139. One should also perform a statistical test to compare a model with year to a simpler model without year.
Before excluding year as a variable, it would be better to first try a model where you group the years into fewer categories, for example: 2010-2011, 2012, 2013-2014.
(We may not have enough data from each year in order to be statistically confident of separate coefficients by year.)
Then if after fitting the new model the revised coefficients for years are still not significant, one could exclude year from the model.
Comment: The actuary would want to determine whether the pattern between years of the fitted coefficients makes any sense to him given his knowledge of the situation being modeled.
Statistical tests are important, but just one tool. Actuarial judgement is also important.

3.140. The third plotted point is: (0.0353, 0.0079).
The last plotted point is: (0.9997, 0.9884).



Comment: In my graph, I have had the computer join the plotted points.
Information was taken from the 1998 Massachusetts Wage Distribution Table.
The Gini index is twice the area between the Lorenz Curve and the Line of Equality.

3.141. Simple quantile plots are created via the following steps:
1. Sort the dataset based on the Model A predicted loss cost (from smallest to largest).
2. Bucket the data into quantiles, such that each quantile has the same volume of exposures.
Common choices are quintiles (5 buckets), deciles (10 buckets), or vigintiles (20 buckets).
3. Within each bucket, calculate the average predicted pure premium (predicted loss per unit of exposure) based on the Model A predicted loss cost, and calculate the average actual pure premium.
4. Plot, for each quantile, the actual pure premium and the pure premium predicted by Model A.
5. Repeat steps 1 through 4 using the Model B predicted loss costs.
There are now two quantile plot; one for Model A and one for Model B.
6. Compare the two quantile plots to determine which model provides better lift.
In order to determine the "winning" model, consider the following 3 criteria:
1. **Predictive accuracy.** How well each model is able to predict the actual pure premium in each quantile.
2. **Monotonicity.** By definition, the predicted pure premium will monotonically increase as the quantile increases, but the actual pure premium should also increase (though small reversals are okay).
3. **Vertical distance between the first and last quantiles.** The first quantile contains the risks that the model believes will have the best experience, and the last quantile contains the risks that the model believes will have the worst experience. A large difference (also called "lift") between the actual pure premium in the quantiles with the smallest and largest predicted loss costs indicates that the model is able to maximally distinguish the best and worst risks.
Comment: See Section 7.2.1 of Generalized Linear Models for Insurance Rating.

3.142. (a) For example, using a discrimination threshold of 25%, one would be predicting fraud for any claim for which the GLM says the probability of fraud is greater than 25%.
Alternately, choose a specific probability level, called the discrimination threshold, above which we will investigate the claim for fraud and below which we will not. This determination may be thought of as the model's "prediction" in a binary (i.e., fraud/no fraud) sense.
(b) Using a lower threshold would detect more of the fraudulent claims, at the cost of also having to investigate more claims which turned out not to be fraudulent. Using a higher threshold would detect fewer of the fraudulent claims, but we would have to investigate fewer claims which turned out not to be fraudulent.
Alternately, there is trade-off: a lower threshold results in a higher sensitivity (true positive rate), while a higher threshold results in a higher specificity (and thus a lower false positive rate).
Comment: Similar to 8, 11/17, Q.6d.
See Section 7.3.1 of Generalized Linear Models for Insurance Rating.
"The selection of a discrimination threshold involves a trade-off: a lower threshold will result in more true positives and fewer false negatives than a higher threshold, but at the cost of more false positives and fewer true negatives."

3.143. The mean modeled claim counts are:

	Terr. A	Terr. B
Male	24,000 $\exp[\beta_0]$	15,000 $\exp[\beta_0 + \beta_2]$
Female	20,000 $\exp[\beta_0 + \beta_1]$	13,000 $\exp[\beta_0 + \beta_1 + \beta_2]$

The likelihood function of a Poisson is : $\sum \ln f(y_i;\ \mu_i) = \sum \{-\mu_i + y_i \ln[\mu_i] - \ln[y_i!]\}$.

The loglikelihood ignoring terms that do not depend on the betas is:
$-24{,}000 \exp[\beta_0] + 1200\ (\beta_0) - 20{,}000 \exp[\beta_0 + \beta_1] + 800\ (\beta_0 + \beta_1)$
$- 15{,}000 \exp[\beta_0 + \beta_2] + 1100\ (\beta_0 + \beta_2) - 13{,}000 \exp[\beta_0 + \beta_1 + \beta_2] + 900\ (\beta_0 + \beta_1 + \beta_2)$.

Setting the partial derivative of the loglikelihood with respect to β_1 equal to zero:

$- 20{,}000 \exp[\beta_0 + \beta_1] + 800 - 13{,}000 \exp[\beta_0 + \beta_1 + \beta_2] + 900 = 0$.

Given $\beta_0 = -3.0300$: $1700 = 966.3 \exp[\beta_1] + 628.1 \exp[\beta_1] \exp[\beta_2]$.

Setting the partial derivative of the loglikelihood with respect to β_2 equal to zero:

$- 15{,}000 \exp[\beta_0 + \beta_2] + 1100 - 13{,}000 \exp[\beta_0 + \beta_1 + \beta_2] + 900 = 0$.

$\Rightarrow 2000 = 724.7 \exp[\beta_2] + 628.1 \exp[\beta_1] \exp[\beta_2]$.

Subtracting two equations: $300 = 724.7 \exp[\beta_2] - 966.3 \exp[\beta_1]$.

$\Rightarrow \exp[\beta_2] = 0.4140 + 1.3334 \exp[\beta_1]$.

$\Rightarrow 1700 = 966.3 \exp[\beta_1] + 628.1 \exp[\beta_1]\ (0.4140 + 1.3334 \exp[\beta_1])$.

Let $x = \exp[\beta_1]$. $\Rightarrow 1700 = 966.3\ x + 628.1\ x\ (0.4140 + 1.3334\ x)$.

$\Rightarrow 837.5x^2 + 1226.3\ x - 1700 = 0$.

$$\Rightarrow x = \frac{-1226.3 \pm \sqrt{1226.3^2 - (4)(837.5)(-1700)}}{(2)(837.5)} = 0.8697, \text{ taking the positive root.}$$

$\Rightarrow \beta_1 = \ln(0.8697) = -0.1396$.

$\Rightarrow \exp[\beta_2] = 0.4140 + 1.3334 \exp[\beta_1] = 0.4140 + (1.3334)(0.8697) = 1.5737$.

$\Rightarrow \beta_2 = \ln(1.5737) = 0.4534$.

Expected frequency of a female risk in Territory B is:
$\exp[\beta_0 + \beta_1 + \beta_2] = \exp[-3.0300 - 0.1396 + 0.4534] =$ **6.61%**.

Comment: Similar to 8, 11/15, Q.3.
Using a computer, without being given β_0, the maximum likelihood fit is:

$\hat\beta_0 = -3.02999$, $\hat\beta_1 = -0.139599$, $\hat\beta_2 = 0.453335$.

The mean modeled frequencies are:

	Territory A	Territory B
Male	exp[-3.02999] = 4.83%	exp[-3.02999 + 0.453335] = 7.60%
Female	exp[-3.02999 - 0.139599] = 4.20%	exp[-3.02999 - 0.139599 + 0.453335] = 6.61%

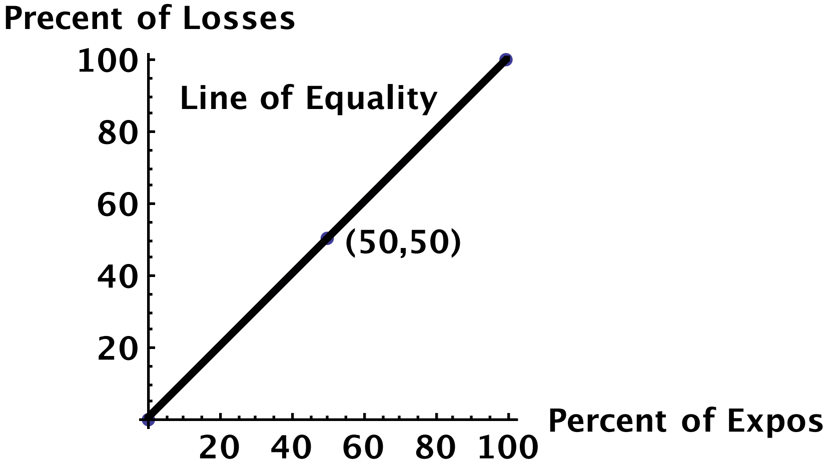
3.144. When the effect of one predictor depends on the level of another predictor, and vice-versa, such an relationship is called an interaction.
An example of an interaction term: X_1X_2.

In this example, $g(\mu) = \beta_0 + \beta_1X_1 + \beta_2X_2 + \beta_3 X_1X_2 + ...$

The effect of X_1 depends on the level of X_2 and vice-versa.

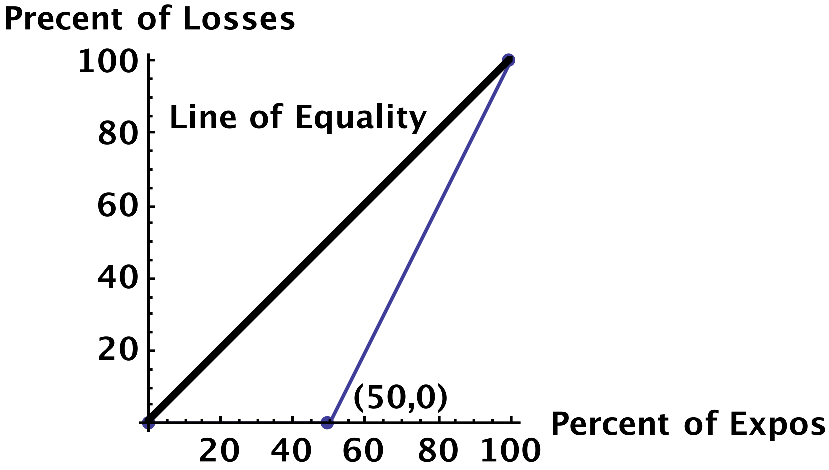
Comment: See Section 5.6 of Generalized Linear Models for Insurance Rating.
The actuary can use the GLM significance statistics in order to determine whether the inclusion of an interaction significantly improves the model.

3.145. (a) The percent of losses for A are 50%. So the Lorenz Curve has the point (50, 50).



The Lorenz curve is equal to the line of equality, and thus the area between them is zero. The Gini Index is twice that, or **zero**.

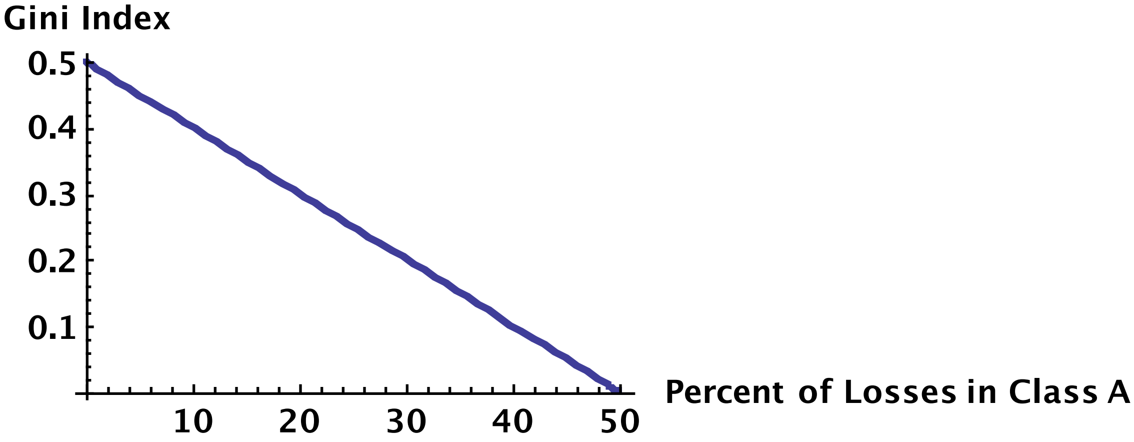
(b) The percent of losses for A are 0%. So the Lorenz Curve has the point (50, 0).



The region between the Lorenz curve and the line of equality is a triangle of base 50% and height 100%, and thus area: (1/2)(50%)(100%) = 0.25. The Gini Index is twice that, or **50%**.

Comment: We looked at the two extreme cases, which will not occur in practice.

Here is a graph of the Gini Index versus the percent of total actual losses in Class A:



3.146. The curve corresponding to the text labeled VA has more area under it, so it is better than the test labeled NE.

3.147. The fitted parameter(s) are the same, while the standard errors are multiplied by $\sqrt{3.071}$.

The standard error of $\hat{\beta}_1$ is: $0.1978 \sqrt{3.071} = 0.3466$.

95% confidence interval for β_1: 5.624 ± (1.96) (0.3466) = **5.624 ± 0.679**.

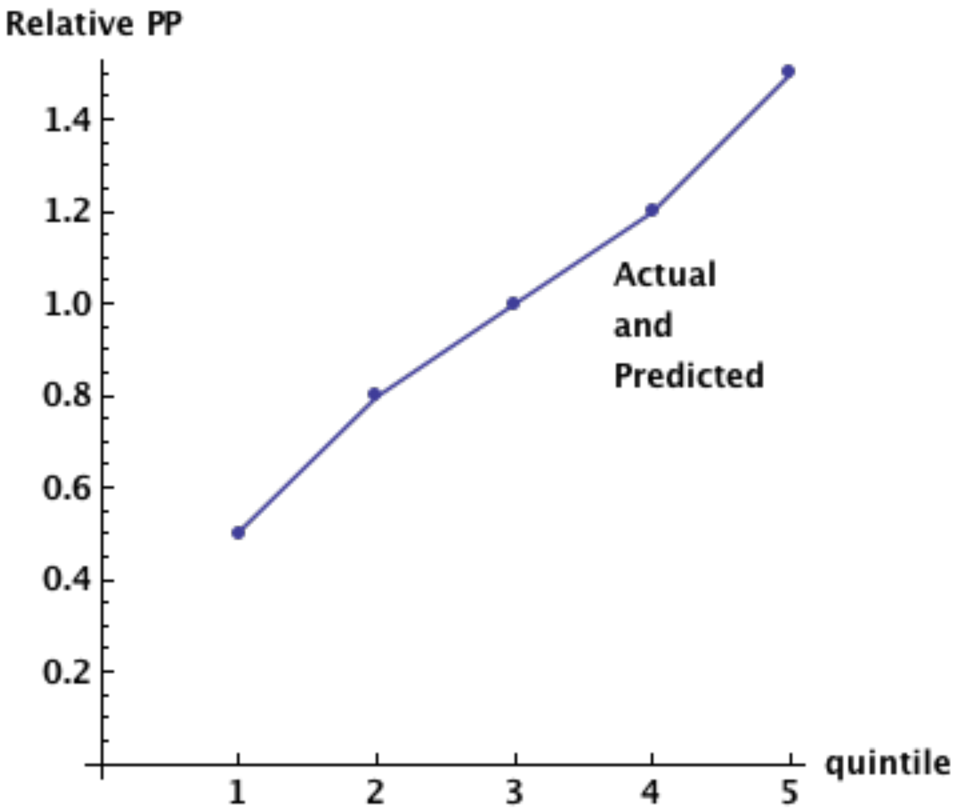
Comment: One could instead use: 5.624 ± (2) (0.3466) = 5.624 ± 0.693.

3.148. A simple quintile plot is a simple quantile plot with 5 buckets.

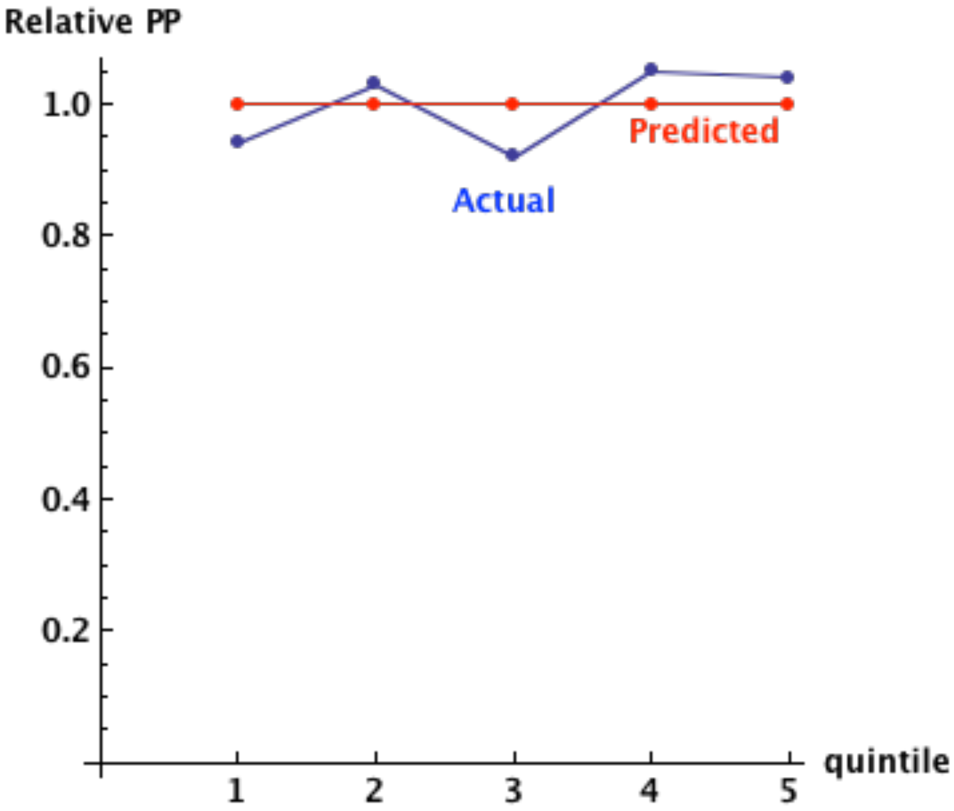
- Sort the dataset based on the model predicted pure premium from smallest to largest.
- Group the data into 5 buckets with equal volume.
- Within each group, calculate the average predicted pure premium based on the model, and the average actual pure premium.
- Plot for each group, the actual pure premium and the predicted pure premium.

Since we are not given the overall average pure premium, I will plot the pure premiums relative to average.

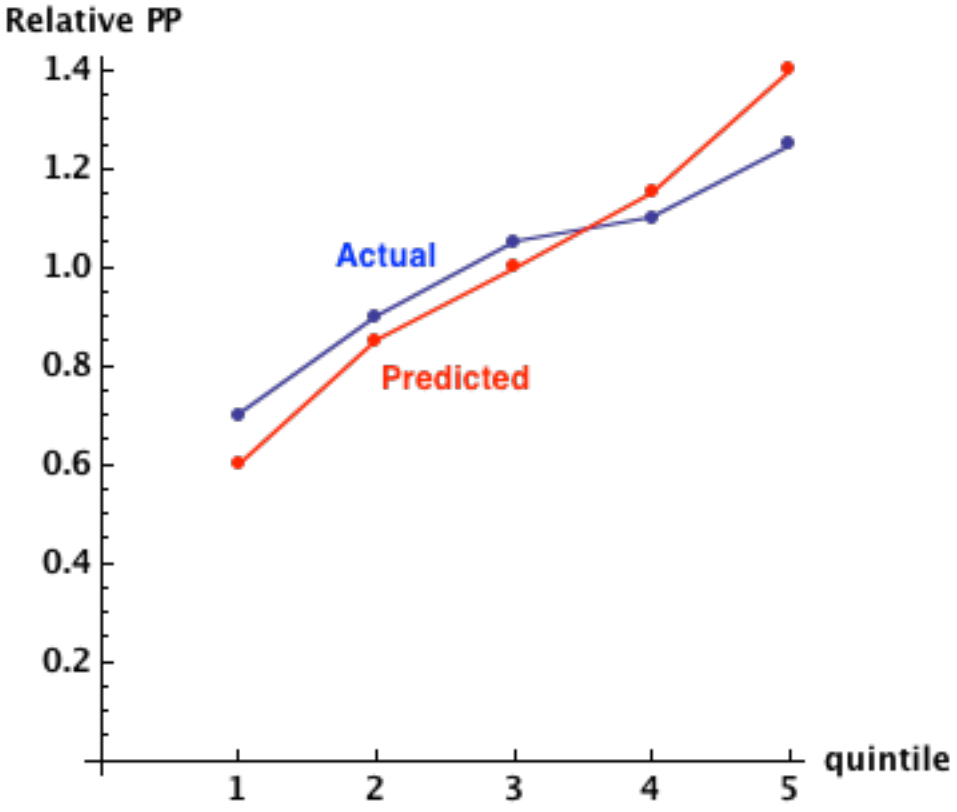
The saturated model has as many predictors as data points. Thus for the saturated model, the predictions exactly match the observations for each record.
The simple quintile plot:



The null model, has no predictors, only an intercept. Thus for the null model the prediction is the same for every record: the grand mean.
Since every risk has the same prediction, one would assign them to buckets at random.
Thus all of the actuals by quintile should be close to the grand mean, with small differences due to the randomness of assignments. The simple quintile plot:



“A model that could be used in practice”, would have the actuals increase monotonically, have good but not perfect predictive accuracy, and a reasonably large vertical distance between the actuals in the first and last quintiles. A simple quintile plot:



Comment: Similar to 8, 11/07, Q. 5.
There are many possible examples of the last plot.
Since the records are ordered by predicted values, the records in each bucket change for each graph. Thus, actuals are not the same between the graphs.
Quintile plots are sorted by predicted values from smallest to largest value. Thus the predicted values must be monotonically increasing (or in the case of the null model equal). Actuals need not be monotonically increasing, although that is desirable.
In every graph, the average of the actuals should be the grand mean.
In the final plot, the average of the predicteds should be close to if not equal to the grand mean; the GLM may have a small bias.
In the final plot, the predicted and actuals for the final quintile should each be less than in the saturated model. In the final plot, the predicted and actuals for the final quintile should each be more than in the null model.

3.149. I prefer the Gamma model, since the standardized deviance residuals are much closer to being Normally Distributed.

3.150. Since the proposed model is not able to segment the data into lower and higher loss ratio buckets, the proposed model is not significantly outperforming the current rating plan.
Comment: See Section 7.2.3 of Generalized Linear Models for Insurance Rating.

3.151. AIC = (-2) (maximum loglikelihood) + (number of parameters)(2).
For example, AIC = (-2)(-9844.16) + (5)(2) = 19,698.32.

Model	Number of Parameters	Loglikelihood	AIC
A	5	-9844.16	19,698.32
B	10	-9822.48	19,664.96
C	15	-9815.70	19,661.40

Since AIC is smallest for model C, model C is preferred.

3.152. BIC = (-2) (maximum loglikelihood) + (number of parameters) ln[number of data points].
For example, BIC = (-2)(-9844.16) + 5 ln[5000] = 19730.91.

Model	Number of Parameters	Loglikelihood	BIC
A	5	-9844.16	19,730.91
B	10	-9822.48	19,730.13
C	15	-9815.70	19,759.16

Since BIC is smallest for model B, model B is preferred.
Comment: Similar to 8, 11/16, Q.7.
See Section 6.2.2 in Generalized Linear Models for Insurance Rating.
Most actuarial GLMs are fit to many more than 5000 data points.
“As most insurance models are fit on very large datasets, the penalty for additional parameters imposed by BIC tends to be much larger than the penalty for additional parameters imposed by AIC. In practical terms, the authors have found that AIC tends to produce more reasonable results. Relying too heavily on BIC may result in the exclusion of predictive variables from your model.”

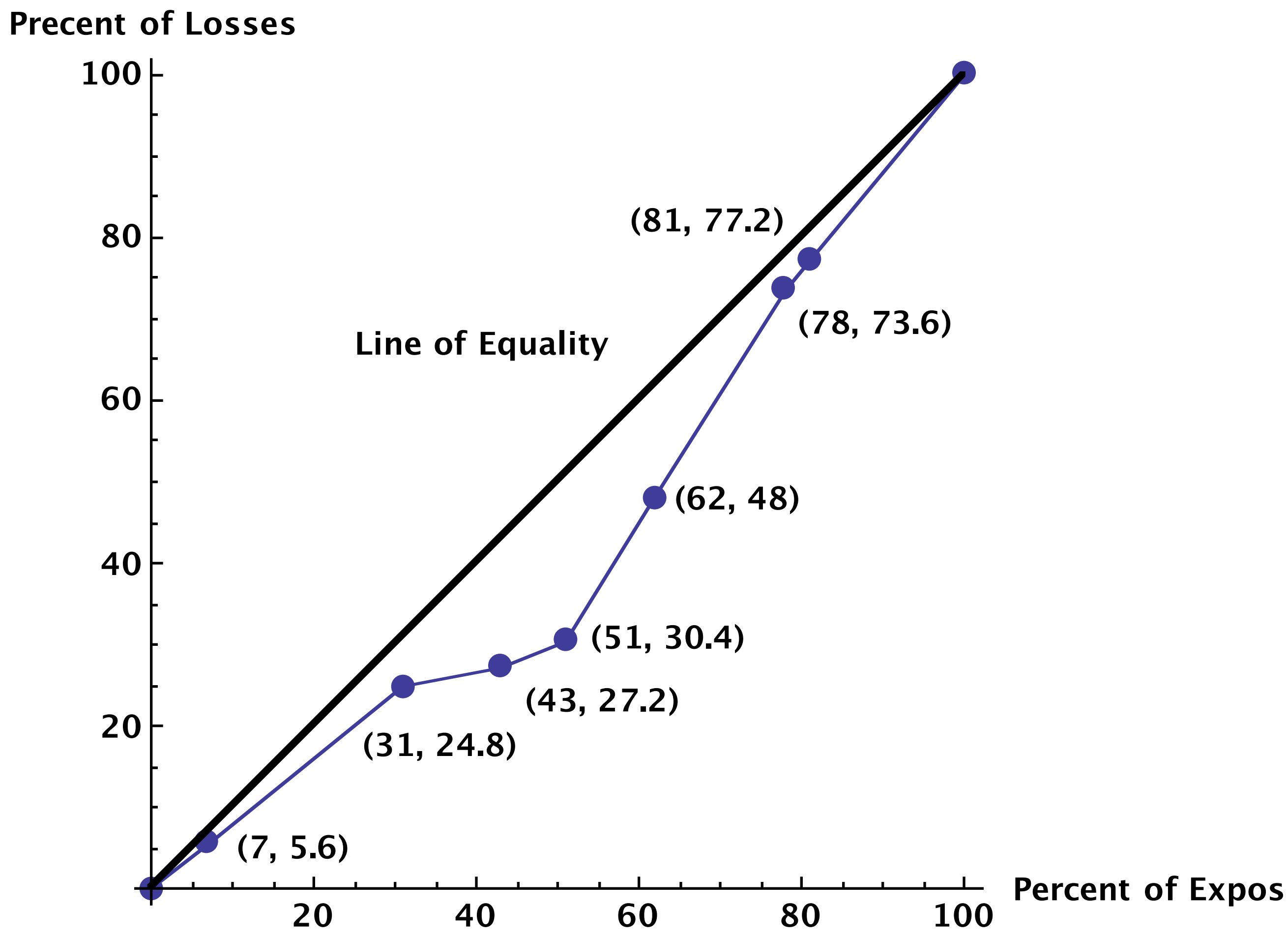
3.153. The first model does a better job of fitting the data and is thus preferred.

3.154. Sort the risks from best to worst based on the model predicted pure premium.

Risk	Model P.P.	Exposures	Cumulative Exposures	Cumulative % of Exposures
2	1000	7	7	7%
8	2000	24	31	31%
5	3000	12	43	43%
3	4000	8	51	51%
4	5000	11	62	62%
6	6000	16	78	78%
1	7000	3	81	81%
7	8000	19	100	100%

Risk	Exposures	Actual P.P.	Actual Losses	Cumulative Losses	Cumulative % of Losses
2	7	4000	28,000	28,000	5.6%
8	24	4000	96,000	124,000	24.8%
5	12	1000	12,000	136,000	27.2%
3	8	2000	16,000	152,000	30.4%
4	11	8000	88,000	240,000	48.0%
6	16	8000	128,000	368,000	73.6%
1	3	6000	18,000	386,000	77.2%
7	19	6000	114,000	500,000	100.0%

On the x-axis, plot the cumulative percentage of exposures.
On the y-axis, plot the cumulative percentage of actual losses.
The plotted points are: (0, 0), (7%, 5.6%), (31%, 24.8%), ... , (81%, 77.2%), (100%, 100%).



<u>Comment</u>: Similar to 8, 11/16, 5a.
The Gini index is twice the area between the Lorenz Curve and the line of equality.
The higher the Gini Index, the better the rating plan is at identifying risk differences.

3.155.

Variable	Number of Parameters
Vehicle Price	3
Vehicle Age	8 - 1 = 7
Driver age	2 - 1 = 1
Number of drivers	3 - 1 = 2
Gender	2 - 1 = 1
Interaction Gender & Driver Age	1

Number of parameter is: 3 + 7 + 1 + 2 + 1 + 1 = **15**.
<u>Comment</u>: Similar to CAS S, 11/15, Q.35.
A model with only Vehicle Price would involve: $\beta_0 + \beta_1$ (vp) + β_2 (vp)^2.
The interaction of gender and driver age only uses one parameter since each of gender and driver age only use one parameter.

3.156. A double lift chart compares the current rating plan to a proposed model.
Sort data by ratio of model prediction to current premium.
Subdivide sorted data into quantiles with equal exposure.
For each quantile calculate average actual loss cost, average model predicted loss cost and the average loss cost underlying the current manual premium .
Index the quantile averages to the overall averages.
Plot the results.
Comment: The "winning" model is the one that more closely matches the actual pure premiums.

3.157. The difference in degrees of freedom is: 18,175 - 18,169 = 6; we add 6 parameters.

Test statistic is: $F = \frac{D_S - D_B}{(\text{number of added parameters})\ \hat{\phi}_B} = \frac{8{,}905.6226 - 8{,}901.4414}{(6)\ (0.4523)} = 1.541.$

The number of degrees of freedom in the numerator is 6.
The number of degrees of freedom in the denominator is:
number of degrees of freedom for the bigger model = 18,169.
We compare the test statistic to the appropriate F-distribution.
The null hypothesis is to use the simpler model.
We reject the null hypothesis if the test statistic is sufficiently big.
Comment: Using a computer, the p-value is 16.0%. Thus at for example a 10% significance level, we do not reject the null hypothesis to use the simpler model.

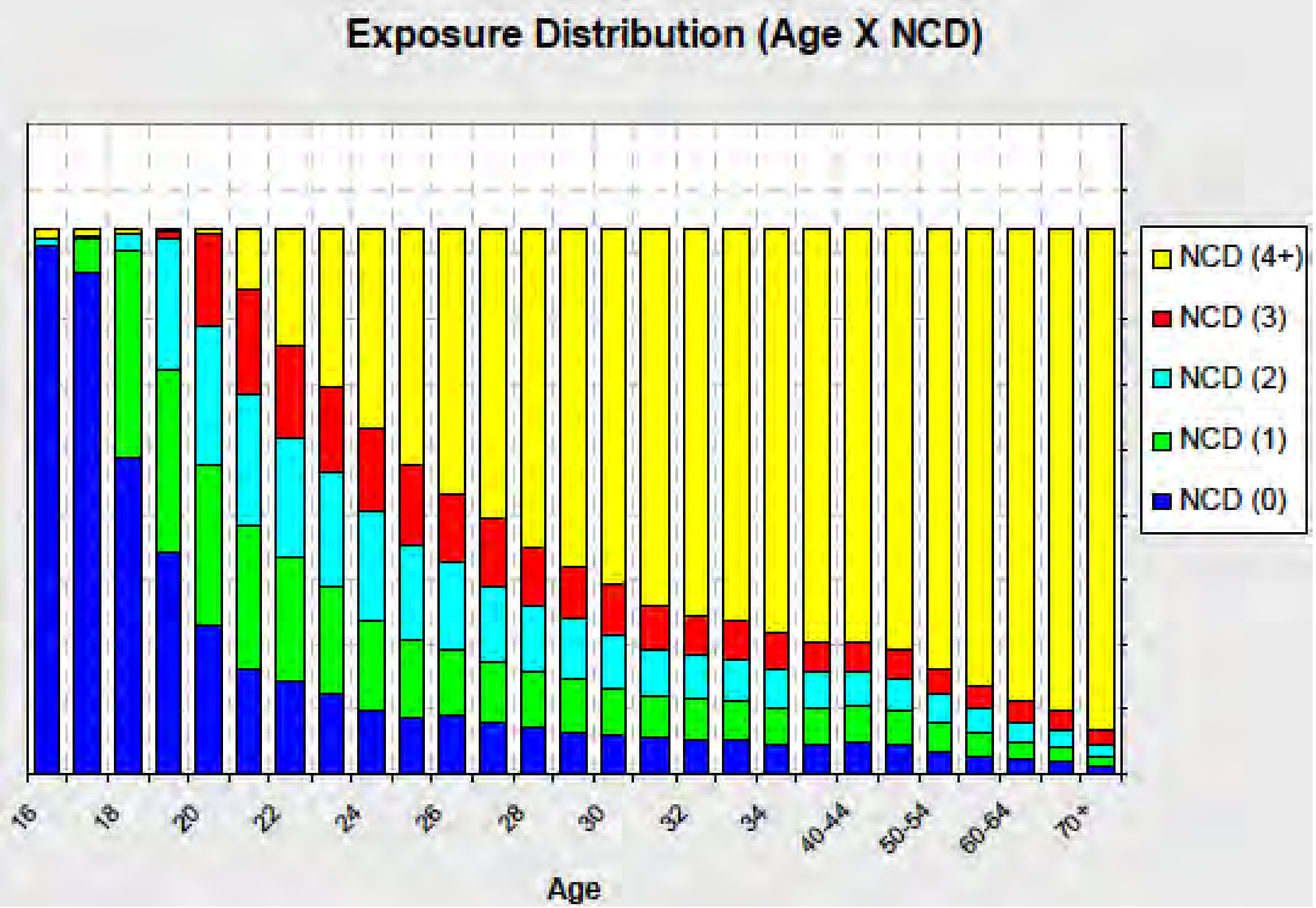
3.158. mean = exp[5.07 + 0.48 - 0.36] = 179.5.
Variance = $\text{mean}^2 / \alpha = 179.5^2 / 2.2 =$ **14,646**.
Comment: Similar to CAS S, 5/16, Q.32.

3.159. The Gini index can be used to measure the lift of an insurance rating plan by quantifying its ability to segment the population into the best and worst risks.
The larger the Gini index, the better job the rating plan does of segmenting.
Thus the rating plan used in Model 1 has more lift than the rating plan used in Model 2.

3.160. One works with loss ratios with respect to the premiums for the current plan.
To create a loss ratio chart:
1. Sort the dataset based on the model prediction, in other words modeled loss ratios.
2. Group the data into quantiles with equal volumes of exposures.
3. Within each group, calculate the actual loss ratio (under the current plan).
Comment: If the proposed model is able to segment the data into lower and higher loss ratio buckets, then the proposed model is better than the current model.

3.161. Driver age and number of years claims-free are positively correlated. Older drivers are likely to be claims-free for more years than younger drivers. Thus in order to avoid double counting effects, the GLM lessens the effect of each variable somewhat compared to a model that just used one of the two variables.

Comment: A graph of number of years claims-free versus driver age:



Graph taken from "GLM II: Basic Modeling Strategy," by Claudine Modlin,
CAS Predictive Modeling Seminar, October 2008.

If two variables are very highly correlated, which is not the case here, then the GLM will have trouble converging and the parameter estimates may be unreliable.

3.162. If the current rating plan were perfect, then all risks should have the same loss ratio. The fact that the proposed model is able to segment the data into lower and higher loss ratio buckets is a strong indicator that it is outperforming the current rating plan.

Comment: Graph taken from "Goodness of Fit vs. Goodness of Lift," by Glenn Meyers
and David Cummings, August 2009 Actuarial Review.

If one insurer were to use the current rating plan, while another insurer were to use the proposed rating plan, the second insurer should be able to attract better risks from the first insurer. The first insurer who continued to use the current plan would be subject to adverse selection.

3.163. The offset is: ln(0.8) = -0.223.
The linear component is: 6 + (0.1)(13) + (-0.2)(3) = 6.7.
Modeled pure premium = exp[6.7 - 0.223] = **650**.
Alternately, the modeled pure premium is: (0.8) exp[6 + (0.1)(13) + (-0.2)(3)] = **650.**

3.164. Laurel is proposing the saturated model; it is overfit.
Hardy is proposing the null model; it is underfit.
A model with a number of parameters between the two would make sense.

3.165. The average relativity is: (1.50)(9%) + (1.35)(4%) + (1.18)(5%) + (1)(82%) = 1.0680.
Thus compared to average, 3 or more years claims-free has a relativity of:
1.00/1.0680 = 0.9363.
Thus three years of data has a credibility of: 1 - 0.9363 = **6.4%**.
Comment: Bailey and Simon look at the credibility of cars rather than drivers.
Also, Bailey and Simon were dealing with a very simple classification system.
The credibility depends on how refined the class system is; the more refined the classification system, the less credibility is given to individual experience.
In order to estimate a credibility for one year of data, one would have to group drivers into those that had at least one year claims free.

3.166. These two variables are likely significantly positively correlated.
(Those policies with three or more operators, usually list parents and one or more children as drivers. The listed child will be a teenager or young adult living at home. Very few teenagers have their own car on their own separate policy.)
Including two such variables in a model, can produce anomalous results.
In any case, due to the interaction of these two variables, it will be difficult to interpret the relativities for the different levels of each variable.

3.167. Both AIC and BIC are penalized measures of fit; in each case a penalty is added to twice the negative loglikelihood. In the case of AIC the penalty is twice the number of parameters in the model, while in the case of BIC the penalty is p log(n), where p is the number of parameters, and n is the number of data points that the model is fit on
In both cases, a smaller statistic is better.
“In practical terms, the authors have found that AIC tends to produce more reasonable results. Relying too heavily on BIC may result in the exclusion of predictive variables from your model.”
In other words, they believe that the use of BIC will tend to produce models that are underfit.

3.168. (a) Working Residual is: $wr_i = (y_i - \mu_i)\ g'(\mu_i)$.

(b) Most insurance models have thousands or even millions of observations, making plots of residuals much less useful. Therefore, for such models, it is critical to bin the residuals before analyzing them. (Binning the residuals aggregates away the volume and skewness of individual residuals, and allows us to focus on the signal.) The advantage of working residuals is that they can be aggregated in a way that preserves the common properties of residuals – that is, they are unbiased (i.e., have no predictable pattern in the mean) and homoscedastic (i.e.,have no pattern in the variance) for a well-fit model.

(c) working weights: $ww_i = \frac{\omega_i}{V(\mu_i)\ g'(\mu_i)^2}$.

g is the link function. V is the variance function for the distribution used.

ω_i is the weight given in the model to the ith observation.

(d) For each bin, the binned working residual is calculated by taking the weighted average of the working residuals of the individual observations within the bin, weighted by the working weights. It is these binned working residual that will be plotted.
(e) 1. Plotting Residuals over the Linear Predictor
2. Plotting Residuals over the Value of a Predictor Variable
3. Plotting Residuals over the Weight
Comment: See Section 6.3.2 of Generalized Linear Models for Insurance Rating.

3.169. The “winning” model is the one that more closely matches the actual pure premiums. The proposed model does a much better job than the current rating plan; thus the proposed model is preferred.

3.170. 1. Model A does a better job of matching the actual than does Model B. Thus based on the criterion of predictive accuracy I prefer Model A.
Both models satisfy the criterion of monotonicity; the actuals increase with quintile.
Model A has a larger vertical distance between the actuals for the first and last quintiles than does Model B. Thus based on this criterion I prefer Model A.
Thus overall I prefer Model A to Model B.
Comment: In order to determine the “winning” model, consider the following 3 criteria:
1. Predictive accuracy. How well each model is able to predict the actual pure premium in each quantile.
2. Monotonicity. By definition, the predicted pure premium will monotonically increase as the quantile increases, but the actual pure premium should also increase (though small reversals are okay).
3. Vertical distance between the first and last quantiles. The first quantile contains the risks that the model believes will have the best experience, and the last quantile contains the risks that the model believes will have the worst experience. A large difference (also called “lift”) between the actual pure premium in the quantiles with the smallest and largest predicted loss costs indicates that the model is able to maximally distinguish the best and worst risks.

3.171. I prefer the Inverse Gaussian model, since the standardized deviance residuals are much closer to being Normally Distributed.

3.172. $X\beta = 1 + (8)(0.31) = 3.48$.

The odds are: $e^{3.48}$ = **32.5**.

Alternately, for the logistic model: $\hat{\pi} = e^{3.48} / (1 + e^{3.48}) = 0.9701$.

The odds are: $\hat{\pi} / (1 - \hat{\pi}) = 0.9701 / (1 - 0.9701)$ = **32.4**.

Comment: Similar to MAS-1, 5/18, Q.27.

We have estimated that the probability of renewal is 32.5 times the probability of a nonrenewal.

3.173.

			30% Threshold			60% Threshold	
Claim #	Fraud		Predict.			Predict.	
1	N		Y	False Pos.		N	True Neg.
2	N		Y	False Pos.		N	True Neg.
3	N		N	True Neg.		N	True Neg.
4	N		N	True Neg.		N	True Neg.
5	Y		Y	True Pos.		Y	True Pos.
6	N		N	True Neg.		N	True Neg.
7	Y		N	False Neg.		N	False Neg.
8	N		Y	False Pos.		Y	False Pos.
9	Y		Y	True Pos.		Y	True Pos.
10	Y		Y	True Pos.		Y	True Pos.
11	N		Y	False Pos.		N	True Neg.
12	Y		Y	True Pos.		N	False Neg.
13	N		Y	False Pos.		N	True Neg.
14	N		Y	False Pos.		Y	False Pos.
15	N		Y	False Pos.		N	True Neg.

(a)

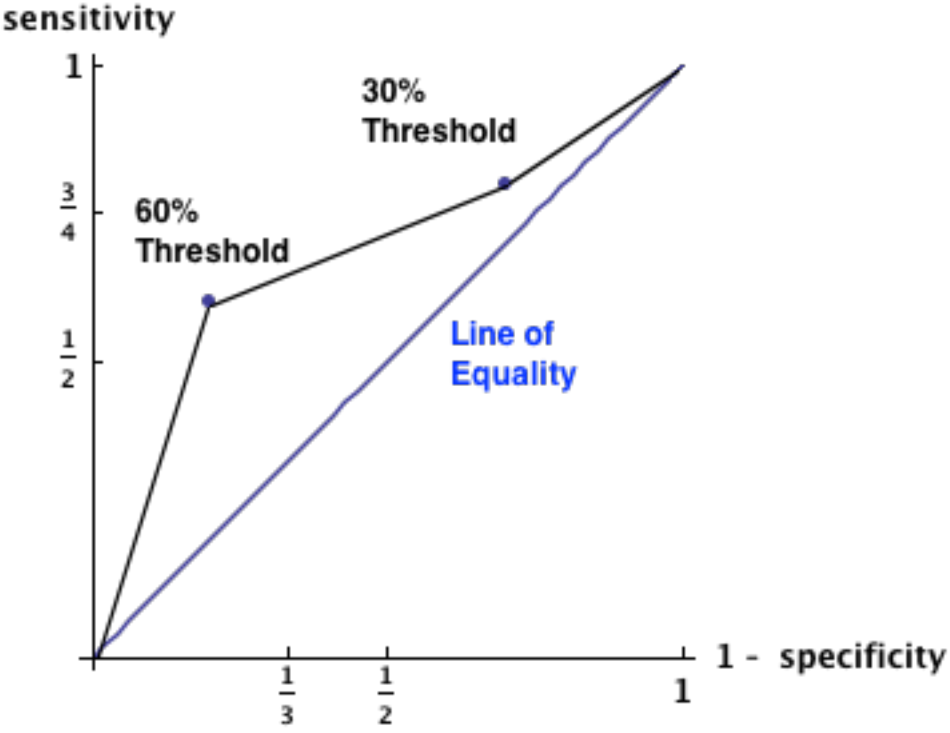
	30% Threshold		
	Predicted		
Actual	Fraud	No Fraud	Total
Fraud	true pos.: 4	false neg.: 1	5
No Fraud	false pos.: 7	true neg.: 3	10
Total	11	4	15

	60% Threshold		
	Predicted		
Actual	Fraud	No Fraud	Total
Fraud	true pos.: 3	false neg.: 2	5
No Fraud	false pos.: 2	true neg.: 8	10
Total	5	10	15

(b) Sensitivity = $\frac{\text{True Positives}}{\text{Total Number of Events}} = \frac{\text{Correct Predictions of Fraud}}{\text{Total Number of Fraudulent Claims}}$.

Specificity = $\frac{\text{True Negatives}}{\text{Total Number of Non-Events}} = \frac{\text{Correct Predictons of No Fraud}}{\text{Total Number of Nonfraudulent Claims}}$.

30% threshold: sensitivity = 4/5, and specificity = 3/10. Graph (1 - 3/10, 4/5).
60% threshold: sensitivity = 3/5, and specificity = 8/10 = 4/5. Graph (1 - 4/5, 3/5).
The ROC Curve, plus the 45-degree comparison line:



<u>Comment</u>: Similar to 8, 11/07, Q. 6a&b.

3.174. Let t_i be the territory relativity for a given insured. Then the offset is $\ln(t_i)$.
As per Section 2.6 of Generalized Linear Models for Insurance Rating:
$g(\mu_i) = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + ...+ \beta_p x_{ip}$ + offset.
Olaf would have a set of predictors of class relativity, and the model would be:
$\ln[\text{class relativity}_i] = \beta_0 + \beta_1 x_{i1} + ...+ \beta_p x_{ip} + \ln(t_i)$.
class relativity = $\exp[\beta_0 + \beta_1 x_{i1} + ...+ \beta_p x_{ip} + \ln(t_i)] = \exp[\beta_0 + \beta_1 x_{i1} + ...+ \beta_p x_{ip}]\ t_i$.
The GLM would be fit as usual, where each t_i is a known constant.

Comment: $\dfrac{\text{class relativity}}{\text{known territory relativity}} = \exp[\beta_0 + \beta_1 x_{i1} + ...+ \beta_p x_{ip}]$.

Territories are not a good fit for the GLM framework.
See Section 9.2 of Generalized Linear Models for Insurance Rating.
One should include the territory relativities produced by the separate model as an offset in the GLM used to determine classification relativities. Similarly, one should include classification relativities produced by the GLM as an offset in the separate model used to determine territory relativities.

3.175. A model with approximately 6 degrees of freedom has the right balance, since it has the smallest test MSE.
A model with approximately 2 degrees of freedom is too simple, it has a larger test MSE.
A model with approximately 20 degrees of freedom is too complex, it has a larger test MSE; while the training MSE is smaller that is due to this model being overfit.
Comment: Similar to 8, 11/17, Q 4b.
See Figure 7 in Generalized Linear Models for Insurance Rating.
We are interested in how the GLM will perform at predicting the response variable on some future set of data rather than on the set of past data with which we are currently working.
Our goal in modeling is to find the right balance where we pick up as much of the signal as possible with minimal noise, represented in this case by model with about 6 degrees of freedom.

3.176. (a) i. One can bin driver age into groups.
ii. Adding polynomial terms, such as X^2.
iii. One can use hinge functions. A hinge function is of the form: $(X - c)_+ = \max(0, X - c)$.
This will result in a piecewise linear function, with a change in slope at each breakpoint c_i.
(b) i. With binning: Continuity is not guaranteed.
Variation within intervals is ignored.
There may not be enough data in each bin to be credible.
There could be non-intuitive results, such as reversals.
ii. It is hard to interpret models that include powers.
"One potential downside to using polynomials is the loss of interpretability. From the coefficients alone it is often very difficult to discern the shape of the curve; to understand the model's indicated relationship of the predictor to the target variable it may be necessary to graph the polynomial function."
iii. Using hinge functions: The breakpoints must be selected by the user.
<u>Comment</u>: Similar to 8, 11/18, Q.5.
In all cases, more parameters are added to the model; the principal of parsimony states that we prefer a simpler model with fewer parameters, all else being equal.

3.177. $\dfrac{\exp[2.182 + 1.137 - 0.422]}{1 + \exp[2.182 + 1.137 - 0.422]}$ = **94.77%**.
<u>Comment</u>: Similar to CAS S, 5/16, Q.33.

3.178. (a) Adding driver age using 5 bins, adds 4 parameters.
i. Unscaled Deviance =
ϕ 2 {(loglikelihood for the saturated model) - (loglikelihood for the fitted model)}.
D_A = (0.61) (2) {-100 - (-130)} = 36.6. D_B = (0.63) (2) {-100 - (-123)} = 28.98.

$$F = \frac{(D_A - D_B) \,/\, (\text{number of added parameters})}{\hat{\phi}_B} = \{(36.6 - 28.98) / 4\} / 0.63 = 3.024.$$

We compare to the given critical value of 2.600.
Since 3.024 > 2.600, model B is significantly better than model A.
Driver age should be included in the rating plan.
ii. AIC_A = (-2)(-130) + (2)(5) = 270.
AIC_B = (-2)(-123) + (2)(5+4) = 264.
Since $AIC_B < AIC_A$, model B is better than model A.
Driver age should be included in the rating plan.
iii. BIC_A = (-2)(-130) + (5) ln(50) = 279.56.
BIC_B = (-2)(-123) + (5+4) ln(50) = 281.21.
$BIC_A < BIC_B$, and model A is better than model B.
Driver age should <u>not</u> be included in the rating plan.
<u>Comment</u>: Similar to 8, 11/18, Q. 6a.
The degrees of freedom for Model B =
number of observations minus number of fitted parameters for model B = 50 - 5 - 4 = 41.
The F-statistic has degrees of freedom 4 and 41.
The given critical value is the 5% critical value for 4 and 41 degrees of freedom.

3.179. $X\beta = 1 + (2)(0.50) + (-5)(0.29) = 0.55$.
The Gamma has the inverse as its canonical link function.
$1/(X\beta) = 1/0.55 = 1.818$.
The Poisson has the log as its canonical link function.
$\exp[X\beta] = e^{0.55} = 1.733$.
The Normal has the identity as its canonical link function.
$X\beta = 0.55$.
$0.55 < 1.733 < 1.818$. Thus the correct ordering is: **III < II < I**.
Comment: Similar to MAS-1, 5/18, Q.25.
The Binomial has the logit as its canonical link function.

3.180. (a) As per Section 2.6 of Generalized Linear Models for Insurance Rating:

$g(\mu_i) = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + ... + \beta_p x_{ip}$ + offset.

Thus the offset for Policy 1 is: $\ln\left(\frac{0.023}{1 - 0.023}\right) = \mathbf{-3.749}$.

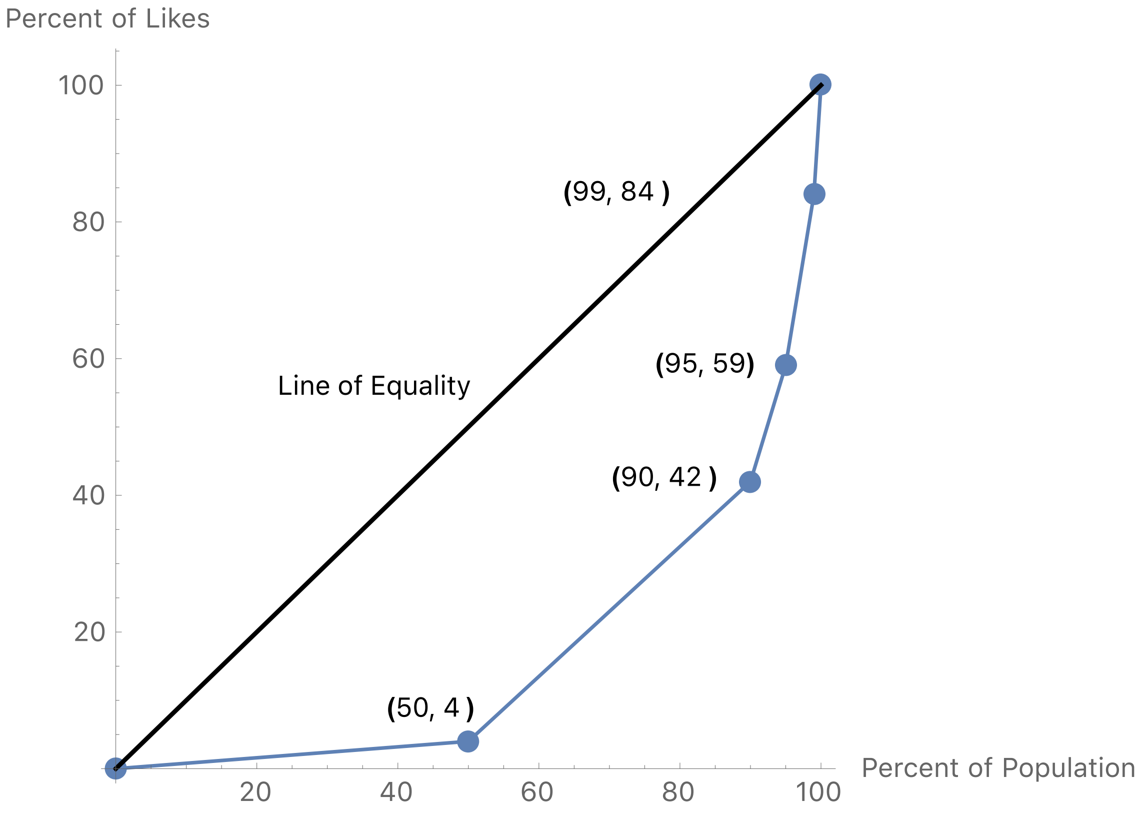
The offset for Policy 2 is: $\ln\left(\frac{0.112}{1 - 0.112}\right) = \mathbf{-2.070}$.

The offset for Policy 3 is: $\ln\left(\frac{0.045}{1 - 0.045}\right) = \mathbf{-3.055}$.

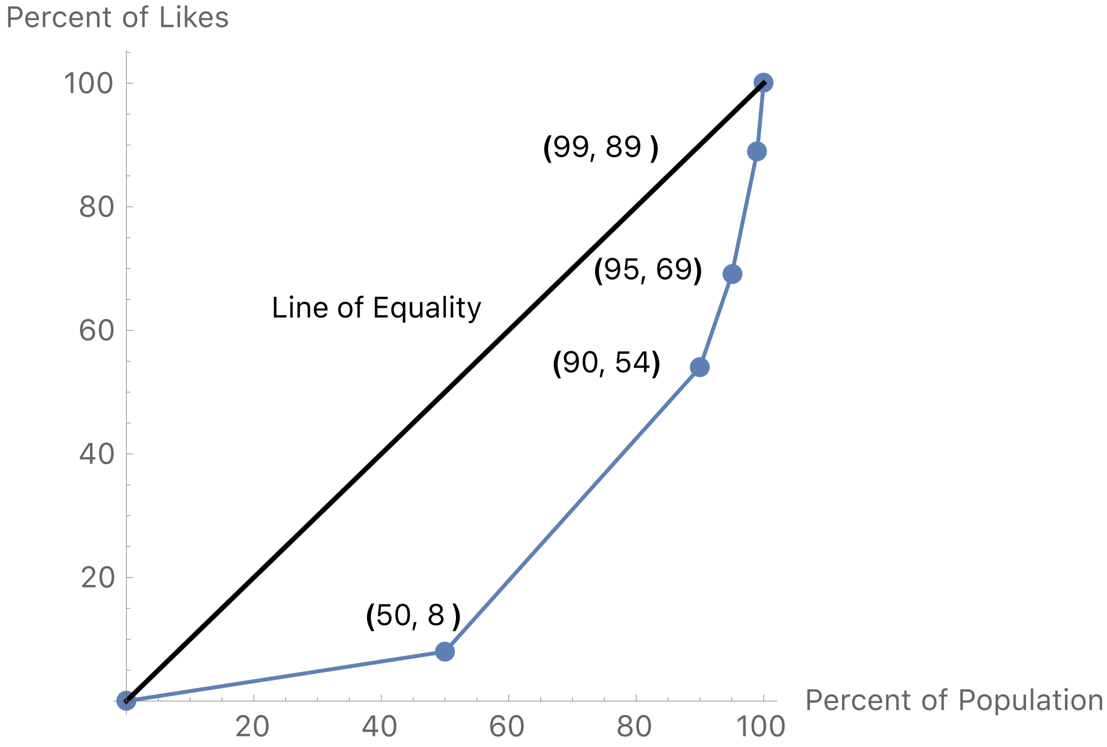
(b) In each case we add the offset to the linear component from the insurance score.
For Policy 1: $1.581 + (-0.032)(34) - 3.749 = -3.256$.
Probability of a claim is: $\frac{\exp[-3.256]}{1 + \exp[-3.256]} = \mathbf{3.71\%}$.
For Policy 2: $1.581 + (-0.032)(66) - 2.070 = -2.601$.
Probability of a claim is: $\frac{\exp[-2.601]}{1 + \exp[-2.601]} = \mathbf{6.91\%}$.
For Policy 3: $1.581 + (-0.032)(88) - 3.055 = -4.290$.
Probability of a claim is: $\frac{\exp[-4.290]}{1 + \exp[-4.290]} = \mathbf{1.35\%}$.
Comment: Similar to 8, 11/18, Q. 7a and 7b.

3.181. Variable X_3 has a nonlinear relationship with the target variable that is not being adequately addressed. This issue may be fixed with a hinge function.
Comment: See Figure 19 in Generalized Linear Models for Insurance Rating.

3.182. The Lorenz curve for men:



The Lorenz curve for women:



The Gini index, twice the area between the Lorenz curve and the line of equality, is higher for men than women, indicating more inequality among men on this dating app.

3.183. (a) $\dfrac{\exp[-8.2 + 0.3 + 0.7 + 0.4 \ln[150{,}000] - 0.1 \ln[150{,}000]]}{\exp[-8.2 + 0.3 + 0.4 \ln[150{,}000]]}$

$= \exp[0.7 - 0.1 \ln[150{,}000]] = \mathbf{0.6115}$.

(b) In order to center AOI, we will divide the AOI by the base AOI of 300,000 prior to logging and including it in the model. The two forms of the model produce the same results.
For example, for Occupancy class 1, non-sprinklered property, with AOI = 300,000, the given model has: $\exp[-8.2 + 0.4 \ln[300{,}000]]$.
With intercept β_0, the revised model would have for this same risk:
$\exp[\beta_0 + 0.4 \ln[300{,}000/300{,}000]] = \exp[\beta_0]$.
$\Rightarrow \exp[-8.2 + 0.4 \ln[300{,}000]] = \exp[\beta_0]$.

$\Rightarrow \beta_0 = -8.2 + 0.4 \ln[300{,}000] = \mathbf{-3.155}$.

(c) For example, for Occupancy class 1, sprinklered property, with AOI = 300,000, the given model has: $\exp[-8.2 + 0.4 \ln[300{,}000] + 0.7 - 0.1 \ln[300{,}000]]$.
With an intercept of -3.155 and coefficient for sprinklered of β_S, the revised model would have for this same risk:
$\exp[-3.155 + 0.4 \ln[300{,}000/300{,}000] + \beta_S - 0.1 \ln[300{,}000/300{,}000]] = \exp[-3.155 + \beta_S]$.
$\Rightarrow -8.2 + 0.4 \ln[300{,}000] + 0.7 - 0.1 \ln[300{,}000] = -3.155 + \beta_S$.

$\Rightarrow \beta_S = \mathbf{-0.562}$.

Alternately, given that $\beta_0 = -8.2 + 0.4 \ln[300{,}000]$, we want:
$0.7 - 0.1 \ln[300{,}000] = \beta_S$. $\Rightarrow \beta_S = \mathbf{-0.561}$.

(d) 1. If all continuous variables are divided by their base values prior to being logged and included in the model, then the intercept term after exponentiating yields the indicated frequency at the base case when all variables are at their base levels. This is both more intuitive and easier to interpret.
2. When terms are not centered, you can have unintuitive results. In the given example, the sprinkler coefficient is positive which can appear to indicate a higher frequency for sprinklered buildings than for non-sprinklered buildings. (However, when taking into account the interaction term, this is not true for values of log(AOI) for insured buildings.) This would not happen if AOI had been centered at its base level; the coefficients are more intuitive to understand when variables are centered.
3. With the AOI predictor in this form, the sprinklered coefficient has a more natural interpretation: it is the (log) sprinklered relativity for a risk with the base AOI.
<u>Comment</u>: Similar to 8, 11/19, Q.6.
The calculated ratio in part (a) does not depend on the occupancy class.

3.184. Model documentation serves at least three purposes:
- To serve as a check on your own work, and to improve your communication skills
- To facilitate the transfer of knowledge to the next owner of the model
- To comply with the demands of internal and external stakeholders

<u>Comment</u>: Quoted from Section 8.1 of <u>Generalized Linear Models for Insurance Rating</u>.

3.185. If one treats a “positive” result as predicting spam, then

sensitivity = $\frac{\text{True Positives}}{\text{Number times there is an event}} = \frac{\text{Number of Correct Indentifications of Spam}}{\text{Number times there is Spam}}$

= $\frac{36.3\%}{36.3\% + 3.0\%}$ = **92.4%**.

specificity = $\frac{\text{True Negatives}}{\text{Number times there is not an event}}$ =

$\frac{\text{Number of Correct Indentifications of Legitimate Email}}{\text{Number times there is Legitimate Email}} = \frac{58.2\%}{58.2\% + 2.5\%}$ = **95.9%**.

If instead one treats a “positive” result as predicting a legitimate email, then

sensitivity = $\frac{\text{True Positives}}{\text{Number times there is an event}}$ =

$\frac{\text{Number of Correct Indentifications of Legitimate Email}}{\text{Number times there is Legitimate Email}} = \frac{58.2\%}{58.2\% + 2.5\%}$ = **95.9%**.

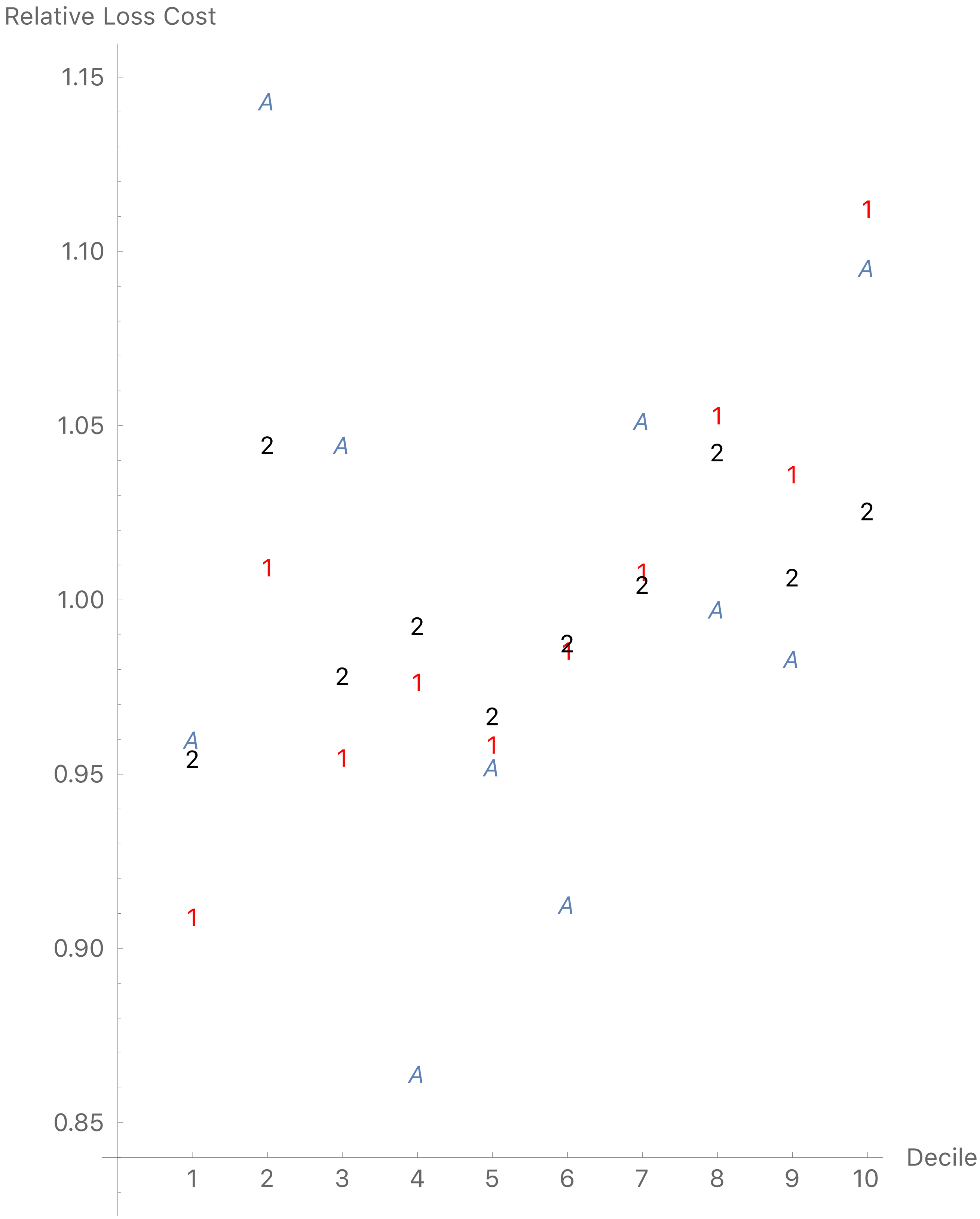
specificity = $\frac{\text{True Negatives}}{\text{Number times there is not an event}} = \frac{\text{Number of Correct Indentifications of Spam}}{\text{Number times there is Spam}}$

= $\frac{36.3\%}{36.3\% + 3.0\%}$ = **92.4%**.

3.186. Compute the relative loss costs. For example, 118.88/123.88 = 0.9596.

Decile	Actual Pure Premium	Actual Relativity	Model 1 Pure Premium	Model 1 Relativity	Model 2 Pure Premium	Model 2 Relativity
1	$118.88	0.9596	$109.62	0.9088	$115.08	0.9541
2	$141.58	1.1428	$121.73	1.0091	$125.95	1.0442
3	$129.37	1.0442	$115.13	0.9545	$117.95	0.9779
4	$107.00	0.8637	$117.76	0.9762	$119.68	0.9923
5	$117.91	0.9517	$115.58	0.9582	$116.57	0.9664
6	$113.02	0.9123	$118.84	0.9852	$119.08	0.9873
7	$130.21	1.0511	$121.57	1.0078	$121.11	1.0041
8	$123.52	0.9970	$126.99	1.0527	$125.70	1.0421
9	$121.75	0.9828	$124.94	1.0358	$121.36	1.0062
10	$135.65	1.0950	$134.13	1.1120	$123.65	1.0252
Total	$123.88		$120.62		$120.61	

Now plot these relative loss costs by decile:



Comment: The data was taken from “GLM III” presentation by Brent Petzoldt, CAS Ratemaking, Product and Modeling Seminar 2019.

3.187. The larger the average severity, the more worthwhile it is for the insurer to spend money to investigate cases of possible fraud. If claims are more severe, then the insurer will be more concerned about false negatives (cases where there is fraud but the modeled probability of fraud is below the threshold), than it would be about false positives (cases where there is not fraud but the modeled probability of fraud is above the threshold). Therefore, **the more severe the claims, the lower the discrimination threshold that should be selected**.
Comment: Similar to 8, 11/19, Q. 5d.
See page 84 of Generalized Linear Models for Insurance Rating.
"The ROC curve allows us to select a threshold we are comfortable with after weighing the benefits of true positives against the cost of false positives. Different thresholds may be chosen for different claim conditions; for example, we may choose a lower threshold for a large claim where the cost of undetected fraud is higher. Determination of the optimal threshold is typically a business decision that is out of the scope of the modeling phase."

3.188. The residuals tend be larger for smaller and larger values of the linear predictor; there is a curvature. This may be caused by a non-linear effect that may have been missed.
Comment: See Figure 18 in Generalized Linear Models for Insurance Rating.

3.189.

- The first and second derivatives of the fitted curve function are continuous—which in a practical sense means that the curve will appear fully "smooth" with no visible breaks in the pattern.
- The fits at the edges of the data (i.e., before the first selected breakpoint and after the last) are restricted to be linear, which curtails the potential for the kind of erratic edge behavior exhibited by regular polynomial functions.
- The use of breakpoints makes it more suitable than regular polynomial functions for modeling more complex effect responses, such as those with multiple rises and falls.

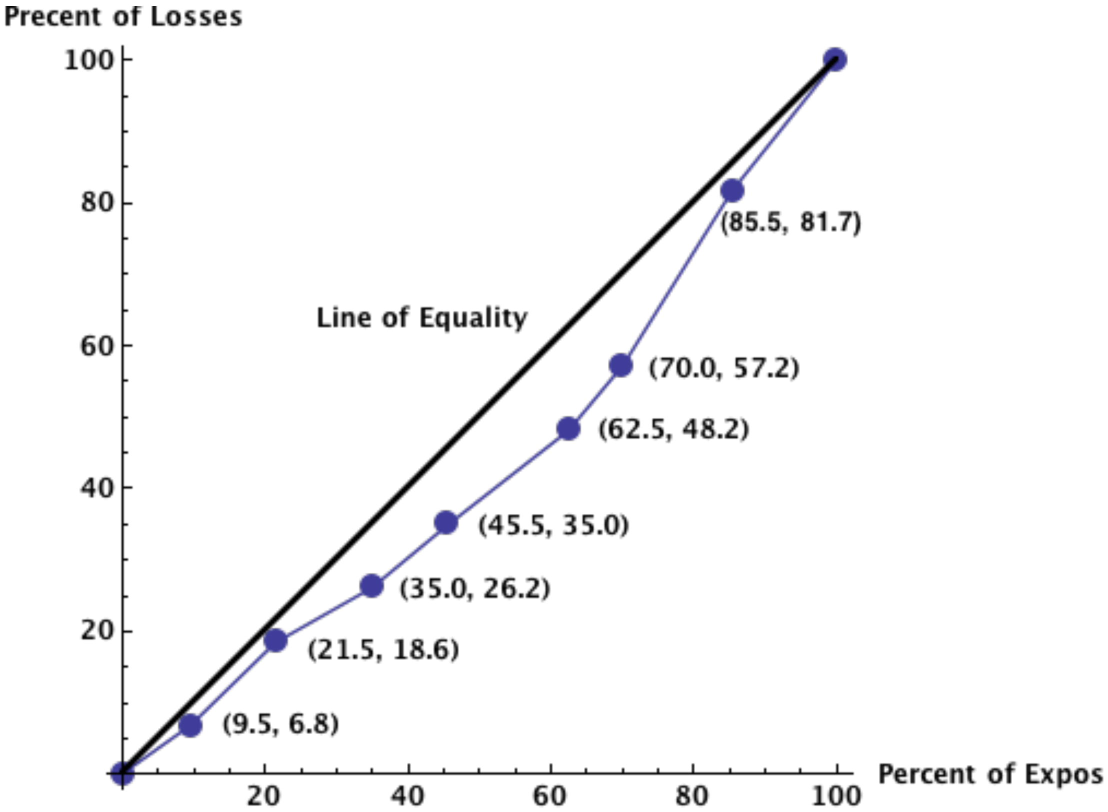
Comment: Quoted from Section 5.4.5 of Generalized Linear Models for Insurance Rating.
Also, the spline is continuous at each of the breakpoints.
Between the breakpoints (knots), a cubic spline follows a cubic polynomial.
The linearity at the edges is what distinguishes a natural cubic spline from a cubic spline.

3.190. Sort the risks from best to worst based on the model predicted pure premium.

Risk	Model P.P. (000)	Exposures	Cumulative Exposures	Cumulative % of Exposures
2	30	19	19	9.5%
4	38	24	43	21.5%
5	43	27	70	35.0%
3	49	21	91	45.5%
8	52	34	125	62.5%
1	56	15	140	70.0%
7	64	31	171	85.5%
6	77	29	200	100.0%

Risk	Exposures	Actual P.P. (000)	Actual Losses (000)	Cumulative Losses (000)	Cumulative % of Losses
2	19	36	684	684	6.8%
4	24	49	1,176	1,860	18.6%
5	27	28	756	2,616	26.2%
3	21	42	882	3,498	35.0%
8	34	39	1,326	4,824	48.2%
1	15	60	900	5,724	57.2%
7	31	79	2,449	8,173	81.7%
6	29	63	1,827	10,000	100.0%

On the x-axis, plot the cumulative percentage of exposures.
On the y-axis, plot the cumulative percentage of actual losses.
The plotted points are: (0, 0), (9.5%, 6.8%), (21.5%, 18.6%), ... , (85.5%, 81.7%), (100%, 100%).



Comment: Similar to 8, 11/16, 5a.
The Gini index is twice the area between the Lorenz Curve and the line of equality.
The higher the Gini Index, the better the rating plan is at identifying risk differences.

3.191. Both models are monotone increasing, which is good.
Model 1 has a larger vertical distance between the first and last deciles than does Model 2; Model 1 has more “lift” than Model 2. All else being equal, larger lift is better, indicating that the model is able to maximally distinguish the best and worst risks.
Based on the first graph, Model 1 does a better job of matching the training data.
However, based on the second graph, Model 1 does a worse job of matching the test data, particularly for the high deciles. Model 1 is overfit; the model picks up too much of the random fluctuation (noise) in the training data.
Based on the first graph, Model 2 does a worse job than Model 1 of matching the training data.
However, based on the second graph, Model 2 does a good job of matching the test data.
Model 2 seems to do a good job of modeling this situation.
I would recommend using Model 2 rather than Model 1.
Comment: Similar to 5, 5/19, Q.9.
In general, we do not want a model to be either underfit (not picking up enough of the signal) nor overfit (picking up too much of the noise).

3.192. working residual: $wr_i = (y_i - \mu_i)\ g'(\mu_i)$.
For the log link function: $g(\mu) = \ln(\mu)$. ⇒ $g'(\mu) = 1/\mu$. ⇒ $wr_i = (y_i - \mu_i)/\mu_i$.

working weights: $ww_i = \dfrac{\omega_i}{V(\mu_i)\ g'(\mu_i)^2}$.

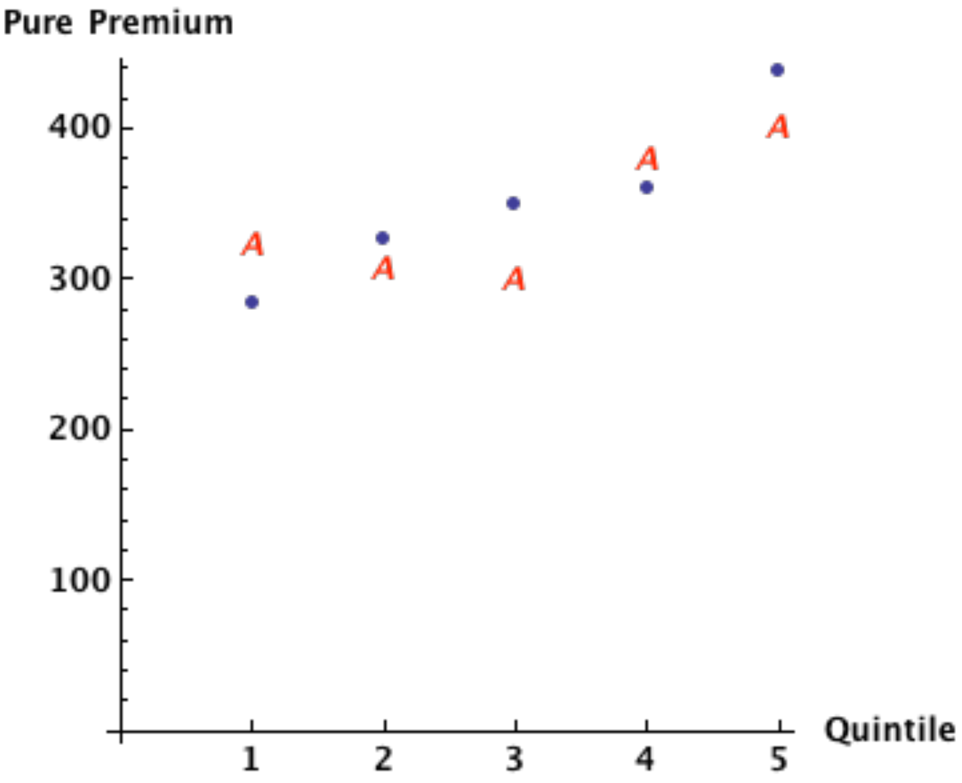
For the Poisson: $V(\mu) = \mu$. ⇒ $ww_i = \omega_i\ \mu_i$.

$wr_i\ ww_i = \omega_i\ (y_i - \mu_i)$.
Thus the numerator of the weighted average is the sum of the product of the working residuals and working weights: (11)(4 - 3.3) + (9)(3 - 3.7) + (15)(6 - 5.5) + (7)(2 - 4.1) + (12)(5 - 5.2) + (8)(4 - 3.4) + (14)(2 - 2.6) + (10)(4 - 3.0) = -1.8.
The denominator of the weighted average is the sum of the working weights:
(11)(3.3) + (9)(3.7) + (15)(5.5) + (7)(4.1) + (12)(5.2) + (8)(3.4) + (14)(2.6) + (10)(3.0) = 336.8.
The binned working residual is: -1.8/336.8 = **-0.00534**.

3.193. The order of predicted pure premiums is: 5, 4, 2, 1, 3.

Insured	Actual Loss Cost	Actual Pure Premium	Model Loss Cost	Model Pure Premium	Exposures
1	$38,000	$380	$36,000	$360	100
2	$36,000	$300	$42,000	$350	120
3	$52,000	$400	$57,000	$438	130
4	$46,000	$307	$49,000	$327	150
5	$58,000	$322	$51,000	$283	180

The corresponding predicted pure premiums are: 283, 327, 350, 360, 438.
The corresponding actual pure premiums are: 322, 307, 300, 380, 400.
The Simple Quantile Plot, with the actual pure premiums shown as A and the predicted pure premiums shown as dots:



Comment: One would construct a similar Simple Quantile Plot for a proposed model, in order to compare that proposed model to the current model.
One would work with many more than 5 observations; I would not draw any conclusions based on such a small amount of data.

3.194. "One potential downside to using polynomials is the loss of interpretability. From the coefficients alone it is often very difficult to discern the shape of the curve; to understand the model's indicated relationship of the predictor to the target variable it may be necessary to graph the polynomial function.
Another drawback is that polynomial functions have a tendency to behave erratically at the edges of the data, particularly for higher-order polynomials."
Comment: Quoted from Section 5.4.3 of Generalized Linear Models for Insurance Rating.

3.195. Working Residual is: $wr_i = (y_i - \mu_i)\, g'(\mu_i)$.

(a) $g(\mu) = \mu$. ⇒ $g'(\mu) = 1$. ⇒ $wr_i = y_i - \mu_i$ = ordinary residual.

0.4 - 0.5 = **-0.1**.

(b) $g(\mu) = \ln(\mu)$. ⇒ $g'(\mu) = 1/\mu$. ⇒ $wr_i = (y_i - \mu_i)/\mu_i$.

(0.4 - 0.5)/0.5 = **-0.2**

(c) $g(\mu) = \ln\left(\frac{\mu}{1-\mu}\right)$. ⇒ $g'(\mu) = \frac{1}{\left(\frac{\mu}{1-\mu}\right)} \frac{(1-\mu) - (\mu)(-1)}{(1-\mu)^2} = \frac{1}{\mu\,(1-\mu)}$. ⇒ $wr_i = \frac{y_i - \mu_i}{\mu_i\,(1-\mu_i)}$.

(0.4 - 0.5) / {(0.5)(1 - 0.5)} = **-0.4**.

3.196. The offsets for deductibles are: ln(1) = 0, ln(1 - 0.06) = -0.0618, ln(1 - 0.11) = -0.1165.
The offsets for territories are: ln(400) = 5.991, ln(600) = 6.397, ln(900) = 6.802.
Thus the offsets for the nine combinations are:

Territory	500 Ded.	1000 Ded.	2500 Ded.
A	5.991	5.930	5.875
B	6.397	6.335	
C	6.802	6.741	6.686

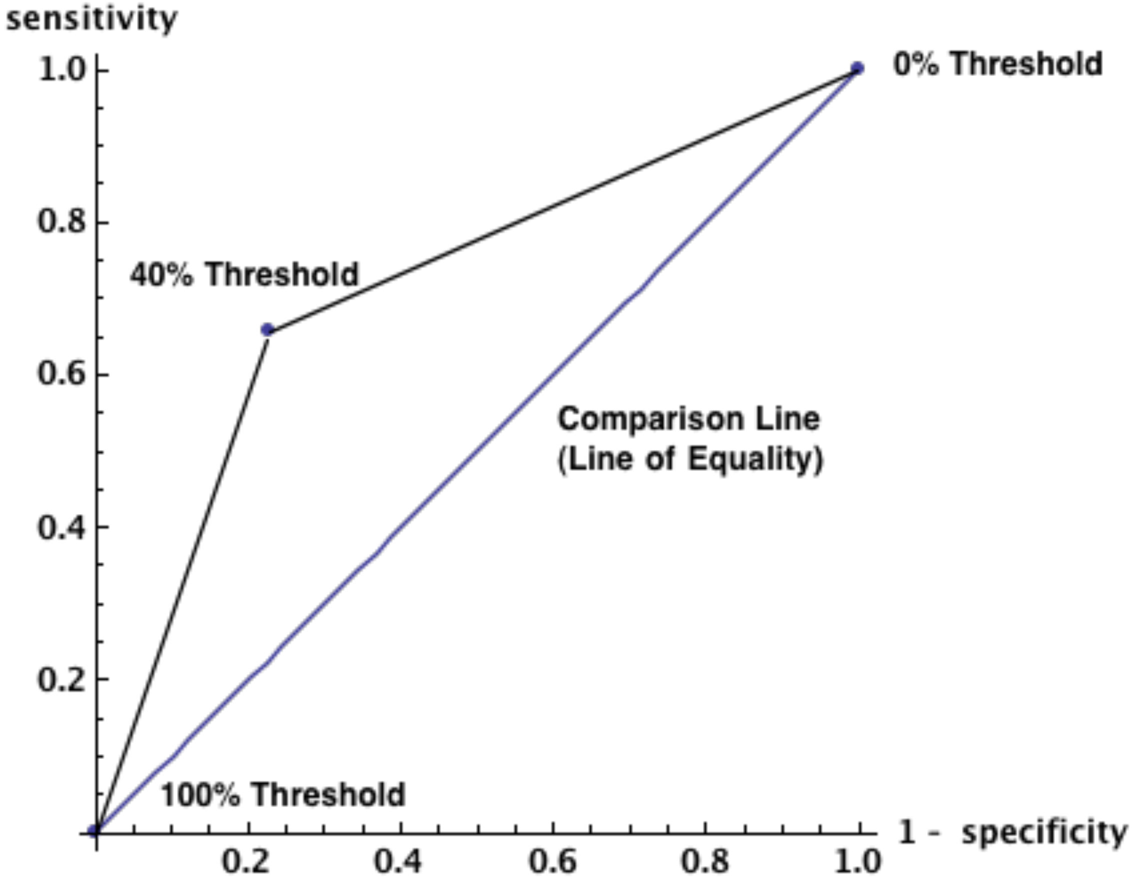
For example, for a 1000 deductible in Territory B, offset = ln(1 - 0.06) + ln(600) = 6.335.
Alternately, one can instead work with the territory relativities:
1, 600/200 = 1.5, 900/400 = 2.25.
The offsets for territories are: ln(1) = 0, ln(1.5) = 0.4055, ln(2.25) = 0.8109.
Thus the offsets for the nine combinations are:

Territory	500 Ded.	1000 Ded.	2500 Ded.
A	0.000	-0.062	-0.117
B	0.405	0.344	0.289
C	0.811	0.749	0.694

For example, for a 1000 deductible in Territory B, offset = ln(1 - 0.06) + ln(1.5) = 0.344.
Comment: Using the territory relativities, all of the offsets are lower by ln(400).
Therefore, the fitted intercept will be larger by ln(400). The predictions of the fitted models would be the same regardless of which way Hari chooses to treat the offsets for territory.

3.197. There are several ways the p parameter may be determined:
• Some model-fitting software packages provide the functionality to estimate p as part of the model-fitting process. (Note that using this option may increase the computation time considerably, particularly for larger datasets.)
• Several candidate values of p can be considered and tested with the goal of optimizing a statistical measure such as log-likelihood or using cross-validation.
• Alternatively, many modelers simply judgmentally select some value that makes sense (common choices being 1.6, 1.67 or 1.7). This may be the most practical in many scenarios, as the fine-tuning of p is unlikely to have a very material effect on the model estimates.
Comment: Quoted from Section 2.7.3 from Generalized Linear Models in Insurance Rating.

3.198. (a) sensitivity = (true positives) / (all positives) = 172 / (172 + 90) = **65.6%**.
specificity = (true negatives) / (all negatives) = 302 / (302 + 88) = **77.4%**.
(b) The 40% threshold corresponds to the point: (1 - 0.774, 0.656). The ROC curve:



For a threshold of 100% the model always predicts no; there are no false negatives and thus 1 - specificity is zero. For a threshold of 100% the model never predicts yes; there are no true positives and thus the sensitivity is zero.
For a threshold of 0% the model never predicts no; there are no false negatives and thus 1 - specificity is one. For a threshold of 0% the model always predicts yes; the true positives are equal to all positives and thus the sensitivity is one.
(c) A model with with no predictive power would follow the comparison line (line of equality).
A perfect model would be at (0, 1) in the upper lefthand corner; sensitivity = 1 and specificity = 1.
Comment: Similar to 8, 11/19, Q. 5 b-c.

3.199. working weights: $\text{ww}_i = \dfrac{\omega_i}{V(\mu_i)\, g'(\mu_i)^2}$.

(a) For the log link function: $g(\mu) = \ln(\mu)$. ⇒ $g'(\mu) = 1/\mu$.

For the Poisson: $V(\mu) = \mu$.
⇒ $\text{ww}_i = \omega_i\ \mu_i = (60)(0.7) = \mathbf{42}$.

(b) For the Gamma: $V(\mu) = \mu^2$. ⇒ $\text{ww}_i = \omega_i = \mathbf{60}$.

(c) For the Tweedie: $V(\mu) = \mu^p$. ⇒ $\text{ww}_i = \omega_i\ \mu_i^{2-p} = (60)(0.7^{2-1.6}) = \mathbf{52.0}$.

(d) For the identity link, $g(\mu) = \mu$. ⇒ $g'(\mu) = 1$.

For the Normal, $V(\mu) = 1$. ⇒ $\text{ww}_i = \omega_i = \mathbf{60}$.

(e) $g(\mu) = \ln(\dfrac{\mu}{1 - \mu})$. ⇒ $g'(\mu) = \dfrac{1}{\left(\dfrac{\mu}{1 - \mu}\right)}\ \dfrac{(1 - \mu) - (\mu)(-1)}{(1 - \mu)^2} = \dfrac{1}{\mu\ (1 - \mu)}$.

For the Bernoulli, $V(\mu) = \mu(1 - \mu)$. ⇒ $\text{ww}_i = \omega_i\ \mu_i(1 - \mu_i) = (60)(0.7)(1 - 0.7) = \mathbf{12.6}$.

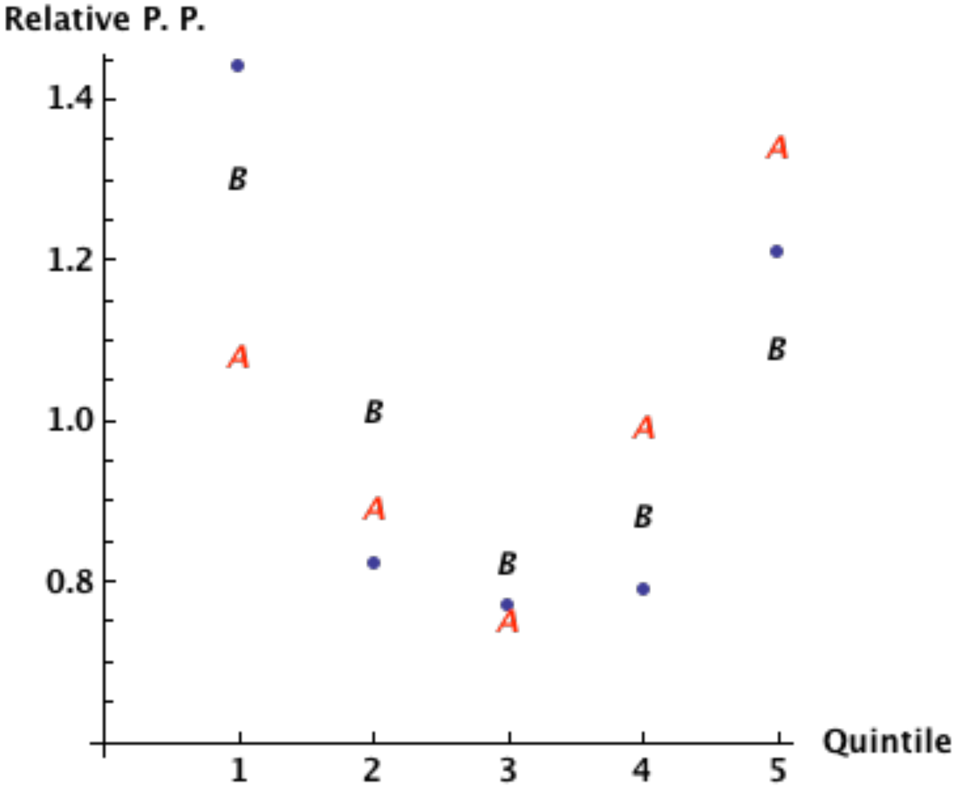
(f) For the reciprocal link, $g(\mu) = 1/\mu$. ⇒ $g'(\mu) = -1/\mu^2$.

For the Inverse Gaussian: $V(\mu) = \mu^3$. ⇒ $\text{ww}_i = \omega_i\ \mu_i = (60)(0.7) = \mathbf{42}$.

3.200. Sort the data based on the ratio:
(Model A Predicted Loss Cost) / (Model B Predicted Loss Cost).

Obs	Actual Loss Cost	Actual Pure Premium	Model A Loss Cost	Model A Pure Premium	Model B Loss Cost	Model B Pure Premium	Exposures	Sort Ratio
1	$15,000	$300	$16,000	$320	$18,000	$360	50	0.89
2	$20,000	$286	$25,000	$357	$22,000	$314	70	1.14
3	$42,000	$525	$31,000	$388	$37,000	$463	80	0.84
4	$44,000	$440	$48,000	$480	$39,000	$390	100	1.23
5	$39,000	$279	$38,000	$271	$41,000	$293	140	0.93
Tot.	$160,000	$364	$158,000	$359	$157,000	$357	440	

The sort ratios from smallest to largest give the order: 3, 1, 5, 2, 4.
In each case, we divide the individual pure premiums by the total pure premium.
The Actual P.P. relativities are: (525, 300, 279, 286, 440) / 364 = 1.44, 0.82, 0.77, 0.79, 1.21.
Model A P.P. relativities are: (388, 320, 271, 357, 480) / 359 = 1.08, 0.89, 0.75, 0.99, 1.34.
Model B P.P. relativities are: (463, 360, 293, 314, 390) / 357 = 1.30, 1.01, 0.82, 0.88, 1.09.
The double lift chart, with actual shown as dots, Model A shown as A, and Model B shown as B:



Comment: One would work with many more than 5 observations; I would not draw any conclusions based on such a small amount of data.

3.201. (a) External: insurance regulators, outside auditors, and risk managers.
Internal: executives, underwriters, claims adjusters, other actuaries, and IT personnel.
(b) Your documentation should have:

- Include everything needed to reproduce the model from source data to model output
- Include all assumptions and justification for all decisions
- Disclose all data issues encountered and their resolution
- Discuss any reliance on external models or external stakeholders
- Discuss model performance, structure, and shortcomings
- Comply with ASOP 41 or local actuarial standards on communications

Comment: See Section 8.3 of Generalized Linear Models for Insurance Rating.

3.202. 200 people have the antibodies, while 800 people do not.
90% specificity ⇔ 10% false positives.

Thus of the 800 people who do not have the antibodies, on average (10%)(800) = 80 will be tested as falsely positive. The remaining 720 people will be correctly tested as negative.
95% specificity ⇔ 5% false negatives.

Thus of the 200 people who have the antibodies, on average (5%)(200) = 10 people will be tested as falsely negative. The remaining 190 people will be correctly tested as positive.
The confusion matrix:

	Result of Test		
	Antibodies	No Antibodies	Total
Antibodies	190	**10**	200
No Antibodies	**80**	720	800
Total	270	730	1000

3.203. The variance of residuals seems to decrease as the weight increases.
This violates the expectation of homoscedasticity, i.e., we want no pattern in the variance.
This indicates that the weights being used in the model are not properly adjusting for differences in variance as is desired. Perhaps some other form of weights would work better,
Comment: See Figure 20 in Generalized Linear Models for Insurance Rating.

3.204. Assume for example 1000 people. 20 people are expected to have this type of cancer.

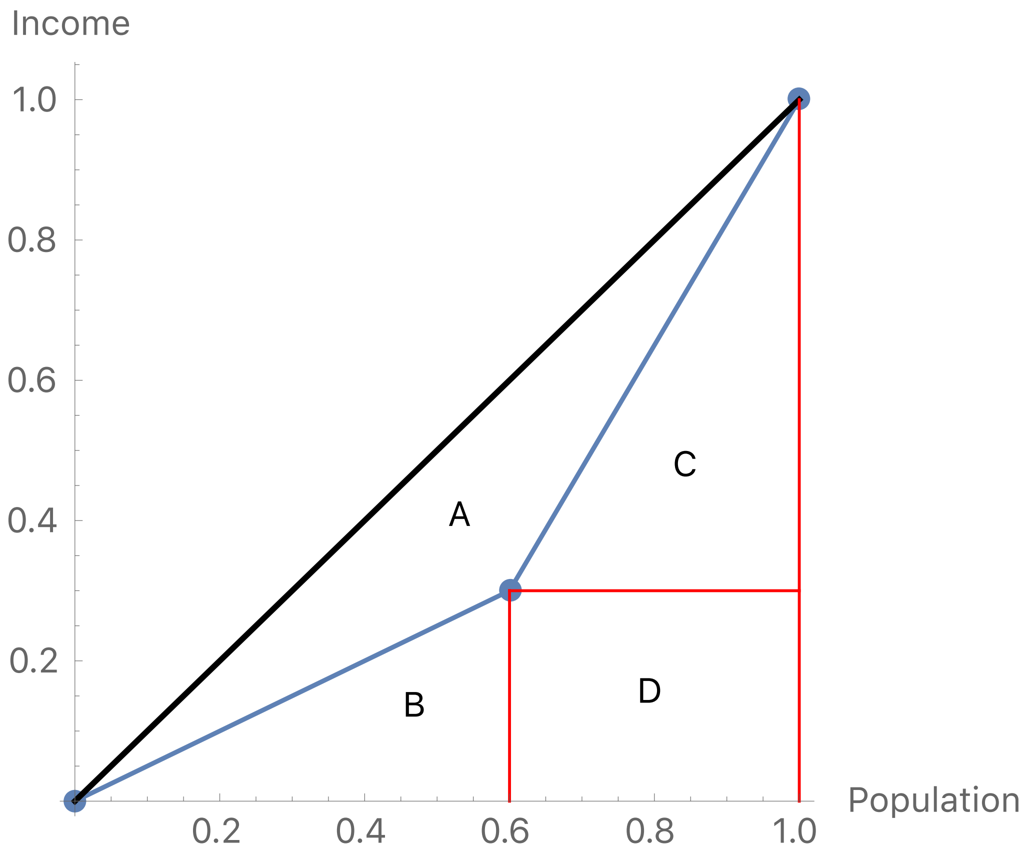
$$95\% = \text{sensitivity} = \frac{\text{True Positive}}{\text{Number with Cancer}}.$$

Thus 19 of 20 with this cancer are expected to test positive.

$$90\% = \text{specificity} = \frac{\text{True Negative}}{\text{Number without Cancer}}.$$

Thus 882 of 980 without this cancer are expected to test negative, and 98 to test positive.
(a) If a person of this age tests positive for this type of cancer, then the probability that they have this type of cancer is: 19 / (19 + 98) = **16.24%.**
(b) If a person of this age tests negative for this type of cancer, then the probability that they do not have this type of cancer is: 882 / (1 + 882) = **99.887%.**

3.205. Here is the Lorenz curve, with various areas labeled:



Area B is a right triangle with width 0.6 and height 0.3; area = (1/2)(0.6)(0.3) = 0.09.
Area C is a right triangle with width 0.4 and height 0.7; area = (1/2)(0.4)(0.7) = 0.14.
Area D is a rectangle with width 0.4 and height 0.3; area = (0.4)(0.3) = 0.12.
A + B + C + D is a right triangle with width 1 and height 1; area = (1/2)(1)(1) = 0.50.
Thus Area A = 0.50 - 0.09 - 0.14 - 0.12 = 0.15.
Area A is the area between the Lorenz curve and the line of equality.
The Gini Index is twice this area: (2)(0.15) = **0.30**.

3.206. One-way or univariate analysis does not accurately take into account the effect of other rating variables. It does not consider exposure correlations with other rating variables.

3.207. a. Linear Model:

● Random Component: Each component of Y is independent and normally distributed. Their means may differ, but they have common variance.

● Systematic Component: The covariates are combined to produce the linear predictor $\eta = X\beta$.

● Link Function: The relationship between the random component and the systematic component is specified with the identity link function: $E(Y) = \mu = \eta$.
(if g is the identity link function, $g^{-1}(\eta) = \eta$.)

Generalized Linear Model:

● Random Component: Each component of Y is independent and a member of an exponential family. (While the Normal is one possibility, there are others.)

● Systematic Component: The covariates are combined to produce the linear predictor $\eta = X\beta$.

● Link Function: The relationship between the random component and the systematic component is specified with the link function, which is differentiable and monotonic such that: $E(Y) = \mu = g^{-1}(\eta)$.
(While the identity link function is one possibility, there are others.)

b) 1) The assumption of normality with common variance is often not true.

2) Sometimes the response variable may be restricted to be positive, but normality with the identity link function violates this.

3.208. i. Classical Linear Model: Response variable is normally distributed.

Generalized Linear Model: Response variable is from the exponential family.

ii) Classical Linear Model: The variance is constant but the mean is allowed to vary.

Generalized Linear Model: The variance is a function of the mean (exponential family).

3.209. a. Intrinsic aliasing is a linear dependency between covariates due to the definition.
For example, if we have only black, red and blue cars, the red cars can be determined from [total cars] - black - blue. As another example, age of vehicle would alias with model year, since if you know one you can determine the other.
Extrinsic aliasing is a linear dependency between covariates that arises due to the nature of the data
rather than inherent properties of the covariates themselves. For example, if in the data all cars with unknown color also have an unknown number of doors, and vice-versa.
b. We have that [all cars] - large cars - medium cars = small cars, so we can say that
$X_{small} = 1 - X_{large} - X_{medium}$.
If we do not have a base level, then we could have two size variables such as Large and Medium, plus all four territories.
We have that [all cars] - North - South - West = East, so we can say that
$X_{East} = 1 - X_{North} - X_{South} - X_{West}$.
If we do not have a base level, then we could have three territory variables such as North, South, and West, plus all three sizes.
Alternately, we can eliminate β_{small} and β_{East} from the model and include an intercept term;
Small / East would be the base level. Intercept plus 2 size and 3 territory variables.
Comment: The current syllabus reading does not distinguish between intrinsic and extrinsic aliasing.
In part (b) we should end up with 6 variables in total.
If we have an intercept term, we would have in addition three territory levels and two size levels.
Aliasing occurs when there is a linear dependency among the observed covariates. Equivalently, aliasing can be defined as a linear dependency among the columns of the design matrix X.
Near aliasing is a common problem and occurs when two or more factors contain levels that are almost, but not quite, perfectly correlated. This same problem comes up when performing multiple linear regressions.

3.210. There are many possible ways to set this up.
Taking North and Medium as the base levels as instructed.
Let X_1 correspond to the intercept term. It is one for all cells.
Let X_2 correspond to South. $X_2 = 1$ if South and 0 otherwise.
Let X_3 correspond to East. $X_3 = 1$ if East and 0 otherwise.
Let X_4 correspond to West. $X_4 = 1$ if West and 0 otherwise.
Let X_5 correspond to Small. $X_5 = 1$ if Small and 0 otherwise.
Let X_6 correspond to Large. $X_6 = 1$ if Large and 0 otherwise.
Then the design matrix, X, and response vector Y are:

$$X = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \end{pmatrix} \begin{array}{c} \text{North Small} \\ \text{North Medium} \\ \text{North Large} \\ \text{South Small} \\ \text{South Medium} \\ \text{South Large} \\ \text{East Small} \\ \text{East Medium} \\ \text{East Large} \\ \text{West Small} \\ \text{West Medium} \\ \text{West Large} \end{array} \quad Y = \begin{pmatrix} 100 \\ 150 \\ 250 \\ 80 \\ 110 \\ 290 \\ 90 \\ 170 \\ 200 \\ 180 \\ 260 \\ 540 \end{pmatrix}$$

The vector of parameters is:

$$\begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_6 \end{pmatrix}$$

Comment: While one could fit a Poisson to pure premiums, and treat the result as a discrete approximation, it is more common to fit a Tweedie Distribution.
Using a computer the fitted parameters are:
$\beta_1 = 4.95978$, $\beta_2 = -0.040822$, $\beta_3 = -0.0833816$, $\beta_4 = 0.672944$, $\beta_5 = -0.427444$, $\beta_6 = 0.617924$.
We have a multiplicative model with relativities:
South: exp[-0.040822] = 0.960, East: exp[-0.0833816] = 0.920, West: exp[0.672944] = 1.960,
Small: exp[-0.427444] = 0.652, Large: exp[0.617924] = 1.855.
For example, the fitted value for South and Small is:
$\exp[\beta_1 + \beta_2 + \beta_5] = \exp[4.95978 - 0.040822 - 0.427444] = 89.26$.

The fitted values are:

	Vehicle Size			
Territory	Small	Medium	Large	Total
North	92.98	142.56	264.46	500
South	89.26	136.86	253.88	480
East	85.54	131.16	243.31	460.01
West	182.23	279.42	518.35	980
Total	450.01	690	1280	2420.01

Subject to rounding, the totals for the fitted match those for the data.
In general, the estimates will be in balance as they were here, when one uses the canonical link function; the canonical link function for the Poisson is the log link function.
See “A Systematic Relationship Between Minimum Bias and Generalized Linear Models,” by Stephen Mildenhall, PCAS 1999.

3.211. a) $Y = \beta_1 X1 + \beta_2 X2 + \beta_3 X3 + e$.

$Y_1 = 400 = \beta_1 + 0 + \beta_3 + e_1$.

$Y_2 = 250 = \beta_1 + 0 + 0 + e_2$.

$Y_3 = 200 = 0 + \beta_2 + \beta_3 + e_3$.

$Y_4 = 100 = 0 + \beta_2 + 0 + e_4$.

Sum of Squared Errors = $e_1^2 + e_2^2 + e_3^2 + e_4^2$

$= (400 - \beta_1 - \beta_3)^2 + (250 - \beta_1)^2 + (200 - \beta_2 - \beta_3)^2 + (100 - \beta_2)^2$.

Set equal to zero the partial derivatives with respect to betas:

$2(400 - \beta_1 - \beta_3)(-1) + 2(250 - \beta_1)(-1) = 0. \Rightarrow 2\beta_1 + \beta_3 = 650$.

$2(200 - \beta_2 - \beta_3)(-1) + 2(100 - \beta_2)(-1) = 0. \Rightarrow 2\beta_2 + \beta_3 = 300$.

$2(400 - \beta_1 - \beta_3)(-1) + 2(200 - \beta_2 - \beta_3)(-1) = 0. \Rightarrow \beta_1 + \beta_2 + 2\beta_3 = 600$.

Solve for the betas.

b) i. constant variance. However, the variance is often a function of the mean.

ii. The components of the response variable are normally distributed.

For example, the response variable may be restricted to non-negative values, violating normality.

iii. Additivity of effects. Many factors in reality have multiplicative effects.

Comment: GLMs relax all of the three assumptions in part (b).

In the additive model in the question, we are taking Rural as the base; we have three categorical variables that each can take on the values zero or one, although when X1 = 1 we must have X2 = 0 and vice-versa.

The solution to the three equations is: $\beta_1 = 525/2$, $\beta_2 = 175/2$, and $\beta_3 = 125$.

The resulting estimates are:

Gender	Urban	Rural
Male	525/2 + 125 = 387.5	525/2 = 262.5
Female	175/2 + 125 = 212.5	175/2 = 87.5

The corresponding minimum sum of squared errors is:

$(400 - 387.5)^2 + (250 - 262.5)^2 + (200 - 212.5)^2 + (100 - 87.5)^2 = 625$.

3.212. a. As per the exam question, take Male (X_1), Female (X_2), Urban (X_3).
Then the design matrix, X, and response vector Y are:

$$X = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{array}{c} \text{Male Urban} \\ \text{Male Rural} \\ \text{Female Urban} \\ \text{Female Rural} \end{array} \quad Y = \begin{pmatrix} 400 \\ 250 \\ 200 \\ 100 \end{pmatrix}.$$

The vector of parameters is: $\begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}$.

b. For the Gamma Distribution, $f(y) = \theta^{-\alpha} y^{\alpha-1} e^{-y/\theta} / \Gamma(\alpha)$.

$\ln f(y) = (\alpha-1)\ln(y) - y/\theta - \alpha\ln(\theta) - \ln[\Gamma(\alpha)] = (\alpha-1)\ln(y) - y/(\mu/\alpha) - \alpha\ln(\mu/\alpha) - \ln[\Gamma(\alpha)]$

$= (\alpha-1)\ln(y) - \alpha y/\mu - \alpha\ln(\mu) + \alpha\ln(\alpha) - \ln[\Gamma(\alpha)]$.

With the identity link function: $\mu = \beta_1X_1 + \beta_2X_2 + \beta_3X_3$.

Thus the loglikelihood is:

$(\alpha-1)\ln(400) - \alpha 400/(\beta_1 + \beta_3) - \alpha n(\beta_1 + \beta_3) + \alpha\ln(\alpha) - \ln[\Gamma(\alpha)] +$

$(\alpha-1)\ln(250) - \alpha 250/(\beta_1) - \alpha\ln(\beta_1) + \alpha\ln(\alpha) - \ln[\Gamma(\alpha)] +$

$(\alpha-1)\ln(200) - \alpha 200/(\beta_2 + \beta_3) - \alpha\ln(\beta_2 + \beta_3) + \alpha\ln(\alpha) - \ln[\Gamma(\alpha)] +$

$(\alpha-1)\ln(100) - \alpha 100/(\beta_2) - \alpha\ln(\beta_2) + \alpha\ln(\alpha) - \ln[\Gamma(\alpha)]$.

Setting the partial derivative with respect to α_1 equal to zero:

$0 = \alpha 400/(\beta_1 + \beta_3)^2 - \alpha/(\beta_1 + \beta_3) + \alpha 250/(\beta_1)^2 - \alpha/(\beta_1). \Rightarrow$

$400/(\beta_1 + \beta_3)^2 + 250/\beta_1^2 = 1/(\beta_1 + \beta_3) + 1/\beta_1$.

Setting the partial derivative with respect to β_2 equal to zero:

$0 = \alpha 200/(\beta_2 + \beta_3)^2 - \alpha/(\beta_2 + \beta_3) + \alpha 100/(\beta_2)^2 - \alpha/(\beta_2). \Rightarrow$

$200/(\beta_2 + \beta_3)^2 + 100/\beta_2^2 = 1/(\beta_2 + \beta_3) + 1/\beta_2$.

Setting the partial derivative with respect to β_3 equal to zero:

$0 = \alpha 400/(\beta_1 + \beta_3)^2 - \alpha/(\beta_1 + \beta_3) + \alpha 200/(\beta_2 + \beta_3)^2 - \alpha/(\beta_2 + \beta_3). \Rightarrow$

$400/(\beta_1 + \beta_3)^2 + 200/(\beta_2 + \beta_3)^2 = 1/(\beta_1 + \beta_3) + 1/(\beta_2 + \beta_3)$.

c. For the Gamma Distribution, $f(y) = \theta^{-\alpha} y^{\alpha-1}\ e^{-y/\theta} / \Gamma(\alpha)$.

$\ln f(y) = (\alpha\text{-}1)\ln(y) - y/\theta - \alpha\ln(\theta) - \ln[\Gamma(\alpha)] = (\alpha\text{-}1)\ln(y) - y/(\mu/\alpha) - \alpha\ln(\mu/\alpha) - \ln[\Gamma(\alpha)]$

$= (\alpha\text{-}1)\ln(y) - \alpha y/\mu - \alpha\ln(\mu) + \alpha\ln(\alpha) - \ln[\Gamma(\alpha)]$.

With the inverse link function: $1/\mu = \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3$.

Thus the loglikelihood is:

$(\alpha\text{-}1)\ln(400) - \alpha 400(\beta_1 + \beta_3) + \alpha\ln(\beta_1 + \beta_3) + \alpha\ln(\alpha) - \ln[\Gamma(\alpha)] +$

$(\alpha\text{-}1)\ln(250) - \alpha 250(\beta_1) + \alpha\ln(\beta_1) + \alpha\ln(\alpha) - \ln[\Gamma(\alpha)] +$

$(\alpha\text{-}1)\ln(200) - \alpha 200(\beta_2 + \beta_3) + \alpha\ln(\beta_2 + \beta_3) + \alpha\ln(\alpha) - \ln[\Gamma(\alpha)] +$

$(\alpha\text{-}1)\ln(100) - \alpha 100(\beta_2) + \alpha\ln(\beta_2) + \alpha\ln(\alpha) - \ln[\Gamma(\alpha)]$.

Setting the partial derivative with respect to β_1 equal to zero:

$0 = -\alpha 400 + \alpha/(\beta_1 + \beta_3) - \alpha 250 + \alpha/(\beta_1)$. $\Rightarrow 650 = 1/(\beta_1 + \beta_3) + 1/\beta_1$.

Setting the partial derivative with respect to β_2 equal to zero:

$0 = -\alpha 200 + \alpha/(\beta_2 + \beta_3) - \alpha 100 + \alpha/(\beta_2)$. $\Rightarrow 300 = 1/(\beta_2 + \beta_3) + 1/\beta_2$.

Setting the partial derivative with respect to β_3 equal to zero:

$0 = -\alpha 400 + \alpha/(\beta_1 + \beta_3) - \alpha 200 - \alpha/(\beta_2 + \beta_3)$. $\Rightarrow 600 = 1/(\beta_1 + \beta_3) + 1/(\beta_2 + \beta_3)$.

(d) $f(x) = \sqrt{\dfrac{\theta}{2\pi}}\ \dfrac{\exp\left[-\dfrac{\theta\left(\dfrac{x}{\mu}-1\right)^2}{2x}\right]}{x^{1.5}}$.

Ignoring terms that do not involve μ, $\ln f(x) = -\dfrac{\theta\left(\dfrac{x}{\mu}-1\right)^2}{2x} = -\dfrac{\theta}{2x}\left(\dfrac{x^2}{\mu^2} - 2\dfrac{x}{\mu} + 1\right)$

$= -\dfrac{\theta x}{2\mu^2} + \dfrac{\theta}{\mu} - \dfrac{\theta}{2x}$.

Using the squared reciprocal link function: $1/\mu^2 = \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3$.

Thus ignoring terms that do not include μ, the loglikelihood is:

$\dfrac{-\theta}{2}\{400(\beta_1 + \beta_3) + 250(\beta_1) + 200(\beta_2 + \beta_3) + 100(\beta_2)\} + \theta\{\sqrt{\beta_1 + \beta_3} + \sqrt{\beta_1} + \sqrt{\beta_2 + \beta_3} + \sqrt{\beta_2}\}$.

Setting the partial derivative with respect to β_1 equal to zero:

$0 = \dfrac{-\theta}{2}\{400 + 250\} + \dfrac{\theta}{2}\{1/\sqrt{\beta_1 + \beta_3} + 1/\sqrt{\beta_1}\}. \Rightarrow 650 = 1/\sqrt{\beta_1 + \beta_3} + 1/\sqrt{\beta_1}$.

Setting the partial derivative with respect to β_2 equal to zero:

$0 = \dfrac{-\theta}{2}\{200 + 100\} + \dfrac{\theta}{2}\{1/\sqrt{\beta_2 + \beta_3} + 1/\sqrt{\beta_2}\}. \Rightarrow 300 = 1/\sqrt{\beta_2 + \beta_3} + 1/\sqrt{\beta_2}$.

Setting the partial derivative with respect to β_3 equal to zero:

$0 = \dfrac{-\theta}{2}\{400 + 200\} + \dfrac{\theta}{2}\{1/\sqrt{\beta_1 + \beta_3} + 1/\sqrt{\beta_2 + \beta_3}\}. \Rightarrow 600 = 1/\sqrt{\beta_1 + \beta_3} + 1/\sqrt{\beta_2 + \beta_3}$.

Comment: Using a computer, the fitted parameters in part b are:
$\beta_1 = 263.236$, $\beta_2 = 98.160$, $\beta_3 = 110.129$.
For example, the fitted value for Female and Urban is: 98.160 + 110.129 = 208.29.
The fitted values in part b are:

Gender	Urban	Rural
Male	373.36	263.24
Female	208.29	98.16

Using a computer, the fitted parameters in part c are:
$\beta_1 = 0.00447623$, $\beta_2 = 0.00789904$, $\beta_3 = -0.0021321$.
For example, the fitted value for Female and Urban is: 1/ (0.00789904 - 0.0021321) = 173.40.

The fitted values in part c are:

Gender	Urban	Rural	Total
Male	426.60	223.40	650
Female	173.40	126.60	300
Total	600	350	950

The totals for the fitted match those for the data.
These were the equations that needed to be solved for this model in part c.
In general, the estimates will be in balance as they were here, when one uses the canonical link function; the canonical link function for the Gamma is the reciprocal link function.
See “A Systematic Relationship Between Minimum Bias and Generalized Linear Models,” by Stephen Mildenhall, PCAS 1999.
Note that when the weights differ by cell, this balance involves weighted averages.
Using a computer, the fitted parameters in part d are:
β_1 = 0.0000218789, β_2 = 0.000053899, β_3 = -0.0000166235.
For example, the fitted value for Female and Urban is:
$1 / \sqrt{0.000053899 - 0.0000166235} = 163.79$.
The fitted values in part d are:

Gender	Urban	Rural	Total
Male	436.21	213.79	650
Female	163.79	136.21	300
Total	600	350	950

Since the canonical link function for the Inverse Gaussian is the squared reciprocal link function, again the totals for the fitted match those for the data.

3.213. a. Let β_1 represent territory A. Let β_2 represent territory B.

Let β_3 represent private passenger. Let β_4 represent light trucks.

(These are not the only choices. We have chosen medium trucks as the base level.)

$$\text{Design matrix} = X = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

The first row of (1, 0, 1, 0) corresponds to the cell for Territory A (β_1) and private passenger (β_3).

The second row of (1, 0, 0, 1) corresponds to the cell for Territory A (β_1) and light truck (β_4).

The third row of (1, 0, 0, 0) corresponds to the cell for Territory A (β_1) and medium truck.

(There are other ways to arrange the design matrix.)
The corresponding vector of betas (parameters) is:
$(\beta_1 + \beta_3, \beta_1 + \beta_4, \beta_1, \beta_2 + \beta_3, \beta_2 + \beta_4, \beta_2)$.

b. For a poisson error structure variance is a function of the expected value, while under the gamma error structure the variance of an observation is a function of its mean squared.

c. Determine the form of the density for the chosen error structure (distribution of errors.)
Using this density and the chosen link function take a product of the chances of the observations; this is the likelihood as a function of the parameters.
Maximize the log of the likelihood function by setting the partial derivatives with respect to each of the parameters equal to zero.
Solve the resulting system of equations for the fitted parameters.
Compute the predicted values.

Comment: In part (a) one could have instead for example taken:

Let β_1 be an intercept. Let β_2 represent territory A.

Let β_3 represent light trucks. Let β_4 represent medium trucks.

In that case, we have taken Territory B / Heavy Trucks as the base level.
Instead, other combinations of territory and truck weight could have been chosen as the base level.
If we use an intercept term, then we can have only one coefficient for territory, and two coefficients for vehicle type. Including the intercept, we still have a total of four coefficients in our model.
In more complicated situations one would not be able to solve the equations for the parameters in closed form. Fortunately, there are commercial packages of computer software specifically designed to solve and analyze GLMs.

3.214. One-way analysis doesn't consider:
1. Correlations between rating variables, in other words correlations of exposures by cell.
For example, young people drive older cars more often. Worse loss ratios for older cars can be partially driven by the larger proportion of youthful drivers.
For example, age may be correlated with territory if a greater proportion of senior citizens live in certain parts of a state. The relative loss ratios for such territories will be better due to the higher proportion of drivers who are senior citizens.
2. Interdependencies among rating variables.
For example, the rate differentials between male and female drivers vary by age.
For example, young drivers who have expensive cars may be poor risks, but old drivers who have expensive cars may be good risks.

3.215. a) In the absence of any other information, I would choose a Poisson error function which is commonly used for frequency.
The frequencies look like they might follow a multiplicative model; the ratios of the columns look kind of similar and the ratios of the rows look kind of similar.
(In contrast, the differences in the columns look kind of different and the differences in the rows look kind of different. Thus I would not choose an additive model and the identity link function.)
Therefore, I will use a log link function corresponding to a multiplicative model.
$g(x) = \ln(x)$. $g^{-1}(x) = e^x$.
One needs to pick a base level.
It is likely that Yes/Yes has the most exposures, so I will pick that as the base level.
The vector of model parameters:
Let β_0 be the intercept term, which is a parameter which applies to all observations.
Let β_1 correspond to no for homeowners.
Let β_2 correspond to no for auto policy.

Then the response vector would be: $\begin{pmatrix} \text{Yes HO / Yes Auto} \\ \text{No HO / Yes Auto} \\ \text{Yes HO / No Auto} \\ \text{No HO / No Auto} \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 8 \\ 12 \end{pmatrix}$.

The design matrix would be: $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. $X_1 \Leftrightarrow \begin{cases} \text{No H.O.} = 1 \\ \text{Otherwise} = 0 \end{cases}$. $X_2 \Leftrightarrow \begin{cases} \text{No Auto} = 1 \\ \text{Otherwise} = 0 \end{cases}$.

Alternately, the vector of model parameters: Let β_1 correspond to yes for auto.
Let β_2 correspond to no for auto. Let β_3 correspond to yes for homeowners.

Then the response vector would be: $\begin{pmatrix} \text{Yes HO / Yes Auto} \\ \text{No HO / Yes Auto} \\ \text{Yes HO / No Auto} \\ \text{No HO / No Auto} \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 8 \\ 12 \end{pmatrix}$.

The design matrix would be: $\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

$X_1 \Leftrightarrow \begin{cases} \text{Yes Auto} = 1 \\ \text{Otherwise} = 0 \end{cases}$. $X_2 \Leftrightarrow \begin{cases} \text{No Auto} = 1 \\ \text{Otherwise} = 0 \end{cases}$. $X_3 \Leftrightarrow \begin{cases} \text{Yes H.O.} = 1 \\ \text{Otherwise} = 0 \end{cases}$.

b) Missing data can lead to aliasing. For the missing data, if in most cases both whether it had an auto and homeowner policy is missing, then there is the potential problem of aliasing or near aliasing. No data auto and no data homeowners would be either perfectly or highly correlated. With aliasing the model parameters make no sense. Near aliasing creates problems with convergence of the model.
A solution would be to exclude these missing data records from modeling.
Another solution is to eliminate the unknown level from one of the factors so there are no linear dependencies. In other words, one further covariate needs to be removed; this could either be the "unknown" auto covariate or the "unknown" homeowners covariate.
Comment: One can make other choices in part (a) for the vector of parameters and get full credit provided the design matrix is consistent.

3.216. a) Increase of 7.0% ⇔ -0.4172. 2 phone calls ⇔ -0.4239.
-0.4172 - 0.4239 + 1.793 = 0.9519.
Using the logistic model, probability of renewal is: exp(0.9519) / {1 + exp(0.9519)} = **72.15%.**
b) I would <u>not</u> use this strategy:
1. There is no reasonable connection between insurance losses and the number of times an insured calls an insurer. Charging those who call the insurer more would not be acceptable to the public.
Insurance regulators are very unlikely to allow the use of this rating variable.
While causality is not required, this is an extreme case of the opposite of causality.
2. The proposed variable is easily manipulated by the insured.
3. The proposed variable lacks constancy; the number of phone calls from an insured is likely to change from year to year.
4. We do not know why the rate of renewal decreases with number of phone calls made by the insured to the insurer. Could this be because when they call, insureds get impolite or incompetent service? In that case, a better strategy would be to improve the insurer's service so that they do not lose so many customers.
5. The number of phone calls is likely related to other variables which are more directly related to renewal probability, such as moving or age. The actuary should go back and try to find variables that are the underlying reasons for the model results.
6. If many of these calls are from insureds who are making claims, then perhaps the lower renewal is due to poor claims service. It would be a better strategy to improve claims service.
7. By raising the rate of insureds who made phone calls, you are making their future renewal rate even lower. The insurer is likely to lose a lot of insureds if it followed this strategy.
Alternately, I would use this strategy:
1. When pricing based on the lifetime of a policy, it makes sense to take into account the expected renewal rate. Those with a lower expected renewal rate, such as those who make several phone calls to the insurer, should be charged more, all else being equal.
2. Those who call the insurer more often are likely to be reporting a claim. Those with claims in the past, have a higher expected future claim frequency. So it makes sense to charge those with more calls more, since their future average claim frequency is higher than average.
3. Those who call more often in the past are more likely to call more often in the future, resulting in higher expense for the insurer.
4. Those who call the insurer more often are more likely to make a small claim when they suffer a small loss, and thus have higher expected future loses.
<u>Comment</u>: I found it much, much easier in this case, to argue against using the strategy.
(My reasons in favor other than the first, have nothing to do with the given model.)
On your exam, pick whichever side of the argument allows you to quickly come up with two good reasons.
Without diagnostics there is no way to check the statistical significance of the modeled result.
Some of the extra phone calls may be from insureds who got big increases and are calling to complain or to see if this insurer will match a quote from another insurer. Thus the two variables in the model may be correlated.

3.217. (a) The first column refers to β_1 whether or not we have a male,
the second column refers to β_2 whether or not we are in Territory A,
the third column refers to β_3 the intercept, and thus is all ones.

$$X = \begin{array}{cl} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} & \begin{array}{c} \text{A, Male} \\ \text{A, Female} \\ \text{B, Male} \\ \text{B, Female} \end{array} \end{array} \qquad \text{or } X = \begin{array}{cl} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} & \begin{array}{c} \text{A, Male} \\ \text{B, Male} \\ \text{A, Female} \\ \text{B, Female} \end{array} \end{array}$$

$$\text{(b) } Y = \begin{pmatrix} 700/1400 \\ 400/1000 \\ 600/1000 \\ 420/1200 \end{pmatrix} = \begin{array}{cl} \begin{pmatrix} 0.50 \\ 0.40 \\ 0.60 \\ 0.35 \end{pmatrix} & \begin{array}{c} \text{A, Male} \\ \text{A, Female} \\ \text{B, Male} \\ \text{B, Female} \end{array} \end{array} \qquad \text{or } Y = \begin{array}{cl} \begin{pmatrix} 0.50 \\ 0.60 \\ 0.40 \\ 0.35 \end{pmatrix} & \begin{array}{c} \text{A, Male} \\ \text{B, Male} \\ \text{A, Female} \\ \text{B, Female} \end{array} \end{array}$$

(c) With a Normal error function and an identity link function, this is the same as a multiple regression.
Assuming $\beta_3 = 0.35$, then the squared error is:

$$1400\ (\beta_1 + \beta_2 + 0.35 - 0.5)^2 + 1000\ (\beta_2 + 0.35 - 0.4)^2$$
$$+\ 1000\ (\beta_1 + 0.35 - 0.6)^2 + 1200\ (0.35 - 0.35)^2 =$$
$$1400\ (\beta_1 + \beta_2 - 0.15)^2 + 1000\ (\beta_2 - 0.05)^2 + 1000\ (\beta_1 - 0.25)^2.$$

Setting the partial derivative with respect to β_1 equal to zero:

$0 = 2800(\beta_1 + \beta_2 - 0.15) + 2000(\beta_1 - 0.25). \Rightarrow 4800\ \beta_1 + 2800\ \beta_2 = 920.$

Setting the partial derivative with respect to β_2 equal to zero:

$0 = 2800(\beta_1 + \beta_2 - 0.15) + 2000(\beta_2 - 0.05). \Rightarrow 2800\ \beta_1 + 4800\ \beta_2 = 520.$

$\Rightarrow \beta_2 = (520 - 2800\ \beta_1)\ /\ 4800.$

Plugging back into the first equation: $4800\ \beta_1 + 2800\ (520 - 2800\ \beta_1)\ /\ 4800 = 920.$

$\Rightarrow \beta_1$ **= 0.1947**. $\Rightarrow \beta_2 = -0.0052.$

Alternately, without taking into account exposures by cell, the squared error is:
$(\beta_1 + \beta_2 + 0.35 - 0.5)^2 + (\beta_2 + 0.35 - 0.4)^2 + (\beta_1 + 0.35 - 0.6)^2 + (0.35 - 0.35)^2 =$
$(\beta_1 + \beta_2 - 0.15)^2 + (\beta_2 - 0.05)^2 + (\beta_1 - 0.25)^2$.

Setting the partial derivative with respect to β_1 equal to zero:

$0 = 2(\beta_1 + \beta_2 - 0.15) + 2(\beta_1 - 0.25). \Rightarrow 4\ \beta_1 + 2\ \beta_2 = 0.8.$

Setting the partial derivative with respect to β_2 equal to zero:

$0 = 2(\beta_1 + \beta_2 - 0.15) + 2(\beta_2 - 0.05). \Rightarrow 2\ \beta_1 + 4\ \beta_2 = 0.4.$

$\Rightarrow \beta_2 = (0.4 - 2\ \beta_1) / 4 = 0.1 - 0.5\beta_1.$

Plugging back into the first equation: $4\ \beta_1 + 2\ (0.1 - 0.5\beta_1) = 0.8$.

$\Rightarrow \beta_1$ **= 0.2**. $\Rightarrow \beta_2$ **= 0**.

Alternately, in either case one can fit via maximum likelihood and get the same result as by minimizing the squared errors.

For the Normal Distribution, $f(x) = \dfrac{\exp[-\frac{(x-\mu)^2}{2\sigma^2}]}{\sigma\sqrt{2\pi}}$. $\ln[f(x)] = -\dfrac{(x-\mu)^2}{2\sigma^2} - \ln[\sigma] - \ln[2\pi]/2$.

Without taking info account exposures by cell, the loglikelihood is:
$- (\beta_1 + \beta_2 + 0.35 - 0.5)^2 / (2\sigma^2) - (\beta_1 + 0.35 - 0.4)^2 / (2\sigma^2) - (\beta_2 + 0.35 - 0.6)^2 / (2\sigma^2)$
$- (0.35 - 0.35)^2 / (2\sigma^2) - 4 \ln[\sigma] - \ln[2\pi]/2.$

Setting the partial derivative of the loglikelihood with respect to β_1 equal to zero:

$0 = -(\beta_1 + \beta_2 - 0.15) / \sigma^2 - (\beta_1 - 0.25) / \sigma^2. \Rightarrow 2\ \beta_1 + \beta_2 = 0.4.$

Setting the partial derivative with respect to β_2 equal to zero:

$0 = -(\beta_1 + \beta_2 - 0.15) / \sigma^2 - (\beta_2 - 0.05) / \sigma^2. \Rightarrow \beta_1 + 2\ \beta_2 = 0.2.$

Solving two equations in two unknowns: β_1 **= 0.2**, and β_2 **= 0**.

Setting the partial derivative with respect to σ equal to zero:
$0 = (\beta_1 + \beta_2 - 0.15)^2 / \sigma^3 - (\beta_1 - 0.05)^2 / \sigma^3 - (\beta_2 - 0.25)^2 / \sigma^3 - 4 / \sigma.$
$\Rightarrow \sigma^2 = \{(0.2 + 0 - 0.15)^2 + (0.2 + 0 - 0.05)^2 + (0.2 + 0 - 0.25)^2\} / 4 = 0.006875.$

(d) 1. The Normal Distribution allows negative values, while the Poisson Distribution does not. Since claim frequencies are never negative, the Poisson error structure is preferable here.

2. The Normal error structure assumes that the process variances of the frequency in the four cells are equal. In contrast, a Poisson error structure assumes that the process variances of the frequency in the four cells are equal to their means. I would expect that those cells with higher expected frequencies would have higher process variances than those with lower expected frequencies, and thus would prefer the Poisson error structure to the Normal.

3. The log link function would assume multiplicative relativities, while the identity link function assumes additive relativities. If the relationship is approximately multiplicative, then the log link function would do a better job than the identity link function.

Comment: In parts (a) and (b), the order in which one lists the rows is arbitrary;
it would be a good idea to label what you did.
In part (c), if one includes the exposures in the sum of squared errors, that is equivalent to using exposures as the prior weights in the GLM, or using exposures in an offset term.
Including the exposures is equivalent to doing a weighted multiple regression.
The observed frequencies are:

Gender	Territory A	Territory B		Difference
Male	0.50	0.60		0.10
Female	0.40	0.35		-0.05
Difference	-0.10	-0.25		

The differences between territories are not similar for the two genders.
The differences between genders are not similar for the two territories.

Gender	Territory A	Territory B		Ratio
Male	0.50	0.60		1.2
Female	0.40	0.35		0.875
Ratio	0.80	0.583		

The ratios between territories are not similar for the two genders.
The ratios between genders are not similar for the two territories.
Thus perhaps neither an additive nor a multiplicative relationship is appropriate.

3.218. (a) ϕ is the scale or dispersion parameter, which scales the variance.
ω_i is a (prior) weight, representing the amount of data we have for observation i; the variance is inversely proportional to the volume of data.
(b) i. Gamma Distribution is most commonly used to model the error structure for severity; it works well in many situations based on diagnostics.
The Gamma is continuous with support from zero to infinity.
The gamma distribution also has an intuitively attractive property for modeling claim amounts since it is invariant to measures of currency. In other words measuring severities in dollars and measuring severities in cents will yield the same results using a gamma multiplicative GLM. (This is not true of some other distributions such as Poisson, but would be for the Inverse Gaussian.)
For the Gamma: $V(\mu_i) = \mu_i^2$.
ii. For policy renewal a Bernoulli or Binomial is used, since policy renewal is a yes/no process.
For the Bernoulli: $V(\mu_i) = \mu_i (1 - \mu_i)$.
For the Binomial representing µ trials (µ policies): $V(\mu_i) = \mu_i (1 - \mu_i) / m$.
(c) 1. For severity, ω_i would be the number of claims, the measure of how much data we have.
2. For policy renewal, if using the Bernoulli, ω_i would be the number of policies.
If using the Binomial, $\omega_i = 1$.
Comment: The current syllabus reading discusses weights in its Section 2.5. However, unlike the previous syllabus reading, it does not discuss the weights to use for the case of modeling policy renewals. If using a Binomial, then m is the number of policies; we were given the observed renewal rate for a set of m similar polices. The weight is already implicitly included; the larger m the more weight to the observation from that set of policies. The Bernoulli instead would model each individual policy; we would need to specifically weight the data from a larger set of policies more heavily.

3.219. Smaller Bayesian Information Criterion is better.
BIC = -2 (maximum loglikelihood) + p ln(n),
where n = 1000 is the sample size and p is the number of parameters.
Since the scaled deviance = (2) (saturated max. loglikelihood - maximum likelihood for model), we can compare between the models:
Scaled Deviance + p ln(n) = Scaled Deviance + p ln(1000).
(The maximum likelihood for the saturated model is the same in each case.)

Model #	p	Scaled Deviance	Scaled Deviance + p ln(1000)
1	2	1085.0	1098.82
2	3	1084.8	1105.52
3	3	1083.0	1103.72
4	4	1081.9	1109.53
5	5	1081.6	1116.14

The smallest Scaled Deviance + p ln(n) is for **Model 1**.

3.220. Smaller Akaike Information Criterion is better.
AIC = -2 (maximum loglikelihood) + (number of parameters)(2).
Since the scaled deviance = (2) (saturated max. loglikelihood - maximum likelihood for model), we can compare between the models:
Scaled Deviance + p 2 = Scaled Deviance + (number of parameters)(2).
(The maximum likelihood for the saturated model is the same in each case.)

Model #	p	Scaled Deviance	Scaled Deviance + (number of parameters)(2)
1	2	1085.0	1089.0
2	3	1084.8	1090.8
3	3	1083.0	1089.0
4	4	1081.9	1089.9
5	5	1081.6	1091.6

The smallest AIC is a tie between **Model 1 and Model 3**.

3.221. Estimated mean severity for a rural male is: exp[2.32 - 0.64 + 0.76] = 11.473.
For the Gamma Distribution, Var[Y] = $\phi\mu^2$ = (2) (11.473^2) = **263.3**.

3.222. exp[βx] = exp[-1.485 + 0 - 1.175 - 0.101] = $e^{-2.761}$ = 0.06323.
For the logit link function: $\mu = \frac{e^{\beta x}}{e^{\beta x} + 1}$ = 0.06323 / (0.06323 + 1) = **5.95%.**

3.223. μ = exp[-2.633 + 0.132 + 0] = **0.07957**.

3.224.

Variable	Number of Parameters
Vehicle Price	4
Driver age	2 - 1 = 1
Number of drivers	4 - 1 = 3
Gender	2 - 1 = 1
Interaction Gender & Driver Age	1

Maximum number of parameter is: 4 + 1 + 3 + 1 + 1 = **10**.
<u>Comment</u>: A model with only Vehicle Price would involve: $\beta_0 + \beta_1$ (vp) + β_2 $(vp)^2$ + β_3 $(vp)^3$.
The interaction of gender and driver age only uses one parameter since each of gender and driver age only use one parameter.

3.225. Smaller AIC is better, so we prefer Model 1.
exp[βx] = exp[-3.264 + (12)(0.212) + 0.727] = $e^{0.007}$ = 1.007.
For the logit link function: $\mu = \frac{e^{\beta x}}{e^{\beta x} + 1}$ = 1.007 / (1.007 + 1) = **50.2%.**

3.226. Let x be the number of additional parameters for the new model.
Let ℓ_1 be the loglikelihood for the original model, and ℓ_2 be the loglikelihood for the model including the new variable.
Scaled Deviance = (2) (saturated max. loglikelihood - maximum likelihood for model).
Thus the change in model scaled deviance is: $-2(\ell_2 - \ell_1) = -53$.

AIC = (-2) (maximum loglikelihood) + (number of parameters)(2).
Thus the change in AIC is: $(-2)(\ell_2 - \ell_1) + 2x = -53 + 2x = -47$. ⇒ x = 3.

BIC = (-2) (maximum loglikelihood) + (number of parameters) ln(number of data points).
Thus the change in BIC is: $(-2)(\ell_2 - \ell_1) + x \ln(n) = -53 + 3 \ln(n) = -32$. ⇒ $n = e^7$ = **1097**.

3.227. We have to assume equal exposures in each of the four cells.
The mean modeled frequencies are:

	State A	State B
Male	$\exp[\beta_1 + \beta_3]$	$\exp[\beta_1]$
Female	$\exp[\beta_2 + \beta_3]$	$\exp[\beta_2]$

The loglikelihood ignoring terms that do not depend on the betas is:
$-\exp[\beta_1 + \beta_3] + 0.0920 (\beta_1 + \beta_3) - \exp[\beta_2 + \beta_3] + 0.1500 (\beta_2 + \beta_3)$
$- \exp[\beta_1] + 0.0267 \beta_1 - \exp[\beta_2] + 0.0500 \beta_2$.

Setting the partial derivative of the loglikelihood with respect to β_1 equal to zero:
$-\exp[\beta_1 + \beta_3] + 0.0920 - \exp[\beta_1] + 0.0267 = 0$.
Given $\beta_3 = 1.149$: $-\exp[\beta_1] e^{1.149} + 0.0920 - \exp[\beta_1] + 0.0267 = 0$.
⇒ $\exp[\beta_1] = (0.0920 + 0.0267) / (1 + e^{1.149}) = 0.02857$.
⇒ $\exp[\beta_1 + \beta_3] = 0.02857 e^{1.149}$ = **0.0901** = expected frequency of a male risk in State A.

Comment: Similar to 8, 11/13, Q.2c.
What the exam questions calls "the likelihood function" is the loglikelihood function.
$\hat{\beta}_1 = \ln(0.02857) = -3.555$.

Setting the partial derivative of the loglikelihood with respect to β_2 equal to zero:
$-\exp[\beta_2 + \beta_3] + 0.1500 - \exp[\beta_2] + 0.0500$.
Given $\beta_3 = 1.149$: $-\exp[\beta_2] e^{1.149} + 0.1500 - \exp[\beta_2] + 0.0500 = 0$.
⇒ $\exp[\beta_2] = (0.1500 + 0.0500) / (1 + e^{1.149}) = 0.04813$. ⇒ $\hat{\beta}_2 = -3.034$.

Using a computer, without being given β_3, the maximum likelihood fit is:
$\hat{\beta}_1 = -3.5555$, $\hat{\beta}_2 = -3.0338$, and $\hat{\beta}_3 = 1.1490$.

The mean modeled frequencies are:

	State A	State B
Male	exp[-3.5555 + 1.1490] = 9.01%	exp[-3.5555] = 2.86%
Female	exp[-3.0338 + 1.1490] = 15.19%	exp[-3.0338] = 4.81%

3.228. In order to solve for the unknown intercept, we use the given probability of accident for a driver in age group 2, from area C and with vehicle body type Other.
0.22 = exp[x + 0.064 - 0.371] / {exp[x + 0.064 - 0.371] + 1}.
⇒ exp[x + 0.064 - 0.371] = 0.22 / (1 - 0.22) = 0.28205.
⇒ x + 0.064 - 0.371 = ln[0.28205] = -1.2657. ⇒ x = -0.9587.
Thus for a driver in age group 3, from area C and with vehicle body type Sedan, the odds (ratio) is: $\pi / (1 - \pi)$ = exp[-0.9857 + 0 + 0 + 0] = **0.3834**.
Comment: The probability of having an accident for a driver in age group 3, from area C and with vehicle body type Sedan is: 0.3834 / (1 + 0.3834) = 0.277.
Note that 0.277 / (1 - 0.277) = 0.383.

3.229. $\dfrac{\text{Exp}[-2.358 + 0.905]}{1 + \text{Exp}[-2.358 + 0.905]} = \mathbf{0.190}$.

3.230. exp[2.100 + 1.336 + 1.406 + 1.800] = **766.63**.

3.231. For an observation from Zone 4, with Vehicle Class Sedan and Driver Age Middle age, the mean is: exp[2.1] = 8.166.
For the Gamma Distribution the variance is: $\phi\ \mu^2 = (1)\ (8.166^2)$ = **66.7**.

3.232. $\dfrac{\exp[1.530 + 0.735 - 0.031]}{1 + \exp[1.530 + 0.735 - 0.031]} = \mathbf{90.33\%}$.

3.233. Since Model 2 has one fewer parameter than model 3,
model 2 has 9 degrees of freedom.
AIC = (-2) (maximum loglikelihood) + (number of parameters)(2).
BIC = (-2) (maximum loglikelihood)) + (number of parameters) ln(number of data points).
Therefore from Model 1: 95,473.61182 = (-2)(-47,704) + (5) ln(n). ⇒ n = 500,000.
For Model 2, AIC = -47,495 + (9)(2).
For Model 2, BIC = -47,495 + (9) ln(500,000).
The absolute difference between the AIC and the BIC for Model 2 is:
$|$ (9) ln(500,000) - 18 $|$ = **100.1**.

3.234. Graph one shows an increasing variance with fitted value.
Homoscedasticity would be constant variance, so statement 1 is false; statement 2 is true.
The residuals in Graph 2 are not symmetric around zero; there are more extreme positive values than there are extreme negative values. This indicates that the residuals are not normally distributed.
Statement 3 is true.
Comment: In Graph 2 it is not clear the meaning of the horizontal lines.
A Normal Q-Q Plot would have been much more useful than Graph 2.

3.235. The model with the smallest AIC is usually the best model in model selection process, all other things being equal. Statement A is not true.
The model with the smallest BIC is usually the best model in model selection process, all other things being equal. Statement B is not true.
The model with the smallest scaled deviance is usually the best model in model selection process, all other things being equal. Statement C is not true.
AIC = (-2) (maximum loglikelihood) + (number of parameters)(2).
BIC = (-2) (maximum loglikelihood)) + (number of parameters) ln(number of data points).
The penalty for AIC is (number of parameters)(2).
The penalty for BIC is (number of parameters) ln(number of data points).
So the penalties are equal for: 2 = ln(number data Points). ⇒ number of data points = e^2 = 7.4.
Thus, other things equal, when number of observations ≥ 8, BIC penalizes more for the number of parameters used in the model than AIC. **Thus statement E is true**.
Comment: Since statements D and E are opposites, it is likely that one of them is true.

3.236. Change in AIC is: (2) (number of parameters added).
Change in BIC is: ln(1500) (number of parameters added).
We want: ln(1500) (number of parameters added) > (2) (number of parameters added) + 25.
⇒ Number of parameters added > 4.7. ⇒ Number of added parameters is at least 5.
⇒ Minimum possible number of levels in the new categorical variable is: 5 + 1 = **6**.

3.237. 100,000 exp[-15 - 1.2 + (0.15)(25) + (0.004)(25^2) + (0.012)(25)]
= 100,000 $e^{-9.65}$ = **6.44** deaths.

3.238. i. Where the variable in question relates to a policy option selected by the insured, having its factor reflect anything other than the excess losses due to higher limit is not a good idea. One can get counterintuitive results such as charging less for more coverage.
Even if the indicated result is not counterintuitive, to the extent that the factor differs from the pure effect on loss potential, it will affect the way insureds choose coverage options in the future. Thus, the selection dynamic will change and the past results would not be expected to replicate for new policies. For this reason it is recommended that factors for coverage options such as increased limit factors be estimated outside the GLM, using traditional actuarial techniques. (The resulting factors should then be included in the GLM as an offset.)

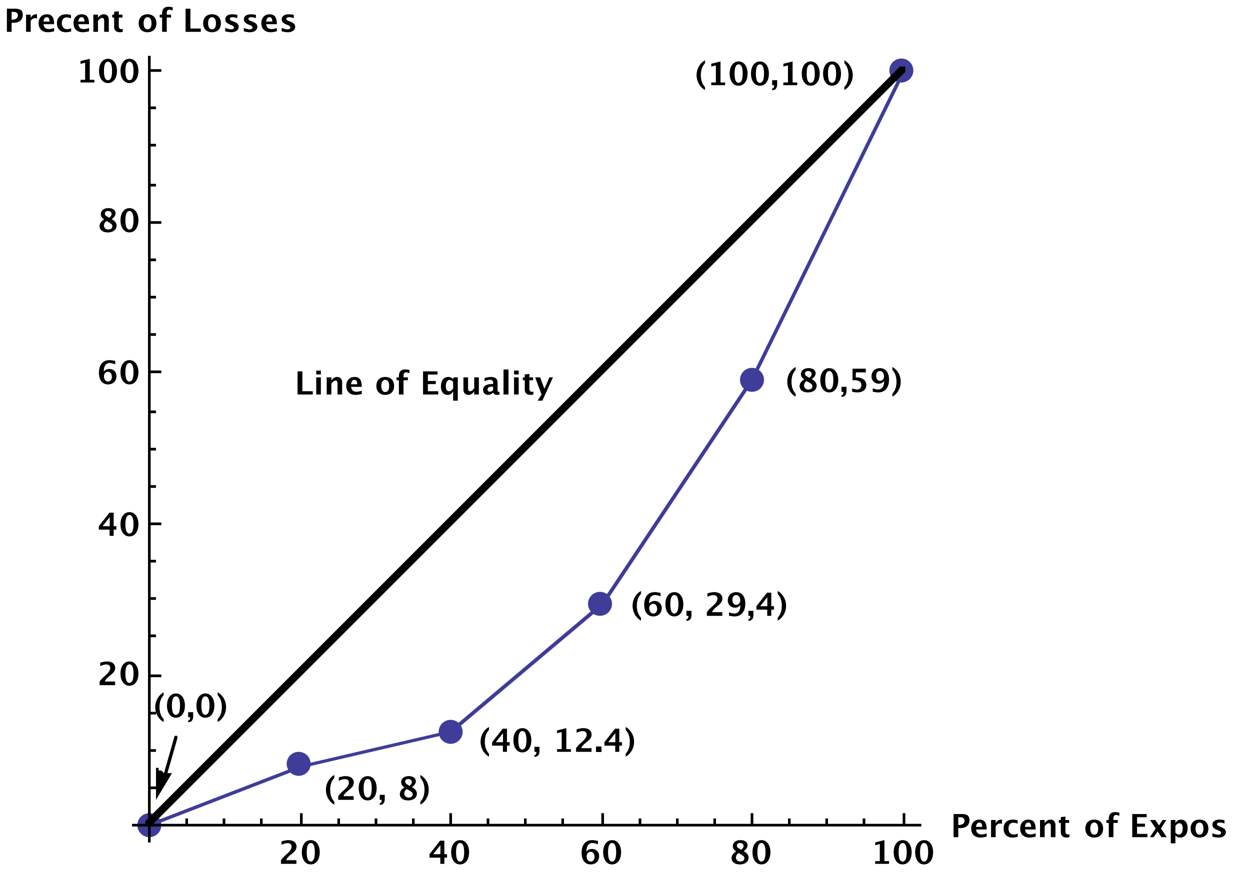
ii. I assume what is intended is that the number of coverage changes during the current policy period will be used to help rate the policy during its next policy period. (We are not given any information on whether the number of coverage changes in a policy period is related to the insurance costs the following period compared to otherwise similar insureds.)
The number of changes during a given policy period is not a good classification variable.
It is something that is likely to be zero for many policy periods, and vary somewhat randomly over time. If those with more coverage changes are changed more it is unlikely to be acceptable to insurance regulators and the public. If those with more coverage changes are changed more, then it will give insureds less incentive to make necessary coverage changes during a policy period; some of these coverage changes would have resulted in additional premiums for the insurer.
Alternately, the information will not be available for new business since we are building a GLM for the prospective period.
Alternately, the number of coverage changes is likely to change from what it is in the current policy period and thereafter year by year.
iii. Territories are not a good fit for the GLM framework. You may have thousands of zipcodes to consider and aggregating them to a manageable level will cause you to lose a great deal of important signal. If one does not aggregate the large number of zipcodes, then there are too many parameters which can lead to overfitting.
Using a spatial smoothing technique would be a more appropriate technique; one would then include the value determined for ZIP code as an offset term in the GLM.

(b) 1. One can get counterintuitive results such as charging more for less coverage.
2. Even if the indicated result is not counterintuitive, to the extent that the factor differs from the pure effect on loss potential, it will affect the way insureds choose coverage options in the future. Thus, the selection dynamic will change and the past results would not be expected to replicate for new policies.
3. Deductibles should lower frequency (small losses below deductible not reported) but usually increase severity (since claims that do get reported are higher average cost). This violates the assumption for the Tweedie Distribution, that a lower pure premium is due to both a lower frequency and a lower severity.
(c) One can calculate deductible relativities from loss elimination ratios.
Deductible Relativity = (1 - LER for chosen deductible) / (1 - LER for Base Deductible).
Loss elimination ratios can be estimated from size of loss data.
Loss elimination ratio = (Limited Expected Value at Deducible Amount) / Mean.
In the GLM, one would then include an offset of ln[deductible relativity].
<u>Comment</u>: While the average size of non-zero payment, equal to the mean residual life, usually increases as the size of deductible increases, this is <u>not</u> always the case.
Deductible factors may produce higher relativities at higher deductibles due to factors other than pure losses elimination:
1. Insureds at high loss potential and high premiums may be more likely to elect high deductibles in order to reduce their premium.
2. Underwriters may force high deductibles on riskier insureds.

3.239. a. Sort the risks from best to worst based on the model predicted loss.

Risk	Model Predicted Loss	Actual Loss	Cumulative Losses	Cumulative % of Losses
5	200	400	400	8.0%
2	500	220	620	12.4%
4	800	850	1,470	29.4%
3	1,500	1,480	2,950	59.0%
1	2,000	2,050	5,000	100.0%
Total		5000		

On the x-axis, plot the cumulative percentage of exposures.
I will assume that each risk has the same number of exposures.
On the y-axis, plot the cumulative percentage of actual losses.



b. The Gini index is twice the area between the Lorenz Curve and the line of equality.
The higher the Gini Index, the better the rating plan is at identifying risk differences, in other words the rating plan has more lift.
“The Gini index can also be used to measure the lift of an insurance rating plan by quantifying its ability to segment the population into the best and worst risks.”
Comment: See Section 7.2.4 in Generalized Linear Models for Insurance Rating.
Usually, one would be working with thousands of risks.

3.240. (a) exp[0.910 + (3)(0.013) + ln[25,000] (-0.187) + (8)(0.062)] = $e^{-0.4357}$ = **64.7%.**

(b) One could take the coefficients of the new business model as a given, other than b_0, which will be re-estimated.
Let the prior year claim count be x for renewal business.
Then the renewal business model is:
$\mu = \exp[\beta_0 + 0.013 \text{ age} + (-0.187) \text{ logprem} + 0.62 \text{ locont} + \beta_4 x]$.
We would fit the model via maximum likelihood to the data for renewal business, taking into account the form of density for the Tweedie Distribution.
Alternately, one can fit a single model to the data for new and renewal business.
Let the prior year claim count be x for renewal business
Let D = 0 if new business and 1 if renewal business.
Then the combined model is: $\mu = \exp[\beta_0 + \beta_1 \text{ age} + \beta_2 \text{ logprem} + \beta_3 \text{ locont} + D\ \beta_4 + D\ \beta_5\ x]$.
We would fit the model via maximum likelihood to the combined data, taking into account the form of density for the Tweedie Distribution.

(c) 1. Time-consistency. One can fit the model to the data for separate years and compare the coefficients. If the fitted coefficients are similar, that indicates stability over time.
Alternately, one could introduces dummy variables into the model for the various years of data.
For example, if we have data from 2012, 2013 and 2104,
then we could take 2012 = base year, x_5 = 1 if 2013, x_6 =1 if 2014.
Then test whether the coefficients of these variables are significantly different from zero. If one or more of the fitted coefficients are significantly different than zero, that indicates instability over time.
2. Bootstrapping. Create multiple datasets from the initial dataset by sampling with replacement. Run the model on each sampled set. Assess stability of estimates of coefficients by comparing the results from each run.
3. Cross-Validation. Split the data into k parts and run the model on the (k-1) parts, then validate the result on the remaining part. Compare how similar the estimates are from the k iterations to assess variable stability.
4. Validation on Holdout Dataset. Split the data into two subsets, training and holdout. Determine the best model on the training set. Ideally, this model should fit well the holdout data.
5. Cook's Distance. Sort the observations based on their Cook's Distance value (higher distance = more influence on the model.) Remove one or more of the most influential observations and rerun the model on this new set of data to see the effect on estimated parameters.

3.241. (a) For the base model:
AIC = (-2)(-750) + (2)(10) = **1520**.
BIC = (-2)(-750) + 10 ln[1,000,000] = **1638.2**.
For the new model:
AIC = (-2)(-737.5) + (2)(15) = **1505**.
BIC = (-2)(-737.5) + 15 ln[1,000,000] = **1682.2**.
(b) AIC is preferable. As here, most actuarial models involve a lot of data points. Therefore, the penalty for more parameters is very large for the BIC. Using BIC will tend to result in too simple models. In contrast, AIC does not depend on the number of data points.
(c) Based on part (b), I will rely on AIC.
Smaller AIC is better, so I will recommend the new model.
Comment: See Section 6.2.2 in Generalized Linear Models for Insurance Rating.
If one instead relied on BIC, the base model would be preferred.
Scaled Deviance = 2 (loglikelihood of saturated model - loglikelihood of model).
Thus equivalently to using AIC, one could compare models using: Scaled Deviance + 2p.
For the base model, Scaled Deviance + 2p = 500 + (2)(10) = 520.
For the new model, Scaled Deviance + 2p = 475 + (2)(15) = 505.
Since 505 < 530, we prefer the new model based on this criterion.
Equivalently to using BIC, one could compare models using: Scaled Deviance + p ln[N]..
For the base model, Scaled Deviance + 2p = 500 + 10 ln[1 million] = 638.16.
For the new model, Scaled Deviance + 2p = 475 + 15 ln[1 million] = 682.23.
Since 638.16 < 682.23, we prefer the base model based on this criterion.

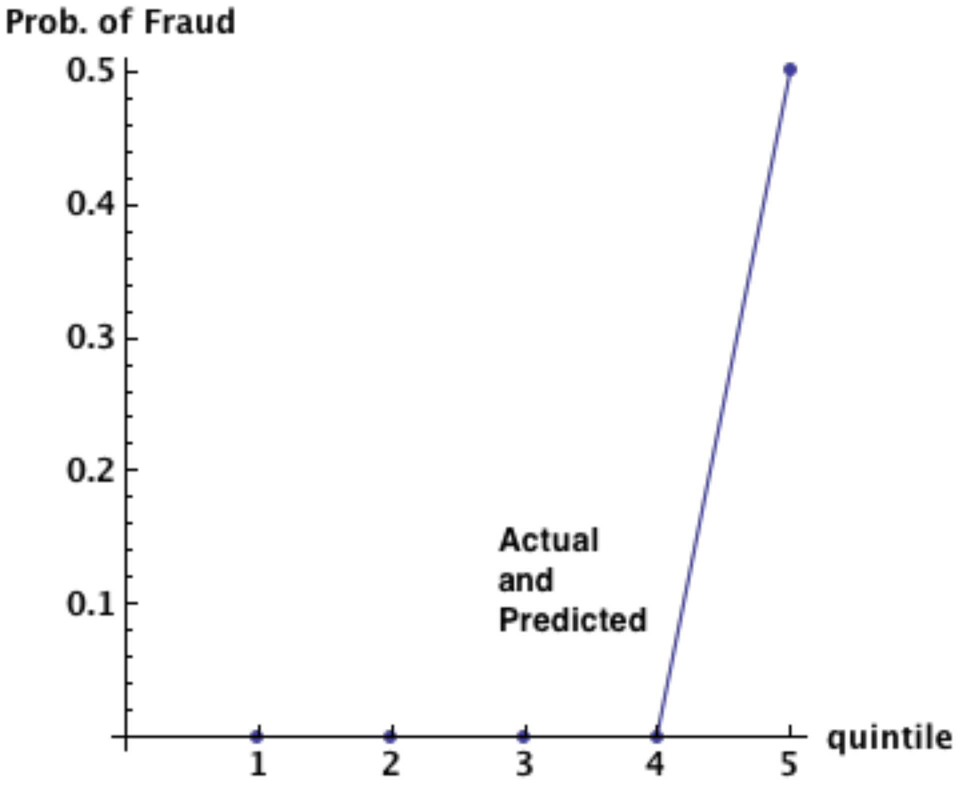
3.242. (a) 1. Attempting to test the performance of any model on the same set of data on which the model was built will produce overoptimistic results. Using the training data to compare this model to any model built on different data would give our model an unfair advantage.
2. As we increase the complexity of the model, the fit to the training data will always get better. In contrast, for data the model fitting process has not seen, additional complexity may not improve the performance of a model; as the model gets more complex its performance on the holdout data (test data) will eventually get worse, as shown in the figure in this question.
(b) Model 2 has the right balance, since it has the smallest test MSE.
Model 1 is too simple (fewer degrees of freedom than Model 2), while model 3 is too complex (more degrees of freedom than Model 2).
(c) "Out-of-time validation is especially important when modeling perils driven by common events that affect multiple policyholders at once. An example of this is the wind peril, for which a single storm will cause many incurred losses in the same area. If random sampling is used for the split, losses related to the same event will be present in both sets of data, and so the test set will not be true unseen data, since the model has already seen those events in the training set. This will result in overoptimistic validation results. Choosing a test set that covers different time periods than the training set will minimize such overlap and allow for better measures of how the model will perform on the completely unknown future."
Alternately, as in Couret and Venter, one may select either the even or odd years of data as the training set and the other as the holdout set, in order to be neutral with respect to trend and maturity.
Comment: See Section 4.3 of Generalized Linear Models for Insurance Rating.
The figure shown is very similar to Figure 7 in Generalized Linear Models for Insurance Rating.
We are interested in how the GLM will perform at predicting the response variable on some future set of data rather than on the set of past data with which we are currently working.
Our goal in modeling is to find the right balance where we pick up as much of the signal as possible with minimal noise, represented in this case by Model 2.

3.243. A simple quintile plot is a simple quantile plot with 5 buckets.

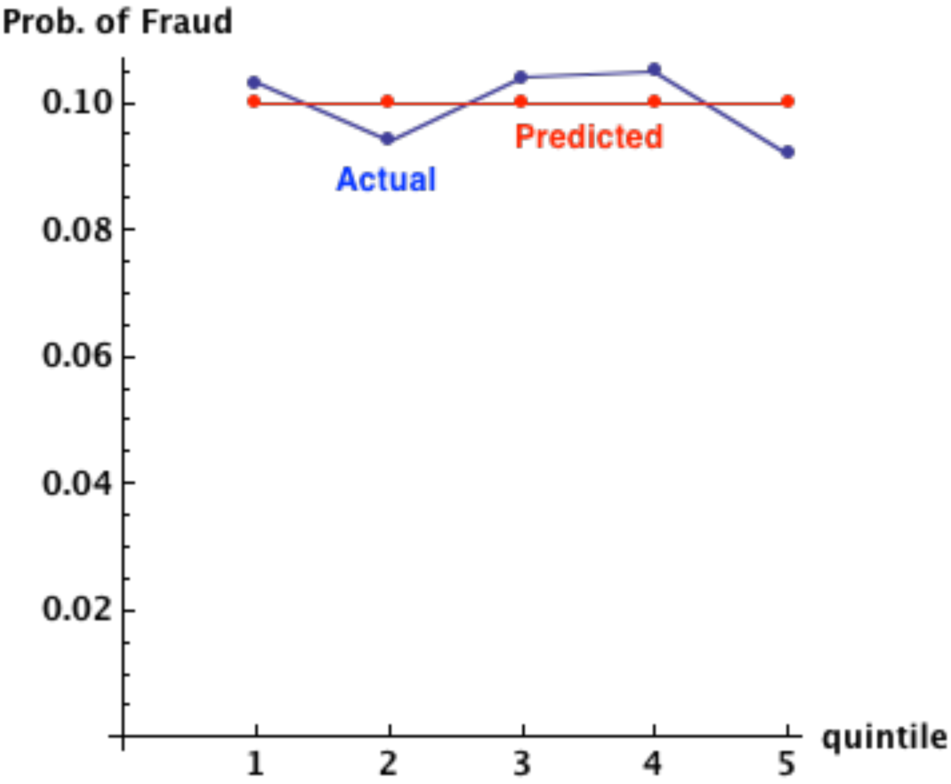
- Sort the dataset based on the model predicted fraud rate from smallest to largest.
- Group the data into 5 buckets with equal volume. (In this case 2000 claims in each.)
- Within each group, calculate the average predicted fraud rate based on the model, and the average actual fraud rate.
- Plot for each group, the actual fraud rate and the predicted fraud rate.

The saturated model has as many predictors as data points. Thus for the saturated model, the predictions exactly match the observations for each claim. In this case, 1000 of the claims involve fraud, and would all be placed in the last quintile. Thus the last quintile would consist of 1000 claims with fraud and 1000 claims without fraud.

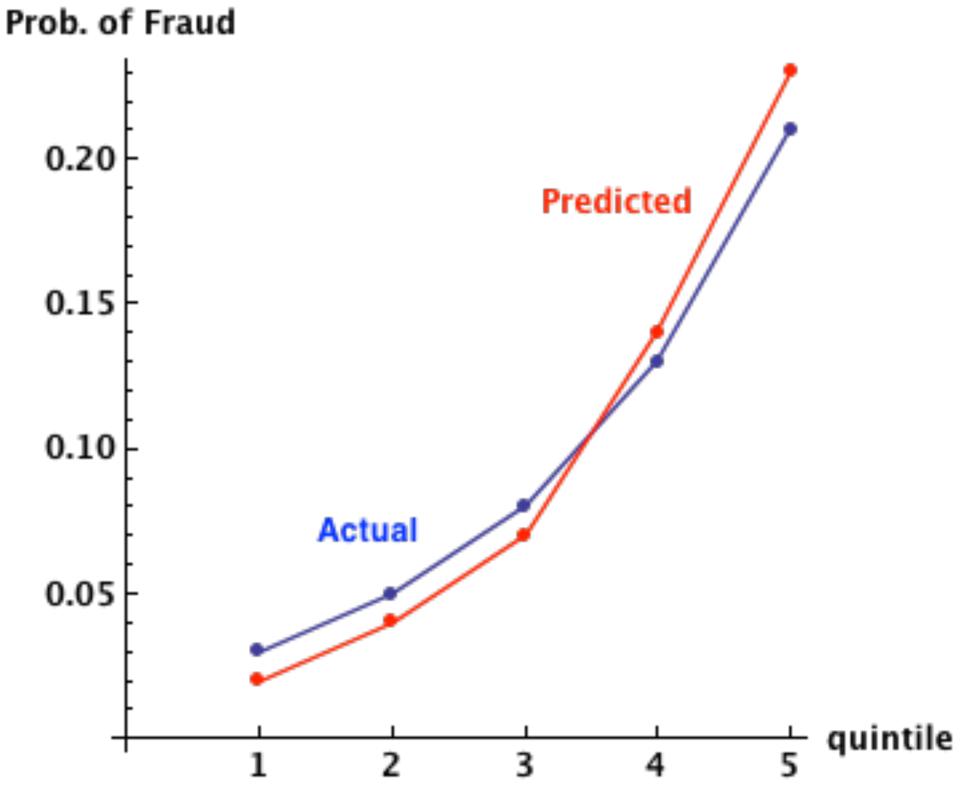
The simple quintile plot:



The null model, has no predictors, only an intercept. Thus for the null model the prediction is the same for every record: the grand mean.
In this case, the overall probability of fraud is: 1,000/10,000 = 10%.
Since every risk has the same prediction, one would assign them to buckets at random.
Thus all of the actuals by quintile should be close to the grand mean, with small differences due to the randomness of assignments. The simple quintile plot:



“A model that could be used in practice”, would have the actuals increase monotonically, have good but not perfect predictive accuracy, and a reasonably large vertical distance between the actuals in the first and last quintiles. A simple quintile plot:



Comment: See Sections 6.1.1 and 7.2.1 of GLMs for Insurance Rating.
Combines separate ideas in the syllabus reading.
There are many possible examples of the last plot.
Since the records are ordered by predicted values, the records in each bucket change for each graph. Thus, actuals are not the same for each graph.
Quintile plots are sorted by predicted values from smallest to largest value. Thus the predicted values must be monotonically increasing (or in the case of the null model equal). Actuals need not be monotonically increasing, although that is desirable.
In every graph, the average of the actuals should be the grand mean of 10%.
In the final plot, the average of the predicteds should be close to if not equal to 10%; the GLM may have a small bias.
In the final plot, the predicted and actuals for the final quintile should each be less than the 50% in the saturated model. In the final plot, the predicted and actuals for the final quintile should each be more than the 10% in the null model.

3.244.

			25% Threshold			50% Threshold	
Claim #	Fraud		Predict.			Predict.	
1	Y		N	False Neg.		N	False Neg.
2	N		N	True Neg.		N	True Neg
3	N		N	True Neg.		N	True Neg.
4	N		Y	False Pos.		Y	False Pos.
5	Y		Y	True Pos.		Y	True Pos.
6	Y		Y	True Pos.		N	False Neg.
7	N		N	True Neg.		N	True Neg.
8	Y		Y	True Pos.		Y	True Pos.
9	N		Y	False Pos.		Y	False Pos.
10	N		Y	False Pos.		N	True Neg.

(a)

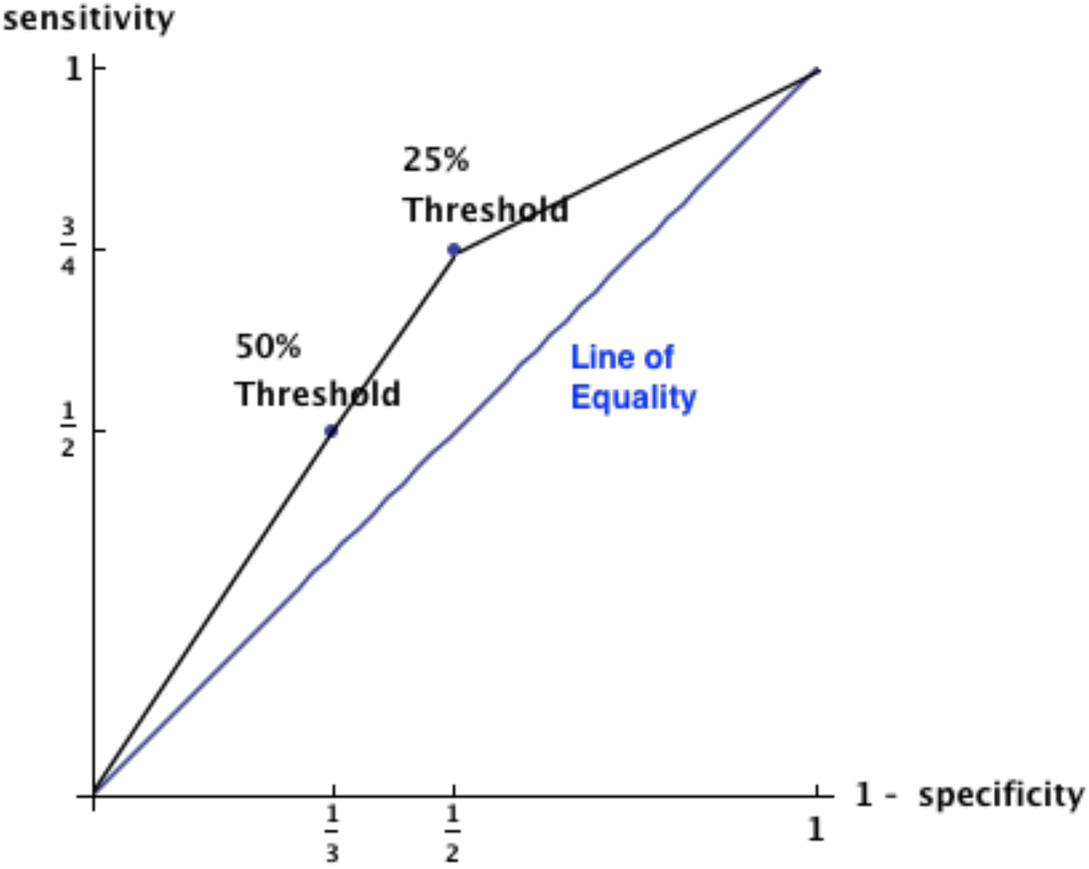
	25% Threshold		
	Predicted		
Actual	Fraud	No Fraud	Total
Fraud	true pos.: 3	false neg.: 1	4
No Fraud	false pos.: 3	true neg.: 3	6
Total	6	4	10

	50% Threshold		
	Predicted		
Actual	Fraud	No Fraud	Total
Fraud	true pos.: 2	false neg.: 2	4
No Fraud	false pos.: 2	true neg.: 4	6
Total	6	4	10

(b) Sensitivity = $\frac{\text{True Positives}}{\text{Total Number of Events}} = \frac{\text{Correct Predictions of Fraud}}{\text{Total Number of Fraudulent Claims}}$.

Specificity = $\frac{\text{True Negatives}}{\text{Total Number of Non-Events}} = \frac{\text{Correct Predictons of No Fraud}}{\text{Total Number of Nonfraudulent Claims}}$.

25% threshold: sensitivity = 3/4, and specificity = 3/6 = 1/2. Graph (1 - 1/2, 3/4).
50% threshold: sensitivity = 2/4 = 1/2, and specificity = 4/6 = 2/3. Graph (1 - 2/3, 1/2).
The ROC Curve, plus the 45-degree comparison line:



(c) Using a 25% threshold results in more predictions of fraud than using a 50% threshold. Therefore, the 25% threshold has greater sensitivity, more true positives, which is good; however, this is at the cost of lower specificity, more false positives, which is bad.
Alternately, Advantage: You will catch more actual fraud claims because you will have a higher true positive rate. Disadvantage: You will have a higher false positive rate as well, which means you will waste resources to review claims that are not fraudulent.

(d) There are few claims, but they are large. Thus we are very willing to spend money investigating claims for possible fraud; we do not want to miss any true positives and are willing to live with false positives. Therefore, we would prefer the lower threshold of 25%, which has greater sensitivity.
Alternately, a threshold of 0.25 is more appropriate. The high severity makes the cost of not investigating a fraudulent claim very high. The low frequency means that the number of additional claims that will need to be investigated is not very large. The cost of investigating these few additional claims is far less than the cost of potentially missing a few fraudulent claims at a higher discrimination threshold.
Comment: See Table 13 and Figure 26 in GLMs for Insurance Rating.
According to the CAS Examiner's Report, in part (a) one was required to show a table similar to the one I have, showing the origin of the true positives, false positives, true negatives, and false negatives.

3.245. $X\beta = 2 + (2)(1) + (1)(-1) = 3$.
The Gamma has the inverse as its canonical link function.
$1/ (X\beta) = 1/3$.
The Poisson has the log as its canonical link function.
$\exp[X\beta] = e^3 = 20.09$.
The Binomial has the logit as its canonical link function.
$\exp[X\beta] / \{1 + \exp[X\beta]\} = e^3 / (1 + e^3) = 0.9526$.
$1/3 < 0.9525 < 20.09$. Thus the correct ordering is: **I < III < II**.
Comment: The Normal has the identity as its canonical link function.

3.246. The base rate is charged to Male and Territory Q.
Using the log link function, $\beta_{\text{Terr}} = \ln[545/148] = 1.3036$.
$\beta_{\text{Gend}} = \ln[446/148] = 1.11031$.
$\ln[4024/148] = \beta_{\text{Terr}} + \beta_{\text{Gend}} + \beta_{\text{Inter}}$.
$\Rightarrow 3.3028 = 1.3036 + 1.11031 + \beta_{\text{Inter}}$. $\Rightarrow \beta_{\text{Inter}} = \mathbf{0.8961}$.

Comment: $148 \exp[1.3036 + 1.11031 + 0.8961] = 4024$.
$\ln(\mu) = \beta + \beta_{\text{Terr}} X_R + \beta_{\text{Gend}} X_F + \beta_{\text{Inter}} X_R X_F$,
where $X_R = 1$ if territory R and zero otherwise,
and $X_F = 1$ if Female and zero otherwise.

3.247. $X\beta = 5 + (-0.65)(5) = 1.75$. The odds are: $e^{1.75} = \mathbf{5.75}$.
Alternately, for the logistic model: $\hat{\pi} = e^{1.75} / (1 + e^{1.75}) = 0.852$.
The odds are: $\hat{\pi} / (1 - \hat{\pi}) = 0.852 / (1 - 0.852) = \mathbf{5.75}$.
Comment: We have estimated that the probability of renewal is 5.75 times the probability of a nonrenewal.

3.248. (a) i. One can bin driver age into groups.
For this example three bins may work well: 18 to 25, 26 to 80, above 80.
ii. One can use hinge functions. A hinge function is of the form: $(X - c)_+ = \max(0, X - c)$.
This will result in a piecewise linear function, with a change in slope at each breakpoint c_i.
For this example, I would choose breakpoints at 25 and 80.
(b) i. With binning: Continuity is not guaranteed.
Variation within intervals is ignored.
There may not be enough data in each bin to be credible.
There could be non-intuitive results, such as reversals.
ii. Using hinge functions:
The breakpoints must be selected by the user.
Comment: In both cases, more parameters are added to the model; the principal of parsimony states that we prefer a simpler model with fewer parameters, all else being equal.

3.249. (a) Assuming model A has an intercept, adding driver age using a second order polynomial adds two parameters.
i. Unscaled Deviance =
ϕ 2 {(loglikelihood for the saturated model) - (loglikelihood for the fitted model)}.
D_A = (1.75)(2) {-1000 - (-1500)} = 1750. D_B = (1.75)(2) {-1000 - (-1465)} = 1627.5.

$$F = \frac{(D_A - D_B) / (\text{number of added parameters})}{\hat{\phi}_B} = \{(1750 - 1627.5) / 2\} / 1.75 = 35.$$

We compare to the given critical value of 3.183.
Since 35 > 3.183, model B is significantly better than model A.
Driver age should be included in the rating plan.
ii. Let p be the number of fitted parameters for Model A.
$AIC_A = (-2)(-1500) + 2p = 3000 + 2p.$
$AIC_B = (-2)(-1465) + 2(p+2) = 2934 + 2p.$
Since $AIC_B < AIC_A$, model B is better than model A.
Driver age should be included in the rating plan.
iii. Let n be the number of data points (for each of the models).
$BIC_A = (-2)(-1500) + p \ln(n) = 3000 + p \ln(n).$
$BIC_B = (-2)(-1465) + (p+2) \ln(n) = 2930 + (p+2) \ln(n).$
$BIC_A - BIC_B = 70 - 2 \ln(n)$. This difference is positive for $n < e^{35} = 1.586 \times 10^{15}$.
Thus $BIC_B < BIC_A$, and model B is better than model A.
Driver age should be included in the rating plan.
(b) When parameters are added to a model, the deviance improves (gets smaller). Thus using deviance alone would lead to overfitting. The issue is whether the deviance gets significantly better. Using AIC or BIC is more appropriate, as they penalize for adding new parameters.
Comment: The degrees of freedom for Model B =
number of observations minus number of fitted parameters for model B.
The F-statistic has degrees of freedom equal that of 2 and Model B.
The given critical value is the 5% critical value for 2 and 50 degrees of freedom.
According to the CAS Examiner's report, for part b common mistakes included: "Giving some of the limitations of deviance such as needing to have the same underlying dataset with the same distribution. This limitation is not restricted to deviance alone."
Section 6.1.3 of Generalized Linear Models for Insurance Rating says with respect to either scaled or unscaled deviance:
"Firstly, when comparing two models using log-likelihood or deviance, the comparison is valid only if the datasets used to fit the two models are exactly identical. ...
For any comparisons of models that use deviance, in addition to the caveat above, it is also necessary that the assumed distribution must be identical as well. This restriction arises from deviance being based on the amount by which log-likelihood deviates from the perfect log-likelihood; changing any assumptions other than the coefficients would alter the value of the perfect log-likelihood as well the model log-likelihood, muddying the comparison."
However, the question asked why the deviance statistic alone should not be used to assess model fit. As seen in the solution to part (a), deviance together with the difference in number of parameters and the estimated dispersion parameter can be used to assess model fit.

3.250. (a) As per Section 2.6 of Generalized Linear Models for Insurance Rating:
$g(\mu_i) = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + ...+ \beta_p x_{ip}$ + offset.

Thus the offset for Policy 1 is: $\ln(\frac{0.013}{1 - 0.013})$ = **-4.330**.

The offset for Policy 2 is: $\ln(\frac{0.203}{1 - 0.203})$ = **-1.368**.

The offset for Policy 3 is: $\ln(\frac{0.025}{1 - 0.025})$ = **-3.664**.

(b) In each case we add the offset to the linear component from the insurance score.
For Policy 1: 1.250 + (-0.020)(78) - 4.330 = -4.640.

Probability of a claim is: $\frac{\exp[-4.640]}{1 + \exp[-4.640]}$ = **0.96%**.

For Policy 2: 1.250 + (-0.020)(92) - 1.368 = -1.958.

Probability of a claim is: $\frac{\exp[-1.958]}{1 + \exp[-1.958]}$ = **12.37%**.

For Policy 3: 1.250 + (-0.020)(35) - 3.664 = -3.114.

Probability of a claim is: $\frac{\exp[-3.114]}{1 + \exp[-3.114]}$ = **4.25%**.

(c) The logit function is: $\ln(\frac{x}{1 - x})$, for $0 < x < 1$.

The logit function has range from $-\infty$ to ∞.

The logistic function is: $\frac{e^x}{1 + e^x}$, for $-\infty < x < \infty$.

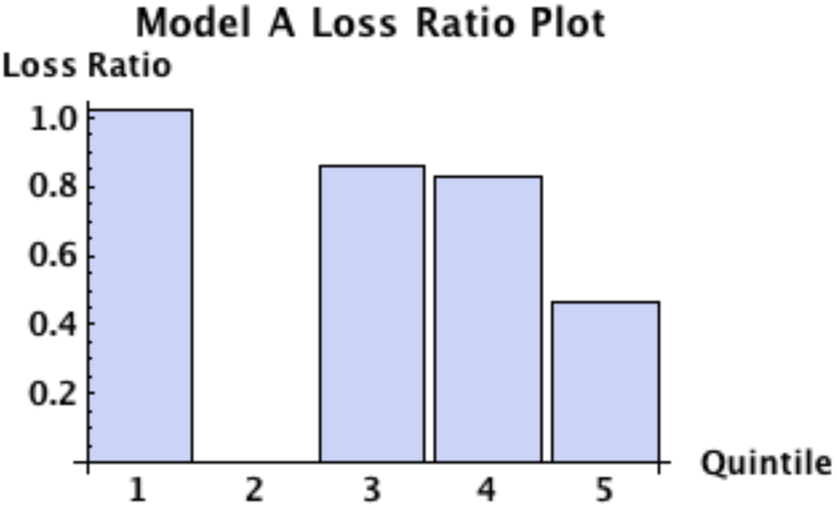
The logistic function has range from 0 to 1.
(d) Since the range of the logistic function is 0 to 1, using its inverse the logit as a link function guarantees that the response is in the correct range for probabilities, zero to one.

3.251. Based on the first graph, Model 1 does a very good job of matching the training data. However, based on the second graph, Model 1 does a very poor job of matching the test data, particularly for the high deciles. Model 1 is overfit; the model picks up too much of the random fluctuation (noise) in the training data.
Based on the first graph, Model 2 does a poor job of matching the training data.
Based on the second graph, Model 2 also does a poor job of matching the test data, although not as poor of a job as Model 1.
Model 2 is probably underfit; the model does not pick up enough of the signal in the training data.
Model 2 is monotone increasing, which is good. In the second graph, Model 1 is not monotone increasing; there are reversals, which is bad.
Model 1 has a larger vertical distance between the first and last deciles than does Model 2; Model 1 has more “lift” than Model 2. All else being equal, larger lift is better, indicating that the model is able to maximally distinguish the best and worst risks.
Comment: According to the CAS Examiner’s “recommendation of one model over the other was not required.” I would not recommend using either model.
See Section 7.2.1 of Generalized Linear Models for Insurance Rating.

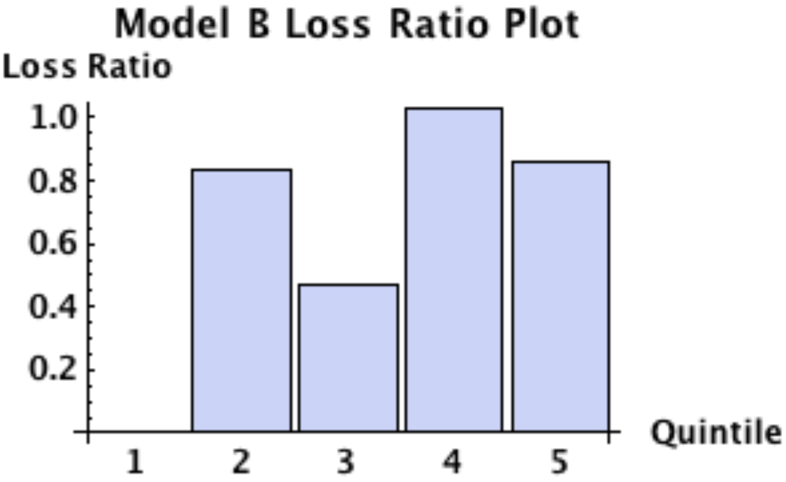
3.252. (a) Sort the data based on the loss ratio predicted by the given model.

Obser.	Actual Loss Cost	Actual Loss Ratio	Model A Loss Cost	Model A Loss Ratio	Model B Loss Cost	Model B Loss Ratio	Earned Premium
1	1,500	83.3%	825	45.8%	900	50.0%	1,800
2	675	46.6%	765	52.8%	800	55.2%	1,450
3	0	0.0%	615	25.9%	350	14.7%	2,375
4	2,250	85.7%	900	34.3%	3,000	114.3%	2,625
5	5,000	102.6%	1,050	21.5%	3,700	75.9%	4,875

For Model A, the order of predicted loss ratios is: 5, 3, 4, 1, 2.
The corresponding actual loss ratios are: 102.6%, 0%, 85.7%, 83.3%, 46.6%.

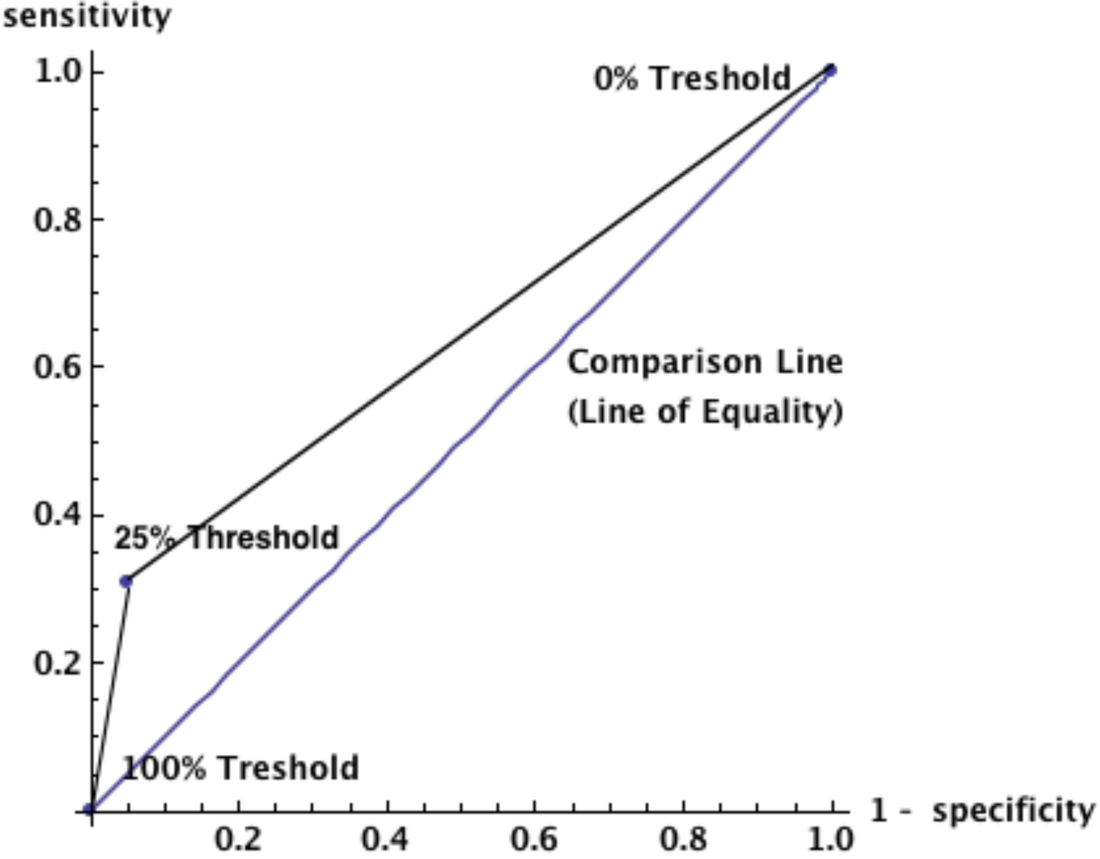


For Model B, the order of predicted loss ratios is: 3, 1, 2, 5, 4.
The corresponding actual loss ratios are: 0%, 83.3%, 46.6%, 102.6%, 85.7%.



(b) A loss ratio chart tells one whether a proposed model is outperforming the current rating plan by identifying differences in risks, but not whether the proposed model's predictions are accurate. The loss ratio chart compares a proposed plan to the current plan, rather than directly comparing two proposed plans to each other as in a double lift plot.
Simple quintile plots use loss costs (pure premiums) rather than loss ratios, which make them somewhat harder to understand and explain than loss ratio plots. Unlike double lift charts, simple quantile plots of two models would have graphs that are on separate charts and we can only compare the models by looking at two charts.
Double Lift Plots are sorted based on the ratio = $\frac{\text{Model A Predicted Loss Cost}}{\text{Model B Predicted Loss Cost}}$; this is unintuitive and harder to explain (to non-actuaries.) Double Lift Plots compare where model A disagrees with model B most, since they are sorted based on the ratio $\frac{\text{Model A Predicted Loss Cost}}{\text{Model B Predicted Loss Cost}}$, so Double Lift Plots can be harder to interpret. Double Lift Plots do not provide information about actual loss dollars. Double lift plots can only be used to compare two models; for the other plots we can create one plot for each of several models in order to compare among these models.
(c) The loss ratio plots should be monotonically increasing. Neither loss ratio plot is good, but Model A is worse than Model B.
In the simple quintile plots: Model B does a better job at predictive accuracy, both plots are monotonically increasing, and Model B has a much greater vertical distance between the actuals in the first and last quantiles (which is good). Thus based on these simple quintile plots, Model B is preferred to Model A.
In the double lift plot: Model B more closely matches the actual than Model A does, particularly for the first and last quintiles.
In all three cases, the plot indicates that **one prefers Model B** to Model A.
<u>Comment</u>: One would apply these plots to a much larger set of data than the five observations shown in the question; normally one would not draw any conclusions based on such a small amount of data.
To create a loss ratio chart:
1. Sort the dataset based on the model prediction, in other words modeled loss ratios.
2. Group the data into quantiles with equal volumes of exposures.
3. Within each group, calculate the actual loss ratio.
Nevertheless, in part (a) the CAS also allowed plots where the data was sorted by modeled loss <u>costs</u> rather than modeled loss <u>ratios</u>. In that case, the plot will be the same for both models, since they rank the observations in the same order: 3, 2, 1, 4, 5.
The premium used to produce a Loss Ratio Plot should be at present rates, reflecting the current model.
"In a double lift chart, the first quantile contains those risks which Model A thinks are best relative to Model B. In other words, the first and last quantiles contain those risks on which Models A and B disagree the most (in percentage terms)."
Part (b) is not fully discussed in the syllabus reading. "The advantage of loss ratio charts over quantile plots and double lift charts is that they are simple to understand and explain. Loss ratios are the most commonly-used metric in determining insurance profitability, so all stakeholders should be able to understand these plots."

3.253. (a) For the logit link function, $\beta x = \ln(\frac{\mu}{1 - \mu}) \Leftrightarrow \mu = \frac{\exp[\beta x]}{1 + \exp[\beta x]}$. The logit link function is appropriate, since it maps the real line into the interval [0, 1]; probabilities are between 0 and 1.
(b) sensitivity = (true positives) / (all positives) = 72 / (72 + 162) = **30.8%**.
specificity = (true negatives) / (all negatives) = 1203 / (1203 + 63) = **95.0%**.
(c) The 25% threshold corresponds to the point: (1 - 0.950, 0.308). The ROC curve:



For a threshold of 100% the model always predicts no; there are no false negatives and thus 1 - specificity is zero. For a threshold of 100% the model never predicts yes; there are no true positives and thus the sensitivity is zero.
For a threshold of 0% the model never predicts no; there are no false negatives and thus 1 - specificity is one. For a threshold of 0% the model always predicts yes; the true positives are equal to all positives and thus the sensitivity is one.
A model with with no predictive power would follow the comparison line (line of equality).
A perfect model would be at (0, 1) in the upper lefthand corner; sensitivity = 1 and specificity = 1.
(d) The larger the average severity, the more worthwhile it is for the insurer to spend money to investigate cases of possible fraud. If claims are more severe, then the insurer will be more concerned about false negatives (cases where there is fraud but the modeled probability of fraud is below the threshold), than it would be about false positives (cases where there is not fraud but the modeled probability of fraud is above the threshold).
Therefore, **the more severe the claims, the lower the threshold that should be selected**.
Comment: The probit link function and the complementary log-log link function would also work in part (a). See An Introduction to Generalized Linear Models by Dobson and Barnett, not on the syllabus of this exam.

3.254. (a) $\frac{\exp[-8.4607 + 0.2714 + 0.7228 + 0.4311 \ln[200{,}000] - 0.0960 \ln[200{,}000]]}{\exp[-8.4607 + 0.2714 + 0.4311 \ln[200{,}000]]}$

$= \exp[0.7228 - 0.0960 \ln[200{,}000]\ ] =$ **0.6383**.

(b) In order to center AOI, we will divide the AOI by the base AOI of 200,000 prior to logging and including it in the model. The two forms of the model produce the same results.
For example, for Occupancy class 1, non-sprinklered property, with AOI = 200,000, the given model has: $\exp[-8.4607 + 0.4311 \ln[200{,}000]\ ]$.
With intercept β_0, the revised model would have for this same risk:
$\exp[\beta_0 + 0.4311 \ln[200{,}000/200{,}000]\ ] = \exp[\beta_0]$.
$\Rightarrow \exp[-8.4607 + 0.4311 \ln[200{,}000]\ ] = \exp[\beta_0]$.

$\Rightarrow \beta_0 = -8.4607 + 0.4311 \ln[200{,}000] =$ **-3.1987**.

(c) 1. If all continuous variables are divided by their base values prior to being logged and included in the model. then the intercept term after exponentiating yields the indicated frequency at the base case when all variables are at their base levels. This is both more intuitive and easier to interpret.
2. When terms are not centered, you can have unintuitive results. In the given example, the sprinkler coefficient is positive which can appear to indicate a higher frequency for sprinklered buildings than for non-sprinklered buildings. (However, when taking into account the interaction term, this is not true for values of log(AOI) for insured buildings.) This would not happen if AOI had been centered at its base level; the coefficients are more intuitive to understand when variables are centered.
3. With the AOI predictor in this form, the sprinklered coefficient has a more natural interpretation: it is the (log) sprinklered relativity for a risk with the base AOI.
Comment: See Section 5.6.2 of Generalized Linear Models for Insurance Rating.
The calculated ratio in part (a) does not depend on the occupancy class.